The Source Location Problem with Local 3-Vertex-Connectivity Requirements [⋆]

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Abstract

Let $G = (V, E)$ be a simple undirected graph with a set V of vertices and a set E of edges. Each vertex $v \in V$ has an integer valued demand $d(v) \geq 0$. The source location problem with vertex-connectivity requirements in a given graph G asks to find a set S of vertices with the minimum cardinality such that there are at least $d(v)$ vertex disjoint paths between S and each vertex $v \in V - S$. In this paper, we show that the problem with $d(v) \leq 3$, $v \in V$ can be solved in linear time. Moreover, we show that in the case where $d(v) \geq 4$ for some vertex $v \in V$, the problem is NP-hard.

Key words: undirected graph, source location problem, local vertex-connectivity, deficient set.

1 Introduction

Problems of selecting the best location of facilities in a given network to satisfy a certain property are called *location problems* [13]. Recently, the location

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problems with requirements measured by a network-connectivity were studied extensively [2,4,9–12,15–17].

Connectivity and/or flow-amount are very important factors in applications to control and design of multimedia networks. In a multimedia network, some vertices of the network, such as the so-called mirror servers, may have functions of offering the same services for users. Let us call a vertex that can offer the service i a *source*, and let S be a set of sources, where we can locate more than one source in a network. A user at vertex v can use a service i by communicating with at least one source s through a path between s and v (or a set of paths between S and v). The flow-amount (which is the capacity of paths between S and v) affects the maximum data amount that can be transmitted from S to a user at a vertex v. Also, the edge-connectivity or the vertex-connectivity between a source set S and a vertex v measures the robustness of the service against network failures. Actually, such connectivity and/or flow-amount between a vertex and a set of specified vertices was defined in some telephone company, considering design of a reliable telephone network with plural switching apparatuses [8]. Moreover, recently, not only location problems but also connectivity augmentation problems based on this connectivity have been studied [6,7,14].

In this paper, we consider the problem of finding the best location of a source set S under connectivity and/or flow-amount requirements from each vertex to a source set S. We introduce the *source location problem* which is formulated as follows.

Problem 1 (*Source location problem*)
Input : A graph $G = (V, E)$ with a set V of vertices and a set E of edges capacitated by nonnegative reals, a cost function $w : V \rightarrow R_+$ (where R_+ denotes the set of nonnegative reals), and a demand function $d : V \rightarrow R_+$.
Output : A vertex set $S \subseteq V$ such that $\psi(S, v) \geq d(v)$ for every vertex $v \in V - S$ and $\sum\{w(v) \mid v \in S\}$ is minimum, where $\psi(S, v)$ is a measurement based on the edge-connectivity, the vertex-connectivity or the flow-amount between S and a vertex v in a graph G.

For such measurements $\psi(S, v)$, one may consider the minimum capacity $\lambda(S, v)$ of an edge cut $C \subseteq E$ that separates v from S, the minimum size $\kappa(S, v)$ of a vertex cut $C \subseteq V - S - v$ that separates S and v, or the maximum number $\hat{\kappa}(S, v)$ of vertex-disjoint paths between S and v such that no two paths meet at the same vertex in S.

Source location problems with $\psi(S, v) = \lambda(S, v)$ in undirected graphs were treated by Tamura et al. [16,17], Ito et al. [11,12] and Arata et al. [2]. They gave polynomial time algorithms for uniform costs $w(v) = 1$, $v \in V$, while the problem with general costs $w(v)$, $v \in V$ is shown to be weakly NP-hard [2].

Ito et al. [10] considered the source location problem with uniform capacities, uniform costs, and demand $d(v) = k$ in digraphs, and showed that the problem can be solved in polynomial time if k is fixed.

Ito et al. treated the source location problem for undirected graphs with unit capacities, a measurement "$\kappa(S, v) \geq k$ and $\lambda(S, v) \geq l$ for all $v \in V - S$", and uniform costs $w(v) = 1$, $v \in V$ [9]. They presented an $O(m + n^2 + n\min\{m, ln\}\min\{l, n\})$ time algorithm for $k \leq 2$ and showed the NP-hardness of the problem for $k \geq 3$ even if $l = 0$, where $n = |V|$, $m = |\{\{u, v\} \mid (u, v) \in E\}|$.

Thus, the problems with $\psi(S, v) = \kappa(S, v)$ are intractable, but Nagamochi et al. [15] showed that for a given integer k, the problem with $\psi(S, v) = \hat{\kappa}(S, v)$ and $d(v) = k$ can be solved in polynomial time. For this problem, they gave an $O(\min\{k, \sqrt{n}\}nm)$ time algorithm for digraphs and an $O(\min\{k, \sqrt{n}\}kn^2)$ time algorithm for undirected graphs (notice that if $\psi() = \kappa()$ or $\psi() = \hat{\kappa}()$ then edge capacities are assumed to be unit without affecting the problem). Furthermore, they showed that the source location problem for a measurement "$\hat{\kappa}^+(S, v) \geq l$ and $\hat{\kappa}^-(S, v) \geq k$" in digraphs can be solved in polynomial time, where $\hat{\kappa}^+(S, v)$ (resp. $\hat{\kappa}^-(S, v)$) is the maximum number of vertex-disjoint directed paths from S to v (resp. from v to S) such that no two paths meet at the same vertex in S. However, for the problems with general demands, it is not known whether it can be solved in polynomial time or not.

In this paper, we consider the source location problem with $\psi(S, v) = \hat{\kappa}(S, v)$, uniform costs, demand $d(v) \in \{0, 1, \ldots, k\}$ in undirected graphs (we call this problem with local k-vertex-connectivity requirements kLSLP). By establishing a min-max formula for the 3LSLP, we give a linear time algorithm for solving 3LSLP. Moreover, we clear the border between NP-hard and polynomially solvable classes of kLSLP by showing that kLSLP is NP-hard for any fixed integer $k \geq 4$.

The rest of the paper is organized as follows. Some definitions and preliminaries are described in Section 2. Also in Section 2, we consider lower bounds on the optimal value to kLSLP and we state our main result that a min-max formula to 3LSLP is established and that 3LSLP can be solved in linear time. In Section 3, we describe an algorithm, called 3-LVC_SLP, for solving 3LSLP and prove its correctness. In Section 4, we show the NP-hardness of 4LSLP. Finally, we give some concluding remarks and future researches in Section 5.

2 Preliminaries

Let $G = (V, E)$ be a simple undirected graph with a set V of *vertices* and a set E of *edges*, where we denote $|V|$ by n and $|E|$ by m. A singleton set $\{x\}$ may be simply written as x, and "$\subset$" implies proper inclusion while "$\subseteq$" means "$\subset$" or "$=$". A vertex set and an edge set of graph G is denoted by $V(G)$ and $E(G)$, respectively. For a vertex subset $V' \subseteq V$, $G[V']$ means the subgraph induced by V'. For a vertex set $X \subseteq V$, $N_G(X)$ is defined as a set of all vertices in $V - X$ which are adjacent to some of vertices in X. A *partition* $\mathcal{X} = \{X_1, \ldots, X_p\}$ of the vertex set V means a family of nonempty mutually disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset V' of V.

By Menger's theorem, the following lemma holds (see Section 1 for the definition of $\hat{\kappa}(X, v)$).

Lemma 2 *For a vertex $v \in V$ and a vertex set $X \subseteq V - \{v\}$, $\hat{\kappa}(X, v) \geq k$ if and only if $|N_G(W)| \geq k$ for every vertex set $W \subseteq V - X$ with $v \in W$.* $\square$

In this paper, each vertex $v \in V$ has a *demand* $d(v)$ of nonnegative integer. A vertex set $S \subseteq V$ is called a *source set* if it satisfies

$$\hat{\kappa}(S, v) \geq d(v) \text{ for all vertices } v \in V - S, \tag{1}$$

and we call each vertex $v \in S$ a *source*.

Problem 3 (kLSLP)
Input : *An undirected graph $G = (V, E)$ and a demand function $d : V \rightarrow \{0, 1, \ldots, k\}$.*
Output : *A source set $S \subseteq V$ with the minimum cardinality.*

For a vertex set $X \subseteq V$, $d(X)$ denotes the maximum demand among all vertices in X, i.e., $d(X) = \max_{v \in X} d(v)$. A vertex subset $W \subseteq V$ with $d(W) > |N_G(W)|$ is called a *deficient set*. In what follows, we show some properties to derive a lower bound on the optimal value to kLSLP.

Lemma 4 *A vertex set $S \subseteq V$ satisfies $W \cap S \neq \emptyset$ for every deficient set W if and only if S is a source set.*

PROOF. Assume that a vertex set $S \subseteq V$ satisfies $W \cap S \neq \emptyset$ for every deficient set W. This implies that if $Y \subseteq V - S$, then Y is not deficient, and hence $|N_G(Y)| \geq d(Y)$. By Lemma 2, we then have that S satisfies (1).

Assume that $S \subseteq V$ is a source set and there is a deficient set W with $W \subseteq V - S$. Let $v^* \in W$ be a vertex with $d(v^*) = d(W)$. As $\hat{\kappa}(S, v^*) \leq |N_G(W)| < d(v^*)$ by Lemma 2, it follows that $\hat{\kappa}(S, v^*) < d(v^*)$. This contradicts the assumption that S is a source set. $\square$

For a vertex $v \in V$, a deficient set $W \subseteq V$ with $v \in W$ is called a *minimal deficient set with respect to $v \in V$*, if no vertex set $W' \subset W$ with $v \in W'$ is a deficient set. A minimal deficient set has the following property.

Lemma 5 (*i*) *For a vertex $v \in V$, every minimal deficient set W with respect to $v \in W$ satisfying $d(v) = d(W)$ induces a connected graph.* (*ii*) *Let W_i (resp. W_j) be a minimal deficient set with respect to a vertex v_i with $d(v_i) = d(W_i)$ (resp. a vertex v_j with $d(v_j) = d(W_j)$). Then if $W_i \cap W_j \neq \emptyset$ and $W_i - W_j \neq \emptyset \neq W_j - W_i$, then $W_i \cap N_G(W_j) \neq \emptyset \neq W_j \cap N_G(W_i)$.*

PROOF. (i) Assume that there exists a partition $\{W', W''\}$ of W such that $|N_{G[W]}(W')| = 0$. Without loss of generality, let $v \in W'$ and $d(W) = d(W')$. We have $d(W') = d(W) > |N_G(W)| \geq |N_G(W')|$. Hence W' is also a deficient set, which contradicts the minimality of W. (ii) Assume that $W_i \cap N_G(W_j) = \emptyset$ and $W_i \cap W_j \neq \emptyset$. Then $|N_{G[W_i]}(W_i \cap W_j)| = 0$, i.e, $G[W_i]$ is not connected, which contradicts (i). $\square$

Moreover, we characterize a vertex set $X \subseteq V$ that must include at least two sources.

Lemma 6 *Let S be a source set in G. If a vertex set $X \subseteq V$ satisfies one of the following conditions (a)-(c), then we have $|S \cap X| \geq 2$. If X satisfies one of the conditions (d)-(f), then we have $|S \cap X| \geq 3$.*
(*a*) $|N_G(X)| = 1$ *and* $|\{v \in X \mid d(v) \geq 3\}| \geq 2$.
(*b*) $|N_G(X)| = 1$ *and there exists a vertex set $X' \subset X$ with $|N_G(X')| = 1$, $d(X') \geq 2$, and $d(X - X') \geq 3$.*
(*c*) $X = V$ *and* $|\{v \in X \mid d(v) \geq 2\}| \geq 2$.
(*d*) $X = V$ *and* $|\{v \in X \mid d(v) \geq 3\}| \geq 3$.
(*e*) $X = V$ *and there exist two vertices $v_1, v_2 \in X$ with $v_1 \neq v_2$, $d(v_1) \geq 3$, and $d(v_2) \geq 3$, and a deficient set W with $W \cap \{v_1, v_2\} = \emptyset$.*
(*f*) $X = V$ *and there exist a vertex $v_1 \in X$ with $d(v_1) \geq 3$ and two deficient sets W_1, W_2 with $W_1 \cap W_2 = \emptyset$ and $v_1 \notin W_1 \cup W_2$.*

PROOF. (a) From $|N_G(X)| = 1$ and $d(X) \geq 3$, X is a deficient set and by Lemma 4, S contains a vertex $u \in X$. Now we have $|N_G(X - u)| \leq 2$ and $d(X - u) \geq 3$ by $|\{v \in X \mid d(v) \geq 3\}| \geq 2$. Hence $X - u$ is deficient and thus S also contains a source in $X - u$.

(b) This can be proved along a similar way as in the proof of (a), so we here omit its proof.

(c) Assume that $|S| \leq 1$. Then there exists a vertex $u \in V - S$ with $d(u) \geq 2$, by which we have $d(V - S) \geq 2$. As $|N_G(V - S)| \leq |S| \leq 1$, we have $V - S$ is deficient. As $(V - S) \cap S = \emptyset$, by Lemma 4 it follows that S is not a source set, a contradiction.

(d) This can be proved along a similar way as in the proof of (c), so we here omit its proof.

(e) Assume that $|S| \leq 2$. If $v_1 \notin S$ or $v_2 \notin S$, then $d(V - S) \geq 3$ and we have $|N_G(V - S)| \leq |S| \leq 2 < d(V - S)$. This contradicts the fact that S is a source set. So $S = \{v_1, v_2\}$, and hence by our assumption, $W \cap S = \emptyset$. As W is deficient, we have a contradiction.

(f) Assume that $|S| \leq 2$. If $v_1 \notin S$, then $d(V - S) \geq 3$ and we have $|N_G(V - S)| \leq |S| \leq 2 < d(V - S)$. This contradicts the fact that S is a source set. So $v_1 \in S$. From $W_1 \cap W_2 = \emptyset$ and $v_1 \notin W_1 \cup W_2$, we have $S \cap W_1 = \emptyset$ or $S \cap W_2 = \emptyset$, a contradiction. $\square$

Hence, the following lemma holds.

Lemma 7 *Let $\mathcal{W} = \{W_1, \ldots, W_p, W_{p+1}, \ldots, W_{p+a}, W_{p+a+1}, \ldots, W_{p+a+b}\}$ be a subpartition of V such that each W_i is a deficient set. Suppose that each W_i, $i = p+1, p+2, \ldots, p+a$ (resp. $i = p+a+1, p+a+2, \ldots, p+a+b$) satisfies Lemma 6(a) (resp. Lemma 6(b)). Let $f(\mathcal{W}) = p + 2a + 2b$. Then every source set S satisfies $|S| \geq f(\mathcal{W})$.* $\square$

Let $f(G) = \max\{f(\mathcal{W}) \mid \mathcal{W}$ is a family of deficient sets and a subpartition of $V\}$, where $f(\mathcal{W})$ is a function on $\mathcal{W}$ which is a subpartition of V and a family of deficient sets, as defined in Lemma 7. Let $g(G) = 2$ if G satisfies (c) and none of (d) – (f) in Lemma 6, $g(G) = 3$ if G satisfies (d), (e), or (f) in Lemma 6, and $g(G) = 0$ otherwise.

In this paper, we prove the following min-max theorem and we show in consequence that 3LSLP can be solved in linear time.

Theorem 8 *(i) For 3LSLP and a source set $S \subseteq V$, we have $\min |S| = \max\{f(G), g(G)\}$.*
(ii) For 3LSLP, a source set S^* with the minimum cardinality can be found in linear time, and in the case of $|S^*| \geq 4$, so can a family $\mathcal{W}$ of deficient sets with $f(\mathcal{W}) = f(G)$. $\square$

3 Linear Time Algorithm for 3LSLP

In this section, we give an algorithm, called 3-LVC_SLP, for 3LSLP. If a given graph is disconnected, then we can consider the problem separately for each connected component. Hence we suppose that G is a connected graph. Also assume that there exists a vertex $v \in V$ with $d(v) \geq 2$ since the problem with $d : V \to \{0, 1\}$ is trivial. Algorithm 3-LVC_SLP consists of two steps. We first describe the procedure of Step I in Section 3.1, analyze the properties of feasible solutions obtained in Step I in Section 3.2, and finally describe the procedure of Step II in Section 3.3.

3.1 Step I

Step I of algorithm 3-LVC_SLP starts from a source set $S := V$ and updates S greedily as follows.

Algorithm 3-LVC_SLP
Input: An undirected connected graph $G = (V, E)$ and a demand function $d : V \to \{0, 1, 2, 3\}$.
Output: A source set $S \subseteq V$ with the minimum cardinality which satisfies (1).

Step I
(I-0) Number vertices of V such as $d(v_1) \leq \cdots \leq d(v_n)$.
(I-1) Initialize $j := 1$, $S := V$, and $\mathcal{W} := \emptyset$.
(I-2) If $S - \{v_j\}$ satisfies (1) then let $S := S - \{v_j\}$. Otherwise select a minimal deficient set $W' \subseteq V - (S - \{v_j\})$ with respect to v_j, and let $\mathcal{W} := \mathcal{W} \cup \{W'\}$.
(I-3) If $j \neq n$, then $j := j + 1$ and go to Step (I-2).
(I-4) If $\mathcal{W}$ is a subpartition of V then output S and halt; otherwise go to Step II.

We first claim that in the case where $S - \{v_j\}$ does not satisfy (1) in Step I-2, there exists a minimal deficient set $W' \subseteq V - (S - \{v_j\})$ with respect to v_j. Before deleting v_j from S, S is feasible and hence by Lemma 4, every deficient set contains a source in S. On the other hand, $S - \{v_j\}$ is infeasible. Again by Lemma 4, there is a deficient set W' with $W' \cap (S - \{v_j\}) = \emptyset$. Therefore it follows that $W' \cap S = \{v_j\}$ and thus $W' \subseteq V - (S - \{v_j\})$.

Let $S = \{s_1, \ldots, s_p\}$ be the source set finally obtained after v_n is checked. Then we have a family of minimal deficient sets $W_1, \ldots, W_p$ such that W_i is a minimal deficient set with respect to source $s_i \in S$ for $i = 1, \ldots, p$.

If $\{W_1, \ldots, W_p\}$ is a subpartition of V, then S is optimal. This follows because every source set consists of at least p vertices by Lemma 7, and we have $|S| = p$.

Otherwise we go to Step II, in which we update S while preserving (1). For example, for two sources s_i and s_j with $W_i \cap W_j \neq \emptyset$, let $S := S - \{s_i, s_j\} \cup \{s'\}$ for some vertex $s' \in W_i \cap W_j$ and $\mathcal{W} := \mathcal{W} - \{W_i\}$. Repeat such operations until the current source set turns out to be optimal by using Lemmas 6 and 7.

3.2   Properties of a source set obtained by Step I

Assume that algorithm 3-LVC_SLP does not halt in Step I. Let $S_0 = \{s_1, \ldots, s_p\}$ be a source set obtained by Step I and $\mathcal{W}_0 = \{W_1, \ldots, W_p\}$ be a family of the corresponding minimal deficient sets. Since $\mathcal{W}_0$ is not a subpartition of V, we have $|S_0| \geq 2$.

Definition 1 (*i*) *For a source set S, we say that a deficient set W satisfies property (P1) with respect to S, if there is $s \in S$ such that $W \cap S = \{s\}$, $d(W) = d(s)$ and W is minimal with respect to s.*
(*ii*) *We say that a source set $S = \{s_1, \ldots, s_p\}$ and a family $\mathcal{W} = \{W_1, \ldots, W_p\}$ of deficient sets $W_i \subseteq V$ satisfy property (P2), if for each $W_i \in \mathcal{W}$, there is $s_i \in S$ such that $W_i \cap S = \{s_i\}$, $d(W_i) = d(s_i)$ and W_i is minimal with respect to s_i.* $\square$

Note that if a source set S and a family $\mathcal{W}$ of deficient sets satisfy property (P2), then each $W \in \mathcal{W}$ satisfies property (P1) with respect to S. We see that S_0 and $\mathcal{W}_0$ satisfy property (P2) by the following lemma.

Lemma 9 *The source set S_0 and the family $\mathcal{W}_0$ obtained by Step I satisfy property (P2).*

PROOF. At Step I-2, assume that v_j cannot be deleted. As S is a source set and W' is deficient, by Lemma 4 it follows that $W' \cap S = \{v_j\}$. Then all vertices in $W' - \{v_j\}$ have been already deleted, and $d(v_j) = \max\{d(v) \mid v \in W'\}$ holds by the sorting in Step I-0. Hence $W_i \cap S_0 = \{s_i\}$ and $d(W_i) = d(s_i)$ for $i = 1, \ldots, p$. $\square$

Lemma 10 *Let S be a source set and W_i be a minimal deficient set which satisfies property (P1) with respect to S. If $|S| \geq 2$, then we have $1 \leq |N_G(W_i)| \leq 2$.*

PROOF. $|N_G(W_i)| > 0$ clearly holds, since $|N_G(W_i)| = 0$ implies $V = W_i$, contradicting $|S| \geq 2$ and $|S \cap W_i| = 1$. As $d(W_i) \leq 3$, we have $|N_G(W_i)| \leq 2$. $\square$

Lemma 11 *Let a source set S and a family $\mathcal{W}$ of minimal deficient sets satisfy property (P2). If $W_i, W_j \in \mathcal{W}$, $i \neq j$ satisfy $W_i \cap W_j \neq \emptyset$ and $d(W_i) = d(W_j) = 2$, then $V = W_i \cup W_j$.*

PROOF. By Lemma 10 and $d(W_i) = d(W_j) = 2$, we have $|N_G(W_i)| = |N_G(W_j)| = 1$. By Lemma 5, we obtain $|N_G(W_i \cup W_j)| = 0$ and hence $W_i \cup W_j = V$. $\square$

By the following lemma, we see that if a source set S_0 obtained by Step I satisfies $|S_0| \leq 3$, then S_0 is optimal.

Lemma 12 *If $|S_0| \leq 3$, then S_0 is an optimal solution.*

PROOF. By Lemma 9, S_0 and $\mathcal{W}_0$ satisfy property (P2). From (P2), we have the property that $d(W_i) = d(s_i)$ for $i = 1, \ldots, p$. If $\mathcal{W}_0$ is a subpartition of V, then the statement follows by Lemma 7. Assume that $\mathcal{W}_0$ is not a subpartition of V. Hence $|S_0| \geq 2$. For each $W_i \in \mathcal{W}_0$, we have $1 \leq |N_G(W_i)| \leq 2$ from Lemma 10, and $d(W_i) > |N_G(W_i)|$ from the definition of a deficient set. Thus, $d(W_i) = d(s_i)$ implies that $d(s_i) \geq 2$. Hence if $|S_0| = 2$, then by Lemma 6(c) it follows that S_0 is optimal.

We consider the case where $|S_0| = 3$. Without loss of generality, assume that $d(s_1) \geq d(s_2) \geq d(s_3)$. If $d(s_i) = 2$ for all $i = 1, 2, 3$, then Lemma 11 and $|S_0| = 3$ imply that $\mathcal{W}_0$ is a subpartition of V, a contradiction. If $d(s_1) = d(s_2) = d(s_3) = 3$, then S_0 is optimal by Lemma 6(d). If $d(s_1) = d(s_2) = 3$ and $d(s_3) = 2$, then we have $W_3 \cap \{s_1, s_2\} = \emptyset$ by the property (P2), and hence by Lemma 6(e), it follows that S_0 is optimal. If $d(s_1) = 3$ and $d(s_2) = d(s_3) = 2$, then we have $s_1 \notin W_2 \cup W_3$ by the property (P2), and $W_2 \cap W_3 = \emptyset$ by $|S_0| = 3$ and Lemma 11. Hence in this case, by Lemma 6(f), it follows that S_0 is optimal. $\square$

Furthermore, by Lemmas 11 and 12, in the case of $d : V \to \{0, 1, 2\}$ we can prove that a solution obtained by Step I is optimal.

Lemma 13 *If $d : V \to \{0, 1, 2\}$, then S_0 is an optimal solution.*

PROOF. By Lemma 9, S_0 and $\mathcal{W}_0$ satisfy property (P2). The lemma follows if $\mathcal{W}_0$ is a subpartition of V. If this is not the case, then by Lemma 11, we have $|S_0| = 2$, from which S_0 is optimal by Lemma 12. $\square$

Hereafter, suppose that $|S_0| \geq 4$ and $\max\{d(v) \mid v \in V\} = 3$.

Definition 2 *A family $\mathcal{W}' = \{W_1, \ldots, W_t\} \subseteq \mathcal{W}$ ($t \geq 2$) of deficient sets is called a chain if it satisfies the following conditions.*
(a) $W_i \cap W_{i+1} \neq \emptyset$ for $i = 1, \ldots, t - 1$.
(b) $W_i \cap W_h = \emptyset$ for two distinct $i, h \in \{1, 2, \ldots, t\}$ with $2 \leq |i - h| \leq t - 2$. $\square$

Here we decompose S_0 to $S_0^\ell \subseteq S_0$, $\ell = 1, \ldots, q$ to clarify its structure as follows. First, we define a new graph H with the vertex set S_0. In H, (s_i, s_j) is an edge for $s_i, s_j \in S_0$ if and only if $W_i \cap W_j \neq \emptyset$ for the corresponding two deficient sets $W_i, W_j \in \mathcal{W}_0$. Then $S_0^\ell \subseteq S_0$, $\ell = 1, \ldots, q$ is defined as a connected component of the graph H. A family of deficient sets corresponding to the sources in S_0^ℓ is denoted by $\mathcal{W}_0^\ell$. Now in the following Lemma 14, we prove that each $\mathcal{W}_0^\ell$ with $|\mathcal{W}_0^\ell| \geq 2$ is a chain. If $\mathcal{W}_0^\ell$ is a chain, then we can observe from the definition of chains that $\mathcal{W}_0^\ell$ consists of two subpartitions of V. Hence, if each $\mathcal{W}_0^\ell$ with $|\mathcal{W}_0^\ell| \geq 2$ is a chain, then intuitively, the cardinality of S_0 is at most twice the optimal. Actually, in the sequel, for each $\mathcal{W}_0^\ell$, we will replace some two sources $s_i, s_j \in S_0^\ell$ satisfying $W_i \cap W_j \neq \emptyset$ with one vertex $s' \in W_i \cap W_j$ in order to attain an optimal solution.

Lemma 14 *Each $\mathcal{W}_0^\ell = \{W_1, \ldots, W_t\}$ with $t \geq 2$ is a chain. Moreover, if $t \geq 3$, then each $W_i \in \mathcal{W}_0^\ell$ with $i = 2, \ldots, t-1$ satisfies $N_G(W_i) = \{s_{i-1}, s_{i+1}\}$.* $\square$

Before proving Lemma 14, we give the following lemma.

Lemma 15 *Let S be a source set, and W_i and W_j be minimal deficient sets which satisfy property (P1) with respect to S and satisfy $W_i \cap S = \{s_i\}$ and $W_j \cap S = \{s_j\}$ with $s_i \neq s_j$. If $W_i \cap W_j \neq \emptyset$, then the following properties hold.*
(i) $G[W_i \cup W_j]$ contains at least two vertex-disjoint paths between s_i and s_j.
(ii) If $|W_i \cap N_G(W_j)| = 1$, then we have $W_i \cap N_G(W_j) = \{s_i\}$.

PROOF. (i) Since $G[W_i]$ and $G[W_j]$ are connected by Lemma 5 and $W_i \cap W_j \neq \emptyset$, there exists a path between two sources s_i and s_j in $G[W_i \cup W_j]$. Assume that (i) does not hold. Then there exists a partition $\{X_i, X_j, \{z\}\}$ of $W_i \cup W_j$ with $N_{G[W_i \cup W_j]}(X_i) = N_{G[W_i \cup W_j]}(X_j) = \{z\}$, $s_i \in X_i$, and $s_j \in X_j$. Suppose without loss of generality that $z \in W_i$. Then as $N_G(X_i) \cap X_j = \emptyset$, we have $N_G(W_i \cap X_i) - \{z\} \subseteq N_G(W_i) - X_j$. Now we have $X_j - W_i \neq \emptyset$ since $s_j \in X_j - W_i$ holds by $W_i \cap S = \{s_i\}$. Hence $N_G(W_i) \cap (X_j - W_i) \neq \emptyset$ holds since $G[W_i \cup W_j]$ is connected. Therefore, we have $|N_G(W_i \cap X_i)| = |N_G(W_i) - X_j| + |\{z\}| \leq |N_G(W_i) - X_j| + |N_G(W_i) \cap X_j| = |N_G(W_i)|$. Hence, $W_i \cap X_i$ is a

deficient set by $s_i \in W_i \cap X_i$ and $d(W_i) = d(s_i)$. As $z \in W_i - X_i$, $W_i \cap X_i \subset W_i$, which contradicts the minimality of W_i. (ii) Let $W_i \cap N_G(W_j) = \{z\}$. Assume that $z \neq s_i$. Then as $W_j \cap S = \{s_j\}$, we have $s_i \in W_i - W_j - N_G(W_j)$, and any path between s_i and s_j in $G[W_i \cup W_j]$ contains z. This contradicts (i). □

For proving Lemma 14, it suffices to show the following Lemma 16.

Lemma 16 *Let S be a source set with $|S| \geq 4$. Let $\mathcal{W}$ be a family of minimal deficient sets W_i satisfying property $(P1)$ with respect to S such that $S \cap W_i = \{s_i\}$. If there exist two distinct $W_h, W_j \in \mathcal{W} - \{W_i\}$ with $W_i \cap W_h \neq \emptyset \neq W_i \cap W_j$, then we have $N_G(W_i) = \{s_h, s_j\}$ (hence, the number of $W \in \mathcal{W} - \{W_i\}$ with $W_i \cap W \neq \emptyset$ is at most two).*

PROOF. For each $W_i \in \mathcal{W}$, we have $|N_G(W_i)| \in \{1, 2\}$ by Lemma 10. Denote $N_G(W_i)$ by $\{x_i, y_i\}$ (possibly $x_i = y_i$). Assume that there exist two distinct $W_h, W_j \in \mathcal{W} - \{W_i\}$ with $W_i \cap W_h \neq \emptyset \neq W_i \cap W_j$. By Lemma 5, we have $W_j \cap N_G(W_i) \neq \emptyset \neq W_h \cap N_G(W_i)$.

Without loss of generality, assume that $W_j \cap \{x_i, y_i\} = \{x_i\}$. Then by Lemma 15(ii), we have $x_i = s_j$. Since $s_j \notin W_h$ holds by the property (P1), we have $W_h \cap \{x_i, y_i\} = \{y_i\}$, from which we have $y_i = s_h$. If another $W_k \in \mathcal{W} - \{W_i, W_j, W_h\}$ satisfies $W_k \cap W_i \neq \emptyset$, then the property (P1) implies $s_k \in W_k - W_i$ and $N_G(W_i) \cap W_k = \{s_h, s_j\} \cap W_k = \emptyset$, from which $G[W_k]$ is not connected, contradicting Lemma 5.

Assume that $\{x_i, y_i\} \subseteq W_j$ and $\{x_i, y_i\} \subseteq W_h$. By Lemma 5, we have $W_j \cap W_h \neq \emptyset$ and $N_G(W_j) \cap W_i \neq \emptyset \neq N_G(W_h) \cap W_i$. If $V = W_i \cup W_j \cup W_h$, then we have $|S| = 3$ by the property (P1), which contradicts $|S| \geq 4$. Otherwise $N_G(W_i \cup W_j \cup W_h) \neq \emptyset$. Denote $N_G(W_j)$ by $\{x_j, y_j\}$ (possibly $x_j = y_j$). Without loss of generality, we can assume $y_j \in N_G(W_i \cup W_j \cup W_h)$ by $N_G(W_i) \subseteq W_j \cup W_h$. By Lemma 5, we have $x_j \in N_G(W_j) \cap W_i$. Then by Lemma 15(ii), we have $s_i = x_j$. Hence since the property (P1) implies that $s_i = x_j \notin W_h$, we have $W_h \cap N_G(W_j) = \emptyset$. This contradicts Lemma 5. □

Next, we give the following lemma about updating a source set.

Lemma 17 *Let S be a source set, and W_i and W_j minimal deficient sets which satisfy property $(P1)$ with respect to S and satisfy $W_i \cap S = \{s_i\}$ and $W_j \cap S = \{s_j\}$ with $s_i \neq s_j$. Suppose that $V - W_i - W_j \neq \emptyset$ and $W_i \cap W_j \neq \emptyset$. If there is no vertex set $X \subset V$ with*

$$W_i \cup W_j \subseteq X, \ X \cap S = \{s_i, s_j\}, \ |N_G(X)| = 1, \tag{2}$$

then we have $|S| \geq 3$, and $S' = (S - \{s_i, s_j\}) \cup \{s_{ij}\}$ is a source set for any vertex $s_{ij} \in W_i \cap W_j$.

PROOF. Assume that no vertex set $X \subseteq V$ satisfies (2). From $V - W_i - W_j \neq \emptyset$ and the connectedness of G, $|N_G(W_i \cup W_j)| \geq 1$ holds. Hence, if $|S| = 2$, then we have $S = \{s_i, s_j\}$ and for a vertex $x \in V - W_i - W_j$, $X = V - \{x\}$ would satisfy (2), a contradiction. Therefore we have $|S| \geq 3$. Moreover, we can assume that $|N_G(W_i \cup W_j)| = 2$, since if $|N_G(W_i \cup W_j)| = 1$, then $X = W_i \cup W_j$ would satisfy (2). By Lemma 5, $W_i \cap N_G(W_j) \neq \emptyset \neq W_j \cap N_G(W_i)$ holds, from which we have $|N_G(W_i)| = |N_G(W_j)| = 2$ by Lemma 10. Without loss of generality, we can assume $N_G(W_i) = \{x_i, y_i\}$, $N_G(W_j) = \{x_j, y_j\}$, $\{x_i, x_j\} \subseteq V - W_i - W_j$, $y_i \in W_j$ and $y_j \in W_i$. By Lemma 15(ii), $y_j = s_i$ and $y_i = s_j$ hold. Then by $\{s_i\} = N_G(W_j) \cap W_i$, $\{s_j\} = N_G(W_i) \cap W_j$, and the connectedness of $G[W_i]$ and $G[W_j]$, we see that $N_G(W_i \cap W_j) = \{s_i, s_j\}$ holds and all of $G[(W_i \cap W_j) \cup \{s_i\}]$, $G[(W_i \cap W_j) \cup \{s_j\}]$, and $G[(W_i \cap W_j) \cup \{s_i, s_j\}]$ are connected. Let s_{ij} be an arbitrary vertex in $W_i \cap W_j$.

Assume by contradiction that $S' = (S - \{s_i, s_j\}) \cup \{s_{ij}\}$ is not a source set. Then there is a deficient set W' with $S' \cap W' = \emptyset$, i.e., $s_{ij} \notin W'$ and $S \cap W' = \{s_i, s_j\} \cap W' \neq \emptyset$. Hence, we have $N_G(W') \cap (W_i \cap W_j) \neq \emptyset$ since $G[(W_i \cap W_j) \cup \{s_i, s_j\}]$ is connected. Let $N_G(W') = \{x', y'\}$ (possibly $x' = y'$), where $x' \in W_i \cap W_j$.

We consider the case where $\{s_i, s_j\} \subseteq W'$. If $N_G(W') \subseteq W_i \cap W_j$, then we have $V - (W_i \cap W_j) \subseteq W'$ and $S = \{s_i, s_j\}$, contradicting $|S| \geq 3$ (note that $\{s_i, s_j\} \subseteq W'$, $N_G(W') - (W_i \cap W_j) = \emptyset$, and the connectedness of G, $G[W_i]$, and $G[W_j]$ imply $V - (W_i \cap W_j) \subseteq W'$). Hence $N_G(W') - (W_i \cap W_j) \neq \emptyset$. Let $y' \in N_G(W') - (W_i \cap W_j)$. If $y' \in V - W_i - W_j$, then $X = W' \cup W_i \cup W_j$ satisfies (2), a contradiction (note that $\{s_i, s_j\} \subseteq W'$, $N_G(W') - (W_i \cup W_j) = \{y'\}$, and the connectedness of G, $G[W_i]$, and $G[W_j]$ imply $(W_i - W_j) \cup (W_j - W_i) \subseteq W'$ and $N_G(W' \cup W_i \cup W_j) = \{y'\}$). Without loss of generality, assume $y' \in W_i - W_j$. We also assume that $x_i \notin W'$, since $x_i \in W'$ would contradict $|S| \geq 3$ along a similar way as in the case where $N_G(W') \subseteq W_i \cap W_j$. Then $\{s_i, s_j\} \subseteq W'$, $N_G(W') \subseteq W_i$, $x_i \notin W'$, and the connectedness of G and $G[W_j]$ imply $W_j - W_i \subseteq W'$ and $N_G(W' \cup (W_i \cap W_j)) = \{y'\}$. Hence $W_i' = (W_i \cap W') \cup (W_i \cap W_j)$ satisfies $W_i' \subset W_i$, $N_G(W_i') = \{s_j, y'\}$ and $s_i \in W_i \cap W'$, which contradicts the minimality of W_i.

We consider the case where $\{s_i, s_j\} \cap W' = \{s_i\}$ without loss of generality. We have $\{s_j, s_{ij}\} \cap W' = \emptyset$. We claim that $|N_G(W') \cap ((W_i \cap W_j) \cup \{s_j\})| \geq 2$. Assume by contradiction that $|N_G(W') \cap ((W_i \cap W_j) \cup \{s_j\})| = 1$ (note $|N_G(W') \cap (W_i \cap W_j)| > 0$ as seen above). Since there exist at least two vertex-disjoint paths between s_i and s_j in $G[(W_i \cap W_j) \cup \{s_i, s_j\}]$ by Lemma 15(i), we have $\{s_j\} = N_G(W') \cap ((W_i \cap W_j) \cup \{s_j\})$. $s_{ij} \in W_i \cap W_j - W'$ means

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that there exists a vertex set $Y \subseteq W_i \cap W_j$ with $s_{ij} \in Y$ and $N_G(Y) = \{s_j\}$, which contradicts the fact that $G[(W_i \cap W_j) \cup \{s_i\}]$ is connected. Thus, we have $|N_G(W') \cap ((W_i \cap W_j) \cup \{s_j\})| \geq 2$. As $|N_G(W')| \leq 2$, $N_G(W') \subseteq (W_i \cap W_j) \cup \{s_j\}$ holds. This implies $W_i - W_j \subseteq W'$, $N_G(W' \cup (W_i \cap W_j)) = \{s_j\}$, and $N_G(W_i \cup W_j \cup W') = \{x_j\}$. Therefore $X = W_i \cup W_j \cup W'$ satisfies (2), a contradiction. $\square$

Next, for each chain $\mathcal{W}_0^\ell$ with $|\mathcal{W}_0^\ell| \geq 3$, we consider sufficient conditions which allow us to update a source set by using Lemma 17. Intuitively, we will show that the number of sources in each chain $\mathcal{W}_0^\ell$ can be reduced to almost the half by pairing up all minimal deficient sets in $\mathcal{W}_0^\ell$ and applying Lemma 17 to each pair. We define the following three types of chain.

Definition 3 *Let a source set S and a family $\mathcal{W}$ of minimal deficient sets satisfy property (P2). A chain $\mathcal{W}^\ell = \{W_1, \ldots, W_t\} \subseteq \mathcal{W}$ $(t \geq 3)$ is said to be of type (A) if it satisfies the following conditions (i) and (ii), of type (B) if it satisfies neither (i) nor (ii), and of type (C) otherwise. (In the case of type (C), assume that $\mathcal{W}^\ell$ satisfies (i) and does not satisfies (ii) without loss of generality.) Then*
(i) There exists $Z_1 \subseteq V$ with $W_1 \cup W_2 \subseteq Z_1$, $N_G(Z_1) = \{s_3\}$, and $Z_1 \cap S = \{s_1, s_2\}$.
(ii) There exists $Z_t \subseteq V$ with $W_{t-1} \cup W_t \subseteq Z_t$, $N_G(Z_t) = \{s_{t-2}\}$, and $Z_t \cap S = \{s_{t-1}, s_t\}$.
(Note that if $t \geq 3$, then we have $N_G(W_2) = \{s_1, s_3\}$ and $N_G(W_{t-1}) = \{s_{t-2}, s_t\}$ by Lemma 14.)

Lemma 18 *Let a source set S and a family $\mathcal{W}$ of minimal deficient sets satisfy property (P2) and $|S| \geq 4$. Let $\mathcal{W}^\ell = \{W_1, \ldots, W_t\} \subseteq \mathcal{W}$ $(t \geq 3)$ be a chain of type (A). Then, $S \subseteq \bigcup_{W \in \mathcal{W}^\ell} W$. Moreover, for any set $\{s_r' \in W_{2r} \cap W_{2r+1} \mid r = 1, \ldots, \lfloor t/2 \rfloor - 1\}$ of vertices, $S' = (S - \{s_2, s_3, \ldots, s_{2\lfloor t/2 \rfloor - 1}\}) \cup \{s_r' \in W_{2r} \cap W_{2r+1} \mid r = 1, \ldots, \lfloor t/2 \rfloor - 1\}$ is a source set.*

PROOF. First, by Lemma 16, $N_G(W_i) = \{s_{i-1}, s_{i+1}\}$ holds for each W_i with $i = 2, \ldots, t-1$. By the property of type (A), we have $V = Z_1 \cup Z_t \cup (\bigcup_{W \in \mathcal{W}^\ell} W)$, and hence $S \subseteq \bigcup_{W \in \mathcal{W}^\ell} W$. We have $t \geq 4$ from $|S| \geq 4$. We prove the lemma as follows. Let $S_0' = S$. We show that for each $r = 1, \ldots, \lfloor t/2 \rfloor - 1$, there is no vertex set X which satisfies (2) of Lemma 17 for the source set S_{r-1}' and $\{W_{2r}, W_{2r+1}\}$, and $S_r' := (S_{r-1}' - \{s_{2r}, s_{2r+1}\}) \cup \{s_r'\}$ is also a source set for an arbitrary vertex $s_r' \in W_{2r} \cap W_{2r+1}$.

As $1 \leq r \leq \lfloor t/2 \rfloor - 1$, we have $\{W_{2r}, W_{2r+1}\} \cap \{W_1, W_t\} = \emptyset$ for each $\{W_{2r}, W_{2r+1}\}$. To show that there is no vertex set X satisfying (2) for $\{W_{2r}, W_{2r+1}\}$ and S_{r-1}', it suffices to prove that in the graph G_r obtained from G by

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contracting $W_{2r} \cup W_{2r+1}$ to a vertex w_r, there exist two mutually vertex-disjoint paths from w_r to two distinct vertices in $S'_{r-1} - W_{2r} - W_{2r+1}$ in G_r. Now we have $N_{G_r}(w_r) = \{s_{2r-1}, s_{2r+2}\}$. By the construction of S'_{r-1}, $s_{2r+2} \in S'_{r-1}$. Moreover, there exists a path P' from s_{2r-1} to a vertex s'_{r-1} in W_{2r-1} (let $s_0' = s_1$ in the case of $r = 1$). Thus, we obtain two paths $\{(w_r, s_{2r+2})\}$ and $\{(w_r, s_{2r-1})\} \cup P'$. Hence, by Lemma 17, for an arbitrary vertex $s'_r \in W_{2r} \cap W_{2r+1}$, $S'_r := (S'_{r-1} - \{s_{2r}, s_{2r+1}\}) \cup \{s'_r\}$ is a source set. $\quad\square$

Lemma 19 *Let S be a source set with $|S| \geq 4$. Let $\mathcal{W}^\ell = \{W_1, \ldots, W_t\}$ ($t \geq 3$) be a chain of type (B) such that each W_i satisfies property $(P1)$ with respect to S and satisfies $W_i \cap S = \{s_i\}$. If $t = 4$ and $S - (\bigcup_{i=1}^4 W_i) = \emptyset$, then for any vertex $s'_1 \in W_1 \cap W_2$, $S' = (S - \{s_1, s_2\}) \cup \{s'_1\}$ is a source set. Otherwise, for any set $\{s'_r \in W_{2r-1} \cap W_{2r} \mid r = 1, \ldots, \lfloor t/2 \rfloor\}$ of vertices, $S' = (S - \{s_1, s_2, \ldots, s_{2\lfloor t/2 \rfloor}\}) \cup \{s'_r \in W_{2r-1} \cap W_{2r} \mid r = 1, \ldots, \lfloor t/2 \rfloor\}$ is a source set.*

PROOF. First, we consider the case where $t \neq 4$ or $S - (\bigcup_{W \in \mathcal{W}^\ell} W) \neq \emptyset$. Let $W^* = \bigcup_{W \in \mathcal{W}^\ell} W$. Since $\mathcal{W}^\ell$ is a chain of type (B), we have $N_G(W_1) - W^* \neq \emptyset$. Let $N_G(W_1) = \{x_1, y_1\}$, $x_1 \notin W^*$, and $y_1 \in W_2$. There exists a path P_1 from x_1 to a vertex s^* in $S - (W_1 \cup W_2 \cup W_3)$ which goes through only vertices in $V - (W_1 \cup W_2 \cup W_3)$ (this is possible by $t \geq 3$ and $|S| \geq 4$). Similarly, we have $N_G(W_t) - W^* \neq \emptyset$ where $N_G(W_t) = \{x_t, y_t\}$, $x_t \notin W^*$, and $y_t \in W_{t-1}$, and there exists a path P_t from x_t to a vertex s^{**} in $S - (W_{t-2} \cup W_{t-1} \cup W_t)$ which goes through only vertices in $V - (W_{t-2} \cup W_{t-1} \cup W_t)$. We can choose s^{**} such that $s^{**} \in S - W^*$ if $S - W^* \neq \emptyset$ holds, or $s^{**} = s_1$ if $S - W^* = \emptyset$ holds (note that this is possible by the definition of type (B)). Along a similar way as in the proof of Lemma 18, let $S'_0 = S$, and we show that for each $r = 1, \ldots, \lfloor t/2 \rfloor$, there is no vertex set X which satisfies (2) of Lemma 17 for the source set S'_{r-1} and $\{W_{2r-1}, W_{2r}\}$, and $S'_r := (S'_{r-1} - \{s_{2r-1}, s_{2r}\}) \cup \{s'_r\}$ is also a source set for an arbitrary vertex $s'_r \in W_{2r-1} \cap W_{2r}$. For this, we prove that there exist two mutually vertex-disjoint paths from w_r to distinct two vertices in $S'_{r-1} - W_{2r-1} - W_{2r}$ in the graph G_r obtained from G by contracting $W_{2r-1} \cup W_{2r}$ to a vertex w_r. Now, by Lemmas 15(ii) and 16, we have $N_{G_1}(w_1) = \{x_1, s_3\}$ and $N_{G_r}(w_r) = \{s_{2r-2}, s_{2r+1}\}$ for $2 \leq r \leq \lfloor t/2 \rfloor - 1$. We consider the case where $r \leq \lfloor t/2 \rfloor - 1$. Then, by the construction of S'_{r-1}, we obtain $s_{2r+1} \in S'_{r-1} \cap N_{G_r}(w_r)$. There exists a path P' from s_{2r-2} to a vertex s'_{r-1} in W_{2r-2} if $r \geq 2$ holds, and a path P_1 from x_1 to s^* if $r = 1$ holds. Thus, we obtain two paths $\{(w_r, s_{2r+1})\}$, and $\{(w_r, s_{2r-2})\} \cup P'$ if $r \geq 2$, or $\{(w_r, x_1)\} \cup P_1$ if $r = 1$. In the case where $r = \lfloor t/2 \rfloor$ holds and t is odd, as $N_{G_{\lfloor t/2 \rfloor}}(w_{\lfloor t/2 \rfloor}) = \{s_{2\lfloor t/2 \rfloor - 2}, s_{2\lfloor t/2 \rfloor + 1} = s_t\}$, we can see this property along a similar way as in the above case. Assume that $r = \lfloor t/2 \rfloor$ holds and t is even. Then we have $N_{G_{t/2}}(w_{t/2}) = \{s_{t-2}, x_t\}$. In the case of $S - W^* \neq \emptyset$, P_t does not share a vertex with the path P' from s_{t-2} to $s'_{t/2-1}$, since the path P_t from x_t to s^{**} does not contain any vertex in W_{t-2}. Assume $S - W^* = \emptyset$. As $t \neq 4$, we have $t \geq 6$, and hence $W_2 \neq W_{t-2}$. Therefore we can replace P_t with a

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path P'_t from x_t to s'_1 which contains no vertex in $W_{t-2} \cup W_{t-1} \cup W_t$. We see that P' and P'_t share no vertex except $w_{t/2}$ in $G_{t/2}$. Therefore, by Lemma 17, $S' = S'_{\lfloor t/2 \rfloor}$ is a source set.

In the case of $t = 4$ and $S - (\bigcup_{i=1}^4 W_i) = \emptyset$, we can prove the lemma along a similar way as in the case of $r = 1$. $\quad\square$

Along a similar way as in Lemmas 18 and 19, we can prove the following.

Lemma 20 *Let S be a source set with $|S| \geq 4$. Let $\mathcal{W}^\ell = \{W_1, \ldots, W_t\}$ ($t \geq 3$) be a chain of type (C) such that each W_i satisfies property $(P1)$ with respect to S and satisfies $W_i \cap S = \{s_i\}$. Then for any set $\{s'_r \in W_{2r} \cap W_{2r+1} \mid r = 1, \ldots, \lfloor (t+1)/2 \rfloor - 1\}$ of vertices, $S' = (S - \{s_2, s_3, \ldots, s_{2\lfloor (t+1)/2 \rfloor - 1}\}) \cup \{s'_r \in W_{2r} \cap W_{2r+1} \mid r = 1, \ldots, \lfloor (t+1)/2 \rfloor - 1\}$ is a source set. $\quad\square$*

Finally, we give lower bounds on the number of sources contained in each chain $\mathcal{W}^\ell_0$.

Lemma 21 *Let a source set S and a family $\mathcal{W}$ of minimal deficient sets satisfy property $(P2)$ and $|S| \geq 4$. Let $\mathcal{W}^\ell = \{W_1, \ldots, W_t\}$ ($t \geq 3$) be a chain, and $W^* = \bigcup_{W \in \mathcal{W}^\ell} W$. For any source set S', the following properties hold.*
(i) If $\mathcal{W}^\ell$ be of type (A), then we have $|S' \cap W^*| \geq \lceil t/2 \rceil + 1$.
(ii) If $\mathcal{W}^\ell$ be of type (B), and we have $t \neq 4$ or $S - W^* \neq \emptyset$, then we have $|S' \cap W^*| \geq \lceil t/2 \rceil$.
(iii) If $\mathcal{W}^\ell$ be of type (B) and $t = 4$ and $S - W^* = \emptyset$, then we have $|S' \cap V| \geq 3$.
(iv) If $\mathcal{W}^\ell$ be of type (C), then we have $|S' \cap W^*| \geq \lceil (t+1)/2 \rceil$.

PROOF. (i) Define $\mathcal{W}' = \{Z_1, W_4\}$ in the case of $t = 4$, $\mathcal{W}' = \{Z_1, Z_t\}$ in the case of $t = 5$ or $t = 6$, and $\mathcal{W}' = \{Z_1, Z_t\} \cup \{W_{2r} \mid r = 2, 3, \ldots, \lceil t/2 \rceil - 2\}$ in the case of $t \geq 7$. From the definition of a chain, $\mathcal{W}'$ is a subpartition of V. Since Z_1 satisfies Lemma 6(b) by $d(s_1) \geq 2$ and $d(s_2) = 3$, we have $|S' \cap Z_1| \geq 2$. Similarly, since $d(s_t) \geq 2$ and $d(s_{t-1}) = 3$ hold, we have $|S' \cap Z_t| \geq 2$. Hence $|S' \cap W^*| \geq \lceil t/2 \rceil + 1$.

(ii) From the definition of a chain, in the cases where t is even or $W_1 \cap W_t = \emptyset$ holds, a set $\{W_{2r-1} \mid r = 1, 2, \ldots, \lceil t/2 \rceil\}$ is a subpartition of V, so we have $|S' \cap W^*| \geq \lceil t/2 \rceil$. We consider the case where t is odd and $W_1 \cap W_t \neq \emptyset$ holds. By Lemma 16, for any vertex $v \in W^*$, we have $|\{W_i \in \mathcal{W}^\ell \mid v \in W_i\}| \leq 2$. Since we have $W_i \cap S' \neq \emptyset$ for each deficient set $W_i \in \mathcal{W}^\ell$, $|S' \cap W^*| \geq \lceil t/2 \rceil$ must hold.

(iii) From the property of type (B), we see $|N_G(W_i)| = 2$ ($i = 1, 2, 3, 4$). Hence $|\{v \in V \mid d(v) = 3\}| \geq 3$. By Lemma 6(d), we have $|S' \cap V| \geq 3$.

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(iv) Define $\mathcal{W}' = \{Z_1\}$ in the case of $t = 3$, and $\mathcal{W}' = \{Z_1\} \cup \{W_{2r} \mid r = 2, 3, \ldots, \lceil (t-1)/2 \rceil\}$ in the case of $t \geq 4$. From the definition of a chain, $\mathcal{W}'$ is a subpartition of V. Since Z_1 satisfies Lemma 6(b) by $d(s_1) \geq 2$ and $d(s_2) = 3$, we have $|S' \cap Z_1| \geq 2$. Hence, we have $|S' \cap W^*| \geq 2 + (\lceil (t-1)/2 \rceil - 1) = \lceil (t+1)/2 \rceil$. $\quad\square$

3.3 Step II

Now we describe the procedure of Step II of algorithm 3-LVC_SLP based on the properties given in Section 3.2.

Step II
(II-0) Let $S := S_0$. If $|S| \leq 3$, then output S as an optimal solution and halt.
(II-1) For each chain $\mathcal{W}_0^\ell = \{W_1, \ldots, W_t\}$ $(l = 1, \ldots, q)$ which is defined in Section 3.2, do the following operations.
(II-1-0) In the case of $t = 2$, if there is no vertex set X which satisfies (2) for S and $\{W_1, W_2\}$, then let $S := (S - \{s_1, s_2\}) \cup \{s'\}$ for some vertex $s' \in W_1 \cap W_2$, and $\mathcal{W} := \mathcal{W} - \{W_2\}$. If such X exists, then let W_X be a vertex set satisfying (2) which is inclusionwise minimal subject to this property, and $\mathcal{W} := (\mathcal{W} - \{W_1, W_2\}) \cup \{W_X\}$.
(II-1-1) In the case where $\mathcal{W}_0^\ell$ is of type (A), according to Lemma 18, let $S_1' := \{s_2, s_3, \ldots, s_{2\lfloor t/2 \rfloor - 1}\}$, $S_2' := \{s_r' \in W_{2r} \cap W_{2r+1} \mid r = 1, \ldots, \lfloor t/2 \rfloor - 1\}$. Let $S := (S - S_1') \cup S_2'$ and $\mathcal{W} := (\mathcal{W} - \{W \mid W \in \mathcal{W}_0^\ell\}) \cup \{(\bigcup_{W \in \mathcal{W}_0^\ell} W)\}$.
(II-1-2) In the case where $\mathcal{W}_0^\ell$ is of type (B) and we have $t = 4$ and $S - (\bigcup_{W \in \mathcal{W}_0^\ell} W) = \emptyset$, then according to Lemma 19, let $S := (S - \{s_1, s_2\}) \cup \{s_1' \in W_1 \cap W_2\}$ and $\mathcal{W} := \{V\}$.
(II-1-3) In the case where $\mathcal{W}_0^\ell$ is of type (B) and we have $t \neq 4$ or $S - (\bigcup_{W \in \mathcal{W}_0^\ell} W) \neq \emptyset$, then according to Lemma 19 let $S_1' := \{s_1, s_2, \ldots, s_{2\lfloor t/2 \rfloor}\}$ and let $S_2' := \{s_r' \in W_{2r-1} \cap W_{2r} \mid r = 1, \ldots, \lfloor t/2 \rfloor\}$. Let $S := (S - S_1') \cup S_2'$ and $\mathcal{W} := (\mathcal{W} - \{W \mid W \in \mathcal{W}_0^\ell\}) \cup \{(\bigcup_{W \in \mathcal{W}_0^\ell} W)\}$.
(II-1-4) In the case where $\mathcal{W}_0^\ell$ is of type (C), according to Lemma 20, let $S_1' := \{s_2, s_3, \ldots, s_{2\lfloor (t+1)/2 \rfloor - 1}\}$ and $S_2' := \{s_r' \in W_{2r} \cap W_{2r+1} \mid r = 1, \ldots, \lfloor (t+1)/2 \rfloor - 1\}$. Let $S := (S - S_1') \cup S_2'$ and $\mathcal{W} := (\mathcal{W} - \{W \mid W \in \mathcal{W}_0^\ell\}) \cup \{(\bigcup_{W \in \mathcal{W}_0^\ell} W)\}$.
(III) Output the resulting source set S^* and halt.

By Lemmas 17, 18, 19, and 20, a set S^* of sources obtained by this algorithm is a source set. We prove the correctness of the algorithm by showing that S^* is optimal. First, the following property holds for a deficient set W_X obtained by Step II-1-0.

Lemma 22 *Let W_X be a deficient set with $W_1 \cup W_2 \subseteq W_X$ which is obtained by Step II-1-0, and $N_G(W_X) = \{w_X\}$.*

(*i*) W_X *satisfies Lemmas 6(a) or 6(b).*
(*ii*) *If* $w_X \notin N_G(W_1 \cup W_2)$*, then we have* $|N_G(W_1 \cup W_2)| = 2$*, and there exists a simple cycle* C *which contains* s_1*,* s_2*,* x_1*,* x_2*, and* w_X *in* $G[W_X \cup \{w_X\}]$*, where* $\{x_1, x_2\} = N_G(W_1 \cup W_2)$*.*

PROOF. (i) If $d(s_1) = d(s_2) = 2$, then we have $V = W_1 \cup W_2$ by Lemma 11, contradicting $|S_0| \geq 4$. Hence let $d(s_1) = 3$. If $d(s_2) = 3$, then W_X satisfies Lemma 6(a). If $d(s_2) = 2$, then we have $|N_G(W_1 \cup W_2)| = 1$, and hence $W_X = W_1 \cup W_2$, from which W_X satisfies Lemma 6(b) (note that $s_1 \notin W_2$ holds since W_2 satisfies property (P1) with respect to S_0).

(ii) If $|N_G(W_1 \cup W_2)| = 1$, then we have $W_X = W_1 \cup W_2$, which contradicts $w_X \notin N_G(W_1 \cup W_2)$. Hence $|N_G(W_1 \cup W_2)| = 2$. Since $W_1 \cap N_G(W_2) \neq \emptyset \neq W_2 \cap N_G(W_1)$ holds by Lemma 5, we have $|N_G(W_1)| = |N_G(W_2)| = 2$. Let $N_G(W_i) = \{x_i, y_i\}$, where $x_i \in N_G(W_1 \cup W_2)$ and $y_i \in W_1 \cup W_2$ ($i = 1, 2$). We have $y_1 = s_2$ and $y_2 = s_1$ by Lemma 15(ii). Hence there exists a simple path P between x_1 and x_2 which contains s_1 and s_2 in $G[W_1 \cup W_2 \cup \{x_1, x_2\}]$. If there exist two mutually vertex-disjoint paths P_1 and P_2 such that P_i, $i = 1, 2$ connects x_i and w_X in $G[W_X \cup \{w_X\} - (W_1 \cup W_2)]$, then $P_1 \cup P_2 \cup P$ is a simple cycle which satisfies the statement (ii) of this lemma (note that $x_1 \neq x_2$ holds by $|N_G(W_1 \cup W_2)| = 2$). If there are no such paths P_1 and P_2, then there exists a subpartition $\{Y_1, Y_2, \{z\}\}$ of $W_X \cup \{w_X\} - (W_1 \cup W_2)$ with $w_X \in Y_1$, $\{x_1, x_2\} \subseteq Y_2$, and $N_{G'}(Y_1) = N_{G'}(Y_2) = \{z\}$ in $G' = G[W_X \cup \{w_X\} - (W_1 \cup W_2)]$, from which we can see that $Y_2 \cup W_1 \cup W_2$ contradicts the minimality of W_X. □

Lemma 23 *A source set* S^* *obtained by algorithm 3-LVC_SLP is an optimal solution.*

PROOF. The case of $|S_0| \leq 3$ has been proved in Lemma 12. We consider the case where $|S_0| \geq 4$. Let $\mathcal{W}^*$ be a family of deficient sets obtained by algorithm 3-LVC_SLP. It suffices to show that $\mathcal{W}^*$ is a subpartition of V, since if $\mathcal{W}^*$ is a subpartition of V, then $|W \cap S^*|$ is equal to lower bounds shown in Lemmas 7, 21, and 22 for each $W \in \mathcal{W}^*$.

Assume by contradiction that $\mathcal{W}^*$ is not a subpartition of V. By the construction of a chain $\mathcal{W}_0^\ell \subseteq \mathcal{W}_0$, a family $\{\bigcup_{W \in \mathcal{W}_0^\ell} W \mid l = 1, \ldots, q\}$ is a subpartition of V. Hence, there is a deficient set $W_X \supseteq W_1 \cup W_2$ obtained from some two deficient sets $W_1, W_2 \in \mathcal{W}_0$ at Step II-1-0 such that $w_X \notin N_G(W_1 \cup W_2)$ for $N_G(W_X) = \{w_X\}$, and $W_X \cap W' \neq \emptyset$ for some $W' \in \mathcal{W}^* - \{W_X\}$.

We claim that no $W_i \in \mathcal{W}_0 - \{W_1, W_2\}$ satisfies $W_i \cap W_X = \emptyset$. Assume by contradiction that there is a minimal deficient set $W_i \cap W_X \neq \emptyset$ for some $W_i \in \mathcal{W}_0 - \{W_1, W_2\}$. Note that $W_i \cap W_1 = W_i \cap W_2 = \emptyset$ holds from the construction of a chain. Also note that $G[W_X]$ is connected from the minimality of W_X.

Hence $N_G(W_i) \cap W_X \neq \emptyset$. Now $W_i - W_X \neq \emptyset$ holds, since $W_i \subseteq W_X$ would contradict the construction of W_X and the fact that W_i always contains a source s' with $s_1 \neq s' \neq s_2$. Hence, Lemma 5 implies $w_X \in W_i$. If $N_G(W_i) \subseteq W_X$, then we see $V = W_X \cup W_i$, which contradicts $|S_0| \geq 4$. Hence we can assume that $N_G(W_i) = \{x_i, y_i\}$ for $x_i \in V - W_X$ and $y_i \in W_X$. As $W_i \cap (W_1 \cup W_2) = \emptyset$, we can assume without loss of generality that $s_1 \in W_X - W_i - \{y_i\}$. Then any path connecting w_X and s_1 in $G[W_X \cup \{w_X\}]$ contains the vertex y_i, which contradicts Lemma 22(ii).

Hence we see that $W' = W'_X$ is also obtained from some two sets $W_3, W_4 \in \mathcal{W}_0$ in Step II-1-0, and moreover, we have $w'_X \notin N_G(W_3 \cup W_4)$, where $\{w'_X\} = N_G(W'_X)$. Since $G[W_X]$ is connected, we have $w'_X \in W_X$. From the construction of W_X, we have $\{s_3, s_4\} \subseteq W'_X - W_X$. Without loss of generality, let $s_3 \in W'_X - W_X - \{w_X\}$. Then any path connecting w'_X and s_3 in $G[W'_X \cup \{w'_X\}]$ contains the vertex w_X, which contradicts Lemma 22(ii). $\square$

For an efficient implementation of algorithm 3-LVC_SLP, we use 2-vertex-connected components [18] and 3-vertex-connected components [5], and their tree structure. The analysis of its time complexity is given in Appendix.

Lemma 24 *Algorithm 3-LVC_SLP can be implemented to run in linear time.* $\square$

Lemmas 12 and 23 imply that $|S^*| = \max\{f(G), g(G)\}$ holds if $|S_0| \leq 3$ or if $|S_0| = 4$ and $\mathcal{W}_0$ is a chain of type (B), and $|S|^* = f(G)$ holds otherwise. Summarizing the arguments given so far, Theorem 8 is now established.

4 NP-hardness of 4LSLP

In this section, we prove the next result.

Theorem 25 *Given an undirected graph $G = (V, E)$ and a demand function $d : V \rightarrow \{0, 3, 4\}$, the problem of testing whether there is a solution S to the 4LSLP with cardinality $\leq k$ for a specified value k is NP-hard.* $\square$

A graph is called *k-regular* if the degree of every vertex is exactly k. For a graph $G = (V, E)$, a set $V' \subseteq V$ of vertices is called a *vertex cover* if every edge $e = (u, v) \in E$ satisfies $\{u, v\} \cap V' \neq \emptyset$. For a vertex set $X \subset V$ in $G = (V, E)$, we denote by $E_G(X, V - X)$ the set of edges $e = (u, v)$ such that $u \in X$ and $v \in V - X$. Let $deg_G(v)$ denote the degree of a vertex v in G. Here we define a class of graphs obtained from some 3-regular simple graphs, as follows.

Definition 4 *We say that a graph G satisfies* property (Q) *if it is obtained from a 3-regular simple graph H by replacing each edge $e = (v_i, v_j) \in E(H)$ with three edges $(v_i, v_{i,j})$, $(v_{i,j}, v_{j,i})$, and $(v_{j,i}, v_j)$ introducing two new vertices $v_{i,j}$ and $v_{j,i}$. Then we denote $V_E(H) = \cup_{(v_i,v_j) \in E(H)}\{v_{i,j}, v_{j,i}\}$, $F_1(H) = \cup_{(v_i,v_j) \in E(H)}\{(v_{i,j}, v_{j,i})\}$, $F_2(H) = \cup_{(v_i,v_j) \in E(H)}\{(v_i, v_{i,j}), (v_{j,i}, v_j)\}$, and $G = (V(H) \cup V_E(H), F_1(H) \cup F_2(H))$.* □

Note that for any graph G with property (Q), a 3-regular simple graph H with $G = (V(H) \cup V_E(H), F_1(H) \cup F_2(H))$ is uniquely determined. A graph with property (Q) satisfies the following properties.

Lemma 26 *Let a graph $G = (V, E) = (V(H) \cup V_E(H), F_1(H) \cup F_2(H))$ satisfy property (Q), where H is a 3-regular simple connected graph, and let X be an arbitrary vertex cover in G.*
(i) We have $|X| \geq 6$.
(ii) Let $Y \subset V$ be a vertex set in G such that $G[Y]$ contains an edge $(u, v) \in E$ and the edge cut $E_G(Y, V - Y)$ satisfies $|E_G(Y, V - Y)| \leq 2$. Then we have $|X \cap Y| \geq 3$, except the case where $Y = \{u, v\}$ and $(u, v) \in F_1(H)$.

PROOF. It is not difficult to see that the graph H satisfies the following properties by the 3-regularity of H.

Claim 1 *(i) We have $|E(H)| \geq 6$.*
(ii) Assume that H has a vertex set $Z \subset V(H)$ such that the edge cut $E_H(Z, V(H) - Z)$ satisfies $|E_H(Z, V(H) - Z)| \leq 2$. Then we have $|E(H[Z])| \geq 3$ and $|E(H[V(H) - Z])| \geq 3$. □

Now, from the definition of a vertex cover,

$$\{u, v\} \cap X \neq \emptyset \text{ for each edge } e = (u, v) \in F_1(H). \tag{3}$$

(i) By (3), we have $|X| \geq |E(H)|$. This together with Claim 1(i) imply $|X| \geq 6$.

(ii) Let $Z = Y \cap V(H)$. From property (Q) of G, we see that if $Z = \emptyset$, then $Y = \{a, b\}$ for some edge $e' = (a, b) \in F_1(H)$. As $(u, v) \in E(G[Y])$ it then follows that $Y = \{u, v\}$ and $(u, v) \in F_1(H)$ (note that $|E_G(Y, V - Y)| \leq 2$ and $|E(G[Y])| \geq 1$ hold). If $V(H) = Z$, then $V - Y = \{a, b\}$ must hold for some edge $e' = (a, b) \in F_1(H)$. From (3) and $|E(H[Z])| - 1 = |E(H)| - 1 \geq 5$ (by Claim 1(i)), we have $|X \cap Y| \geq 3$.

We consider the case where $Z \neq \emptyset$ and $V(H) - Z \neq \emptyset$. Note that $|E_H(Z, V(H) - Z)| \leq 2$ holds also in H. Moreover, the connectedness of G and $|E_G(Y, V - Y)| \leq 2$ imply that for each edge $(v_i, v_j) \in E(H[Z])$, the edge $(v_{i,j}, v_{j,i}) \in F_1(H)$ is also contained in $G[Y]$. From this, Claim 1(ii), and (3), we get $|X \cap Y| \geq 3$. □

In this section, we show the NP-hardness of 4LSLP by reducing from the following problem which is a special case of the vertex cover problem (we call this problem VCQ).

Vertex-cover problem in a graph with property (Q) (VCQ)

INSTANCE: $(G = (V, E), k)$: A graph $G = (V, E)$ satisfying property (Q) and an integer k.

QUESTION: Is there a vertex cover X with $|X| \leq k$ in G?   □

Lemma 27 *VCQ is NP-hard.*

PROOF. We prove this lemma by reducing from the following problem, denoted by VC3R, which is known to be NP-complete [1,3].

Vertex-cover problem in a 3-regular graph (VC3R)

INSTANCE: $(G = (V, E), k)$: A 3-regular graph $G = (V, E)$ and an integer k.

QUESTION: Is there a vertex cover X with $|X| \leq k$ in G?   □

Take an instance $I_{VC3R} = (G_1 = (V_1, E_1), k)$ of VC3R, where $n_1 = |V_1|$ and $m_1 = |E_1|$. First we convert I_{VC3R} to an instance $I_{VCQ} = (G_2 = (V_2, E_2) = (V(G_1) \cup V_E(G_1), F_1(G_1) \cup F_2(G_1)), k + m_1)$ of VCQ. Clearly, G_2 can be constructed in polynomial time in n_1 and m_1. For proving the lemma, it suffices to show the following claim.

Claim 1 G_1 *has a vertex cover with cardinality at most k if and only if G_2 has a vertex cover with cardinality at most $k + m_1$.*

PROOF. Let X_1 be a vertex cover in G_1 with $|X_1| \leq k$. Then a vertex set $X_2 = X_1 \cup \{v_{i,j} \in V_E(G_1) \mid (v_i, v_j) \in E_1, v_i \notin X_1\} \cup \{v_{i,j} \in V_E(G_1) \mid (v_i, v_j) \in E_1, i < j, \{v_i, v_j\} \subseteq X_1\}$ is a vertex cover in G_2. Since X_2 contains exactly one vertex in $\{v_{i,j}, v_{j,i}\}$ for each edge $(v_i, v_j) \in E_1$, we have $|X_2| = |X_1| + m_1 \leq k + m_1$.

Let X_2 be a vertex cover in G_2 with $|X_2| \leq k + m_1$. For each pair $\{v_{i,j}, v_{j,i}\} \subseteq V_E(G_1)$ of two vertices corresponding to an edge $(v_i, v_j) \in E_1$ with $i < j$, if $\{v_{i,j}, v_{j,i}\} \subseteq X_2$, then we reconstruct $X_2 := (X_2 - v_{i,j}) \cup \{v_i\}$ (note that this operation preserves the property that X_2 is a vertex cover in G_2). Let X_2^* be the resulting vertex cover in G_2. Then $X_1 = X_2^* \cap V_1$ satisfies $|X_1| \leq k$ since X_2^* contains $v_{i,j}$ or $v_{j,i}$ corresponding to each edge $(v_i, v_j) \in E_1$. Moreover, X_1 is a vertex cover in G_1 since if there is an edge $(v_h, v_\ell) \in E_1$ with $\{v_h, v_\ell\} \cap X_1 = \emptyset$,

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then $v_{h,\ell}$ or $v_{\ell,h}$ (say, $v_{h,\ell}$) are not contained in X_2 and we have $\{v_h, v_{h,\ell}\} \cap X_2 = \emptyset$, which contradicts the assumption that X_2 is a vertex cover in G_2. $\quad\square$

$\square$

We shall prove the NP-hardness of 4LSLP as follows. Take an instance $I_{VCQ} = (G = (V, E) = (V(H) \cup V_E(H), F_1(H) \cup F_2(H)), k)$ of VCQ, where $n = |V|$, $m = |E|$, and H is a 3-regular simple graph. For simplicity, assume that G is connected. From the I_{VCQ}, we construct an instance $I_{LSLP} = (G' = (V', E'), d)$ of 4LSLP as follows. For each $v_i \in V$, we construct a complete graph (V_i, E_i) whose vertex set is a set of four copies of the vertex v_i, where $V_i = \{v_1^i, v_2^i, v_3^i, v_4^i\}$ and $E_i = \{(v_j^i, v_\ell^i) \mid \{j, \ell\} \subset \{1, \ldots, 4\}\}$. For each $e = (v_i, v_j) \in E$, we construct one vertex v_{ij}. Let $V_E^2 = \{v_{ij} \mid (v_i, v_j) \in E, \ i < j, \{v_i, v_j\} \subseteq V_E(H)\}$ and $V_E^3 = \{v_{ij} \mid (v_i, v_j) \in E, v_i \in V_E(H), \ v_j \in V - V_E(H)\}$. Note that every vertex $v \in V_E(H)$ satisfies $deg_G(v) = 2$ and every vertex $v \in V - V_E(H)$ satisfies $deg_G(v) = 3$. We construct G' from G by replacing each vertex $v_i \in V$ and each edge $e = (v_j, v_\ell) \in E$ with (V_i, E_i) and the vertex $v_{j\ell}$, respectively, and adding edges connecting $v_{j\ell}$ and $V_j \cup V_\ell$ for each edge $e = (v_j, v_\ell) \in E$. Let $V' = (\cup_{v_i \in V} V_i) \cup V_E^2 \cup V_E^3$ and $E' = (\cup_{v_i \in V} E_i) \cup (\cup_{v_{ij} \in V_E^2 \cup V_E^3} \{(v_{ij}, u) \mid u \in V_i \cup V_j\})$. Let $d(v) = 3$ for each vertex $v \in V_E^2$ and $d(v) = 4$ for each vertex $v \in V_E^3$ and $d(v) = 0$ otherwise. Clearly, G' can be constructed in polynomial time in n and m.

We see that for each edge $(v_i, v_j) \in E$, $V_i \cup V_j \cup \{v_{ij}\}$ is a deficient set in G'. This follows since if an edge $(v_i, v_j) \in E$ satisfies $\{v_i, v_j\} \subseteq V_E(H)$ (resp. $v_i \in V_E(H)$ and $v_j \in V - V_E(H)$), then we have $|N_{G'}(V_i \cup V_j \cup \{v_{ij}\})| = 2$ and $d(v_{ij}) = 3$ (resp. $|N_{G'}(V_i \cup V_j \cup \{v_{ij}\})| = 3$ and $d(v_{ij}) = 4$).

The following lemma completes the proof of Theorem 25.

Lemma 28 *G has a vertex cover with cardinality at most k if and only if G' has a source set with cardinality at most k.*

PROOF. Assume that G' has a source set S with $|S| \leq k$. For each $v_{ij} \in (V_E^2 \cup V_E^3) \cap S$, we reconstruct $S := (S - \{v_{ij}\}) \cup \{v'\}$ for some $v' \in V_i \cup V_j$. For the resulting set S, we have $S \subseteq V' - (V_E^2 \cup V_E^3)$. The set $X = \{v_i \in V \mid V_i \cap S \neq \emptyset\}$ satisfies $|X| \leq k$. Assume by contradiction that X is not a vertex cover in G. Then there is an edge $(v_i, v_j) \in E$ with $\{v_i, v_j\} \cap X = \emptyset$. From the construction of G', we have $(V_i \cup V_j \cup \{v_{ij}\}) \cap S = \emptyset$, which contradicts the assumption that S is a source set (note that $V_i \cup V_j \cup \{v_{ij}\}$ is a deficient set). Hence X is a vertex cover in G.

Assume that G has a vertex cover $X \subseteq V$ with $|X| \leq k$. Let $S = \{v_1^i \in V' \mid v_i \in X\}$. From Lemma 26(i), we have the following property.

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Claim 1 $|X| = |S| \geq 6$ *holds.* □

We claim that S is a source set in G'. Assume by contradiction that S is not a source set. Then there is a deficient set $W \subseteq V'$ with $W \cap S = \emptyset$ which contains a vertex $v_{ab} \in (V_E^2 \cup V_E^3) \cap W$ corresponding to some edge $(v_a, v_b) \in E$ in G. We choose W such that $|N_{G'}(W)|$ is minimum. Note that $|N_{G'}(W)| \leq 3$ holds since $d(W) \leq 4$. We first show that every neighbour of W belongs either to S or to $V_E^2 \cup V_E^3$.

Claim 2 *If $N_{G'}(W) \cap V_i \neq \emptyset$, then we have $W \cap V_i \neq \emptyset$ and $V_i - S \subseteq W$ (hence, we have $N_{G'}(W) \subseteq S \cup V_E^2 \cup V_E^3$).*

PROOF. If $W \cap V_i = \emptyset$, then $|N_{G'}(W)| \geq 4$ holds by $N_{G'}(W) \supseteq V_i$ and $|V_i| \geq 4$, a contradiction (note that this is why the cardinality of V_i is set to four).

Assume that $W \cap V_i \neq \emptyset$ and $V_i - S - W \neq \emptyset$. For each vertex $v_i^j \in V_i - S - W$, every vertex in $\{v_i^j\} \cup N_{G'}(v_i^j)$ belongs to $W \cup N_{G'}(W)$ from the construction of G'. This implies that $N_{G'}(W \cup (V_i - S)) = N_{G'}(W) - (V_i - S - W) \subset N_{G'}(W)$. Hence $W \cup (V_i - S)$ contradicts the minimality of $|N_{G'}(W)|$ (note that $W \cup (V_i - S)$ does not contain any source). □

Let $S_W = S \cap N_{G'}(W)$. Now X is a vertex cover in G and hence we can assume without loss of generality that the edge (v_a, v_b) satisfies $v_a \in X$. By Claim 2 and $v_1^a \in S$, we have $v_1^a \in N_{G'}(W)$ and $V_a - \{v_1^a\} \subseteq W$. Hence $|S_W| \geq 1$.

Claim 3 $N_{G'}(W \cup S_W) = N_{G'}(W) - S_W$.

PROOF. Clearly, $N_{G'}(W \cup S_W) \supseteq N_{G'}(W) - S_W$. We see that for each $v_1^i \in S_W$, every vertex in $N_{G'}(v_1^i)$ belongs to $N_{G'}(W) \cup W$ from the construction of G' and $V_i - \{v_1^i\} \subseteq W$. This implies $N_{G'}(W \cup S_W) \subseteq N_{G'}(W) - S_W$. □

We see that $|N_{G'}(W \cup S_W)| = 0$ would imply $S_W = S$ (by the connectedness of G') and $|S_W| = |S| \geq 6$ (by Claim 1), which contradicts $|S_W| \leq |N_{G'}(W)| \leq 3$. Hence we have $|N_{G'}(W \cup S_W)| \geq 1$. By Claim 3, $|N_{G'}(W)| \leq 3$, $|S_W| \geq 1$, and $|N_{G'}(W \cup S_W)| \geq 1$, we have the following two possible cases (I) $|N_{G'}(W \cup S_W)| = 1$ and $|S_W| \leq 2$ and (II) $|N_{G'}(W \cup S_W)| = 2$ and $|S_W| = 1$.

Let $\{u_1, u_2\} = N_{G'}(W \cup S_W)$ (possibly, $u_1 = u_2$) and $e_i \in E$ denote the edge in G corresponding to u_i (note that $u_i \in V_E^2 \cup V_E^3$). Then the following properties hold.

Claim 4 $V' - (W \cup S_W) - (V_E^2 \cup V_E^3) \neq \emptyset$. *Moreover, each u_i has a neighbour in $V' - (W \cup S_W) - (V_E^2 \cup V_E^3)$.*

PROOF. We have $|S_W| \leq 2$ and $|S| \geq 6$ by Claim 1. This implies $V' - (W \cup S_W) - (V_E^2 \cup V_E^3) \neq \emptyset$ since every vertex in $S - S_W$ is contained in $V' - (W \cup S_W) - (V_E^2 \cup V_E^3)$.

Assume by contradiction that some u_i has no neighbour in $V' - (W \cup S_W) - (V_E^2 \cup V_E^3)$. Now note that no two vertices in $V_E^2 \cup V_E^3$ are adjacent to each other from the construction of G'. Hence, u_i satisfies $N_{G'}(u_i) \subseteq W \cup S_W$ (note that $\{u_1, u_2\} \subseteq V_E^2 \cup V_E^3$). Then the set $W \cup \{u_i\}$ satisfies $(W \cup \{u_i\}) \cap S = \emptyset$ and $|N_{G'}(W \cup \{u_i\})| = |N_{G'}(W)| - |\{u_i\}| < |N_{G'}(W)|$, which contradicts the minimality of $|N_{G'}(W)|$. $\square$

By Claim 4, we see that the edge set $\{e_1, e_2\}$ is an edge cut in G. Let $W_1 \subseteq V$ denote the vertex set in G corresponding to $W \cup S_W$ such that $E_G(W_1, V - W_1) = \{e_1, e_2\}$ and $(v_a, v_b) \in E(G[W_1])$. Therefore the above two cases (I) and (II) are equivalent to (I') $E_G(W_1, V - W_1) = \{e_1\}$ and $|W_1 \cap X| \leq 2$ and (II') $E_G(W_1, V - W_1) = \{e_1, e_2\}$ and $|W_1 \cap X| = 1$, respectively, in G.

By Lemma 26(ii), if $W_1 = \{v_j, v_h\}$ does not hold for any edge $(v_j, v_h) \in F_1(H)$, then we have $|W_1 \cap X| \geq 3$, which implies that neither (I') nor (II') can occur. From this and $(v_a, v_b) \in E(G[W_1])$, we see that $W_1 = \{v_a, v_b\}$ and $(v_a, v_b) \in F_1(H)$ hold. Hence, the case (I') cannot hold and we have $|E_G(W_1, V - W_1)| = 2$ and $|W_1 \cap X| = 1$, from which $|S_W| = 1$ and $|N_{G'}(W \cup S_W)| = 2$. Moreover, we see that $v_{ab} \in V_E^2$ and $d(v_{ab}) = 3$. From this and $|N_{G'}(W)| = |S_W| + |N_{G'}(W \cup S_W)| = 3$, W is not a deficient set in G', a contradiction.

Consequently S is a source set in G'. $\square$

5 Conclusion

In this paper, we have considered the problem that consists, given an undirected graph $G = (V, E)$ and a demand function $d : V \to \{0, 1, \ldots, k\}$, in finding a source set $S \subseteq V$ with the minimum cardinality for which there exist $d(v)$ mutually vertex-disjoint paths between every vertex $v \in V - S$ and S such that no two paths meet at the same vertex in S. We constructed a linear time algorithm for solving the problem when $k \leq 3$ and showed that the problem is NP-hard for any fixed $k \geq 4$.

This paper treated the problems in the case where the cost for locating sources is uniform. It is left open whether the problem with general costs for locating

sources can be solved in polynomial time in the case of $k \leq 3$. It is also a future work to design approximation algorithms for the problem.

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A   Time Complexity of Algorithm 3-LVC_SLP

In this section, we show that algorithm 3-LVC_SLP can be implemented to run in linear time. For this, we first introduce 2-vertex-connected components [18], 3-vertex-connected components [5], and their structure trees, which are known to be very useful for treating graphs with small connectivity.

A connected graph with no cycle is called a *tree*. In a tree T, a vertex $u \in V(T)$ with $|N_T(u)| = 1$ is called a *leaf*. We denote a set of leaves in a tree T by $L(T)$. In a connected graph $G = (V, E)$, a vertex $v \in V$ (resp. a vertex pair $\{v_1, v_2\}$) is called a *cut vertex* (resp. *cut pair*) if $G - v$ is disconnected (resp. $G - \{v_1, v_2\}$ is disconnected and neither v_1 nor v_2 is a cut vertex).

A.1   *2-vertex-connected components and their tree structure*

A vertex set $X \subseteq V$ is called a *2-vertex-connected component* or a *block*, if $G[X]$ is connected and has no cut vertex and X is maximal subject to this

property. In a connected graph G, we create a new vertex x_i associated with each block X_i in G, and let $V_C = \{c_1, \ldots, c_r\} \subseteq V$ be a set of cut vertices. A graph with a set $\{x_1, \ldots, x_s\} \cup V_C$ of vertices and a set $\{(c_j, x_i) \mid c_j \in X_i\}$ of edges is a tree, and we call this tree a *block-cut tree*. A block-cut tree can be constructed in linear time [18].

A.2   3-vertex-connected components and their tree structure

A graph $G = (V, E)$ with $|V| \geq 4$, no cut vertex, and no cut pair is called a *3-vertex-connected graph*.

Let G have no cut vertex. Then, for a cut pair $Y = \{u, v\}$, V can be divided into two vertex sets V_1 and V_2 with $V_1 \cap V_2 = Y$, $V_1 \cup V_2 = V$, and $N_G(V_1 - Y) \cap (V_2 - Y) = \emptyset$. For each graph $G_i = G[V_i]$, $i = 1, 2$, we add a virtual edge (u, v) to G_i. The resulting graphs G_1 and G_2 are called *divided graphs*. We say that G_1 and G_2 have the common virtual edge (u, v). A divided graph obtained by repeating this operation until no such operation can be applied is C_3, B_3, or a 3-vertex-connected simple graph, where C_i denotes a simple cycle with i edges and B_i denotes a graph with i multiple edges between two vertices (we call B_i a *bond graph*). Furthermore, by merging C_p and C_q which have the common virtual edge, we can obtain a larger cycle C_{p+q-2}. By merging B_p and B_q which have the common virtual edge, we can obtain a larger bond graph B_{p+q-2}. By repeating these merge operations until no merge operation can be applied, we have a unique decomposition of G which does not depend on an ordering of dividing and merging operations. The resulting divided graph is called a *3-vertex-connected component*, and such a decomposition can be found in linear time [5].

We create a new vertex z_i associated with each 3-vertex-connected component Z_i. A graph with a set $\{z_1, \ldots, z_s\}$ of vertices and a set of edges (z_i, z_j) such that the corresponding two 3-vertex-connected components Z_1 and Z_2 has the common virtual edge is a tree. We call this tree a *3-block-cut tree*.

A.3   Time complexity of algorithm 3-LVC_SLP

Let $D_i = \{v \in V \mid d(v) = i\}$ for $i = 0, 1, 2, 3$. It is not difficult to see that Step II can be implemented to run in linear time, because we have only to construct chains from the source set obtained in Step I and the family of the corresponding deficient sets, and update the current source set for each chain by using the block-cut tree of G (note that every vertex is contained in at most two minimal deficient sets and chains can be found easily). Hence we here analyze only the complexity of Step I. In Step I-0, the vertices $v_1, \ldots, v_n$

can be numbered in $O(n)$ time by $d : V \to \{0, 1, 2, 3\}$. In Step I-2, if $S - \{v_i\}$ is not a source set, we find a minimal deficient set $W_i \subseteq V - (S - \{v_i\})$ with $W_i \cap S = \{v_i\}$. In the rest of this section, we show that Step I-2 can be executed in linear time. Since G is connected and $D_2 \cup D_3 \neq \emptyset$ holds, $V - D_0 - D_1$ is a source set, and hence we can start from $S := V - D_0 - D_1$.

We consider an efficient implementation of Step I-2 for vertices in D_2. Construct the block-cut tree $\mathcal{T}_1$ from G. Let $\mathcal{T}_1' := \mathcal{T}_1$ and each block in $\mathcal{T}_1'$ unchecked. We repeat the following operations until each leaf in the current tree $\mathcal{T}_1'$ has at least one source in $D_2 \cup D_3$. We pick up one leaf $\ell_T \in L(\mathcal{T}_1')$ in $\mathcal{T}_1'$ which has not been checked. Let $L_G \subseteq V$ be a block in G associated with $\ell_T \in L(\mathcal{T}_1')$, and $v_c \in L_G$ be the cut vertex in G corresponding to $N_{\mathcal{T}_1'}(\ell_T)$. Remark that $\mathcal{T}_1'$ may be obtained from contracting some vertices as described later, and let $L_G^* \subseteq V$ be a set of all vertices which has been contracted to some vertex in L_G so far. If $d(L_G - \{v_c\}) \leq 1$, then contract $L_G - \{v_c\}$ to v_c and update the block-cut tree $\mathcal{T}_1'$. If $d(L_G - \{v_c\}) = 3$, then delete from S every vertex $v \in (L_G - \{v_c\}) \cap D_2$ and check the leaf ℓ_T. If $d(L_G - \{v_c\}) = 2$, then we delete every vertex $v \in (L_G - \{v_c\}) \cap D_2$ except one vertex $v_j \in D_2$ from S, check the leaf ℓ_T, and we have $W_j := L_G^* - \{v_c\}$ as a minimal deficient set with respect to v_j for the vertex $v_j \in (L_G^* - \{v_c\}) \cap S$.

Let S_1' and $\mathcal{T}_1^*$ be the resulting source set and the block-cut tree obtained by executing those operations, respectively. Then we delete from S_1' every vertex $v \in D_2$ which is not contained in $L(\mathcal{T}_1^*)$. We can easily see that the resulting S_1'' is a source set, except the case where all vertices $v \in D_2 \cup D_3$ are contained in the same block in G, and we have (I) $|D_3| = 0$ and $|D_2| \geq 2$, or (II) $|D_3| = 1$ and $|D_2| \geq 1$. In such cases, we have $|S_1''| = 1$, and S_1'' is not a source set. However, for an arbitrary vertex $u \in D_2 - S_1''$, $S_1'' \cup \{u\}$ is a source set and an optimal solution by Lemma 6(c). These special cases can be easily checked in linear time.

We next describe an efficient implementation of Step I-2 for vertices in D_3. For this, we construct a 3-block-cut tree for each block X of G. We denote by $\mathcal{T}_2(X)$ a 3-block-cut tree for a block X of G. Let $\mathcal{T}_1' := \mathcal{T}_1^*$ and $S_2' := S_1''$ (note that each leaf contains a source from the property of $\mathcal{T}_1^*$). We repeat the following operations until all blocks associated with $\mathcal{T}_1^*$ are checked. First, we pick up one leaf $\ell_T \in L(\mathcal{T}_1')$ in $\mathcal{T}_1'$. Let $L_G \subseteq V$, $v_c \in L_G$, and $L_G^* \subseteq V$ be defined as above. In the case of $d(L_G - \{v_c\}) \leq 2$, then contract $L_G - \{v_c\}$ to v_c in G and update the block-cut tree $\mathcal{T}_1'$. Then, since each leaf in $\mathcal{T}_1'$ contains a source by the procedure for D_2, $(L_G^* - \{v_c\}) \cap S_2' \neq \emptyset$ always holds, and hence we turn on a flag on v_c so that we can find out that the vertex set in G contracted to v_c contains a source. In the case of $d(L_G - \{v_c\}) = 3$, we construct the 3-block-cut tree $\mathcal{T}_2(L_G)$. If the number of vertices in $\mathcal{T}_1'$ is at least two, then we turn on a flag on v_c, since $(V - L_G^*) \cap S_2' \neq \emptyset$. Along a similar way as in the operations for D_2 in the block-cut tree $\mathcal{T}_1$, we start from leaves

in $\mathcal{T}_2(L_G)$, check vertices in $D_3 \cap L_G$, and update S_2' while noticing whether a flag exists or not. After the procedure for L_G, we contract $L_G - \{v_c\}$ to v_c, turn on a flag on v_c, and update the block-cut tree $\mathcal{T}_1'$. These operations for L_G can be done in $O(|E(G[L_G]|))$ time. Consequently, it is not difficult to see that Step I-2 can be computed in $O(|E|)$ time in total.