The Boundedness And Compactness Of Extended Cesáro Operators On Mixed Norm Spaces

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Abstract

We study an extended Cesáro operator I_g with holomorphic symbol g in the unit ball B of C^n:

$$I_g(f)(z) := \int_0^1 \Re f(tz) g(tz) \frac{dt}{t},\ g \in H(B),\ z \in B,$$

where $\Re f(z) = \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i}$

Key Words and Phrases: extended Cesáro operator, integration operator, mixed norm space, Bloch space, compactness, boundedness.

§1. Introduction

Let $D = \{z \in C : |z| < 1\}$ denote the open unit disk in the complex plane C.

For $1 \leq p < +\infty$, the Lebesgue space $L^p(D, dA)$ is defined to be the Banach space of Lebesgue measurable functions on the open unit disk D with

$$\|f\|_{L^p(dA)} := \left(\int_D |f(z)|^p dA(z)\right)^{\frac{1}{p}} < +\infty,$$

where $dA(z)$ is the normalized area measure on D. The Bergman space

$L_a^p(D)$ is defined to be the subspace of $L^p(D, dA)$ consisting of analytic functions. For $0 < p < +\infty$, the Hardy space H^p is defined to be the Banach space of analytic functions f on D with

$$\|f\|_p := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < +\infty.$$

And the weighted Dirichlet space D^α is defined to be the space of analytic functions f on D such that

$$\|f\|_{D^\alpha} := |f(0)| + \int_D (1-|z|^2)^\alpha |f'(z)|^2 \, dA(z) < +\infty.$$

If $\alpha = 1$, then D^α is the Hardy space H^2. If $\alpha = 2$, then D^α is the Bergman space $Ł_a^2$.

Bloch space B is defined to be the space of analytic functions f on D such that

$$\|f\|_B := |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)| < +\infty.$$

For g analytic on D, the operators I_g, J_g are defined by the following:

$$(0.1) \quad I_g(h)(z) := \int_0^z g(\zeta) h'(\zeta) d\zeta, \; J_g(f)(z) := \int_0^z f(\zeta) g'(\zeta) d\zeta.$$

If $g(z) = z$, then J_g is the integration operator. If $g(z) = \log \frac{1}{1-z}$, then J_g is the Cesáro operator.

In [2], A. Aleman and A. G. Siskakis studied the operator J_g defined on the weighted Bergman space.

In [6], Ch. Pommerenke proved the following:

Theorem A. *The operator J_g is bounded on $D^1 = H^2$ if and only if*

$$g \in BMOA.$$

In [2], A. Aleman and A. G. Siskakis proved the following:

Theorem B. *Let* $\alpha > 1$. *Then for* g *analytic on* D, *the operator* J_g *is bounded on* D^α *if and only if*

$$\sup_{z\in D}(1-|z|^2)|g'(z)| < +\infty, \ i.e\ g \in B.$$

In [10], we proved the following results:

Theorem C. For $\alpha > 0$ *The operator* I_g *is bounded on* D^α *if and only if*

$$\sup_{z\in D}|g(z)| < +\infty.$$

And I_g *is compact on* D^α *if and only if*

$$g \equiv 0.$$

Let H(B) be the class of all holomorphic functions on the unit ball B of C^n. For $f \in H(B)$ having the homogeneous expansion $f = \sum_{j=1}^{\infty} F_j$, let $\Re f(z) = \sum_{j=1}^{\infty} jF_j(z)$ be the radial derivative of g. It is trivial that $\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}$. For $g \in H(B)$, the operator I_g with symbol g is defined on $H(B)$ as

$$(0.2) \quad I_g(f)(z) := \int_0^1 \Re f(tz) g(tz) \frac{dt}{t}, \ g \in H(B),\ z \in B.$$

It is trivial that when $n=1$, (0.2) is just (0.1).

A positive continuous function φ on $[0, 1)$ is called normal if there are two constants $(0<)a<b$ such that

$$\frac{\varphi(r)}{(1-r)^a} \downarrow 0, \quad \frac{\varphi(r)}{(1-r)^b} \uparrow +\infty$$

as $r \to 1^-$. For normal φ, and for $f \in H(B)$, we put

$$\|f\|_{p,q,\varphi}=\left\{\int_0^1 M_q^p(f,r)\frac{\varphi^p(r)}{1-r}dr\right\}^{\frac{1}{p}}(0<p<+\infty)$$

and

$$\|f\|_{\infty,q,\varphi}:=\sup_{0<r<1} M_q(f,r)\varphi(r)$$

Here

$$M_q(f,r)=\left\{\int_{\partial B}|f(r\zeta)|^q d\sigma(\zeta)\right\}^{\frac{1}{q}}(0<q<+\infty),$$
$$M_\infty(f,r)=\sup_{\zeta\in\partial B}|f(r\zeta)|$$

The mixed norm space $H_{p,q}(\varphi)$, $0<p,\ q\leq\infty$, consists of all $f\in H(B)$ such that $\|f\|_{p,q,\varphi}<\infty$. When $0<p=q<+\infty$, the mixed norm space $H_{p,q}(\varphi)$ is just the weighted Bergman space

$$A_\alpha^p(\varphi)=\{f\in H(B):\|f\|_{A_\alpha^p}=\left(\int_B|f(z)|^p\frac{\varphi^p(|z|)}{1-|z|}dm(|z|)\right)^{\frac{1}{p}}<+\infty\}.$$

For a holomorphic function f, f is called a Bloch function if

$$\|f\|_B=\sup\{|\Re f(z)|(1-|z|^2):\ z\in B\}<+\infty,$$

and f is called a little Bloch function if

$$\lim_{|z|\to 1}|\Re f(z)|(1-|z|^2)=0.$$

The space of all Bloch and little Bloch functions will be denoted by B and B_0, respectively.

In [5], Z. Hu proved the following:

Theorem D. Let $0<p,q\leq+\infty$, and let φ be normal. Then for $g\in H(B)$

(I) J_g is bounded on $H_{p,q}(\varphi)$ if and only if $g\in B$.

(II) J_g is compact on $H_{p,q}(\varphi)$ if and only if $g \in B_0$.

In this paper, we study the operator I_g on the mixed norm space $H_{p,q}(\varphi)$.

Throughout this paper, C, K will denote positive constant whose value is not necessary the same at each occurrence.

§2. The boundedness and compactness of I_g on the mixed norm space $H_{p,q}(\varphi)$.

In this section, we prove the boundedness and compactness of the operator I_g on the mixed norm space $H_{p,q}(\varphi)$. In [5], Z. Hu showed the following:

Theorem 1.([5]) Let $0 < p, q \leq +\infty$, and let φ be normal. Then for $g \in H(B)$

(I) J_g is bounded on $H_{p,q}(\varphi)$ if and only if $g \in B$.

(II) J_g is compact on $H_{p,q}(\varphi)$ if and only if $g \in B_0$.

Theorem 2.([5]) Let $0 < p, q \leq +\infty$ and let m be a positive integer. Then for $f \in H(B)$,

$$\|f\|_{p,q,\varphi} \simeq \sum_{j=0}^{m-1} |grad_j f(0)| + \left\{ \int_0^1 M_q^p(\Re^m f, r)(1-r^2)^{mp} \frac{\varphi^p(r)}{1-r} \, dr \right\}^{\frac{1}{p}}.$$

Lemma 2.([5]) Given $0 < p, q \leq +\infty$, take $\beta > b$ and

$$f_\zeta(z) = \frac{(1-|\zeta|^2)^\beta}{\varphi(|\zeta|)(1- < z, \zeta >)^{\frac{n}{q}+\beta}}, \quad \zeta \in B.$$

Then $\|f_\zeta\|_{p,q,\varphi} \leq C$. Here C is independent of ζ.

By using the above results, we prove the following with respect to the boundedness of the operator I_g on the mixed norm space $H_{p,q}(\varphi)$:

Theorem 1.1. Let $0 < p, q \leq +\infty$, and let φ be normal. Then for $g \in H(B)$,

I_g is bounded on $H_{p,q}(\varphi)$ if and only if $g \in H^\infty$.

Proof. Suppose that $g \in H^\infty$. By the proof of Theorem 1 in [5], we see that for $f, g \in H(B)$,

$$\Re(I_g f)(z) = (f \Re g)(z). \quad (1.1)$$

So by using Thoerem 2 in [5] and (1.1), we have, for any $f \in H(B)$,

$$\begin{aligned}\|I_g f\|_{p,q,\varphi} &\leq c\left\{\int_0^1 M_q^p(\Re(I_g f),\ r)(1-r^2)^p \frac{\varphi^p(r)}{1-r}\,dr\right\}^{\frac{1}{p}} \\ &\leq c\|g\|_\infty \left\{\int_0^1 M_q^p(\Re^1 f,\ r)(1-r^2)^p \frac{\varphi^p(r)}{1-r}\,dr\right\}^{\frac{1}{p}} \\ &\leq c\|g\|_\infty \|f\|_{p,q,\varphi}.\end{aligned}$$

Hence we have that I_g is bounded on $H_{p,q}(\varphi)$.

To prove the converse, suppose that I_g is bounded on $H_{p,q}(\varphi)$.

Let $f_\zeta(z) = \dfrac{(1-|\zeta|^2)^\beta}{\varphi(|\zeta|)(1-<z,\zeta>)^{\frac{n}{q}+\beta}}$, $\zeta \in B$. Then we have

$$\begin{aligned}\Re f_\zeta(z) &= \frac{(1-|\zeta|^2)^\beta}{\varphi(|\zeta|)} \sum_{j=1}^n z_j \frac{\partial\{(1-<z,\zeta>)^{-\frac{n}{q}-\beta}\}}{\partial z_j} \\ &= \frac{(1-|\zeta|^2)^\beta}{\varphi(|\zeta|)}\left(\frac{n}{q}+\beta\right) <z,\zeta> \frac{1}{(1-<z,\zeta>)^{\frac{n}{q}+\beta+1}}\end{aligned}$$

So we have

$$\Re f_\zeta(\zeta)=\frac{(1-|\zeta|^2)^\beta}{\varphi(|\zeta|)}\left(\frac{n}{q}+\beta\right)|\zeta|^2\,\frac{1}{(1-|\zeta|^2)^{\frac{n}{q}+\beta+1}}$$

Notice that $(I_g f_\zeta)(0)=0$. Then for any $\zeta\in B$, by using Theorem 2 and Lemma 2 in [5] and (1.1), we have

$$\begin{aligned}\|I_g f\| &\geq c\|I_g f_\zeta\|_{p,q,\varphi}\\ &\geq c\left\{\int_0^1 M_q^p(\Re(I_g f_\zeta),\ r)(1-r^2)^p\frac{\varphi^p(r)}{1-r}\,dr\right\}^{\frac{1}{p}}\\ &\geq c\left\{\int_{\frac{|\zeta|+1}{2}}^{\frac{|\zeta|+3}{4}} M_q^p(\Re(I_g f_\zeta),\ r)(1-r^2)^p\frac{\varphi^p(r)}{1-r}\,dr\right\}^{\frac{1}{p}}\\ &\geq cM_q\left(\Re(I_g f_\zeta),\ \frac{1+|\zeta|}{2}\right)(1-|\zeta|^2)\varphi(|\zeta|)\\ &\geq c\,|\Re(I_g f_\zeta)(\zeta)|(1-|\zeta|^2)^{\frac{n}{q}}(1-|\zeta|^2)\varphi(|\zeta|)\\ &= c\,|g(\zeta)(\Re f_\zeta)(\zeta)|(1-|\zeta|^2)^{\frac{n}{q}+1}\varphi(|\zeta|)\\ &= c\,|g(\zeta)|\left(\frac{n}{q}+\beta\right)|\zeta|^2\end{aligned}$$

Hence we have

$$g\in H^\infty.\quad\Box$$

By using the above results, we prove the following with respect to the compactness of the operator I_g on the mixed norm space $H_{p,q}(\varphi)$:

Corollary 1.2. Let $0<p,\ q\leq+\infty$, and let φ be normal. Then for $g\in H(B)$,

I_g is compact on $H_{p,q}(\varphi)$ if and only if $g\equiv 0$.

Proof. Suppose that I_g is compact on $H_{p,q}(\varphi)$. Since f_ζ weakly convergence to zero in $H_{p,q}(\varphi)$, by the compactness and the proof of Theorem 1.1

$$|g(\zeta)|\leq C\|I_g f_\zeta\|_{p,q,\varphi}\to 0\,(|\zeta|\to 1^-)$$

Hence we have $g \equiv 0$. It is trivial the converse. □

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