Essential Norms Of Integration Operators And Multipliers On Weighted Dirichlet Spaces

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Abstract

Let g be an analytic function on the open unit disk D in the complex plane C. We study the following operators:

$$J_g(f)(z)=\int_0^z f(\zeta)g'(\zeta)d\zeta,\ I_g(f)(z)=\int_0^z f'(\zeta)g(\zeta)d\zeta.$$

on the weighted Dirichlet spaces. Then we chracterize the essential norm of the operators J_g, I_g, and M_g on the weighted Dirichlet spaces.

Key Words and Phrases: integration operator, compact, essential norm, multiplier.

Introduction

Let $D=\{z\in C:|z|<1\}$ denote the open unit disk in the complex plane C and let

$\partial D=\{z\in C:|z|=1\}$ denote the unit circle. For $z,\ w\in D,\ \varphi_z(w)=\frac{z-w}{1-\bar{z}w}$.

The space $H(D)$ is defined to be the space of analytic functions f on the open unit disk D.

For $1\leq p<+\infty$, the Lebesgue space $L^p(D, dA)$ is defined to be the Banach space of Lebesgue measurable functions on the open unit disk D

with

$$\|f\|_{L^p(dA)} = \left(\int_D |f(z)|^p dA(z)\right)^{\frac{1}{p}} < +\infty,$$

where $dA(z)$ is the normalized area measure on D. The Bergman space $L_a^p(D)$ is defined to be the subspace of $L^p(D, dA)$ consisting of analytic functions. For $f \in L_a^p$, the norm $\|f\|_{L(dA)}^p$ is equivalent to the following norm:

$$\|f\|_p = |f(0)| + \left(\int_D |f'(z)|^p (p+1)(1-|z|^2)^p dA(z)\right)^{\frac{1}{p}} < +\infty.$$

For $0 < p < +\infty$, the Hardy space H^p is defined to be the Banach space of analytic functions f on D with

$$\|f\|_{H^p} = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < +\infty.$$

Let $\alpha > -1$. Then the weighted Dirichlet space D^α is defined to be the space of analytic functions f on D such that

$$\|f\|_{D^\alpha} = |f(0)| + \left(\int_D (1-|z|^2)^\alpha |f'(z)|^2 \, dA(z)\right)^{\frac{1}{2}} < +\infty.$$

If $\alpha = 1$, then the space D^α is the Hardy space H^2. If $\alpha = 2$, then the space D^α is the Bergman space L_a^2.

Let $\omega(r)$, $0 \leq r < 1$, be a positive weight function which is integrable on$(0, 1)$. We extend ω on D by setting $\omega(z) = \omega(|z|)$. And we suppose that the weights ω are normalized so that $\int_D \omega(z) \, dA(z) = 1$. And we suppose that ω is a weight satisfying the following conditions: there is a constant $c_1 > 0$ such that

$$(1) \quad \omega(r) \geq \frac{c_1}{1-r} \int_r^1 \omega(u) du, \; 0 < r < 1,$$

and there is $s \in (0, 1)$ and $c_2 > 0$ such that

$$(2) \quad \omega(sr+1-s) \geq c_2\omega(r),\ 0 < r < 1.$$

And we will suppose sequences $\gamma = \{\gamma_n\}_{n\geq 0}$ of positive numbers, with $\gamma_0 =$ 1 and with the property that

$$\lim_{n\to\infty} \frac{\gamma_n + 1}{\gamma_n} = 1$$

, where

$$\gamma_n = \int_0^1 r^{2n+1}\,\omega(r)\,dr.$$

For $1 \leq p < +\infty$, the weighted Bergman space $L_a^p(\omega)$ is the space of all analytic functions $f: D \to C$ such that

$$\|f\|_{p,\omega} = \left(\int_D |f(z)|^p\,\omega(z)\,dA(z)\right)^{\frac{1}{p}} < +\infty.$$

Standard estimates show that point evaluations are bounded linear functionals on $L_a^p(\omega)$ and $L_a^p(\omega)$ is a Banach space. And $L_a^2(\omega)$ is a Hilbert space.

Let $\alpha > 0$. Then α-Bloch space B^α is defined to be the space of analytic functions f on D such that

$$\|f\|_{B^\alpha} = |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha |f'(z)| < +\infty.$$

And the little α-Bloch space, denoted B_0^α, is the closed subspace of B^α consisting of functions f with$(1-|z|^2)^\alpha f'(z) \to 0(|z| \to 1^-)$. Note that B^1, B_0^1 are the Bloch space B, the little Bloch space B_0, respectively.

Let X and Y be Banach spaces. Then a function f on D is a multiplier of X into Y if $fg \in Y$ for all g in X. In the case, we write $fX \subset Y$.

For g analytic on D, the operators J_g, I_g, and M_g are defined by the following:

$$J_g(f)(z)=\int_0^z g'(\zeta)f(\zeta)d\zeta,\ I_g(f)(z)=\int_0^z f'(\zeta)g(\zeta)d\zeta,\ M_g(f)(z)=g(z)f(z).$$

If $g(z)=z$, then J_g is the integration operator. If $g(z)=\log\frac{1}{1-z}$, then J_g is the Cesáro operator.

And the operator I_g is the companion operator of the operator J_g. And the relation $J_g=M_g-I_g$ gives some sort of cancellation property. There are symbols g such that M_g and I_g are not bounded on the Bergman space or Hardy space but their difference J_g is bounded. In fact, in some cases it is advantageous to think of J_g and I_g as distant cousins of Hankel operator and Toeplitz operator, respectively.

In [5], Ch. Pommerenke proved the following result with respect to the operator J_g:

Theorem A. *For g analytic on D, the operator J_g is bounded on $D^1=H^2$ if and only if*

$$g\in BMOA.$$

In [1], A. Aleman and A. G. Siskakis proved the following result with respect to the operator J_g:

Theorem B. *For g analytic on D, for $p\geq 1$, the operator J_g is bounded on H^p if and only if*

$$g\in BMOA.$$

And the operator J_g is compact on H^p if and only if

$$g\in VMOA.$$

In [2], A. Aleman and A. G. Siskakis proved the following result with respect to the operator J_g:

Theorem C. *Let $p\geq 1$. Then for g analytic on D, the operator J_g*

is bounded on $L^p_a(\omega)$ *if and only if* $g \in B$. And *the operator* J_g *is compact on* $L^p_a(\omega)$ *if and only if* $g \in B_0$.

As a result, we find that these results correspond to the results of Hankel operator.

In [9], we proved the following result with respect to the operator I_g:

Theorem D. *Let* $\alpha > 1$. *For* g *analytic on* D, *the operator* I_g *is bounded on* D^α *if and only if*

$$\sup_{z \in D} |g(z)| < +\infty, \ i.e\ g \in H^\infty.$$

We also find that this result correspond to the result of Toeplitz operator.

In [7], A. G. Siskakis and R. Zhao studied the boundedness and compactness of J_g on *BMOA*.

In [11], we also characterized the essential norm of J_g and I_g on the weighted Bloch spaces.

In this paper, we do characterize the essential norm of the operators J_g, I_g and M_g on the (weighted) Bergman spaces that were not known so far. As a result, we get the results with respect to the boundedness and the compactness of J_g, I_g and M_g on the (weighted) Bergman spaces.

Throughout this paper, C, K will denote positive constant whose value is not necessary the same at each occurrence.

The essential norm of the operators J_g, I_g, and M_g on the weighted Dirichlet spaces $L^2_a(\omega)$

In this section we study the essential norms of the operators J_g, I_g, and M_g on the weighted Dirichlet spaces $L^2_a(\omega)$.

Lemma 1.1. *For* $w\in D$, *let* k_w *be the normalized reproducing kernels of* $L^2_a(\omega)$. *Then for* $f\in L^2_a(\omega)$,

$$|f(w)|\leq|k_w(w)|\left(\int_D|f(z)|^2\,\omega(z)\,dA(z)\right)^{\frac{1}{2}}.$$

Proof. Since k_w are the normalized reproducing kernels, the normalized reproducing kernel k_w for the space $L^2_a(\omega)$ is given by

$$k_w(z)=\frac{K(z,\ w)}{K(w,\ w)},$$

where $K(z,\ w)$ be the reproducing kernel of $L^2_a(\omega)$. So we have

$$|k_w(w)|^{-1}f(w)=<f,\ k_w>.$$

By using Schwarz inequality,

$$\begin{aligned}|k_w(w)|^{-1}|f(w)|&=|<f,\ k_w>|\\&\leq\int_D|f(u)||k_w(u)|\omega(u)dA(u)\\&\leq\left(\int_D|f(u)|^2\omega(u)dA(u)\right)^{\frac{1}{2}}\left(\int_D|k_w(u)|^2\omega(u)dA(u)\right)^{\frac{1}{2}}\\&=\left(\int_D|f(u)|^2\omega(u)dA(u)\right)^{\frac{1}{2}}.\quad\square\end{aligned}$$

In [2], A. Aleman and A. G. Siskakis proved the following lemma:

Lemma 1.2. *Suppose* X *is a Banach space, satisfying the following conditions*:

(1) *For each* $\lambda\in D$, $L_\lambda(f)=f(\lambda)$ *is a bounded linear functional on* X.

(2) *For each* $\sigma\in\partial D$, *the operator* $U_\sigma(f)(z)=f(\sigma z)$ *is bounded on* X *and* $\sup_{\sigma\in\partial D}\|U_\sigma\|<\infty$.

(3) *For some* $s\in(0,\ 1)$ *and* $\psi_s(z)=sz+1-s$, *the composition operator*

$C_s(f)=f\circ\psi_s$ *is bounded on* X.

Let $B(X)=\{f\in X:\|f\|\leq 1\}$ *be the closed unit ball. Then there is* $C>0$ *such that*

$$\sup_{f\in B(X)}|f'(\lambda)|\leq\frac{C}{1-|\lambda|}\sup_{f\in B(X)}|f(\lambda)|.$$

Proof. See [2]. □

By using Lemma 1.1 and Lemma 1.2, we get the following result:

Theorem 1.3. *Suppose that* J_g *is a bounded operator on* $L_a^2(\omega)$. *Then for* g *analytic on* D, *the essential norm of the operator* J_g *on* $L_a^2(\omega)$ *have the following*:

$$\|J_g\|_e\sim\lim_{s\to 1^-}\sup_{|z|>s}(1-|z|^2)|g'(z)|$$

i.e. $C_1\lim_{s\to 1^-}\sup_{|z|>s}(1-|z|^2)|g'(z)|\leq\|J_g\|_e\leq C_2\lim_{s\to 1^-}\sup_{|z|>s}(1-|z|^2)|g'(z)|$ *for some constants* C_1, $C_2>0$.

Proof. Let $\{z_n\}$ be sequence of points in D such that

$$\lim_{n\to\infty}(1-|z_n|^2)|g'(z_n)|=\lim_{s\to 1^-}\sup_{|z|>s}(1-|z|^2)|g'(z)|$$

and $|z_n|\to 1\,(n\to\infty)$. Let k_z be the normalized reproducing kernel of $L_a^2(\omega)$. Let $f_n(z):=k_{z_n}(z)$.

Then $\{f_n\}_n$ have unit norm and tend to zero pointwise. Since the space $L_a^2(\omega)$ is reflexive, we have $f_n\to 0$ weakly (see [6, p318]). For any K compact operator on $L_a^2(\omega)$, we see that $Kf_n\to 0$ in $L_a^2(\omega)$. So we have

$$\begin{aligned}\|J_g-K\|&\geq\lim_{n\to\infty}\sup\|J_gf_n-Kf_n\|\\&\geq\lim_{n\to\infty}\sup\big|\|J_gf_n\|_{L_a^2(\omega)}-\|Kf_n\|_{L_a^2(\omega)}\big|\end{aligned}$$

$$= \limsup_{n\to\infty} \|J_g f_n\|_{L^2_a(\omega)}.$$

By using Lemma 1.1 and Lemma 1.2, we have

$$(1-|\lambda|)\,|(J_g f)'(\lambda)| \le C|k_\lambda(\lambda)|\left(\int_D |(J_g f)(u)|^2\,\omega(u)\,dA(u)\right)^{\frac{1}{2}}.$$

Applying $f = f_n$ and $\lambda = z_n$ to the above inequality, we have

$$\begin{aligned}(1-|z_n|)\,|g'(z_n)k_{z_n}(z_n)| &= (1-|z|)|(J_g f_n)'(z_n)| \\ &\le C|k_{z_n}(z_n)|\left(\int_D |(J_g f_n)(u)|^2\,\omega(u)\,dA(u)\right)^{\frac{1}{2}}.\end{aligned}$$

Thus we have

$$(1-|z_n|^2)\,|g'(z_n)| \le 2C\left(\int_D |(J_g f_n)(u)|^2\ \omega(u)\,dA(u)\right)^{\frac{1}{2}} = 2C\|J_g f_n\|_{L^2_a(\omega)}.$$

Hence we have

$$\begin{aligned}\|J_g - K\| &\ge \limsup_{n\to\infty} \|J_g f_n\|_{L^2_a(\omega)} \\ &\ge \frac{1}{2C}\limsup_{n\to\infty}(1-|z_n|^2)\,|g'(z_n)| \\ &= \frac{1}{2C}\lim_{n\to\infty}(1-|z_n|^2)\,|g'(z_n)| \\ &= \frac{1}{2C}\lim_{s\to 1^-}\sup_{|z|>s}(1-|z|^2)\,|g'(z)|.\end{aligned}$$

Hence we have

$$\|J_g\|_e \ge \frac{1}{2C}\lim_{s\to 1^-}\sup_{|z|>s}(1-|z|^2)\,|g'(z)|.$$

Next, let $g \in B$. And let $g_r(z) = g(rz)$ for $0 < r < 1$. Then $g_r \in B_0$. In fact, we have

$$\begin{aligned}(1-|z|^2)\,|g'_r(z)| &= (1-|z|^2)r|g'(rz)| \\ &= (1-|rz|^2)\,|g'(rz)|r\frac{1-|z|^2}{1-|rz|^2}\end{aligned}$$

$$\leq \|g\|_B \ r\frac{1-|z|^2}{1-|rz|^2}$$
$$\to 0\,(|z|\to 1^-).$$

Hence we see that $g_r \in B_0$. By Corollary 1 of [2], we see that J_{g_r} is compact on $L_a^2(\omega)$. So we have

$$\|J_g\|_e \leq \|J_g - J_{g_r}\| = \|J_{g-g_r}\|.$$

By the boundedness of J_{g-g_r}, we have that $\|J_{g-g_r}\| \leq C\|g-g_r\|_B$ for some positive constant $C>0$. On the other hand, we have

$$\|g-g_r\|_B \leq \sup_{\delta<|z|<1}(1-|z|^2)\,|rg'(rz)-g'(z)| + \sup_{|z|<\delta}(1-|z|^2)|rg'(rz)-g'(z)|.$$

The second term above approaches zero as $r\to 1^-$ since $rf'(rz)\to f'(z)$ uniformly for $|z|\leq\delta$. If $\delta< r<1$ and $\delta<|z|<1$, then $\delta^2< r|z|<1$. So we have

$$\sup_{\delta<|z|<1}(1-|z|^2)\,|rg'(rz)| \leq \sup_{\delta^2<|rz|<1}(1-|rz|^2)\,|g'(rz)| = \sup_{\delta^2<|\zeta|<1}(1-|\zeta|^2)\,|g'(\zeta)|.$$

And we also have

$$\sup_{\delta<|z|<1}(1-|z|^2)\,|g'(z)| \leq \sup_{\delta^2<|z|<1}(1-|z|^2)\,|g'(z)|.$$

Hence we have for any $0<\delta<1$

$$\|g-g_r\|_B \leq \sup_{\delta^2<|z|<1}(1-|z|^2)\,|g'(z)|.$$

Put $s=\delta^2$. Since $s\in(0,\,1)$ is an arbitrary, we have

$$\|J_g\|_e \leq C \lim_{s\to 1^-}\sup_{s<|z|}(1-|z|^2)\,|g'(z)|. \quad \square$$

Corollary 1.4. *Suppose that J_g is a bounded operator on $L_a^2(D)$. Then for g analytic on D, the essential norm of the operator J_g on $L_a^2(D)$ have the following*:

$$\|J_g\|_e \sim \lim_{s\to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)|$$

$$i.e.\ C_1 \lim_{s\to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)| \le \|J_g\|_e \le C_2 \lim_{s\to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)|$$

for some constants $C_1,\ C_2 > 0$.

Lemma 1.5. *Let* k_z *be the normalized reproducing kernel of* $L_a^2(\omega)$. *Then* $(1-|z|^2)|k'_z(z)|$ *is comparable to* $|k_z(z)|$.

Proof. Let k_z be the normalized reproducing kernel of $L_a^2(\omega)$. Then the normalized reproducing kernel for the space $L_a^2(\omega)$ is given by

$$k_w(z) = \frac{K(z,\ w)}{\sqrt{K(w,\ w)}},$$

where $K(z,\ w)$ be the reproducing kernel of $L_a^2(\omega)$. So we need to prove that for some positive constants $C_1,\ C_2 > 0$

$$C_1|K(z,\ z)| \le (1-|z|^2)\,|K'(z,\ z)| \le C_2|K(z,\ z)|.$$

By the direct calculation, we have

$$K(z,\ w) = \frac{1}{2}\sum_{n\ge 0}\frac{1}{\gamma_n}(\overline{w}z)^n,\ \gamma_n = \int_0^1 r^{2n+1}\omega(r)\,dr.$$

So we have

$$K'(z,\ w) = \frac{1}{2}\sum_{n\ge 1}\frac{1}{\gamma_n}(\overline{w})^n n z^{n-1}.$$

Applying $z = w$, we have

$$K'(w,\ w) = \frac{1}{2}\sum_{n\ge 1} n|w|^{-1}\frac{1}{\gamma_n}|w|^{2n}.$$

By using the assumption $\lim_{n\to\infty}\frac{\gamma_{n+1}}{\gamma_n} = 1$, we have

$$(1-|w|^2)\,2\,|w|\ |K'(w,\ w)| = (1-|w|^2)\sum_{n\ge 1}\frac{1}{\gamma_n}n|w|^{2n}$$

$$
\begin{aligned}
&= (1-|w|^2)\Big(\frac{1}{\gamma_1}|w|^2+\frac{2}{\gamma_2}|w|^4+\frac{3}{\gamma_3}|w|^6\cdots\Big)\\
&= \Big(\frac{1}{\gamma_1}|w|^2+\frac{2}{\gamma_2}|w|^4+\frac{3}{\gamma_3}|w|^6\cdots\Big)\\
&\quad -\Big(\frac{1}{\gamma_1}|w|^4+\frac{2}{\gamma_2}|w|^6+\frac{3}{\gamma_3}|w|^8\cdots\Big)\\
&= \frac{1}{\gamma_1}|w|^2+\Big(\frac{2}{\gamma_2}-\frac{1}{\gamma_1}\Big)|w|^4+\Big(\frac{3}{\gamma_3}-\frac{2}{\gamma_2}\Big)|w|^6\cdots\\
&= \frac{1}{\gamma_1}|w|^2+\frac{1}{\gamma_2}(2-\frac{\gamma_2}{\gamma_1}|w|^4\\
&\quad +\frac{1}{\gamma_3}\Big(3-2\frac{\gamma_3}{\gamma_2}\Big)|w|^6\cdots\\
&\sim \sum_{n\geq 0}\frac{1}{\gamma_n}|w|^{2n}=2\,|K(w,\,w)|.
\end{aligned}
$$

Hence we see that $(1-|z|^2)|k'_z(z)|$ is comparable to $|k_z(z)|$. □

By using Lemma 1.5, we get the following result:

Theorem 1.6. *Suppose that I_g is a bounded operator on $L^2_a(\omega)$. Then for g analytic on D, the essential norm of the operator I_g on $L^2_a(\omega)$ have the following*:

$$
\|I_g\|_e \sim \lim_{s\to 1^-}\sup_{|z|>s}|g(z)|\,(=\|g\|_\infty),
$$
$$
i.e.\ C\lim_{s\to 1^-}\sup_{|z|>s}|g(z)|\leq\|I_g\|_e\leq\lim_{s\to 1^-}\sup_{|z|>s}|g(z)|.
$$

Proof. Let $\{z_n\}$ be sequence of points in D such that

$$
\lim_{n\to\infty}|g(z_n)|=\lim_{s\to 1^-}\sup_{|z|>s}|g(z)|
$$

and $|z_n|\to 1\,(n\to\infty)$. Let k_z be the normalized reproducing kernel of $L^2_a(\omega)$. Let $f_n(z):=k_{z_n}(z)$. Then $\{f_n\}_n$ have unit norm and tend to zero pointwise. Since the space $L^2_a(\omega)$ is reflexive, we have $f_n\to 0$ weakly (see [6, p318]). For any K compact operator on $L^2_a(\omega)$, we see that $K f_n\to 0$

in $L_a^2(\omega)$. So we have

$$\begin{aligned}\|I_g - K\| &\geq \limsup_{n\to\infty} \|I_g f_n - K f_n\| \\ &\geq \limsup_{n\to\infty} \big| \|I_g f_n\|_{L_a^2(\omega)} - \|K f_n\|_{L_a^2(\omega)} \big| \\ &= \limsup_{n\to\infty} \|I_g f_n\|_{L_a^2(\omega)}.\end{aligned}$$

By using Lemma 1.1 and Lemma 1.2, we have

$$(1-|\lambda|)\,|(I_g f)'(\lambda)| \leq C\,|k_\lambda(\lambda)| \left(\int_D |(I_g f)(u)|^2 \omega(u)\, dA(u) \right)^{\frac{1}{2}}.$$

Applying $f = f_n$ and $\lambda = z_n$ to the above inequality, we have

$$\begin{aligned}(1-|z_n|^2)\,|g(z_n)k'_{z_n}(z_n)| &= (1-|z_n|^2)\,|(I_g f_n)'(z_n)| \\ &\leq 2C\,|k_{z_n}(z_n)| \left(\int_D |(I_g f_n)(u)|^2 \omega(u)\, dA(u) \right)^{\frac{1}{2}}.\end{aligned}$$

By Lemma 1.5, we have $(1-|z|^2)\,|k'_z(z)|$ is comparable to $|k_z(z)|$. So we have

$$\begin{aligned}|g(z_n)k_{z_n}(z_n)| &\sim (1-|z_n|^2)|g(z_n)k'_{z_n}(z_n)| \\ &\leq 2C\,|k_{z_n}(z_n)| \left(\int_D |(I_g f_n)(u)|^2 \omega(u)\, dA(u) \right)^{\frac{1}{2}}.\end{aligned}$$

Thus we have for some positive constant $K > 0$,

$$|g(z_n)| \leq 2CK \left(\int_D |(I_g f_n)(u)|^2 \omega(u)\, dA(u) \right)^{\frac{1}{2}} = 2CK \|I_g f_n\|_{L_a^2(\omega)}.$$

Hence we have

$$\begin{aligned}\|I_g - K\| &\geq \limsup_{n\to\infty} \|I_g f_n\|_{L_a^2(\omega)} \\ &\geq \frac{1}{2CK} \limsup_{n\to\infty} |g(z_n)| \\ &= \frac{1}{2CK} \lim_{n\to\infty} |g(z_n)| \\ &= \frac{1}{2CK} \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|.\end{aligned}$$

Thus we have

$$\|I_g\|_e \geq \frac{1}{2CK} \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|.$$

Since the essential norm of the operator I_g on $L_a^2(\omega)$ is less than the operator norm of the operator I_g on $L_a^2(\omega)$, we have $\|I_g\|_e \leq \|I_g\|$. Since $\|I_g\| \leq \sup_{z \in D} |g(z)| = \sup_{|z|>s} |g(z)|$ for any $0 < s < 1$, we have

$$\|I_g\|_e \leq \|I_g\| \leq \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|. \quad \square$$

Corollary 1.7. *Suppose that I_g is a bounded operator on $L_a^2(D)$. Then for g analytic on D, the essential norm of the operator I_g on $L_a^2(D)$ have the following*:

$$\|I_g\|_e \sim \lim_{s \to 1^-} \sup_{|z|>s} |g(z)| \, (=\|g\|_\infty), \ i.e. \ C \lim_{s \to 1^-} \sup_{|z|>s} |g(z)| \leq \|I_g\|_e \leq \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|.$$

In [3], S. Axler, J. B. Conway and G. McDONALD proved the essential norm of Toeplitz operators on the Bergman spaces. We also prove similar result by the different method:

Thorem 1.8. *Suppose that M_g is a bounded operator on $L_a^2(\omega)$. Then for g analytic on D, the essential norm of the operator M_g on $L_a^2(\omega)$ have the following*:

$$\|M_g\|_e = \lim_{s \to 1^-} \sup_{|z|>s} |g(z)| \, (=\|g\|_\infty).$$

Proof. Let $\{z_n\}$ be sequence of points in D such that

$$\lim_{n \to \infty} |g(z_n)| = \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|$$

and $|z_n| \to 1 \, (n \to \infty)$. Let $\{z_n\}$ be sequence of points in D such that

$$\lim_{n \to \infty} |g(z_n)| = \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|$$

and $|z_n| \to 1 \, (n \to \infty)$. Let k_z be the normalized reproducing kernel of L_a^2

(ω). Let $f_n(z) := k_{z_n}(z)$. Then $\{f_n\}_n$ have unit norm and tend to zero pointwise. Since the space $L_a^2(\omega)$ is reflexive, we have $f_n \to 0$ weakly (see [6, p318]). For any K compact operator on $L_a^2(\omega)$, we see that $Kf_n \to 0$ in $L_a^2(\omega)$. So we have

$$\begin{aligned}\|M_g - K\| &\geq \limsup_{n\to\infty} \|M_g f_n - K f_n\| \\ &\geq \limsup_{n\to\infty} \left| \|M_g f_n\|_{L_a^2(\omega)} - \|K f_n\|_{L_a^2(\omega)} \right| \\ &= \limsup_{n\to\infty} \|M_g f_n\|_{L_a^2(\omega)}.\end{aligned}$$

By using Lemma 1.1, we have

$$|M_g f(\lambda)| \leq |k_\lambda(\lambda)| \left(\int_D |(M_g f)(u)|^2 \omega(u)\, dA(u) \right)^{\frac{1}{2}}.$$

Applying $f = f_n$ and $\lambda = z_n$ to the above inequality, we have

$$|g(z_n) k_{z_n}(z_n)| = |M_g f_n(z_n)| \leq |k_{z_n}(z_n)| \left(\int_D |(M_g f_n)(u)|^2 \omega(u)\, dA(u) \right)^{\frac{1}{2}}.$$

So we have

$$|g(z_n)| \leq \left(\int_D |(M_g f_n)(u)|^2 \omega(u)\, dA(u) \right)^{\frac{1}{2}}.$$

Thus we have

$$|g(z_n)| \leq \left(\int_D |(M_g f_n)(u)|^2 \omega(u)\, dA(u) \right)^{\frac{1}{2}} = \|M_g f_n\|_{L_a^2(\omega)}.$$

Hence we have

$$\begin{aligned}\|M_g - K\| &\geq \limsup_{n\to\infty} \|M_g f_n\|_{L_a^2(\omega)} \\ &\geq \limsup_{n\to\infty} |g(z_n)| \\ &= \lim_{n\to\infty} |g(z_n)| \\ &= \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|.\end{aligned}$$

Since the essential norm of the operator M_g on $L_a^2(\omega)$ is less than the operator norm of the operator M_g on $L_a^2(\omega)$, we have $\|M_g\|_e \leq \|M_g\|$.

Since $\|M_g\| \leq \sup_{z\in D}|g(z)|=\sup_{|z|>s}|g(z)|$ for any $0<s<1$, we have

$$\|M_g\|_e \leq \|M_g\| \leq \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|. \quad \square$$

The essential norm of the operators J_g, I_g, and M_g on the Bergman spaces L_a^p

In this section we study the essential norms of the operators J_g, I_g, and M_g on the Bergman spaces L_a^p for $p>1$.

Lemma 2.1. *For $1\leq p<\infty$, let $f\in L_a^p$. Then for any $a\in D$*

$$|f'(a)|\leq \frac{1}{(1-|a|^2)^{\frac{p+2}{p}}}\left(\int_D |f'(z)|^p(p+1)(1-|z|^2)^p\, dA(z)\right)^{\frac{1}{p}}.$$

Proof. Let u be a positive subharmonic function in D. Then we have

$$u(0)\leq \int_D u(z)(p+1)(1-|z|^2)^p\, dA(z)$$

(see [4]). By applying $u=u\circ\varphi_a (a\in D)$, we have

$$\begin{aligned} u(a) &\leq \int_D u\circ\varphi_a(z)(p+1)(1-|z|^2)^p\, dA(z) \\ &\leq \int_D u(z)\frac{(1-|a|^2)^{(2+p)}}{|1-\bar{a}z|^{2(2+p)}}(p+1)(1-|z|^2)^p\, dA(z). \end{aligned}$$

Let $f\in L_a^p$. By applying $u(z)=|f'(z)|^p|\frac{|1-\bar{a}z|^{2(2+p)}}{(1-|a|^2)^{(2+p)}}$ to the above inequality, we have

$$|f'(a)|^p|\frac{(1-|a|^2)^{2(2+p)}}{(1-|a|^2)^{(2+p)}}|\leq \int_D |f'(z)|^p(p+1)(1-|z|^2)^p\, dA(z).$$

So we have

$$|f'(a)| \le \frac{1}{(1-|a|^2)^{\frac{p+2}{p}}} \left(\int_D |f'(z)|^p (p+1)(1-|z|^2)^p \, dA(z) \right)^{\frac{1}{p}}. \quad \square$$

By using Lemma 2.1, we can prove the following results:

Theorem 2.2. *Let $1 < p < \infty$. Suppose that J_g is a bounded operator on L_a^p. Then for g analytic on D, the essential norm of the operator J_g on L_a^p have the following*:

$$\|J_g\|_e \sim \lim_{s \to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)|$$

$$\text{i.e. } C_1 \lim_{s \to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)| \le \|J_g\|_e \le C_2 \lim_{s \to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)|$$

for some constants C_1, $C_2 > 0$.

Proof. Let $\{z_n\}$ be sequence of points in D such that

$$\lim_{n \to \infty} (1-|z_n|^2)\,|g'(z_n)| = \lim_{s \to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)|$$

and $|z_n| \to 1\,(n \to \infty)$. Let $f_n(z) := \left(\frac{1-|z_n|^2}{(1-\overline{z_n}z)^2} \right)^{\frac{2}{p}}$. Then $\{f_n\}_n$ have unit norm and tend to zero pointwise. Since the space L_a^p is reflexive for all $1 < p < \infty$, we have $f_n \to 0$ weakly (see [6, p318]). For any K compact operator on L_a^p, we see that $Kf_n \to 0$ in L_a^p. So we have by using Lemma 2.1,

$$\begin{aligned}
\|J_g - K\| &\ge \lim_{n \to \infty} \sup \|J_g f_n - K f_n\| \\
&\ge \lim_{n \to \infty} \sup \left| \|J_g f_n\|_{L^p(D,dA)} - \|K f_n\|_{L^p(D,dA)} \right| \\
&= \lim_{n \to \infty} \sup \|J_g f_n\|_{L^p(D,dA)} \\
&\ge C \lim_{n \to \infty} \sup \left(\int_D |g'(z) f_n(z)|^p (p+1)(1-|z|^2)^p \, dA(z) \right)^{\frac{1}{p}} \\
&\ge C \lim_{n \to \infty} \sup (1-|z_n|^2)^{\frac{2+p}{p}} \, |g'(z_n) f_n(z_n)| \\
&= C \lim_{n \to \infty} (1-|z_n|^2)\,|g'(z_n)| \\
&= C \lim_{s \to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)|
\end{aligned}$$

Hence we have

$$\|J_g\|_e \geq C \lim_{s\to 1^-} \sup_{|z|>s} (1-|z|^2)\,|g'(z)|.$$

Since we can prove the converse inequality as well as the proof of Theorem 1.3, we omit it.

□

Theorem 2.3. *Let* $1 < p < \infty$. *Suppose that* I_g *is a bounded operator on* L_a^p. *Then for* g *analytic on* D, *the essential norm of the operator* I_g *on* L_a^p *have the following*:

$$\|I_g\|_e \sim \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|\,(=\|g\|_\infty),\ i.e.\ C \lim_{s\to 1^-} \sup_{|z|>s} |g(z)| \leq \|I_g\|_e \leq \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|.$$

Proof. Let $\{z_n\}$ be sequence of points in D such that

$$\lim_{n\to\infty} |g(z_n)| = \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|$$

and $|z_n| \to 1\,(n\to\infty)$. Let $f_n(z) := \left(\frac{1-|z_n|^2}{(1-\overline{z_n}z)^2}\right)^{\frac{2}{p}}$. Then $\{f_n\}_n$ have unit norm and tend to zero pointwise. Since the space L_a^p is reflexive for all $1 < p < \infty$, we have $f_n \to 0$ weakly (see [6, p318]). For any K compact operator on L_a^p, we see that $Kf_n \to 0$ in L_a^p. So we have by using Lemma 2.1,

$$\begin{aligned}
\|I_g - K\| &\geq \lim_{n\to\infty} \sup \|I_g f_n - K f_n\| \\
&\geq \lim_{n\to\infty} \sup \big|\|I_g f_n\|_{L^p(dA)} - \|K f_n\|_{L^p(dA)}\big| \\
&= \lim_{n\to\infty} \sup \|I_g f_n\|_{L^p(dA)} \\
&\geq C \lim_{n\to\infty} \sup \left(\int_D |g(z) f'_n(z)|^p (p+1)(1-|z|^2)^p\, dA(z)\right)^{\frac{1}{p}} \\
&\geq C \lim_{n\to\infty} \sup (1-|z_n|^2)^{\frac{2+p}{p}} |g(z_n) f'_n(z_n)| \\
&= \frac{4}{p} C \lim_{n\to\infty} |g(z_n)|
\end{aligned}$$

$$= \frac{4}{p} C \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|$$

Hence we have for $C_1 := \frac{4}{p} C > 0$

$$\|I_g\|_e \geq C_1 \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|.$$

Since the essential norm of the operator I_g on L_a^p is less than the operator norm of the operator I_g on L_a^p, we have $\|I_g\|_e \leq \|I_g\|$. Since $\|I_g\| \leq \sup_{z \in D} |g(z)| = \sup_{|z|>s} |g(z)|$ for any $0 < s < 1$, we have

$$\|I_g\|_e \leq \|I_g\| \leq \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|. \quad \square$$

Lemma 2.4. ([4, p53]) *For* $1 \leq p < \infty$, *let* $f \in L_a^p$. *Then for any* $a \in D$,

$$|f(a)| \leq \frac{1}{(1-|a|^2)^{\frac{2}{p}}} \|f\|_{L^p(dA)}.$$

Proof. See [4, p53]. $\square$

In [3], S. Axler, J. B. Conway and G. McDONALD proved the essential norm of Toeplitz operators on the Bergman spaces. We also prove similar result by the different method:

Thorem 2.5. *Let* $1 < p < \infty$. *Suppose that* M_g *is a bounded operator on* L_a^p. *Then for g analytic on D, the essential norm of the operator* M_g *on* L_a^p *have the following*:

$$\|M_g\|_e = \lim_{s \to 1^-} \sup_{|z|>s} |g(z)| \, (= \|g\|_\infty).$$

Proof. Let $\{z_n\}$ be sequence of points in D such that

$$\lim_{n \to \infty} |g(z_n)| = \lim_{s \to 1^-} \sup_{|z|>s} |g(z)|$$

and $|z_n| \to 1\,(n \to \infty)$. Let $f_n(z) := \left(\frac{1-|z_n|^2}{(1-\overline{z_n}z)^2}\right)^{\frac{2}{p}}$. Then $\{f_n\}_n$ have unit norm and tend to zero pointwise. Since the space L_a^p is reflexive for all $1 < p < \infty$, we have $f_n \to 0$ weakly (see [6, p318]). For any K compact operator on L_a^p, we see that $Kf_n \to 0$ in L_a^p. So we have by using Lemma 2.4,

$$\begin{aligned}
\|M_g - K\| &\geq \limsup_{n\to\infty} \|M_g f_n - K f_n\| \\
&\geq \limsup_{n\to\infty} \left| \|M_g f_n\|_{L^p(dA)} - \|K f_n\|_{L^p(dA)} \right| \\
&= \limsup_{n\to\infty} \|M_g f_n\|_{L^p(dA)} \\
&\geq \limsup_{n\to\infty} (1-|z_n|^2)^{\frac{2}{p}} |g(z_n) f_n(z_n)| \\
&= \lim_{n\to\infty} |g(z_n)| \\
&= \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|
\end{aligned}$$

Hence

$$\|M_g\|_e \geq \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|.$$

Since the essential norm of the operator M_g on L_a^p is less than the operator norm of the operator M_g on L_a^p, we have $\|M_g\|_e \leq \|M_g\|$. Since $\|M_g\| \leq \sup_{z\in D} |g(z)| = \sup_{|z|>s} |g(z)|$ for any $0 < s < 1$, we have

$$\|M_g\|_e \leq \|M_g\| \leq \lim_{s\to 1^-} \sup_{|z|>s} |g(z)|. \quad \square$$

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