Economic Review (Otaru University of Commerce),
Vol. 71, No. 2 & 3, 35-50, December, 2020.

Supplement to the paper "A unified treatment of agreement coefficients and their asymptotic results: The formula of the weighted mean of weighted ratios"

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This article supplements Ogasawara (2020), and gives (i) the partial derivatives of the sample chance-expected proportions with respect to the sample proportions of the associated multinomial distribution, evaluated at their population values in Section S1 and (ii) additional tables in Section S2.

Reference

Ogasawara, H. (2020). A unified treatment of agreement coefficients and their asymptotic results: The formula of the weighted mean of weighted ratios. *Journal of Classification* (online published) https://doi.org/10.1007/s00357-020-09366-1.

S1. The partial derivatives and associated results

In Section S1, $v_{i_1 \cdots i_m}$ is used rather than $w_{i_1 \cdots i_m} (i_j = 1,...,k; j = 1,...,m)$. In some cases it is more natural to use $w_{i_1 \cdots i_m}$ than $v_{i_1 \cdots i_m}$ as in $\hat{K}_{(\mathrm{G})}$ and $\hat{K}_{(\mathrm{GO})}$ (see Ogasawara, 2020, Appendix A2). In such cases, $v_{i_1 \cdots i_m} = 1 - w_{i_1 \cdots i_m}$ $(i_j = 1,...,k; j = 1,...,m)$ can be used. That is, $v_{i_1 \cdots i_m}$'s in the partial derivatives below can also be read as $1 - w_{i_1 \cdots i_m} (i_j = 1,...,k; j = 1,...,m)$.

S1.1 The first partial derivatives and $\boldsymbol{\pi}' \partial \pi_{\mathrm{e}(\cdot)}^{(v)} / \partial \boldsymbol{\pi}$

(i) $\pi_{\mathrm{e(B)}}^{(v)}$

$$\partial \pi_{\mathrm{e(B)}}^{(v)} / \partial \boldsymbol{\pi} = \mathbf{0} .$$

(ii) $\pi^{(v)}_{\mathrm{e(S)}}$

$$\frac{\partial \pi^{(v)}_{\mathrm{e(S)}}}{\partial \pi_{i_1 \cdots i_m}} = \frac{\partial \sum_{l_1,\cdots,l_m=1}^{k} v_{l_1 \cdots l_m} \prod_{j=1}^{m} \bar{\pi}_{l_j}}{\partial \pi_{i_1 \cdots i_m}}$$

$$= \frac{1}{m} \sum_{b=1}^{m} \left(\sum_{l_2=1}^{k} \cdots \sum_{l_m=1}^{k} v_{i_b l_2 \cdots l_m} \prod_{\substack{a=1 \\ a \neq 1}}^{m} \bar{\pi}_{l_a} + \sum_{l_1=1}^{k} \sum_{l_3=1}^{k} \cdots \sum_{l_m=1}^{k} v_{l_1 i_b l_3 \cdots l_m} \prod_{\substack{a=1 \\ a \neq 2}}^{m} \bar{\pi}_{l_a} \right.$$

$$\left. + \cdots + \sum_{l_1=1}^{k} \cdots \sum_{l_{m-1}=1}^{k} v_{i_1 \cdots l_{m-1} i_b} \prod_{\substack{a=1 \\ a \neq m}}^{m} \bar{\pi}_{l_a} \right)$$

$$= \sum_{b=1}^{m} \sum_{j=1}^{m} \frac{\mathbf{v}^{(i_b,\, j)}_{\pi(\mathrm{S})}{}' \mathbf{1}_{(k^{m-1})}}{m} \quad (i_j = 1, \ldots, k;\ j = 1, \ldots, m),$$

where $\mathbf{v}^{(i_b, j)}_{\pi(\mathrm{S})}$ is the $k^{m-1} \times 1$ vector, whose elements are

$$v_{l_1 \cdots l_{j-1} i_b l_{j+1} \cdots l_m} \prod_{\substack{a=1 \\ a \neq j}}^{m} \bar{\pi}_{l_a} \quad (l_a = 1, \ldots, k;\ a = 1, .., m;\ a \neq j).$$

$$\boldsymbol{\pi}'\frac{\partial \pi_{\mathrm{e(S)}}^{(v)}}{\partial \boldsymbol{\pi}} = \frac{1}{m}\sum_{i_1,\ldots,i_m=1}^{k}\sum_{b=1}^{m}\left\{\sum_{l_2=1}^{k}\cdots\sum_{l_m=1}^{k} v_{i_b l_2\cdots l_m}\left(\prod_{a=1}^{m}\bar{\pi}_{l_a}\right)\frac{1}{\bar{\pi}_{l_1}}\right.$$

$$+\sum_{l_1=1}^{k}\sum_{l_3=1}^{k}\cdots\sum_{l_m=1}^{k} v_{l_1 i_b l_3\cdots l_m}\left(\prod_{a=1}^{m}\bar{\pi}_{l_a}\right)\frac{1}{\bar{\pi}_{l_2}}$$

$$\left.+\cdots+\sum_{l_1=1}^{k}\cdots\sum_{l_{m-1}=1}^{k} v_{l_1\cdots l_{m-1} i_b}\left(\prod_{a=1}^{m}\bar{\pi}_{l_a}\right)\frac{1}{\bar{\pi}_{l_m}}\right\}\pi_{i_1\cdots i_m}$$

$$=\frac{1}{m}\sum_{b=1}^{m}\sum_{i_b=1}^{k}\left\{\sum_{l_2,\ldots,l_m=1}^{k} v_{i_b l_2\cdots l_m}\left(\prod_{a=1}^{m}\bar{\pi}_{l_a}\right)\frac{1}{\bar{\pi}_{l_1}}\right.$$

$$+\sum_{l_1,l_3,\ldots,l_m=1}^{k} v_{l_1 i_b l_3\cdots l_m}\left(\prod_{a=1}^{m}\bar{\pi}_{l_a}\right)\frac{1}{\bar{\pi}_{l_2}}$$

$$\left.+\cdots+\sum_{l_1,,\ldots,l_{m-1}=1}^{k} v_{l_1\cdots l_{m-1} i_b}\left(\prod_{a=1}^{m}\bar{\pi}_{l_a}\right)\frac{1}{\bar{\pi}_{l_m}}\right\}\pi_{i_b}^{(b)}.$$

(iii) $\pi_{\mathrm{e(SO)}}^{(v)}$

$$\frac{\partial \pi_{\mathrm{e(SO)}}^{(v)}}{\partial \pi_{i_1\cdots i_m}} = \frac{\partial \sum_{l_1,\cdots,l_m=1}^{k} v_{l_1\cdots l_m}\bar{\pi}_{l_1\cdots l_m}}{\partial \pi_{i_1\cdots i_m}} = \frac{\partial \sum_{l_1,\cdots,l_m=1}^{k} v_{l_1\cdots l_m}\sum_{b=1}^{m}\left(\prod_{a=1}^{m}\pi_{l_a}^{(b)}\right)/m}{\partial \pi_{i_1\cdots i_m}}$$

$$=\frac{1}{m}\sum_{b=1}^{m}\left(\sum_{l_2=1}^{k}\cdots\sum_{l_m=1}^{k} v_{i_b l_2\cdots l_m}\prod_{\substack{a=1\\ a\neq 1}}^{m}\pi_{l_a}^{(b)}+\sum_{l_1=1}^{k}\sum_{l_3=1}^{k}\cdots\sum_{l_m=1}^{k} v_{l_1 i_b l_3\cdots l_m}\prod_{\substack{a=1\\ a\neq 2}}^{m}\pi_{l_a}^{(b)}\right.$$

$$\left.+\cdots+\sum_{l_1=1}^{k}\cdots\sum_{l_{m-1}=1}^{k} v_{l_1\cdots l_{m-1} i_b}\prod_{\substack{a=1\\ a\neq m}}^{m}\pi_{l_a}^{(b)}\right)$$

$$=\sum_{b=1}^{m}\sum_{j=1}^{m}\frac{\mathbf{v}_{\pi(\mathrm{SO})}^{(i_b,j)}\,'\mathbf{1}_{(k^{m-1})}}{m}\quad (i_j=1,\ldots,k;\ j=1,\ldots,m),$$

where $\mathbf{v}_{\pi(\mathrm{SO})}^{(i_b,j)}$ is the $k^{m-1}\times 1$ vector, whose elements are

$$v_{l_1\cdots l_{j-1}i_b l_{j+1}\cdots l_m}\prod_{\substack{a=1\\a\neq j}}^{m}\pi_{l_a}^{(b)}\quad (l_a=1,\dots,k;\ a=1,\dots,m;\ a\neq j).$$

Lemma S1. $\boldsymbol{\pi}'\partial\pi_{\mathrm{e(SO)}}^{(v)}/\partial\boldsymbol{\pi}=m\,\pi_{\mathrm{e(SO)}}^{(v)}$ *and* $\boldsymbol{\pi}'\partial\pi_{\mathrm{e(SO)}}^{(w)}/\partial\boldsymbol{\pi}$ $=m\,\pi_{\mathrm{e(SO)}}^{(w)}$.

Proof.

$$\boldsymbol{\pi}'\frac{\partial\pi_{\mathrm{e(SO)}}^{(v)}}{\partial\boldsymbol{\pi}}=\frac{1}{m}\sum_{i_1,\dots,i_m=1}^{k}\sum_{b=1}^{m}\left(\sum_{l_2=1}^{k}\cdots\sum_{l_m=1}^{k}v_{i_b l_2\cdots l_m}\prod_{\substack{a=1\\a\neq 1}}^{m}\pi_{l_a}^{(b)}\right.$$

$$+\sum_{l_1=1}^{k}\sum_{l_3=1}^{m}\cdots\sum_{l_m=1}^{k}v_{l_1 i_b l_3\cdots l_m}\prod_{\substack{a=1\\a\neq 2}}^{m}\pi_{l_a}^{(b)}$$

$$\left.+\cdots+\sum_{l_1=1}^{k}\cdots\sum_{l_{m-1}=1}^{k}v_{l_1\cdots l_{m-1}i_b}\prod_{\substack{a=1\\a\neq m}}^{m}\pi_{l_a}^{(b)}\right)\pi_{i_1\cdots i_m}$$

$$=\frac{1}{m}\sum_{b=1}^{m}\sum_{i_b=1}^{k}\left(\sum_{l_2,\dots,l_m=1}^{k}v_{i_b l_2\cdots l_m}\prod_{\substack{a=1\\a\neq 1}}^{m}\pi_{l_a}^{(b)}+\sum_{l_1,l_3,\dots,l_m=1}^{k}v_{l_1 i_b l_3\cdots l_m}\prod_{\substack{a=1\\a\neq 2}}^{m}\pi_{l_a}^{(b)}\right.$$

$$\left.+\cdots+\sum_{l_1,\dots,l_{m-1}=1}^{k}v_{l_1 i_2 l_3\cdots l_m}\prod_{\substack{a=1\\a\neq m}}^{m}\pi_{l_a}^{(b)}\right)\pi_{i_b}^{(b)}.$$

where the first term in braces on the right-hand side gives

$$=\frac{1}{m}\sum_{b=1}^{m}\sum_{i_b=1}^{k}\sum_{l_2,\dots,l_m=1}^{k}v_{i_b l_2\cdots l_m}\pi_{i_b}^{(b)}\prod_{\substack{a=1\\a\neq 1}}^{m}\pi_{l_a}^{(b)}=\frac{1}{m}\sum_{b=1}^{m}\sum_{l_1,\dots,l_m=1}^{k}v_{l_1 l_2\cdots l_m}\prod_{a=1}^{m}\pi_{l_a}^{(b)}$$

$$=\sum_{l_1,\dots,l_m=1}^{k}v_{l_1 l_2\cdots l_m}\sum_{b=1}^{m}\left(\prod_{a=1}^{m}\pi_{l_a}^{(b)}\right)/m=\pi_{\mathrm{e(SO)}}^{(v)}$$

and the remaining terms give the same $\pi_{\mathrm{e(SO)}}^{(v)}$'s, yielding the required result.

The second result of Lemma S1 is similarly given by using $\pi_{\mathrm{e(SO)}}^{(w)}$ and $w_{l_1\cdots l_m}$ $(i_j=1,...,k;\, j=1,...,m)$. Q.E.D.

(iv) $\pi_{\mathrm{e(C)}}^{(v)}$

$$\frac{\partial \pi_{\mathrm{e(C)}}^{(v)}}{\partial \pi_{i_1\cdots i_m}} = \frac{\partial \sum_{l_1\cdots l_m=1}^{k} v_{l_1\cdots l_m} \prod_{a=1}^{m} \pi_{l_a}^{(a)}}{\partial \pi_{i_1\cdots i_m}}$$

$$= \sum_{l_2=1}^{k}\cdots\sum_{l_m=1}^{k} v_{i_1 l_2\cdots l_m} \prod_{\substack{a=1\\ a\neq 1}}^{m} \pi_{l_a}^{(a)} + \sum_{l_1=1}^{k}\sum_{l_3=1}^{m}\cdots\sum_{l_m=1}^{k} v_{l_1 i_2 l_3\cdots l_m} \prod_{\substack{a=1\\ a\neq 2}}^{m} \pi_{l_a}^{(a)}$$

$$+\cdots+ \sum_{l_1=1}^{k}\cdots\sum_{l_{m-1}=1}^{k} v_{l_1\cdots l_{m-1} i_m} \prod_{\substack{a=1\\ a\neq m}}^{m} \pi_{l_a}^{(a)}$$

$$= \sum_{j=1}^{m} \mathbf{v}_{\pi(\mathrm{C})}^{(i_j)}{}'\mathbf{1}_{(k^{m-1})} \quad (i_j=1,...,k;\, j=1,...,m),$$

where $\mathbf{v}_{\pi(\mathrm{C})}^{(i_j)}$ is the $k^{m-1}\times 1$ vector, whose elements are

$$v_{l_1\cdots l_{j-1} i_j l_{j+1}\cdots l_m} \prod_{\substack{a=1\\ a\neq j}}^{m} \pi_{l_a}^{(a)} \ \ (l_a=1,...,k;\ a=1,...,m;\ a\neq j).$$

Lemma S2. $\boldsymbol{\pi}'\partial\pi_{\mathrm{e(C)}}^{(v)}/\partial\boldsymbol{\pi} = m\,\pi_{\mathrm{e(C)}}^{(v)}$ *and* $\boldsymbol{\pi}'\partial\pi_{\mathrm{e(C)}}^{(w)}/\partial\boldsymbol{\pi} = m\,\pi_{\mathrm{e(C)}}^{(w)}$.

Proof.

$$\boldsymbol{\pi}'\frac{\partial \pi_{\mathrm{e(C)}}^{(v)}}{\partial \boldsymbol{\pi}} = \sum_{i_1,..,i_m=1}^{k}\left\{ \sum_{l_2=1}^{k}\cdots\sum_{l_m=1}^{k} v_{i_1 l_2\cdots l_m}\left(\prod_{a=1}^{m}\pi_{l_a}^{(a)}\right)\frac{1}{\pi_{l_1}^{(1)}}\pi_{i_1\cdots i_m} \right.$$

$$+\sum_{l_1=1}^{k}\sum_{l_3=1}^{m}\cdots\sum_{l_m=1}^{k} v_{l_1 i_2 l_3\cdots l_m}\left(\prod_{a=1}^{m}\pi_{l_a}^{(a)}\right)\frac{1}{\pi_{l_2}^{(2)}}\pi_{i_1\cdots i_m}$$

$$\left. +\cdots+\sum_{l_1=1}^{k}\cdots\sum_{l_{m-1}=1}^{k} v_{l_1\cdots l_{m-1} i_m}\left(\prod_{a=1}^{m}\pi_{l_a}^{(a)}\right)\frac{1}{\pi_{l_m}^{(m)}}\pi_{i_1\cdots i_m}\right\},$$

where the first term in braces on the right-hand side gives

$$\sum_{i_1,l_2,\ldots,l_m=1}^{k}\sum_{l_2=1}^{k}\cdots\sum_{l_m=1}^{k} v_{i_1 l_2\cdots l_m}\sum_{i_2,\ldots,i_m=1}^{m}\pi_{i_1\cdots i_m}\prod_{a=2}^{m}\pi_{l_a}^{(a)}$$

$$=\sum_{i_1,l_2,\ldots,l_m=1}^{k} v_{i_1 l_2\cdots l_m}\pi_{i_1}^{(1)}\prod_{a=2}^{m}\pi_{l_a}^{(a)}=\sum_{l_1,l_2,\ldots,l_m=1}^{k} v_{l_1 l_2\cdots l_m}\prod_{a=1}^{m}\pi_{l_a}^{(a)}=\pi_{\mathrm{e(C)}}^{(v)},$$

and the remaining terms give the same $\pi_{\mathrm{e(C)}}^{(v)}$'s, yielding the required result. The second result of Lemma S2 is similarly given by using $\pi_{\mathrm{e(C)}}^{(w)}$ and $w_{l_1\cdots l_m}$ $(i_j=1,\ldots,k;\, j=1,\ldots,m)$. Q.E.D.

(v) $\pi_{\mathrm{e(G)}}^{(v)}$

Note that $\pi_{\mathrm{e(G)}}^{(w)}$ and $\pi_{\mathrm{e(G)}}^{(v)}$ are defined only for unweighted pair-wise (dis)agreement ($m=2$ or $m'=2$; $w_{ij}=\delta_{ij}$, $v_{ij}=1-\delta_{ij}$ $(i,j=1,\ldots,k)$). The following reduced expressions are used for $\pi_{\mathrm{e(G)}}^{(w)}$ and $\pi_{\mathrm{e(G)}}^{(v)}$ for simplicity

$$\pi_{\mathrm{e(G)}}^{(w)}=\sum_{i,j=1}^{k} w_{ij}\frac{\bar{\pi}_i(1-\bar{\pi}_i)}{k-1}=\sum_{i,j=1}^{k}\delta_{ij}\frac{\bar{\pi}_i(1-\bar{\pi}_i)}{k-1}=\sum_{a=1}^{k}\frac{\bar{\pi}_a(1-\bar{\pi}_a)}{k-1}$$

and $\pi_{\mathrm{e(G)}}^{(v)}=1-\pi_{\mathrm{e(G)}}^{(w)}=1-\sum_{a=1}^{k}\frac{\bar{\pi}_a(1-\bar{\pi}_a)}{k-1}$. Then, we have

$$\frac{\partial\pi_{\mathrm{e(G)}}^{(w)}}{\partial\pi_{ij}}=\frac{\partial\sum_{a=1}^{k}\bar{\pi}_a(1-\bar{\pi}_a)/(k-1)}{\partial\pi_{ij}}=\frac{1-2\bar{\pi}_i+1-2\bar{\pi}_j}{m(k-1)}=\frac{2(1-\bar{\pi}_i-\bar{\pi}_j)}{m(k-1)},$$

$$\frac{\partial\pi_{\mathrm{e(G)}}^{(v)}}{\partial\pi_{ij}}=-\frac{2(1-\bar{\pi}_i-\bar{\pi}_j)}{m(k-1)}\ (i,j=1,..,k).$$

Since $\sum_{a=1}^{k}\bar{\pi}_a=1$, the above result can be simplified as

$$\frac{\partial\pi_{\mathrm{e(G)}}^{(w)}}{\partial\pi_{ij}}=-\frac{2(\bar{\pi}_i+\bar{\pi}_j)}{m(k-1)}\ \text{and}\ \frac{\partial\pi_{\mathrm{e(G)}}^{(v)}}{\partial\pi_{ij}}=\frac{2(\bar{\pi}_i+\bar{\pi}_j)}{m(k-1)}\ (i,j=1,\ldots,k).$$

However, for the associated asymptotic variance, the earlier result can also be used since the added terms will give no contribution to the variance due to the singular property of cov(**p**). The equal results with different partial derivatives are associated with the property that π_{ij} $(i,j=1,\ldots,k)$ are not independent mathematical variables with their sum being fixed.

(vi) $\pi^{(v)}_{\mathrm{e(GO)}}$

As for $\pi^{(w)}_{\mathrm{e(G)}}$ and $\pi^{(v)}_{\mathrm{e(G)}}$, $\pi^{(w)}_{\mathrm{e(GO)}}$ and $\pi^{(v)}_{\mathrm{e(GO)}}$ are defined only for unweighted pair-wise (dis)agreement ($m = 2$ or $m' = 2$; $w_{ij} = \delta_{ij}$, $v_{ij} = 1 - \delta_{ij}$; $i, j = 1,\ldots,k$). Then, we have the following results.

$$\pi^{(w)}_{\mathrm{e(GO)}} = \sum_{a=1}^{k} \frac{\bar{\pi}_a - \overline{\pi_a^2}}{k-1} = \sum_{a=1}^{k} \frac{\bar{\pi}_a - \bar{\pi}_{aa}}{k-1},$$

$$\pi^{(v)}_{\mathrm{e(GO)}} = 1 - \sum_{a=1}^{k} \frac{\bar{\pi}_a - \overline{\pi_a^2}}{k-1} = 1 - \sum_{a=1}^{k} \frac{\bar{\pi}_a - \bar{\pi}_{aa}}{k-1},$$

$$\frac{\partial \pi^{(w)}_{\mathrm{e(GO)}}}{\partial \pi_{ij}} = \frac{\partial \sum_{a=1}^{k} (\bar{\pi}_a - \overline{\pi_a^2}) / (k-1)}{\partial \pi_{ij}}$$

$$= \frac{1 - 2\pi_i^{(1)} + 1 - 2\pi_j^{(2)}}{m(k-1)} = \frac{2(1 - \pi_i^{(1)} - \pi_j^{(2)})}{m(k-1)},$$

$$\frac{\partial \pi^{(v)}_{\mathrm{e(GO)}}}{\partial \pi_{ij}} = -\frac{2(1 - \pi_i^{(1)} - \pi_j^{(2)})}{m(k-1)} \quad (i, j = 1,..,k).$$

As before, the following simplified result can also be used

$$\frac{\partial \pi^{(w)}_{\mathrm{e(GO)}}}{\partial \pi_{ij}} = -\frac{2(\pi_i^{(1)} + \pi_j^{(2)})}{m(k-1)} \quad \text{and} \quad \frac{\partial \pi^{(v)}_{\mathrm{e(GO)}}}{\partial \pi_{ij}} = \frac{2(\pi_i^{(1)} + \pi_j^{(2)})}{m(k-1)} \quad (i, j = 1,\ldots,k).$$

S1.2 $(\partial \pi^{(v)}_{\mathrm{e(\bullet)}} / \partial \boldsymbol{\pi}')\ \mathrm{diag}(\boldsymbol{\pi})\ (\partial \pi^{(v)}_{\mathrm{e(\bullet)}} / \partial \boldsymbol{\pi})$

(i) $\pi^{(v)}_{\mathrm{e(B)}}$

$$\frac{\partial \pi^{(v)}_{\mathrm{e(B)}}}{\partial \boldsymbol{\pi}'}\ \mathrm{diag}(\boldsymbol{\pi})\ \frac{\partial \pi^{(v)}_{\mathrm{e(B)}}}{\partial \boldsymbol{\pi}} = 0$$

(ii) $\pi^{(v)}_{\mathrm{e(S)}}$, $\pi^{(v)}_{\mathrm{e(SO)}}$, $\pi^{(v)}_{\mathrm{e(C)}}$, $\pi^{(v)}_{\mathrm{e(G)}}$ and $\pi^{(v)}_{\mathrm{e(GO)}}$

$$\frac{\partial \pi_{\mathrm{e}(\bullet)}^{(v)}}{\partial \boldsymbol{\pi}'} \operatorname{diag}(\boldsymbol{\pi}) \frac{\partial \pi_{\mathrm{e}(\bullet)}^{(v)}}{\partial \boldsymbol{\pi}} = \sum_{i_1,\ldots,i_m=1}^{k} \frac{\partial \pi_{\mathrm{e}(\bullet)}^{(v)}}{\partial \pi_{i_1\cdots i_m}} \pi_{i_1\cdots i_m} \frac{\partial \pi_{\mathrm{e}(\bullet)}^{(v)}}{\partial \pi_{i_1\cdots i_m}}$$

$$= \sum_{i_1,\ldots,i_m=1}^{k} (\partial \pi_{\mathrm{e}(\bullet)}^{(v)} / \partial \pi_{i_1\cdots i_m})^2 \pi_{i_1\cdots i_m}.$$

S1.3 The second partial derivatives

(i) $\pi_{\mathrm{e(B)}}^{(v)}$

$$\partial^2 \pi_{\mathrm{e(B)}}^{(v)} / (\partial \boldsymbol{\pi})^{<2>} = \mathbf{0}.$$

(ii) $\pi_{\mathrm{e(S)}}^{(v)}$

$$\frac{\partial^2 \pi_{\mathrm{e(S)}}^{(v)}}{\partial \pi_{i_1^{(1)}\cdots i_m^{(1)}} \partial \pi_{i_1^{(2)}\cdots i_m^{(2)}}} = \sum_{b1=1}^{m} \sum_{b2=1}^{m} \sum_{j1=1}^{m} \sum_{\substack{j2=1 \\ j1\neq j2}}^{m} \mathbf{v}_{\pi(\mathrm{S})}^{(i_{b1}^{(1)},\, i_{b2}^{(2)},\, j1,\, j2)}{}' \mathbf{1}_{(k^{m-2})}$$

$$(i_j^{(1)}, i_j^{(2)} = 1,\ldots,k;\ j = 1,\ldots,m),$$

where $\mathbf{v}_{\pi(\mathbf{S})}^{(i_{b1}^{(1)},\, i_{b2}^{(2)},\, j1,\, j2)}$ is the $k^{m-2} \times 1$ vector, whose elements when $j1 < j2$ are

$$\frac{1}{m^2} v_{l_1\cdots l_{j1-1} i_{b1}^{(1)} l_{j1+1}\cdots l_{j2-1} i_{b2}^{(2)} l_{j2+1}\cdots l_m} \prod_{\substack{a=1 \\ a\neq j1,j2}}^{m} \bar{\pi}_{l_a} \quad (l_a = 1,\ldots,k;\ a = 1,..,m;\ a \neq j1, j2),$$

where $\prod_{\substack{a=1 \\ a\neq j1,j2}}^{m} \bar{\pi}_{l_a} = 1$ when $m = 2$.

(iii) $\pi_{\mathrm{e(SO)}}^{(v)}$

$$\frac{\partial^2 \pi_{\mathrm{e(SO)}}^{(v)}}{\partial \pi_{i_1^{(1)}\cdots i_m^{(1)}} \partial \pi_{i_1^{(2)}\cdots i_m^{(2)}}} = \sum_{b=1}^{m} \sum_{j1=1}^{m} \sum_{\substack{j2=1 \\ j1\neq j2}}^{m} \mathbf{v}_{\pi(\mathrm{SO})}^{(i_b^{(1)},\, i_b^{(2)},\, j1,\, j2)}{}' \mathbf{1}_{(k^{m-2})}$$

$$(i_j^{(1)}, i_j^{(2)} = 1,\ldots,k;\ j = 1,\ldots,m),$$

where $\mathbf{v}_{\pi(\mathrm{SO})}^{(i_b^{(1)},\, i_b^{(2)},\, j1,\, j2)}$ is the $k^{m-2} \times 1$ vector, whose elements when $j1 < j2$

are

$$\frac{1}{m} v_{l_1 \cdots l_{j1-1} i_b^{(1)} l_{j1+1} \cdots l_{j2-1} i_b^{(2)} l_{j2+1} \cdots l_m} \prod_{\substack{a=1 \\ a \neq j1, j2}}^{m} \pi_{l_a}^{(b)} \quad (l_a = 1,\ldots,k;\ a = 1,..,m;\ a \neq j1, j2),$$

where $\prod_{\substack{a=1 \\ a \neq j1, j2}}^{m} \pi_{l_a}^{(b)} = 1$ when $m = 2$.

(iv) $\pi_{\mathrm{e(C)}}^{(v)}$

$$\frac{\partial^2 \pi_{\mathrm{e(C)}}^{(v)}}{\partial \pi_{i_1^{(1)} \cdots i_m^{(1)}} \partial \pi_{i_1^{(2)} \cdots i_m^{(2)}}} = \sum_{j1=1}^{m} \sum_{\substack{j2=1 \\ j1 \neq j2}}^{m} \mathbf{v}_{\pi(\mathrm{C})}^{(i_{j1}^{(1)}, i_{j2}^{(2)})} {}' \mathbf{1}_{(k^{m-2})}$$

$(i_j^{(1)}, i_j^{(2)} = 1,\ldots,k;\ j = 1,\ldots,m),$

where $\mathbf{v}_{\pi(\mathrm{C})}^{(i_{j1}^{(1)}, i_{j2}^{(2)})}$ is the $k^{m-2} \times 1$ vector, whose elements when $j1 < j2$ are

$$\frac{1}{m} v_{l_1 \cdots l_{j1-1} i_{j1}^{(1)} l_{j1+1} \cdots l_{j2-1} i_{j2}^{(2)} l_{j2+1} \cdots l_m} \prod_{\substack{a=1 \\ a \neq j1, j2}}^{m} \pi_{l_a}^{(a)} \quad (l_a = 1,\ldots,k;\ a = 1,..,m;\ a \neq j1, j2),$$

where $\prod_{\substack{a=1 \\ a \neq j1, j2}}^{m} \pi_{l_a}^{(a)} = 1$ when $m = 2$.

(v) $\pi_{\mathrm{e(G)}}^{(v)}$, $\pi_{\mathrm{e(GO)}}^{(v)}$

$$\frac{\partial^2 \pi_{\mathrm{e(G)}}^{(w)}}{\partial \pi_{i_1 j_1} \partial \pi_{i_2 j_2}} = \frac{\partial 2(1 - \bar{\pi}_{i_1} - \bar{\pi}_{j_1}) / \{m(k-1)\}}{\partial \pi_{i_2 j_2}}$$

$$= -\frac{2(\delta_{i_1 i_2} + \delta_{i_1 j_2} + \delta_{j_1 i_2} + \delta_{j_1 j_2})}{m^2 (k-1)},$$

$$\frac{\partial^2 \pi_{\mathrm{e(G)}}^{(v)}}{\partial \pi_{i_1 j_1} \partial \pi_{i_2 j_2}} = \frac{2(\delta_{i_1 i_2} + \delta_{i_1 j_2} + \delta_{j_1 i_2} + \delta_{j_1 j_2})}{m^2 (k-1)},$$

$$\frac{\partial^2 \pi^{(w)}_{\mathrm{e(GO)}}}{\partial \pi_{i_1 j_1} \partial \pi_{i_2 j_2}} = \frac{\partial 2(1-\pi^{(1)}_{i_1} - \pi^{(2)}_{j_1}) / \{m(k-1)\}}{\partial \pi_{i_2 j_2}} = -\frac{2(\delta_{i_1 i_2} + \delta_{j_1 j_2})}{m(k-1)},$$

$$\frac{\partial^2 \pi^{(v)}_{\mathrm{e(GO)}}}{\partial \pi_{i_1 j_1} \partial \pi_{i_2 j_2}} = \frac{2(\delta_{i_1 i_2} + \delta_{j_1 j_2})}{m(k-1)} \quad (i_1, i_2, j_1, j_2 = 1, ..., k).$$

S1.4 The third partial derivatives

(i) $\pi^{(v)}_{\mathrm{e(B)}}$, $\pi^{(v)}_{\mathrm{e(G)}}$ and $\pi^{(v)}_{\mathrm{e(GO)}}$

$$\partial^3 \pi^{(v)}_{\mathrm{e}(\bullet)} / (\partial \boldsymbol{\pi})^{<3>} = \mathbf{0}.$$

(ii) $\pi^{(v)}_{\mathrm{e(S)}}$ $(m \geq 3)$

$$\frac{\partial^3 \pi^{(v)}_{\mathrm{e(S)}}}{\partial \pi_{i^{(1)}_1 \cdots i^{(1)}_m} \partial \pi_{i^{(2)}_1 \cdots i^{(2)}_m} \partial \pi_{i^{(3)}_1 \cdots i^{(3)}_m}} = \sum_{b1=1}^{m} \sum_{b2=1}^{m} \sum_{b3=1}^{m} \sum_{\substack{j1, j2, j3=1 \\ j1 \neq j2 \neq j3 \neq j1}}^{m} \mathbf{v}^{(i^{(1)}_{b1}, i^{(2)}_{b2}, i^{(3)}_{b3}, j1, j2, j3)\prime}_{\pi(\mathrm{S})} \mathbf{1}_{(k^{m-3})}$$

$$(i^{(1)}_j,\ i^{(2)}_j,\ i^{(3)}_j = 1, ..., k;\ j = 1, ..., m),$$

where $\mathbf{v}^{(i^{(1)}_{b1}, i^{(2)}_{b2}, i^{(3)}_{b3}, j1, j2, j3)}_{\pi(\mathrm{S})}$ is the $k^{m-3} \times 1$ vector, whose elements when $j1 < j2 < j3$ are

$$\frac{1}{m^3} v_{l_1 \cdots l_{j1-1} i^{(1)}_{b1} l_{j1+1} \cdots l_{j2-1} i^{(2)}_{b2} l_{j2+1} \cdots l_{j3-1} i^{(3)}_{b3} l_{j3+1} \cdots l_m} \prod_{\substack{a=1 \\ a \neq j1, j2, j3}}^{m} \bar{\pi}_{l_a}$$

$$(l_a = 1, ..., k;\ a = 1, .., m;\ a \neq j1, j2, j3),$$

where $\prod_{\substack{a=1 \\ a \neq j1, j2, j3}}^{m} \bar{\pi}_{l_a} = 1$ when $m = 3$. When $m = 2$, $\partial^3 \pi^{(v)}_{\mathrm{e(S)}} / (\partial \boldsymbol{\pi})^{<3>} = \mathbf{0}$.

(iii) $\pi^{(v)}_{\mathrm{e(SO)}}$ $(m \geq 3)$

$$\frac{\partial^3 \pi^{(v)}_{\mathrm{e(SO)}}}{\partial \pi_{i^{(1)}_1 \cdots i^{(1)}_m} \partial \pi_{i^{(2)}_1 \cdots i^{(2)}_m} \partial \pi_{i^{(3)}_1 \cdots i^{(3)}_m}} = \sum_{b=1}^{m} \sum_{\substack{j1, j2, j3=1 \\ j1 \neq j2 \neq j3 \neq j1}}^{m} \mathbf{v}^{(i^{(1)}_{b}, i^{(2)}_{b}, i^{(3)}_{b}, j1, j2, j3)\prime}_{\pi(\mathrm{SO})} \mathbf{1}_{(k^{m-3})}$$

$$(i^{(1)}_j,\ i^{(2)}_j,\ i^{(3)}_j = 1, ..., k;\ j = 1, ..., m),$$

where $\mathbf{v}_{\pi(\mathrm{S})}^{(i_b^{(1)},\, i_b^{(2)},\, i_b^{(3)},\, j1,\, j2,\, j3)}$ is the $k^{m-3}\times 1$ vector, whose elements when $j1<j2<j3$ are

$$\frac{1}{m} v_{l_1\cdots l_{j1-1} i_b^{(1)} l_{j1+1}\cdots l_{j2-1} i_b^{(2)} l_{j2+1}\cdots l_{j3-1} i_b^{(3)} l_{j3+1}\cdots l_m} \prod_{\substack{a=1\\ a\neq j1,j2,j3}}^{m} \pi_{l_a}^{(b)}$$

$(l_a = 1,\ldots,k;\ a = 1,..,m;\ a \neq j1, j2, j3),$

where $\prod_{\substack{a=1\\ a\neq j1,j2,j3}}^{m} \pi_{l_a}^{(b)} = 1$ when $m = 3$. When $m = 2$, $\partial^3 \pi_{\mathrm{e(SO)}}^{(v)} / (\partial \boldsymbol{\pi})^{<3>} = \mathbf{0}$.

(iv) $\pi_{\mathrm{e(C)}}^{(v)}$ $(m \geq 3)$

$$\frac{\partial^3 \pi_{\mathrm{e(C)}}^{(v)}}{\partial \pi_{i_1^{(1)}\cdots i_m^{(1)}} \partial \pi_{i_1^{(2)}\cdots i_m^{(2)}} \partial \pi_{i_1^{(3)}\cdots i_m^{(3)}}} = \sum_{\substack{j1,j2,j3=1\\ j1\neq j2\neq j3\neq j1}}^{m} \mathbf{v}_{\pi(\mathrm{C})}^{(i_{j1}^{(1)},\, i_{j2}^{(2)},\, i_{j3}^{(3)})}{}' \mathbf{1}_{(k^{m-3})}$$

$(i_j^{(1)},\ i_j^{(2)},\ i_j^{(3)} = 1,\ldots,k;\ j = 1,\ldots,m),$

where $\mathbf{v}_{\pi(\mathrm{C})}^{(i_{j1}^{(1)},\, i_{j2}^{(2)},\, i_{j3}^{(3)})}$ is the $k^{m-3}\times 1$ vector, whose elements when $j1<j2<j3$ are

$$v_{l_1\cdots l_{j1-1} i_{j1}^{(1)} l_{j1+1}\cdots l_{j2-1} i_{j2}^{(2)} l_{j2+1}\cdots l_{j3-1} i_{j3}^{(3)} l_{j3+1}\cdots l_m} \prod_{\substack{a=1\\ a\neq j1,j2,j3}}^{m} \pi_{l_a}^{(a)}$$

$(l_a = 1,\ldots,k;\ a = 1,..,m;\ a \neq j1, j2, j3).$

where $\prod_{\substack{a=1\\ a\neq j1,j2,j3}}^{m} \pi_{l_a}^{(a)} = 1$ when $m = 3$. When $m = 2$, $\partial^3 \pi_{\mathrm{e(C)}}^{(v)} / (\partial \boldsymbol{\pi})^{<3>} = \mathbf{0}$.

S2. Additional tables

Table A1. Counts and degrees of seriousness of 4-wise disagreement for 2^4 profiles using two rating categories given by dichotomization (different from that in Table 1; see below) for scores 1 to 5 evaluated by 4 raters in classification of the intensity of nuptial coloration for 29 fishes (reconstructed from Gwet (2014, Table A.4))

Profiles and their associated values				Marginal proportions		
Profile	Count	v_{4A}	v_{4B}	Rater	Category 1	Category 2
1111	5	0	0.0	1	.345	.655
1112	1	1	0.5	2	.414	.586
1121	0	1	0.5	3	.379	.621
1122	2	1	1.0	4	.414	.586
1211	1	1	0.5	Average	.388	.612
1212	0	1	1.0			
1221	0	1	1.0			
1222	1	1	0.5			
2111	1	1	0.5			
2112	0	1	1.0			
2121	2	1	1.0			
2122	1	1	0.5			
2211	2	1	1.0			
2212	1	1	0.5			
2221	1	1	0.5			
2222	11	0	0.0			
Total	29					

Dichotomization for scores:
Category 1 = scores 1 and 2
Category 2 = scores 3, 4 and 5

Note. v_{4A} and v_{4B} stand for the degrees of seriousness of 4-wise disagreement in each profile based on Method A with degrees 0 and 1, and Method B with degrees 0.0, 0.5 and 1.0, respectively. In the profiles 1 and 2 denote Categories 1 and 2, respectively, given by Raters 1, 2, 3 and 4.

Table A2. Asymptotic and simulated standard errors of sample coefficients of 4-wise agreement for 4 raters using two rating categories in Table A1 (the number of replications in simulations = 10,000)

4-wise agreement with $k = 2$				$n = 29$			$n = 100$		$n = 200$	
$K_{(\cdot)}$	$\pi_{e(\cdot)}^{(v)}$	$\sqrt{n}$ ASE		ASE	SD/ASE	SD_t	SD/ASE	SD_t	SD/ASE	SD_t
$\pi_o^{(v)} = .448$ with v_{4A} for 4-wise disagreement										
$K_{(B)}$	.488	.875	.568	.106	.996	1.057	1.000	1.015	.996	1.003
$K_{(S)}$	.464	.837	.580	.108	1.008	1.077	1.009	1.024	1.003	1.011
$K_{(SO)}$	.463	.834	.586	.109	1.040	1.053	1.017	1.019	1.008	1.008
$K_{(C)}$	.465	.838	.578	.107	1.000	1.081	1.007	1.026	1.002	1.011
$\pi_o^{(v)} = .328$ with v_{4B} for 4-wise disagreement										
$K_{(B)}$	.476	.625	.640	.119	.996	1.077	.996	1.016	.987	.995
$K_{(S)}$	.443	.588	.670	.124	1.018	1.102	1.011	1.026	.999	1.004
$K_{(SO)}$	.440	.585	.681	.126	1.054	1.069	1.021	1.017	1.003	1.000
$K_{(C)}$	.443	.588	.667	.124	1.008	1.109	1.008	1.028	.997	1.005

Note. $K_{(B)}$ = Bennett et al.-type K, $K_{(S)}$ = Scott-type K, $K_{(SO)}$ = modified Scott-type K, $K_{(C)}$ = Cohen-type K, n = sample size, ASE = asymptotic standard error, SD_t = the standard deviation of studentized estimates of $K_{(\cdot)}$ in a simulation, $K_{(\cdot)} = 1 - \Delta_{(\cdot)} = 1 - (\pi_o^{(v)} / \pi_{e(\cdot)}^{(v)})$.

Table A3. Asymptotic and simulated standard errors of sample coefficients of 3-wise agreement for 4 raters using two rating categories in Table A1 (the number of replications in simulations = 10,000)

3-wise agreement with $k = 2$				$n = 29$			$n = 100$		$n = 200$	
$\pi_{\mathrm{o}}^{(v)} = .388$	$K_{(\cdot)}$	$\pi_{\mathrm{e}(\cdot)}^{(v)}$	$\sqrt{n}$ ASE	ASE	SD/ASE	SD_t	SD/ASE	SD_t	SD/ASE	SD_t
The formula of the ratio of means with $v_{3\mathrm{A}}$ for 3-wise disagreement										
$K_{(\mathrm{B})}$	.482759	.750	.58451	.109	.997	1.060	.997	1.013	.991	.998
$K_{(\mathrm{S})}$	.455191	.712	.60287	.112	1.015	1.083	1.010	1.024	1.001	1.007
$K_{(\mathrm{SO})}$	.453518	.710	.60994	.113	1.046	1.056	1.018	1.017	1.005	1.004
$K_{(\mathrm{C})}$	.456023	.713	.59963	.111	1.002	1.091	1.007	1.026	.999	1.008
The formula of the mean of ratios with $v_{3\mathrm{A}}$ for 3-wise disagreement										
$K_{(\mathrm{B})}$	.482759	*	.58451	.109	.997	1.060	.997	1.013	.991	.998
$K_{(\mathrm{S})}$	.455186	*	.60290	.112	1.016	1.084	1.010	1.024	1.001	1.007
$K_{(\mathrm{SO})}$	.453489	*	.61008	.113	1.048	1.056	1.019	1.017	1.005	1.004
$K_{(\mathrm{C})}$	.456031	*	.59962	.111	1.003	1.092	1.007	1.026	.999	1.008

Note. $K_{(\mathrm{B})}$ = Bennett et al.-type K, $K_{(\mathrm{S})}$ = Scott-type K, $K_{(\mathrm{SO})}$ = modified Scott-type K, $K_{(\mathrm{C})}$ = Cohen-type K, n = sample size, ASE = asymptotic standard error, SD_t = the standard deviation of studentized estimates of $K_{(\cdot)}$ in a simulation, $K_{(\cdot)} = 1 - \Delta_{(\cdot)} = 1 - (\pi_{\mathrm{o}}^{(v)} / \pi_{\mathrm{e}(\cdot)}^{(v)})$. $v_{3\mathrm{A}} = 0$ when a 3-wise profile is 111 or 222 otherwise $v_{3\mathrm{A}} = 1$ (compare $v_{3\mathrm{A}}$ in Table A3 with $v_{4\mathrm{A}}$ in Table A1). The asterisks indicate that the corresponding common values of $\pi_{\mathrm{e}(\cdot)}^{(v)}$ are used.

Table A4. Asymptotic and simulated standard errors of sample coefficients of pair-wise agreement for 4 raters using two rating categories in Table A1 (the number of replications in simulations = 10,000)

Pair-wise agreement with $k = 2$				$n = 29$			$n = 100$		$n = 200$	
$\pi_{\mathrm{o}}^{(v)} = .259$	$K_{(\cdot)}$	$\pi_{\mathrm{e}(\cdot)}^{(v)}$	$\sqrt{n}$ ASE	ASE	SD/ASE	SD$_t$	SD/ASE	SD$_t$	SD/ASE	SD$_t$
The formula of the ratio of means with $v_{2\mathrm{A}}$ for pair-wise disagreement										
$K_{(\mathrm{B})}$	.48276	.500	.58451	.109	.997	1.060	.997	1.013	.991	.998
$K_{(\mathrm{S})}$	.45477	.474	.60457	.112	1.022	1.077	1.012	1.022	1.002	1.006
$K_{(\mathrm{SO})}$	.45352	.473	.60994	.113	1.046	1.056	1.018	1.017	1.005	1.004
$K_{(\mathrm{C})}$	.45602	.475	.59963	.111	1.002	1.091	1.007	1.026	.999	1.008
$K_{(\mathrm{G})}$	.50801	.526	.60954	.113	.976	1.081	.987	1.015	.985	.998
$K_{(\mathrm{GO})}$	.50903	.527	.60560	.112	.960	1.086	.982	1.017	.983	.999
The formula of the mean of ratios with $v_{2\mathrm{A}}$ for pair-wise disagreement										
$K_{(\mathrm{B})}$	.48276	*	.58451	.109	.997	1.060	.997	1.013	.991	.998
$K_{(\mathrm{S})}$	.45474	*	.60476	.112	1.022	1.081	1.012	1.023	1.002	1.006
$K_{(\mathrm{SO})}$	.45343	*	.61040	.113	1.050	1.055	1.019	1.017	1.005	1.004
$K_{(\mathrm{C})}$	.45604	*	.59966	.111	1.002	1.095	1.006	1.026	.999	1.009
$K_{(\mathrm{G})}$	.50783	*	.60983	.113	.977	1.078	.987	1.014	.985	.998
$K_{(\mathrm{GO})}$	.50887	*	.60579	.112	.960	1.084	.982	1.016	.983	.999

Note. $K_{(\mathrm{B})}$ = Bennett et al.-type K, $K_{(\mathrm{S})}$ = Scott-type K, $K_{(\mathrm{SO})}$ = modified Scott-type K, $K_{(\mathrm{C})}$ = Cohen-type K, $K_{(\mathrm{G})}$ = Gwet-type K, $K_{(\mathrm{GO})}$ = modified Gwet-type K, n = sample size, ASE = asymptotic standard error, SD$_t$ = the standard deviation of studentized estimates of $K_{(\cdot)}$ in a simulation, $K_{(\cdot)} = 1 - \Delta_{(\cdot)} = 1 - (\pi_{\mathrm{o}}^{(v)} / \pi_{\mathrm{e}(\cdot)}^{(v)})$. $v_{2\mathrm{A}} = 0$ when a pair-wise profile is 11 or 22 otherwise $v_{2\mathrm{A}} = 1$ (compare $v_{2\mathrm{A}}$ in Table A4 with $v_{3\mathrm{A}}$ in Table A3 and $v_{4\mathrm{A}}$ in Table A1). The asterisks indicate that the corresponding common values of $\pi_{\mathrm{e}(\cdot)}^{(v)}$ are used.

Table A5. Asymptotic and simulated correlations of $p_{\mathrm{o}}^{(v)}$, $p_{\mathrm{e}(\cdot)}^{(v)}$ and $\hat{K}_{(\cdot)}$'s for 3-wise agreement of 4 raters using the formula of the ratio of means and two rating categories in Table 1 (not Table A1; n = 29 and the number of replications in simulations = 10,000)

3-wise agreement

	Asymptotic correlations				Simulated correlations			
$p_{\mathrm{o}}^{(v)}$	1				1			
$p_{\mathrm{e(S)}}^{(v)}$	.0894	1			.0947	1		
$p_{\mathrm{e(SO)}}^{(v)}$	.0136	.9957	1		.0079	.9943	1	
$p_{\mathrm{e(C)}}^{(v)}$	.1265	.9990	.9905	1	.1371	.9986	.9873	1
$\hat{K}_{(\mathrm{B})}$	1				1			
$\hat{K}_{(\mathrm{S})}$	.9072	1			.8442	1		
$\hat{K}_{(\mathrm{SO})}$	.9104	.9997	1		.8468	.9987	1	
$\hat{K}_{(\mathrm{C})}$.	9054	.9999	.9994	1	.8421	.9997	.9972	1

Note. $\hat{K}_{(\mathrm{B})}$ = Bennett et al.-type $\hat{K}$, $\hat{K}_{(\mathrm{S})}$ = Scott-type $\hat{K}$, $\hat{K}_{(\mathrm{SO})}$ = modified Scott-type $\hat{K}$, $\hat{K}_{(\mathrm{C})}$ = Cohen-type $\hat{K}$, n = sample size. $v_{3\mathrm{A}}$ = 0 when a 3-wise profile is 111 or 222 otherwise $v_{3\mathrm{A}}$ = 1. $p_{\mathrm{e(B)}}^{(v)} = 1/k^{m'-1} = 1/4$ (a fixed value for a triplet with $k = 2$ and $m' = 3$).