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A supplementary note on the paper "Higher-order asymptotic standard error and asymptotic expansion in principal component analysis"

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This note is to supplement Ogasawara (2006) as an appendix, which was not included in the article due to space limitation. Section 1 for the formulas of the partial derivatives in implicit functions was first given by Ogasawara (2005a). Section 2 gives the partial derivatives of the discrepancy function in principal component analysis with respect to parameters and sample covariances required in using the results of Section 1.

1. The formulas of the partial derivatives of $\hat{\boldsymbol{\theta}}$ with respect to s (Ogasawara, 2005a)

Let $\hat{\mathbf{H}}_{\mathrm{A}} = \begin{bmatrix} \dfrac{\partial^2 \hat{F}}{\partial\hat{\boldsymbol{\theta}}\partial\hat{\boldsymbol{\theta}}'} & \dfrac{\partial\hat{\mathbf{h}}'}{\partial\hat{\boldsymbol{\theta}}} \\ \dfrac{\partial\hat{\mathbf{h}}}{\partial\hat{\boldsymbol{\theta}}'} & O \end{bmatrix}$, then

$$\begin{bmatrix} \dfrac{\partial\hat{\boldsymbol{\theta}}}{\partial s_{ab}} \\ \mathbf{0} \end{bmatrix} = -\hat{\mathbf{H}}_{\mathrm{A}}^{-1} \begin{bmatrix} \dfrac{\partial^2 \hat{F}}{\partial\hat{\boldsymbol{\theta}}\partial s_{ab}} \\ \mathbf{0} \end{bmatrix}, \qquad \text{(A1)}$$

$$\begin{bmatrix} \dfrac{\partial^2 \hat{\boldsymbol{\theta}}}{\partial s_{ab} \partial s_{cd}} \\ \mathbf{0} \end{bmatrix} = -\hat{\mathbf{H}}_{\mathbf{A}}^{-1} \left\{ \sum_{i,j=1}^{q} \begin{bmatrix} \dfrac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial \hat{\theta}_j} \\ \dfrac{\partial^2 \hat{\mathbf{h}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \end{bmatrix} \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial \hat{\theta}_j}{\partial s_{cd}} + \sum_{i=1}^{q} \begin{bmatrix} \dfrac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial s_{cd}} \\ \mathbf{0} \end{bmatrix} \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \right.$$

$$\left. + \sum_{i=1}^{q} \begin{bmatrix} \dfrac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial s_{ab}} \\ \mathbf{0} \end{bmatrix} \frac{\partial \hat{\theta}_i}{\partial s_{cd}} + \begin{bmatrix} \dfrac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial s_{cd} \partial s_{ab}} \\ \mathbf{0} \end{bmatrix} \right\} \tag{A2}$$

and

$$\begin{bmatrix} \dfrac{\partial^3 \hat{\boldsymbol{\theta}}}{\partial s_{ab} \partial s_{cd} \partial s_{ef}} \\ \mathbf{0} \end{bmatrix} = -\hat{\mathbf{H}}_{\mathbf{A}}^{-1} \left\{ \sum_{i,j,k=1}^{q} \begin{bmatrix} \dfrac{\partial^4 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \\ \dfrac{\partial^3 \hat{\mathbf{h}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \end{bmatrix} \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial \hat{\theta}_j}{\partial s_{cd}} \frac{\partial \hat{\theta}_k}{\partial s_{ef}} \right.$$

$$+ \sum_{(U,V,W)}^{3} \sum_{i,j=1}^{q} \begin{bmatrix} \dfrac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial \hat{\theta}_j} \\ \dfrac{\partial^2 \hat{\mathbf{h}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \end{bmatrix} \frac{\partial \hat{\theta}_i}{\partial s_U} \frac{\partial^2 \hat{\theta}_j}{\partial s_V \partial s_W}$$

$$+ \sum_{(U,V,W)}^{3} \sum_{i,j=1}^{q} \begin{bmatrix} \dfrac{\partial^4 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial \hat{\theta}_j \partial s_U} \\ \mathbf{0} \end{bmatrix} \frac{\partial \hat{\theta}_i}{\partial s_V} \frac{\partial \hat{\theta}_j}{\partial s_W}$$

$$+ \sum_{(U,V,W)}^{3} \sum_{i=1}^{q} \begin{bmatrix} \dfrac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial s_U} \\ \mathbf{0} \end{bmatrix} \frac{\partial^2 \hat{\theta}_i}{\partial s_V \partial s_W} + \sum_{(U,V,W)}^{3} \sum_{i=1}^{q} \begin{bmatrix} \dfrac{\partial^4 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial s_U \partial s_V} \\ \mathbf{0} \end{bmatrix} \frac{\partial \hat{\theta}_i}{\partial s_W}$$

$$\left. + \begin{bmatrix} \dfrac{\partial^4 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial s_{ab} \partial s_{cd} \partial s_{ed}} \\ \mathbf{0} \end{bmatrix} \right\}, \quad (p \ge a \ge b \ge 1; p \ge c \ge d \ge 1; p \ge e \ge f \ge 1),$$

(A3)

where $\sum_{(U,V,W)}^{3}$ denotes the summation over the range $(U,V,W) \in \{(ab,cd,ef),(cd,ef,ab),(ef,ab,cd)\}$.

2. The partial derivatives of the discrepancy functions with respect to the parameters and sample variances and covariances

2.1 $F = (1/2)\mathrm{tr}\{(\boldsymbol{\Sigma}(\boldsymbol{\theta}) - \mathbf{S})^2\}$

Let δ_{ab} be the Kronecker delta and $\hat{\sigma}_{ab} = (\hat{\boldsymbol{\Sigma}})_{ab} = \{\boldsymbol{\Sigma}(\hat{\boldsymbol{\theta}})\}_{ab}$, then

$$\frac{\partial^2 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} = \mathrm{tr}\left\{ \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i} \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j} + (\hat{\boldsymbol{\Sigma}} - \mathbf{S}) \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right\}, \tag{A4}$$

$$\frac{\partial^2 \hat{F}}{\partial \hat{\theta}_i \partial s_{ab}} = -(2 - \delta_{ab}) \frac{\partial \hat{\sigma}_{ab}}{\partial \hat{\theta}_i}, \tag{A5}$$

$$\frac{\partial^3 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} = \mathrm{tr}\left\{ \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i} \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} + \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j} \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_k} + \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_k} \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} + (\hat{\boldsymbol{\Sigma}} - \mathbf{S}) \frac{\partial^3 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \right\}, \tag{A6}$$

$$\frac{\partial^3 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial s_{ab}} = -(2 - \delta_{ab}) \frac{\partial^2 \hat{\sigma}_{ab}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}, \tag{A7}$$

$$\frac{\partial^4 \hat{F}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k\partial\hat{\theta}_l} = \mathrm{tr}\left\{\frac{\partial\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i}\frac{\partial^3\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_j\partial\hat{\theta}_k\partial\hat{\theta}_l} + \frac{\partial\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_j}\frac{\partial^3\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_k\partial\hat{\theta}_l}\right.$$
$$+\frac{\partial\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_k}\frac{\partial^3\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_l} + \frac{\partial\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_l}\frac{\partial^3\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k} + \frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_j}\frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_k\partial\hat{\theta}_l}$$
$$+\frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_k}\frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_j\partial\hat{\theta}_l} + \frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_l}\frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_j\partial\hat{\theta}_k} \tag{A8}$$
$$\left. + (\hat{\mathbf{\Sigma}} - \mathbf{S})\frac{\partial^4\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k\partial\hat{\theta}_l}\right\},$$

$$\frac{\partial^4 \hat{F}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k\partial s_{ab}} = -(2-\delta_{ab})\frac{\partial^3\hat{\sigma}_{ab}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k},$$
$$(i, j, k, l = 1, \ldots, q;\ p \ge a \ge b \ge 1), \tag{A9}$$

where $\hat{\mathbf{\Sigma}} - \mathbf{S}$, when evaluated at the true values, becomes $\mathbf{\Sigma}(\boldsymbol{\theta}) - \mathbf{\Sigma}_{\mathrm{T}} \ne O$.

2.2 $F_\rho = (1/2)\mathrm{tr}\{(\mathbf{P}(\boldsymbol{\theta}) - \mathbf{R})^2\}$

Let $\hat{\rho}_{ab} = (\hat{\mathbf{P}})_{ab} = \{\mathbf{P}(\hat{\boldsymbol{\theta}})\}_{ab}$ and $\hat{\theta}_{\rho i} = (\hat{\boldsymbol{\theta}}_\rho)_i$, then

$$\frac{\partial^2 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}} = \mathrm{tr}\left\{\frac{\partial\hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}}\frac{\partial\hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho j}} + (\hat{\mathbf{P}} - \mathbf{R})\frac{\partial^2\hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}}\right\}, \tag{A10}$$

$$\frac{\partial^2 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial s_{ab}} = -\mathrm{tr}\left(\frac{\partial\mathbf{R}}{\partial s_{ab}}\frac{\partial\hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}}\right)$$

$$= -(2-\delta_{ab})s_{aa}^{-1/2}s_{bb}^{-1/2}\frac{\partial\hat{\rho}_{ab}}{\partial\hat{\theta}_{\rho i}} + \delta_{ab}s_{aa}^{-3/2}\sum_{j=1}^{p}s_{aj}s_{jj}^{-1/2}\frac{\partial\hat{\rho}_{aj}}{\partial\hat{\theta}_{\rho i}}, \tag{A11}$$

$$\frac{\partial^3 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}} = \mathrm{tr}\left\{\frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}}\frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}} + \frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho j}}\frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho k}} + \frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho k}}\frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}} + (\hat{\mathbf{P}} - \mathbf{R})\frac{\partial^3 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}}\right\}, \tag{A12}$$

$$\begin{aligned}\frac{\partial^3 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial s_{ab}} &= -\mathrm{tr}\left(\frac{\partial \mathbf{R}}{\partial s_{ab}}\frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}}\right) \\ &= -(2-\delta_{ab})s_{aa}^{-1/2}s_{bb}^{-1/2}\frac{\partial^2 \hat{\rho}_{ab}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}} \\ &\quad + \delta_{ab}s_{aa}^{-3/2}\sum_{k=1}^{p} s_{ak}s_{kk}^{-1/2}\frac{\partial^2 \hat{\rho}_{ak}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}},\end{aligned} \tag{A13}$$

$$\begin{aligned}\frac{\partial^3 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial s_{ab}\partial s_{cd}} &= -\mathrm{tr}\left(\frac{\partial^2 \mathbf{R}}{\partial s_{ab}\partial s_{cd}}\frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}}\right) \\ &= \frac{1}{2}(2-\delta_{ab})\delta_{cd}(\delta_{ac}s_{aa}^{-3/2}s_{bb}^{-1/2} + \delta_{bc}s_{aa}^{-1/2}s_{bb}^{-3/2})\frac{\partial \hat{\rho}_{ab}}{\partial\hat{\theta}_{\rho i}} \\ &\quad -\frac{3}{2}\delta_{ab}\delta_{cd}\delta_{ac}s_{aa}^{-5/2}\sum_{j=1}^{p} s_{aj}s_{jj}^{-1/2}\frac{\partial \hat{\rho}_{aj}}{\partial\hat{\theta}_{\rho i}} \\ &\quad +\delta_{ab}s_{aa}^{-3/2}\frac{2-\delta_{cd}}{2}\left(\delta_{ac}s_{dd}^{-1/2}\frac{\partial \hat{\rho}_{ad}}{\partial\hat{\theta}_{\rho i}} + \delta_{ad}s_{cc}^{-1/2}\frac{\partial \hat{\rho}_{ac}}{\partial\hat{\theta}_{\rho i}}\right) \\ &\quad -\frac{1}{2}\delta_{ab}s_{aa}^{-3/2}\delta_{cd}s_{ac}s_{cc}^{-3/2}\frac{\partial \hat{\rho}_{ac}}{\partial\hat{\theta}_{\rho i}},\end{aligned} \tag{A14}$$

$$\frac{\partial^4 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}\partial\hat{\theta}_{\rho l}} = \mathrm{tr}\left\{\frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}}\frac{\partial^3 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}\partial\hat{\theta}_{\rho l}} + \frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho j}}\frac{\partial^3 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho k}\partial\hat{\theta}_{\rho l}} + \frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho k}}\frac{\partial^3 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho l}} + \frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho l}}\frac{\partial^3 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}} + \frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}}\frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho k}\partial\hat{\theta}_{\rho l}} + \frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho k}}\frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho l}} + \frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho l}}\frac{\partial^2 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}} + (\hat{\mathbf{P}} - \mathbf{R})\frac{\partial^4 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}\partial\hat{\theta}_{\rho l}}\right\}, \quad \text{(A15)}$$

$$\frac{\partial^4 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}\partial s_{ab}} = -\mathrm{tr}\left(\frac{\partial \mathbf{R}}{\partial s_{ab}}\frac{\partial^3 \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}}\right)$$

$$= -(2-\delta_{ab})s_{aa}^{-1/2}s_{bb}^{-1/2}\frac{\partial^3 \hat{\rho}_{ab}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}} + \delta_{ab}s_{aa}^{-3/2}\sum_{l=1}^{p} s_{al}s_{ll}^{-1/2}\frac{\partial^2 \hat{\rho}_{al}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial\hat{\theta}_{\rho k}}, \quad \text{(A16)}$$

$$\frac{\partial^4 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}\partial s_{ab}\partial s_{cd}} = \frac{1}{2}(2-\delta_{ab})\delta_{cd}(\delta_{ac}s_{aa}^{-3/2}s_{bb}^{-1/2} + \delta_{bc}s_{aa}^{-1/2}s_{bb}^{-3/2})$$

$$\times\frac{\partial^2\hat{\rho}_{ab}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}} - \frac{3}{2}\delta_{ab}\delta_{cd}\delta_{ac}s_{aa}^{-5/2}\sum_{k=1}^{p}s_{ak}s_{kk}^{-1/2}\frac{\partial^2\hat{\rho}_{ak}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}}$$

$$+\delta_{ab}s_{aa}^{-3/2}\frac{2-\delta_{cd}}{2}\left(\delta_{ac}s_{dd}^{-1/2}\frac{\partial^2\hat{\rho}_{ad}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}} + \delta_{ad}s_{cc}^{-1/2}\frac{\partial^2\hat{\rho}_{ac}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}}\right) \quad (A17)$$

$$-\frac{1}{2}\delta_{ab}s_{aa}^{-3/2}\delta_{cd}s_{ac}s_{cc}^{-3/2}\frac{\partial^2\hat{\rho}_{ac}}{\partial\hat{\theta}_{\rho i}\partial\hat{\theta}_{\rho j}},$$

$$\frac{\partial^4 \hat{F}_\rho}{\partial\hat{\theta}_{\rho i}\partial s_{ab}\partial s_{cd}\partial s_{ef}} = -\mathrm{tr}\left(\frac{\partial^3 \mathbf{R}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}\frac{\partial \hat{\mathbf{P}}}{\partial\hat{\theta}_{\rho i}}\right)$$

$$= -\frac{1}{4}(2-\delta_{ab})\delta_{cd}(3\delta_{ac}\delta_{ef}\delta_{ea}s_{aa}^{-5/2}s_{bb}^{-1/2} + \delta_{ac}\delta_{ef}\delta_{eb}s_{aa}^{-3/2}s_{bb}^{-3/2}$$

$$+\delta_{bc}\delta_{ef}\delta_{ea}s_{aa}^{-3/2}s_{bb}^{-3/2} + 3\delta_{bc}\delta_{ef}\delta_{eb}s_{aa}^{-1/2}s_{bb}^{-5/2})\frac{\partial\hat{\rho}_{ab}}{\partial\hat{\theta}_{\rho i}}$$

$$+\frac{15}{4}\delta_{ab}\delta_{cd}\delta_{ac}\delta_{ef}\delta_{ae}s_{aa}^{-7/2}\sum_{j=1}^{p}s_{aj}s_{jj}^{-1/2}\frac{\partial\hat{\rho}_{aj}}{\partial\hat{\theta}_{\rho i}} - \frac{3}{2}\delta_{ab}\delta_{cd}\delta_{ac}s_{aa}^{-5/2}$$

$$\times\left\{\frac{2-\delta_{ef}}{2}\left(\delta_{ae}s_{ff}^{-1/2}\frac{\partial\hat{\rho}_{af}}{\partial\hat{\theta}_{\rho i}} + \delta_{af}s_{ee}^{-1/2}\frac{\partial\hat{\rho}_{ae}}{\partial\hat{\theta}_{\rho i}}\right) - \frac{1}{2}\delta_{ef}s_{ae}s_{ee}^{-3/2}\frac{\partial\hat{\rho}_{ae}}{\partial\hat{\theta}_{\rho i}}\right\}$$

$$-\frac{3}{4}\delta_{ab}\delta_{ef}\delta_{ae}s_{aa}^{-5/2}(2-\delta_{cd})\left(\delta_{ac}s_{dd}^{-1/2}\frac{\partial\hat{\rho}_{ad}}{\partial\hat{\theta}_{\rho i}} + \delta_{ad}s_{cc}^{-1/2}\frac{\partial\hat{\rho}_{ac}}{\partial\hat{\theta}_{\rho i}}\right)$$

$$-\frac{1}{4}\delta_{ab}s_{aa}^{-3/2}(2-\delta_{cd})\delta_{ef}\left(\delta_{ac}\delta_{de}s_{dd}^{-3/2}\frac{\partial\hat{\rho}_{ad}}{\partial\hat{\theta}_{\rho i}} + \delta_{ad}\delta_{ce}s_{cc}^{-3/2}\frac{\partial\hat{\rho}_{ac}}{\partial\hat{\theta}_{\rho i}}\right)$$

$$+\frac{1}{2}\delta_{ab}\delta_{cd}\left\{\frac{3}{2}\delta_{ef}\delta_{ea}s_{aa}^{-5/2}s_{ac}s_{cc}^{-3/2} - s_{aa}^{-3/2}\frac{2-\delta_{ef}}{2}(\delta_{ae}\delta_{cf}+\delta_{af}\delta_{ce})s_{cc}^{-3/2}\right.$$

$$\left.+\frac{3}{2}\delta_{ef}\delta_{ec}s_{aa}^{-3/2}s_{ac}s_{cc}^{-5/2}\right\}\frac{\partial\hat{\rho}_{ac}}{\partial\hat{\theta}_{\rho i}},$$

$$(i,j,k,l=1,\ldots,q;\ p\ge a\ge b\ge 1;\ p\ge c\ge d\ge 1;\ p\ge e\ge f\ge 1), \qquad \text{(A18)}$$

where $\hat{\mathbf{P}}-\mathbf{R}$, when evaluated at the true values, is $\mathbf{P}(\boldsymbol{\theta}_\rho)-\mathbf{P}_{\mathrm{T}}\ne O$. In the above results, s_{aa}'s and s_{ab}'s $(a,b=1,\ldots,p;a\ne b)$ can be replaced by 1 and r_{ab}'s, respectively without loss of generality.

References

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