A Contingent Claim Analysis of Suicide

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Abstract

An option-theoretic model of suicide in the continuous time framework is proposed. Given completeness of the financial market and the associated contingent claim argument, the value of human capital consistent with the no-arbitrage principle is determined as the expected, discounted, present value of the future wage stream under the risk-neutral probability measure. The suicide option - the right but not the obligation to commit suicide - is modelled as an American put option with this human capital stock and a certain reference level of human capital as the underlier and strike price, respectively. The value of underlier falling short of the srtike price doet not induce the option holder's immediate suicide because of the option value to postpone such a fatal and irreversible decision. This value, the delayed exercise premium, is given in a near closed form up to a deterministic exercise boundary. The nearly closed-form nature of this boundry allows one to calibrate the model to the real suicide rates among Japanese male workers from 1998 to 2009. The calibrated value of the strike price roughly amounts to the perpetual annuity value of the 90 percentage of the initial wage earned as of the new entry into the labor market, with the coupon rate given by the spread in market prices of risk between financial and labor market.

Key Words

Suicide, American Put Option, Human Capital, Delayed Exercise Premium, Immediate Exercise Boundary.

JEL Classification

D91, G13, J24.

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Probably no one who attempts suicide .. is fully aware of all his motives, which are usually too complex. At least in my case it is prompted by ... a vague sense of anxiety about my own future.

Ryunosuke Akutagawa, *Suicide Note.*

1 Introduction

Suicide has drawn public attention in Japan over the last decade. One epoch-making year is 1998 in which the number of suicide victims exceeded 30 thousands at first since the record has been made public. The suicide statistics have shown a clear pattern of upheaval from this year and has remained to be so high for more than a decade until 2012. Just from a viewpoint of the socio-economic cost calculation, several reports suggest that this is a substantial welfare loss.

The epigraph is taken from a letter of a Japanese novelist Ryunosuke Akutagawa who committed suicide in 1927 by taking poison pills. His suicide received public attention with a great surprise, because he killed himself because of "a vague sense of anxiety" about his own future. However, several factors indicated in his note are reminiscent of those in the context of financial option pricing in a stochastic dynamic environment- the value of financial security in the future is quite uncertain, and the lack of appropriate control for the risk in the future should be perceived as an actual cost as of today, given the present-value rule: uncertain future cash-flow should be discounted back to the present value using an appropriate discount factor to compare the benefit and cost of taking any actions now.

This study is an attempt to modelling a suicide decision from a financial option-theoretic viewpoint. We will formalize the suicide decision as a problem of optimal exercise of an American put option written on a version of human capital of a person as the underlying asset. Every worker is endowed with this option, namely, the right but not the obligation, to commit suicide. The first target in this study is to articulate what "a vague sense of anxiety about my own future" exactly means. Based on the literature of the American option pricing, we can decompose the value of such option into two. The first component is the European part, namely, the value of option to commit suicide only on the verge of one's terminal day. This value is embedded in the American option allowing for the holder to exercise at any time before the maturity date. Therefore, the value of suicide option should be greater than or at least equal to the European counterpart. The remaining value in the American part is called the early-exercise premium. This part comes from the flexibility to commit suicide at any time before the end of one's life. If the anxiety in one's own future is so dominant, the ultimate way to clear it is to commit suicide. A positive value of the early exercise premium plays a role of great attraction for a desparate person.

Also based on the literature of American option valuation, we derive an alternative decomposition of suicide option value into two parts. The first component is the current pain to be alive, that is measured by the distance between the current level of human capital in short of the strike price. The second component is the premium associated with one's flexibility to reconsider and postpone such a fatal and irreversible suicide decision. This part is called the delayed exercise premium, which is introduced by Carr, Jarrow and Myneni (1992). We interpret this premium as is opposite to the early exercise premium reminiscent of the remark by Akutagawa- this is the option value to be positive about one's own future. This premium partially solves the concern by Dixit and Pindyck (1997) about the earlier economic attempt to modelling suicide by Hamermesh and Soss (1974) which does not pay a careful attention to the option value of postponing suicide.

Nearly closed-form expressions of these premia involve one undetermined object: the immediate exercise boundary dividing the product space of the life time interval and the value of human capital into two parts: the immediate exercise region and the continuation region. A person commits suicide once his/her human capital evolved in the continuation region hits this boundary. We show that this boundary is a continuous and deterministic function of time. More crucially, the boundary lies below the strike price of suicide option, interpreted as the reference level above which human capital is free of suicide risk. We also give a recursive formula to identify the exercise boundary. As long as the values of model parameters are known, therefore, we can obtain an approximate boundary with the precision of a desired level.

We calibrate this reference to reality by matching the simulated and actual suicide rates based on those among Japanese male workers from 1998 to 2009, with the level of wage upon new entry into the market normalized as one. The resulting threshold is $K \approx 47$, i.e., about 47 times as large as the initial wage of a worker in the first working year. This semingly large magnitude is not surprising once $K \approx .9/\zeta$ where ζ is the implicit dividend yield in the evolution of wage as specified in (4) later. In other words, the implied threshold amounts to the perpetual annuity value of the 90% of the initial wage. Surprisingly, the majority of workers in our simulation exercise have the history of human capital evolution strictly below this threshold over the entire life. Therefore, virtually everybody is at the risk of suicide, whereas only a small portion of them actually commit suicide. This discrepancy suggests the importance of exercise premia in the analysis of suicide option.

The outline of the balance of this study is as follows. Section 2 begins with a briefly review of the previous literature about the modelling of suicide behavior and the associated empirical studies. Section 3 deals with the basic setup and motivation of theoretical framework. A few properties of the value function and the immediate exercise region of the suicide option are derived. Section 4 is a small numerical study. Section 5 concludes the study with suggestions for future directions.

2 Literature Review

The earlier economic approach to modeling suicide is given by Hamermesh and Soss (1974), who assume that the level of utility from the permanent income hits some lower bound specified at, e.g., zero. This model is frequently cited as the earliest work to reveal two representative suicidal risk factors, the income level, and age. However, it has not been well emphasized in the context of suicidology that the variable in Hamermeh-Soss model is not the income nor a wage flow but the *permanent income*. We will elaborate more on this point in Section 3.4.

A few continuous-time models have been proposed. Lo and Kwok (2006) consider a signaling index X_t summarizing the current state of affective disorder of a patient. X_t is supposed to follow a geometric Brownian motion process. The drift of $d \ln X_t$ has a negative sign and is interpreted as some preventive treatment. The authors assume that the patient commits suicide if the level of disorder X_t exceeds the threshold X_c, and characterize this decision by the transition probability from the initial value X_0 to X_c, which resembles a structural credit-risk modeling since Merton (1974). Although the authors claim that they apply the contingent claims analysis, they do not consider the optimal choice of the timing of suicide nor the option value of postoponing a suicide decision. More crucially, there is no link between the disorder index and the environmental parameters.

Chan and Lien (2010) propose a continuous-time model of enthanasia in the context of legal and ethical debate after the U.S. Supreme Court's ruling in 1997. They attempt to give a behavioral model of a patient who just knows that he/she suffers from a terminal illness and about to decide if he/she uses the option of euthanasia and therefore to maximize the benefit of euthanasia, V_t net of the cost M_t at time t. M_t is a summary of the medial cost of pain management and related medical/legal/psychological burdens until the end of life. Given the geometric Bronwnian motion assumptions on V_t and M_t, they apply McDonald and Siegel (1986, (4) and (5)) for the immediate exercise boundary. My model in the next section is different from theirs in multiple aspects. First, the underlying asset in my case is more concrete human capital determined as the expected, discounted, present value of the future earning ability and therefore it is reverting to zero toward the end of life. Second, my model has an explicit expiration day of the option, namely, the end of life. In contrast, Chan and Lien (2010) based on McDonald and Siegel (1986) assume the infinite horizon: that is why their immediate exercise boundary turns out to be a constant, independent of the time-to-maturity. Obviously, the assumption of perpetuity for the model of a terminal decision is very strange. Third, their model focuses on the terminal decision after the notification by a physician of the terminal condition. Therefore, the human capital cannot play a significant role: the cost M_t in their model should be covered by the financial asset and a government-sponsored life-assistance. Fourth, they assume that V_t and M_t are driven by two Brownian motion processes, but they do not give any assumption about the completeness of a financial market.

Suzuki (2007) proposes the utility-based real-option approach to the suicide decision. The wage flow is modelled as a jump-diffusion process in the infinite-horizon setup. The average growth rate of the wage flow suffers from down-ward reduction by the small yet positive probability of downside jump, modelling a sudden layoff. However, the assumption of infinite horizon is rather strange for the analysis of suicide. It appears as a constant threshold of the wage level below which a person commits suicide and therefore is silent about the aging effect of a person over time. Moreover, the relation between the threshold wage and the pre-specified threshold of utility is not well discussed.

Writing an American option on human capital is an idea investigated in the context of financing the cost of higher education through the so-called income-contingent loans and through a financially sound design of a government-supported education program. See, e.g., Palacios (2007, Chapter 7, Part III and Appendix B).

3 A Model

3.1 A Complete Financial Market

Suppose the life of a person is described by the closed interval $[0, T]$ with $T < \infty$. We identify this interval with his/her working life by abstracting the retirement age. We will give some discussion about the qualitative validity of our model if we introduce the retirement age in Section 3.4, and will do some robustness check for these more elaborated wealth processes in Section 4.2.

Let us fix a complete and filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$. $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a non-decreasing sequence of sub-σ-fields of $\mathcal{F}$ describing the evolution of public information. P is the "physical" probability measure defining the expectation and conditional expectation operators $E[(\cdot)]$ and $E_t[(\cdot)] = E[(\cdot)|\mathcal{F}_t]$. On this filtered space is defined a one-dimensional Brownian motion process $W = (W_t)_{t\in[0,T]}$. W summarizes the fundamental source of risk in the financial market. It generates the Brownian natural filtration $\mathbb{F}^{(W)}$, which is the minimum σ-field containing information generated by the history of the market risk $\sigma(W_s : s \leq t)$ with the augmentation by the P-null sets. Without loss of generality, we identify $\mathbb{F}$ with $\mathbb{F}^{(W)}$ and therefore the latter notation will not appear henceforth.

Suppose there exist one representative risk-free asset and another representative risky asset with prices denoted by B_t and S_t, respectively. As these letters suggest, the leading examples are a default-free government bond and a stock-market index. Suppose they follow risk-free and risky geometric processes:

$$dB_t/B_t = rdt \tag{1}$$

$$dS_t/S_t = \mu dt + \sigma dW_t = rdt + \sigma dW_t^* \tag{2}$$

where r is the risk-free rate, μ is the average growth rate of the risky asset price and σ is the volatility or the instantaneous standard deviation to scale the market risk. Suppose r,

μ and σ are constant and bounded above, and $\sigma > 0$. The last assumption implies that the financial market is complete: the single risk factor W_t is associated with a single risky asset S_t accoring to the rule $dW_t = \sigma^{-1}(dS_t/S_t - rdt)$, hence is replicable by a portfolio of traded risky and risk-free assets. The condition guarantees the existence of the market price of risk in the risky financial asset

$$\theta := \sigma^{-1}(\mu - r).$$

The rightest side in (2) stems from the definition $dW_t^* := dW_t \mp \theta dt$. At this stage, W^* is just an alternative notation. To assign an economic interpretation to it, define the stochastic exponential of θ: $\eta_{t,s} := \exp\left(-\theta(W_s - W_t) - (\theta^2/2)(s-t)\right) = \eta_{0,t}^{-1}\eta_{0,s}$. Over the interval $[t, s] \subset [0, T]$, there are two discount factors applied to any $\mathcal{F}_s$-measurable variable to convert it back to the value at time t:

$$R_{t,s} := e^{-r(t-s)}, \quad \xi_{t,s} := R_{t,s}\eta_{t,s} = e^{-r(t-s)-\theta(W_t-W_s)-\theta^2(t-s)/2}.$$

$R_{t,s}$ is the objective discount factor based on the risk-free rate of return in the market. $\xi_{t,s}$ is called the stochastic discount factor under the physical probability measure P. Because of the boundedness of coefficients, $(\eta_{0,t})_{t\in[0,T]}$ is a $\mathbb{F}$-martingale by the Novikov's theorem and therefore $E_t[\eta_{0,s}] = \eta_{0,t}$ or $E_t[\eta_{t,s}] = 1$ for $s \geq t$. In other words, η can be used as a density in switching the probability measure from P to another equivalent one, say Q, such that $Q(A) = E[\eta_{0,T}1_{\{A\}}]$ for any $A \in \mathcal{F}$ and $E_t^Q[(\cdot)] = E_t[\eta_{t,T}(\cdot)]$ such that $E_t[\xi_{t,s}Y] = E_t^Q[R_{t,s}Y]$ for any $\mathcal{F}_s$-measurable variable Y. The last property justifies the label of Q as the risk-neutral probability measure because it allows the application of the risk-free discount factor in the present-value formula, like a risk-neutral agent would do. Finally, the Girsanov theorem implies that $dW_t^Q = dW_t + \theta dt$ is the Q-Brownian motion increment. Now the right hand side of (2) characterizes the risk-neutral evolution of the risky asset price. We should keep in mind that the market risk does not disappear from this risk-neutral valuation: it lives in Q.

3.2 The Spanned Wage Process and Human Capital

Suppose the wage rate ω_t, evolves as follows:

$$d\omega_t/\omega_t = \mu^{(w)}dt + \sigma^{(w)}dW_t = (r - \zeta)dt + \sigma^{(w)}dW_t^* \tag{3}$$

where $\mu^{(w)}$ and $\sigma^{(w)} > 0$ are bounded and

$$\zeta := r - \mu^{(\omega)} + \sigma^{(\omega)}\theta. \tag{4}$$

The right hand side is again justified by the Novikov and Girsanov theorems. Notice that the same Brownian motion process (W_t) drives the middle expressions of (2) and (3), and the risk-neutral version W^* does so in the associated rightest sides. Therefore, we can replicate and evaluate the cashflow of the wage process by that of a portfolio of the risky and risk-free assets. Let us define $\theta^{(w)} := (r - \mu^{(w)})/\sigma^{(w)}$ as the market price of risk of the wage flow. The equivalent expression $\zeta = \sigma^{(w)}(\theta - \theta^{(w)})$ means that ζ measures the spread in the market prices of risk between the risky financial asset and the risky wage per unit of the wage risk. An immediate implication is that $sign(\zeta) = sign(\theta - \theta^{(w)})$. For instance, $\zeta > 0$ if and only if the risky financial asset is more expensive than that of the wage in terms of their prices of risk. Subtraction of ζ from the drift in the rightest side of (3) means that the evolution of wage is subject to the implicit continuous dividend payout with ζ as the dividend yield[1].

We can solve (3) for w_s in terms of w_t for $t < s$ as follows:

$$w_s = w_t R_{t,s}^{-1} Z_{t,s} \eta_{t,s}^{(w)}, \tag{5}$$

$$Z_{t,s} := e^{-\zeta(s-t)}, \tag{6}$$

$$\eta_{t,s}^{(w)} := e^{\sigma^{(w)}(W_s^* - W_t^*) - [(\sigma^{(w)})^2/2](s-t)}. \tag{7}$$

Let H_t denote the human capital at time t. Using the risk-neutral valuation principle, it is the expected, discounted, present value of the future cash flow generated by the ownership of the labor hours under the risk-neutral measure Q and the risk-free discount factor R:

$$H_t = \bar{h} E_t^Q \left[\int_t^T R_{t,v} \omega_v dv \right]$$

where $\bar{h}$ is the total amount of labor force of the worker [2].

Let us define the annuity factor associated with ζ:

$$A_{t,T}(\zeta) := \int_t^T Z_{t,s} ds = \frac{1 - e^{-\zeta(T-t)}}{\zeta}.$$

The exponential term in the numerator accounts for the time-to-maturity effect. The risk-neutral valuation of the future cashflow generated from human capital at time $t \in [0, T]$ is given by

$$H_t = \bar{h}\omega_t A_{t,T}(\zeta) = E_t^Q \left[\int_t^T R_{t,s} \bar{h} w_s ds \right]. \tag{8}$$

The same expression is obtained from the classical contingent-claim approach to exploit the no-arbitrage partial differential equation[3]. $\lim_{t \to T} A_{t,T}(\zeta) = 0$ implies $\lim_{t \to T} H_t = 0$ P-a.s. The Ito's lemma applied to $R_{0,t} H_t + \int_0^t R_{0,s} \bar{h} w_s ds$ and a bit of calculation produces the present value formula in the right hand side. We can derive the stochastic differential equation for

[1] If we assume a jump-diffusion model for the wage flow dynamics, as in Suzuki (2007), the drift is subject to yet another subtraction of the probability of a sudden layoff. In that case, we just need to modify the definition of ζ. This approach has a limitation that the whole market is incomplete then- we cannot replicate the jump part by a continuous trading of the risky and risk-free assets evolving continuously as are specified previously. Moreover, it is still an open question if some important properties of the immediate exercise boundary in Section 3.5 hold or not. Consequently, we will not pursue the jump diffusion model in this study.

[2] This does not mean that the worker in our model actually supplies 100% of their endowed time to the labor service. Although we do not specify any model of the labor-leisure choice, this worker can always buy their leisure time back at the wage rate as an opportunity cost of the leisure consumption.

[3] See Bodie, Merton and Samuelson (1992, Section 4).

H_t by applying the Ito's lemma and the definition of ζ as follows:

$$dH_t = (rH_t - \bar{h}w_t)dt + H_t\sigma^{(w)}dW_t^* \quad (9)$$
$$= H_t\left\{(r - A_{t,T}^{-1}(\zeta))dt + \sigma^{(w)}dW_t^*\right\} \quad (10)$$

for $t \in [0, T)$ with the initial value $H_0 = \bar{h}w_0 A_{0,T}(\zeta)$ and the terminal value $H_T = 0$, P-a.s. (9) is reminiscent of the dynamic evolution of a risky asset price with a fixed terminal value, e.g., that of a risky bond price[4]. In fact, the L'hôpital rule delivers $\lim_{\zeta \to 0} A_{t,T}(\zeta) = T - t$. (10) is useful later in Section 3.6 for computing how likely a worker refrains from committing suicide. Notice that the growth rate of human capital depends on time through $A_{t,T}^{-1}(\zeta)$ in (10), even though coefficients in (1), (2) and (3) are constant. As is similar to ζ in the drift of wage flow in (3), $A_{t,T}^{-1}(\zeta)$ is like a continuous dividend yield associated with the depreciation of human capital as time elapses. As long as $\zeta > 0$, this implicit dividend yield on human capital becomes ζ as $T \to \infty$ because $\lim_{T \to \infty} A_{t,T} = 1/\zeta$.

3.3 Suicide as an American Put Option written on the Human Capital

Let us introduce a worker's problem of optimally determine the timing of suicide. Suppose a worker alive at time $t \in [0, T]$ wants to commit suicide at a certain future point in time, $\tau \in [t, T]$, whenever the human capital is sufficiently low relative to a certain pre-specified level of the basic standard of living, say $K > 0$. K would be different from one worker to another. The optimal timing of suicide solves the following problem:

$$\tau_{t,T} := arg \sup_{\tau \in \mathcal{S}_{t,T}} E_t^Q \left[R_{t,\tau}(K - H_\tau)\right] = arg \sup_{\tau \in \mathcal{S}_{t,T}} E_t^Q \left[R_{t,\tau}(K - H_\tau)^+\right] \quad (11)$$
$$V(t, H) := E_t^Q \left[R_{t,\tau_{t,T}}(K - H_{\tau_{t,T}})\right] = E_t^Q \left[R_{t,\tau_{t,T}}(K - H_{\tau_{t,T}})^+\right] \quad (12)$$

where $\tau \in \mathcal{S}_{t,T}$ is a collection of $\mathbb{F}$-stopping times such that $P(\tau \in [t, T]) = 1$ and the second equalities in (11) and (12) are justified by the optimality, because it is suboptimal to commit suicide if $H_t > K$. It is evident now that we model the suicidal behavior of a worker as the optimal exercise policy of an American put option on human capital as the underlying asset.

3.4 Motivating the Model

We use human capital as the underlying asset of the suicide option. This is partially inspired by the earliest economic model of Hamermesh and Soss (1974) in which suicide occurs when the present value of utility from the permanent income hits the lower bound at

[4] A bond-like behavior is shared by the optimal wealth process obtained from the dynamic optimization of consumption and leisure streams in Bodie et. al. (2012). However, if we assume a certain optimization behavior of a worker under consideration, it raises the issue of a simultaneous determination of the optimal consumption-leisure streams *and* of the timing of suicide. Introducing a version of preference in a dynamic stochastic environment and determining the timing of suicide jointly with consumption-saving, labor supply-leisure consumption would be helpful for understanding some gradual deterioration of mental helth of people at risk toward suicide.

zero. The permanent income is the expected, discounted, aggregated present value of any kinds of future cashflow available to a representative household. We let human capital capture this capitalized nature of the underlier for the suicide option. The absence of financial wealth, owing to the lack of modeling consumption-saving decision in our dynamic environment, may sound quite restrictive. Let us give three justifications of our negligence of the financial asset allocation.

First, the main focus of this study is to analyze a premium of the suicide option associated with the flexibility to select a timing of exercise. An option-theoretic approach can isolate such premium in a clear manner. In a later section we will discuss a broad possibility of modelling multiple aspects of suicide by borrowing several valuation formula of exotic options. This is a relatively easy task if we focus on a worker's life from a viewpoint of the cash-flow risk management.

Second, some numerical simulation of the life-cycle model by Bodie et. al. (2012) suggests that the human capital shares the major part of the total wealth, that corresponds to the permanent income in Hamermesh and Soss (1974). Therefore, our bare-bone model should be viewed as a first-order approximation to otherwise complex nature of suicide by focusing on the largest contributor to the total wealth.

Third, the technique to be developed later in our simplified setup is applicable to a more elaborated wealth process emerged from the optimal choice of consumption, saving, labor and leisure in Bodie et. al. (2012, (A.8) to (A.10)). Given $T_r \in (0, T]$ as a fixed timing of retirement, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$, the total wealth is

$$H_t^* = \left(\frac{y^* \xi_{0,t}}{a_{0,t}}\right)^{-1/R} \left\{ f\bar{h} w_t^{(1-\eta)\rho} A_{t\wedge T_r, T_r}(g) + \phi^{1/R} e^{-\bar{g}(T_r - t\wedge T_r)} A_{t\vee T_r, T}(\bar{g}) \right\} \tag{13}$$

with $\lim_{t\to T} H_t^* = 0$, P-a.s., where y^* is the Lagrange multiplier of a static budget constraint, $\xi_{0,t}$ is the stochastic discount factor as defined previously, $a_{0,t} = \exp(-bt)$ is the subjective utility-discount factor with the constant subjective discount rate $b > 0$, η is a weight on the relative importance of consumption goods over leisure, R is the relative risk aversion coefficient, and g, $\bar{g}$ and f are known functions of model parameters[5]. Applying the Itô's lemma, defining $A_{t,T} := A_{t\wedge T_r, T_r}(g) + A_{t\vee T_r, T}(\bar{g})$ and $\sigma_t^{(N)} := R^{-1}\theta + (1 - R^{-1})(1-\eta)\sigma^{(w)} 1_{\{t \le T_r\}}$, and doing a bit of calculation, the dynamic evolution of H_t^* is analogous to (10):

$$dH_t^*/H_t^* = (r - A_{t,T}^{-1})dt + \sigma_t^{(N)} dW_t^*.$$

Therefore, we can apply the same technique developed in our setup to the suicide option based on H_t^*, apart from the different combinations of parameters in the definition of the an-

[5] η here comes from the notation in Bodie et. al. (2012) and is different from the stochastic exponential of θ introduced in this study. $f, \bar{g}, g$ are given by $f := [(1-\eta)/\eta]^{-(1-\eta)(1-R^{-1})}/\eta$, $\bar{g} = g|_{\eta=1}$ and

$$g := R^{-1}\beta + (1 - R^{-1})\left(r + \frac{\theta^2}{2}\right) - (1-R^{-1})(1-\eta)\left(\mu^{(w)} - \frac{(\sigma^{(w)})^2}{2}\right) - \frac{(1-R^{-1})^2[\theta - (1-\eta)\sigma^{(w)}]^2}{2}.$$

nuity factor and the time-dependent yet deterministic volatility of the risk-neutral Brownian increment, g, $\bar{g}$ and $\sigma_t^{(N)}$. Consequently, more elaborate version of human capital does not change the main feature of our model.

The true limitation of our model, which is also common to many previous works, is the lack of simultaneous determination of the timing of suicide and consumption, saving, labor and leisure. Notice that Hamermesh and Soss (1974) implicitly assume the two-stage decision-making of a person in their model: he/she has already chosen a consumption and labor supply decision optimally to obtain a permanent income process without taking the risk of suicide into account, then he/she may feel temptation to commit suicide in a later life by looking at the present value of tility generated from the permanent income. Let us owe this issue to an investigation by Ikeda (2013b). Therefore, we will not pursue it in this study.

Another main component of the suicide option is $K > 0$. This parameter corresponds to the strike price of a financial option. We interpret K as the threshold level of human capital above which a worker is free of suicide risk. Proposition 1 -(c) in the next subsection justifies this interpretation. and therefore the threshold for him/her to start feeling a pain even if he/she gets a positive amount of cash-flow. The constancy of K may not be general enough once we recognize the fact that a person would feel it more painful to endure a lower quality of life if the past life had been good (habit/rachet effects) or would become familiar with the current state of life (acceptance of hardship). We will discuss some possibility of relaxing this assumption later. However, let us emphasize that the key implication from our model is the time-dependent exercise boundary of suicide option even if the threshold is a constant as is specified in (11).

3.5 Properties of the Value Function and Exercise Region

Let us divide the coordinate for the time and human capital, $(t, H) \in [0, T] \times \mathbb{R}_+$, into two disjoint subsets, namely, the immediate $\mathcal{E}$xercise region and the $\mathcal{C}$ontinuation region:

$$\begin{aligned} \mathcal{E} &= \left\{(t,H) \in [0,T] \times \mathbb{R}_+ : V(t,H) = (K-S)^+\right\}, \\ \mathcal{C} &= \left\{(t,H) \in [0,T] \times \mathbb{R}_+ : V(t,H) > (K-S)^+\right\}. \end{aligned}$$

H is some constant level of human capital for a person facing the decision of committing suicide or not as of time t. It is always possible to find $w \in \mathbb{R}_+$ such that $H = \bar{h} w A_{t,T}(\zeta)$.

Proposition 1 *(Properties of the Exercise and Continuation Regions)*

(a) $V(t,H) \in C^{(1,2)}$, *i.e., V is continuously differentiable with respect to $t \in [0,T]$ and twice continuously differentiable with respect to $H \in \mathbb{R}_+$. Moreover, the admitted partial derivatives are locally bounded on $[0,T) \times \mathbb{R}_+$. If $(t,H) \in \mathcal{C}$,*

$$V_{HH}(t,H)H^2(\sigma^{(w)})^2/2 + V_H(t,H)(rH - \bar{h}w) + V_t(t,H) - rV(t,H) = 0.$$

(b) $V(t,H)$ *is non-increasing and convex with respect to* $H \in \mathbb{R}_+$.

(c) For any $t \in [0,T)$, $(t,H) \in \mathcal{E} \Rightarrow H \leq K \cdot \min\{1, rA_{t,T}(\zeta)\}$. $H \leq K \Leftrightarrow (T,H) \in \mathcal{E}$.

(d) $(t,H) \in \mathcal{E}$ implies $(t, H-\epsilon) \in \mathcal{E}$ for any $\epsilon \in [0,H]$.

(Proof) Combine Jaillet, Lamberton and Lapeyre (1990) with Detemple (2006, Chapter 4). See the appendix for details.

Remark 1 *Given generality of Jaillet et.al. (op.ct.), (a) is still valid if $\sigma^{(w)}$ is replaced by a deterministic function of t, as in the alternative wealth dynamics (13). In (c), $(t,H) \in \mathcal{E}$ actually implies $H \leq KrA_{t,T}(\zeta)$ because both the risk-free rate and the annuity factor take values in $[0,1]$ given $\zeta > 0$. Given $t < T$ and $H = \bar{h}wA_{t,T}(\zeta)$, this implication is restated as $(t,H) \in \mathcal{E} \Rightarrow rK - \bar{h}w \geq 0$. The combination of (a) with the continuity of $(K-H)$ with respect to H implies that the immediate exercise region $\mathcal{E}$ is a closed subset of $[0,T] \times \mathbb{R}_+$. (b) means that $\mathcal{E}$ is down-connected in $[0,T] \times \mathbb{R}_+$. They imply the alternative representation of $\mathcal{E}$ as follows:*

$$\mathcal{E} = \{(t,H) \in [0,T] \times \mathbb{R}_+ : H \leq B_t\}, \quad B_t = \sup\{H \in \mathbb{R}_+ : (t,H) \in \mathcal{E}\}. \tag{14}$$

We will call $B = (B_t)_{t \in [0,T]}$ the immediate exercise boundary.

Proposition 2 *B_t is continuous at any $t \in [0,T)$; $B_{T-} = \lim_{t \to T} B_t = 0$ and $B_T = K$. If $r < 1$ and if we re-parametrize $B_t = \beta_t A_{t,T}(\zeta)$ for $t < T$, $\beta_{T-} = \lim_{t \to T} \beta_t = rK$.*

(Proof) A repeated application of the argument by Jacka (1991, Proof of Proposition 2.4) and Detemple (2006, Proof of Proposition 33). See the appendix for details.

Remark 2 *The discrepancy between $B_{T-} = 0$ and $B_T = K > 0$ means that the boundary has a jump at the terminal day. This property is shared by usual American-style derivatives with finite maturity dates: see Detemple (2006, Proposition 33 and the subsequent discussion). Usually, American options in the Black-Scholes environment with constant coefficients have monotonic immediate exercise boundaries: decreasing for the call and increasing for the put, as is described by Detemple (2006, Proposition) combined with the put-call symmetry. However, (10) shows that the drift of the underlying human capital depends on time. Given $D^{(c)}_{a,b} := e^{-\int_a^b A^{-1}_{c+v,T}(\zeta)dv}$, $\eta^{(w)}_{a,b}$ from (7) and doing a bit of calculation,*

$$H_\tau \overset{d}{=} H_t e^{r(\tau-t)} e^{-(\sigma^{(w)})^2(\tau-t)/2+\sigma^{(w)}W^*_{\tau-t}} e^{-\int_0^{\tau-t} A^{-1}_{v+t,T}(\zeta)dv} := H_t R^{-1}_{0,\tau-t}\eta^{(w)}_{0,\tau-t}D^{(t)}_{0,\tau-t}. \tag{15}$$

The first and second exponential factors depend on t and τ only through their difference $\tau - t$. On the other hand, the third exponential factor D does depend on t directly as a decreasing function of t. This is because the implicit dividend yield $A^{-1}_{t,T}(\zeta)$ in (10) accounts for the time-to-maturity effect. They imply

$$\begin{aligned} V(t,H) &= \sup_{\tau \in \mathcal{S}_{t,T}} E^Q_t \left[R_{0,\tau-t}(K - HR^{-1}_{0,\tau-t}\eta^{(w)}_{0,\tau-t}D^{(t)}_{0,\tau-t})^+\right] \\ &= \sup_{\bar{\tau} \in \bar{\mathcal{S}}_{0,T-t}} \bar{E}_0 \left[R_{0,\bar{\tau}}(K - HR^{-1}_{0,\bar{\tau}}\bar{\eta}^{(w)}_{0,\bar{\tau}}D^{(t)}_{0,\bar{\tau}})^+\right] \end{aligned}$$

where $\bar{\tau} := \tau - t \in \bar{\mathcal{S}}_{0,\tau-t}$ if and only if $\tau \in \mathcal{S}_{t,T}$, and the stationary increment property of the Brownian motion induces the expectation $\bar{E}_0$. The criterion $\bar{E}_0[R_{0,\bar{\tau}}(K - HR^{-1}_{0,\bar{\tau}}\bar{\eta}^{(w)}_{0,\bar{\tau}}D_{t,\bar{\tau}})^+]$

is increasing whereas the choice set $\bar{S}_{0,T-t}$ is shrinking, as t becomes larger. Consequently, the effect of t on $V(t,H)$ is not clear. Then, the argument by Detemple (2006, Proposition 39) may not be applicable to B and therefore it may not be monotonic.

3.6 Exercise Premia

According to a similar argument as in Detemple (2006, Theorem 23), we can decompose V into two parts as follows.

Proposition 3 *Given the re-scaling $B_t = \beta_t A_{t,T}(\zeta)$,*

$$V(t,H) = p(t,H) + E_t^Q\left[\int_t^T R_{t,s}1_{\{s=\tau_s\}}(rK - \bar{h}w_s)ds\right] \tag{16}$$

$$= KR_{t,T} + \int_t^T \left\{rKR_{t,s}N\left(-d_{t,s}^{(-)}(\bar{h}w_t,\beta_s)\right) - \bar{h}w_t Z_{t,s}N\left(-d_{t,s}^{(+)}(\bar{h}w_t,\beta_s)\right)\right\}ds \tag{17}$$

where $p(t,H) = E_t^Q\left[R_{t,T}(K-H_T)^+\right] = KR_{t,T}$ is the value of an European put option to commit suicide at the maturity date T, $N(\cdot) = (\sqrt{2\pi})^{-1/2}\int_{-\infty}^{\cdot} e^{-x^2/2}dx$ is the cumulative distribution function of the standard normal variable, and

$$d_{t,s}^{(\pm)}(a,b) := \frac{\ln(a/b) + (r - \zeta \pm (\sigma^{(w)})^2/2)(s-t)}{\sigma^{(w)}\sqrt{s-t}}. \tag{18}$$

(Proof) See the appendix.

The first term in (16) is the value of an European-style put option to commit suicide only at the terminal date T, which is embedded in the American-style put option with exercisability at any time $\tau \in [t,T]$. However, $H_T = 0$ makes the automatic suicide at time T: there is no clear distinction between suicide and natural death at time T in our model. Therefore, the European option value is degenerated to the present value of K. This degeneracy occurs because the suicide option is written on the human capital which depreciates to zero toward the end of life. The second term in (16) is the early exercise premium. We interpret this premium as an economic form of "a vague sense of anxiety about my own future" as cited in the previous epigraph. It is vague because it is related to, but different from, the burden to live a life below the threshold K. It is some anxiety about future because $(s, H = \bar{h}wA_{s,T}(\zeta)) \in \mathcal{E}$ implies $rK - \bar{h}w_s \geq 0$ from Remark 1 and therefore it represents the present value of benefit to avoid anxiety about his/her human capital in the future by committing suicide at $\tau \in [t,T]$.

Plugging $H_t = \bar{h}w_t A_{t,T}(\zeta) = B_t = \beta_t A_{t,T}(\zeta)$ and therefore $\bar{h}w_t = \beta_t$ for $t < T$ to (17) given the boundary condition $\beta_{T-} = rK$, for $t \in [0,T)$,

$$K - \beta_t A_{t,T}(\zeta) = KR_{t,T} + \int_t^T \left\{rKR_{t,s}N\left(-d_{t,s}^{(-)}(\beta_t,\beta_s)\right) - \beta_t Z_{t,s}N\left(-d_{t,s}^{(+)}(\beta_t,\beta_s)\right)\right\}ds. \tag{19}$$

One immediate implication is the following:

Proposition 4 *Given (1), (2) and (3), the immediate exercise boundary B_t and its re-scaled version β_t are deterministic functions of time.*

We will exploit (19) later for the numerical identification of the immediate exercise boundary. We suggest previously that the behavior of human capital is reminiscent of a risky bond price with a finite time horizon. One stark difference is that the former converges to a unit price, say one thousand US dollars in the case of a U.S. Treasury note, whereas the latter converges to zero. Notice that zero is the absorbing state from which the geometric evolution cannot start. On the other hand, the backward recursive equation in the previous works such as Detemple (2006) always involve the logarithm of the exercise boundary. Because $\ln B_{T-} = \ln 0 = -\infty$ is floating around, it may destroy the validity of a formula for B_t then[6]. This is why the rescaling $B_t = \beta_t A_{t,T}(\zeta)$ and the recursive equation for β_t are useful. Because of the property $\beta_{T-} = rK > 0$ as confirmed in Proposition 2, the issue of zero absorbing state is avoided. The recovery of B from β is immediate.

We can obtain another type of premium as follows.

Proposition 5 *Given the rescaling $B_t = \beta A_{t,T}(\zeta)$ and $n(\cdot) = (2\pi)^{-1/2}\exp(-(\cdot)^2/2)$ as the probability density function of the standard normal variable,*

$$V(t,H) = (K - H_t)^+ + \mathcal{D}_{t,T} \tag{20}$$

where $\mathcal{D}_{t,T}$ is the delayed exercise premium with the explicit representation

$$\mathcal{D}_{t,T} = \bar{h}w_t \int_t^T Z_{t,s}\left\{\frac{\sigma^{(w)}A_{s,T}(\zeta)}{2\sqrt{s-t}}n\left(-d_{t,s}^{(+)}(\bar{h}w_t A_{s,T}(\zeta), K)\right) - N\left(-d_{t,s}^{(+)}(\bar{h}w_t, \beta_s)\right)\right\} ds$$
$$-rK\int_t^T R_{t,s}\left\{N\left(-d_{t,s}^{(-)}(\bar{h}w_t A_{s,T}(\zeta), K)\right) - N\left(-d_{t,s}^{(-)}(\bar{h}w_t, \beta_s)\right)\right\} ds.$$

(Proof) Carr, Jarrow and Myneni (1992, Theorem 2.1). See the appendix for details.

The sum of the second components in two integrals is identical to the early-exercise premium in (17). In other words, the remaining parts in $\mathcal{D}_{t,T}$ stem from a decomposition of the European option value.

(20) reveals that the current level of human capital falling short of K does not necessarily induce the immediate suicide. The delayed exercise premium accounts for the benefit to postpone such a fatal and irreversible decision. We can interpret the first integral in $\mathcal{D}_{t,T}$ as the benefit to wait for future upside opportunity in human capital, whereas the second integral is the cost incurred by staying alive with human capital falling short of K. This alternative premium is crucial for the understanding of a remark by Dixit and Pindyck (1994) directed toward the economic model of suicide by Hamermesh and Soss (1974):

> *According to them, an individual will end his or her own life when the expected present value of the utility of the rest of life falls short of a benchmark or cutoff standard "zero". Most people react by saying that the model gives an excessively*

[6] The issue of zero absorbing state in the identification of the immediate exercise boundary is, to our best knowledge, a novel feature of the human-capital option with a finite time horizon.

rational view of what is an inherently irrational action. Our theory suggests exactly the opposite. Whatever its merits or demerits as descriptive theory, the Hamermesh-Soss model is not rational enough from the prescriptive viewpoint, because it forgets the option value of staying alive. Suicide is the ultimate irreversible act, and the future has a lot of ongoing uncertainty. Therefore the option value of waiting to see if "something will improve" should be very large. The circumstances must be far more bleak than the cutoff standard of the Hamermesh-Soss model to justify pulling the trigger. This is true even if the expected direction of life is still downward; all that is needed is some positive probability on the upside. (Dixit and Pindyck 1994, p.24)

However, Dixit and Pindyck do not give any formal model to address their concern and therefore it has been an open question since then. (20) gives a partial answer to it, although we have not introduced any preference of a person with the exception of the weakest one, "the more is better", behind the arbitrage principle[7]. In the next section, we will conduct a small simulation study to confirm the size of this "option value of waiting to see if something will improve". The lack of recognizing such an option value basically means that we assume K to be a hard threshold below which everybody commits suicide. Because human capital depreciates toward zero, any strictly positive value of K would induce that everybody will commit suicide then. Such interpretation and implication are not true in our model.

4 Numerical Identification of the Exercise Boundary

To study several properties of the option value of suicide numerically given a set of realistic parameter values, we need to identify the immediate exercise boundary B_t or its rescaled version β_t. Let us discuss how to draw a curve given a particular set of model parameters.

4.1 The one-dimensional integral equation for the boundary

We will use T in place of $T-$ in the balance of this section for the identification of $(\beta_t)_{t\in[0,T)}$. We can transform (19) into the following form:

$$\beta_t = \frac{K\left[1 - R_{t,T} - r\int_t^T R_{t,s} N\left(-d_{t,s}^{(-)}(\beta_t, \beta_s)\right) ds\right]}{A_{t,T}(\zeta) - \int_t^T Z_{t,s} N\left(-d_{t,s}^{(+)}(\beta_t, \beta_s)\right) ds}. \tag{21}$$

Let us define N equally-spaced grids on $[0,T]$ by $N+1$ partitions: $(t_j)_{j=0,\dots,N}$ such that $t_0 = 0$, $t_N = T$ and $\Delta t_j = t_j - t_{j-1} = T/N$. Accordingly, let $(\beta_{t_j})_{j=0,\dots,N}$ be the discretized version of $(\beta_t)_{t\in[0,T]}$. $\lim_{t\to T}\beta_t = rK$ from Proposition 2 gives the initial condition. Suppose we have already identified $(\beta_{t_k})_{k=j+1,\dots,N}$. Using the trapezoidal rule for the quadrature

[7]For a preference-based approach, see Ikeda (2013b).

approximation of the integrals as in Detemple (2006, Section 8.2), β_{t_j} is determined from the discrete version of (21):

$$\beta_{t_j} = \frac{K\left[1 - R_{t_j,t_N} - r\sum_{k=j}^{N} R_{t_j,t_k} N(-d^{(-)}_{t_j,t_k})\Delta t_k/(1+1_{\{k=j,N\}})\right]}{A_{t_j,t_N}(\zeta) - \sum_{k=j}^{N} Z_{t_j,t_k} N(-d^{(+)}_{t_j,t_k})\Delta t_k/(1+1_{\{k=j,N\}})}. \quad (22)$$

Because the right hand side of (22) depends on β_{t_j} in a highly non-linear fashion, we need some iterative method to obtain a sufficiently accurate estimate. From an initial guess of $\beta^{(0)}_{t_j}$, compute the right hand side of (22) to obtain the updated guess, $\beta^{(1)}_{t_j}$. Iterate this step q times until the size of update, $\beta^{(q)}_{t_j} - \beta^{(q-1)}_{t_j}$, is within a small number of convergence criterion. The last estimate gives a numerical solution $\hat{\beta}_{t_j}$. Given $(\hat{\beta}_{t_j})_{j=1,\ldots,n}$, it is immediate to have $(\hat{B}_{t_j})$ by $\hat{B}_{t_j} = \hat{\beta}_{t_j} A_{t_j,t_N}(\zeta)$.

4.2 A Numerical Study

Let us apply the above method in the setup of Bodie et. al. (2012) for the study of a person with a certain process of human capital. The following is the list of parameters:

[Table 1 Here]

The bare-bone model of the suicide option on human capital $H_t = \bar{h} w_t A_{t,T}(\zeta)$ relies on six parameters in the first three raws with the implication that $\zeta = .019$. We employ the rule of guess $\beta^{(0)}_{t_j} = \alpha\hat{\beta}_{t_{j+1}}$ with $\alpha = .95$. We also try a few values of α such as $\alpha = 1.05$ but estiamtes are almost identical to the former case when we iterate (22) by ten times, namely, $q = 10$. The initial value of the dynamics of w_t is normalized at one. We will generate 100 thousands sample paths of the wage flow and the associated human capital processes by 250 iterations. Then, calculate the number of paths hitting the immediate exercise boundary before the maturity as the simulated numbers of suicides for each iteration, compute the sample mean over these 250 iterations and form an empirical histogram of the number and timing of suicide summarizing the characteristics of the average suicide behavior of a representative worker.

Assumption 3 *The empirical suicide rate obtained as the sample mean of 250 iterations of 100-thousand simulated paths corresponds to the actual suicide rate based on contemporaneous cross-sectional events of suicides.*

The key parameter is K. From the contra-positive argument in Proposition 1-(c), a worker with human capital above K never exercises the suicide option. Owing to Assumption 3, we can calibrate the value of K to mimick the actual suicide rate among Japanese male workers from 1998 to 2009, which is about 36.13 per one hundred thousands population[8]. It turns out

[8]The suicide rates are calculated for seven 10-year age cohorts from 15 to 24, 25 to 34, 35 to 44, 45 to 54, 55 to 64, 65 to 74, and over 74. Because of the dispersion in the size of population in different age cohorts, the suicide rate per 100 thousands are adjusted accordingly. The data come from *Jisatsu Taisakunotameno Jisatushibou no Chiiki Toukei 1973-2009* (Regional Statistics of Suicide Deaths for the Suicide Prevention Policies, 1973-2009, in Japanese). As noted in the Introduction, 1998 is the year of upheaval of the suicide rate in Japan.

that $K \approx 47.1421$, namely, 47 times as large as that of the initial level of wage normalized as one. To understand this puzzling magnitude of K, notice that $K = .8957/\zeta$. Because $1/\zeta = \lim_{T\to\infty} A_{t,T}(\zeta)$, this means that the calibrated value of K amounts to the perpetual annuity value of about 90% of the initial wage to be received forever, like a coupon payment of a console bond in the infinite horizon. A worker wishes she/he held such an attractive insurance to be free of suicide risk. If not, then she/he has to pay attention to the uncertainty in his/her future human capital evolution in a finite life-time horizon to avoid the pitfall of suicide risk.

Figure 1 is the plot of the mean over the 250 iterations of the actual number of suicides in red and their grand mean in starred yellow, the actual numbers of suicides in blue and their grand mean in starred green. Because there are 100 thousands paths simulated in each iteration, the reported number of suicides can be interpreted as the suicide rates in the population of size 100 thousands. The starred lines are almost identical as the calibration exercise is intended. The simulated age pattern of suicide rates is steeper than the actual one. On one hand, the model can capture the average tendency of suicide among male workers. On the other hand, the model gives an under-estimate of suicide among young workers and an over-estimate of suicide among older workers approaching the retirement age. The steeper pattern of age profile of suicide rates from the model may owe to the constant strike price K. If we can adjust K such that it is slightly lower in younger ages and higher in older ages, we may be able to improve the fit. Another reason for this mis-represented pattern is the lack of retirement period: in this simulation study, the end of a working life is the end of life. If we introduce the possibility of life after the retirement, the older workers facing the retirement may not need to rush into the fatal decision.

[Figure 1 Here]

4.3 Robustness to More Elaborated Wealth Process

We have analyzed the suicide option written on the human capital directly. Let us do some robustness check of our methodology if the option is written on more elaborated wealth processes. Suppose the consumption and labor supply are determined optimally with no regard to the possibility of suicide over $[0, T]$. We still assume that $T_r = T$ so that this worker will have kept working until the end of life without retirement age. The implied wealth process is basically given by setting $T = T_r$ in (13), i.e. the first term in the right hand side. We can apply the same methodology as previously for human capital to this case. The calibrated strike value is $K \approx 43.62$, which is not so different from the previous one. (13) with $T = T_r$ indicates that the implicit coupon rate is $g \approx .0234$ as in Footnote-5 given parameter values as specified in Table 1. Now we have $K \approx .9905/g$. Similarly as previously, a worker with the wealth process generated by choosing the consumption and leisure optimally without worrying about suicide risk, feels safe if she/he received the perpetual annuity value of the 99% of her/his initial wealth. A similar pattern emerges from this more elaborated wealth process.

[Figure 2 Here]

5 Conclusion and Future Extensions

This study attempts to model and analyze the suicidal behavior from an option-theoretic viewpoint. We propose a model of commiting suicide as the optimal exercise rule of an American put option written on human capital. We derive the early exercise premium and the delayed exercise premium representations of the value of suicide option in near closed forms. The strike price, or the least acceptable level of the quality of life, does not correspond to a hard threshold for committing suicide- the true immediate exercise boundary lies below this strike price, reflecting the option value to postpone such a fatal and irreversible decision. We also propose an efficient and simple discretization scheme to numerically identify the boundary through re-scaling of the original boundary. Given a set of some realistic values of parameters, the strike price of suicide option is calibrated so that the resulting rate of suicide mimicks the actual one in Japan from 1998 to 2009.

There are many ways to follow subsequently. First, the type of option contract in this study is a plain American put option. However, there are many exotic options proposed in the context of financial derivatives. For instance, we can replace the constant strike price K by a running geometric average of the past human capital until the current time with some proportionality factor, say $K = \kappa \exp\left(t^{-1}\int_0^t \ln H_u du\right)$. The option of this type is called the floating-strike Asian option. The geometric rather than arithmetic nature of the strike price allows some trick to derive a more complicated but conceptually similar early exercise premium representation as in Detemple (2006, Proposition 40). Moreover, the empirical works regarding the covariates of suicide rates have found that the marital status, especially if divorced or not and if widowed or not, and the relative income of a person with respect to some regional median income matter- see, e.g., Daly, Wilson and Johnson (2012). We conjecture that the theory of options on multiple assets as explored in Detemple (2006, Chapter 6) may have something to do with these inter-personal factors of suicide risk.

Another big issue is to relax the complete market assumption. Remember the fact that personal health condition matters, especially for older people, as one of the major reasons to commit suicide. It seems less convincing to claim that the individual's health risk can be completely synthesized by a continuous trading of the financial assets. If we introduce an idiosyncratic risk factor in the human capital formation of each worker, we need to evaluate the suicide option in an incomplete market environment with no unique stochastic discount factor. We may need to re-introduce an utility function for the utility-indifferencing pricing. Another important class of wage-flow evolution is the jump-duiffusion emphasizing the downside risk of layoff and unemployment. The unemployment has been the major economic variable to account for a large variation of suicide rates. Because the jump component cannot be replicated by a continuous trading of financial assets, the model becomes incomplete on this regard. Moreover, even a feq important theoretical properties of the immediate exercise boundary- if it is continuous or not- may not be well explored. The attempt to resolve these theoretical complexity should be worthwhile for a richer analysis of suicide from the option-theoretic viewpoints.

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6 Appendix

6.1 Details of Proof of Proposition 1

(a) Jaillet, Lamberton and Lapeyre (1990, Theorem 3.6).

(b) Detemple (2006, Proposition 32-(ii)).

(c) Suppose $H > K$. The sub-optimality of the immediate exercise is obvious as long as $t < T$ because the human capital exceeds the least acceptable level, K. Suppose $rA_{t,T}(\zeta)K < H \leq K$. The strict inequality implies $\bar{h}w - rK > 0$ given $H = \bar{h}wA_{t,T}(\zeta)$ for $w > 0$ assumed in (a). If $(t, H) \in \mathcal{E}$ holds under the stated assumptions, $V(t, H) = K - H$. The market-based opportunity cost of taking the one-unit long position in this suicide option at time t is $-(K-H)$. The benefit of suicide at time τ, i.e., the value of such long position, is $\max\{(K - H_\tau)^+, V(\tau, H_\tau)\}$. We can take the long position of the replicating portfolio for H at time t by paying $-H$ dollars. Its value at time τ is $H_\tau + \int_t^\tau R_{t,s}\bar{h}w_s ds$. If we borror the cash in the risk-free account for K dollars as of time t, the short position will be worth of $-\left\{K + \int_t^\tau R_{t,s}rKds\right\}$ as of time τ. Therefore, the initial cost of investment is zero in the market-based valuation at time t whereas the value of portfolio at time τ is

$$\max\left\{(K - H_\tau)^+, V(\tau, H_\tau)\right\} + \left\{H_\tau + \int_t^\tau R_{t,s}\bar{h}w_s ds\right\} - \left\{K + \int_t^\tau R_{t,s}rKds\right\}$$
$$= \max\left\{(K - H_\tau)^-, V(\tau, H_\tau) - (K - H_\tau)\right\} + \int_t^\tau R_{t,s}\left\{\bar{h}w_s - rK\right\} ds > 0,$$

which is an arbitrage opportunity[9]. To be consistent with the absence of arbitrage, therefore, we must have $H \leq K \cdot \min\{1, rA_{t,T}(\zeta)\}$. The last claim is obvious.

(d) Let me introduce another notation for this proof. The solution to (9) at $v \in [t, T]$ given the initial condition $(t, H_t = H)$ is denoted by $H_v^{(t,H)}$. Similarly, $(w_v^{(t,w)})_{v\in[t,T]}$ denotes the solution to (3) given the initial condition $(t, w_t = w)$. The mainstream of the proof follows Detemple (2006, Proof of Proposition 31-(ii)) for the case of an American call option. Let us find two arbitrarily values of human capital as of time t: H and $H' \leq H$. Because the case of $H' = H$ is trivial, suppose $H' < H$. The comparison theorem (Karatzas and Shreve 1988, Proposition 5.2.18) guarantees that $H_v^{(t,H')} \leq H_v^{(t,H)}$, Q-a.s., for any $v \in [t, T]$. Define $\tau' = \arg\sup_{\tau\in\mathcal{S}_{t,T}} E_t^Q\left[R_{t,\tau}(K - H_\tau^{(t,H')})^+\right]$ as the optimal exercise time associated with the dynamic evolution of human capital stem from (t, H').

[9]This is the arbitrage opportunity among combinations of assets traded in the market. See Merton (1992, Chapter 12, 13, 21; 1998, Section 1) for the intuition and a variety of justification.

Now we have the chain of equalities/inequalities

$$\begin{aligned} V(t,H') - V(t,H) &\leq E_t^Q\left[R_{t,\tau'}\{(K-H_{\tau'}^{(t,H')})^+ - (K-H_{\tau'}^{(t,H)})^+\}\right] \\ &\leq E_t^Q\left[R_{t,\tau'}\{H_{\tau'}^{(t,H)} - H_{\tau'}^{(t,H')}\}\right] \\ &= (H-H') - \bar{h}E_t^*Q\left[\int_t^{\tau'} R_{t,v}(w_v^{(t,w)} - w_v^{(t,w')})dv\right] \\ &\leq H - H' \end{aligned}$$

The sub-optimality of τ' for (t,H), namely, $V(t,H) \geq E_t^Q[R_{t,\tau'}(K-H_{\tau'}^{(t,H)})^+]$ gives the first inequality. The general result $a^+ - b^+ \leq (a-b)^+$ for the put payoff function imply the second inequality. The present value expression of H_t in (9) justifies the next equality. The last inequality stems from the comparison theorem again, applied to $w' < w$. Given $(t,H) \in \mathcal{E}$ and $H' \in [0,H]$,

$$V(t,H') \leq V(t,H) + (H-H') = (K-H) + (H-H') = K - H'.$$

Therefore, $(t,H') \in \mathcal{E}$ if $(t,H) \in \mathcal{E}$ and $H' \leq H$. This is the desired result.

□

6.2 Details of the Proof of Proposition 2

Define $m_t := \min\{B_{t-}, B_{t+}\}$ and $M_t := \max\{B_{t-}, B_{t+}\}$. Suppose $B_t > M_t$. Then, $M_t \in \mathcal{E}$ because B_t is the supremum of $\mathcal{E}$ and $\mathcal{E}$ is down-connected from Proposition 1-(d). Now we can find an open interval $\mathcal{O} \subset (M_t, B_t) \subset \mathcal{E}$ with a positive Lebesgue measure, and a sequence $(t_n, H_{t_n}, w_{t_n}) \in \mathcal{C}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty}(t_n, H_{t_n}) = (t,H) \in \mathcal{O}$, $H_{t_n} = \bar{h}w_{t_n}A_{t_n,T}(\zeta)$ and $H = \bar{h}wA_{t,T}(\zeta)$. Because $(t_n, H_{t_n}) \in \mathcal{C}$, Proposition 1-(a) guarantees

$$\frac{1}{2}V_{HH}(t_n, H_{t_n})H_n^2\sigma^2 + V_H(t_n, H_{t_n})(rH_{t_n} - \bar{h}w_{t_n}) + V_t(t_n, H_{t_n}) - rV(t_n, H_{t_n}) = 0.$$

By letting $n \to \infty$ under the continuity of the left hand side with respect to (t_n, H_{t_n}),

$$\begin{aligned} &\frac{1}{2}V_{HH}(t,H)H^2\sigma^2 + V_H(t,H)(rH - \bar{h}w) + V_t(t,H) - rV(t,H) = 0 \\ \Leftrightarrow V_{HH}(t,H) &= \frac{2}{H^2\sigma^2}\left\{r(K-H) - (-1)(rH - \bar{h}w) - 0\right\} = \frac{2}{H^2\sigma^2}\left\{rK - \bar{h}w\right\}. \end{aligned}$$

The rightest side is non-negative because $rK - \bar{h}w \geq 0$ for any $(t,H) \in \mathcal{E}$ from Proposition 1-(c) and the fact that $(t,x) \in \mathcal{E}$. Moreover, it is indeed strictly positive for x in a subset of $\mathcal{O}$ with a positive Lebesgue measure. Therefore, $\int_{M_t}^{B_t}\int_y^{B_t} V_{HH}(t,H)dHdy > 0$. On the other hand, $B_t, M_t \in \mathcal{E}$ and a bit of computation give

$$\begin{aligned} \int_{M_t}^{B_t}\int_y^{B_t} V_{HH}(t,H)dHdy &= V_H(t,B_t)[B_t - M_t] - \{V(t,B_t) - V(t,M_t)\} \\ &= (-1)[B_t - M_t] - \{(H - B_t) - (H - M_t)\} = 0, \end{aligned}$$

which is contradiction. Therefore, the initial assumption of $B_t > M_t$ must be incorrect: $B_t \leq M_t$. Beacuse $B_t < M_t$ violates the closedness of $\mathcal{E}$, $B_t = M_t$ is the conclusion. Similarly, $B_t < m_t$ violates the closedness of $\mathcal{E}$ so that $m_t \leq B_t = M_t$. If $m_t < B_t$, we can find an open interval between m_t and B_t and apply the same argument as above. Combining everything, B is continuous at any time $t \in [0, T)$.

The claim of $\lim_{t\to T} B_t = B_{T-} = 0$ follows from $B_t \leq K \cdot \min\{1, rA_{t,T}(\zeta)\}$ and $A_{t,T}(\zeta) \to 0$ as $t \to T$. $B_T = K$ is obvious because $V(T, H) = K - H$. The claim of $\lim_{t\to T} \beta_t = \beta_{T-} = rK$ is shown as follows: from Proposition 1-(c), $(t, H) \in \mathcal{E}$ implies $B_t = \beta_t A_{t,T}(\zeta) \leq rKA_{t,T}(\zeta)$ or $\beta_t \leq rK$ for any $t < T$ so that $\lim_{t\to T} \beta_t = \beta_{T-} \leq rK$. If $\beta_{T-} < rK$, we can find an open interval between rK and β_{T-}, any point x of which satisfies $\beta_{T-} < x$ and therefore $\beta_{T-}A_{t,T}(\zeta) < xA_{t,T}(\zeta)$. Because B_t is guaranteed to be a function and not a correspondence, we can find a positive number ϵ such that $V(t, xA_{t,T}(\zeta)) > K - xA_{t,T}(\zeta)$ for any $t \in (T-\epsilon, T)$ and therefore $\lim_{t\to T} V(t, xA_{t,T}(\zeta)) = V(T-, xA_{T-,T}(\zeta)) > K$. On the other hand, the continuity of $V(t, H)$ with respect to (t, H) from Proposition 1-(a) and of $A_{t,T}(\zeta)$ with respect to t guarantee that $V(T-, xA_{T-,T}(\zeta)) = V(T, xA_{T,T}(\zeta)) = K - xA_{T,T}(\zeta) = K$. This is a contradiction. Therefore, $\lim_{t\to T} \beta_t = \beta_{T-} = rK$ exactly.

□

6.3 Proof of Proposition 3

(16) follows from Detemple (2006, Theorem 21) for the put payoff function $(K - H_t)$ with the drift $-(rH_t - \bar{h}w_t)$ upon the immediate exercise. It is immediate to show $p(t, H) = E_t^Q[R_{t,T}(K - H_T)^+] = R_{t,T}K$ because $H_T = 0$. Because the event $\{s = \tau_s\}$ is equivalently stated as $\{H_s \leq B_s\}$, the first integral in the right hand side of (16) has an alternative expression $rK \int_t^T R_{t,s} Q_t(H_s \leq B_s) ds$. Using

$$H_s = \bar{h}w_s A_{s,T}(\zeta) = \bar{h}w_t \exp\left((r - \zeta - (\sigma^{(w)})^2/2)(s-t) + \sigma^{(w)}(W_s^Q - W_t^Q)\right) A_{smT}(\zeta)$$

and the reparametrization $B_s = \beta_s A_{s,T}(\zeta)$ for $s < T$, a bit of computation gives the desired expression $Q_t(H_s \leq B_s) = N(-d_{t,s}^{(-)}(\bar{h}w_t, \beta_s))$.

For the second integral, combine (5) with the Novikov theorem for $\theta^{(w)} := -\sigma^{(w)}$ playing a role of θ in the stochastic exponential to show that $(\eta_{0,t}^{(w)})_{t\in[0,T]}$ is $\mathbb{F}$-martingale under the risk-neutral probability measure Q with $E_t^Q[\eta_{t,s}^{(w)}] = 1$. Now we can use $\eta^{(w)}$ as the density to change the probability measure from Q to another equivalent one, say $Q^{(w)}$, such that $Q^{(w)}(\cdot) = E^Q[\eta_{0,T}^{(w)} 1_{\{\cdot\}}]$ and $E_t^{(w)}[Y] = E_t^Q[\eta_{t,T}^{(w)} Y]$ is the $\mathcal{F}_t$-conditional expectation under the new measure $Q^{(w)}$ of any $\mathcal{F}_T$-measurable variable Y. The Girsanov theorem ensures that $dW_t^{(w)} = dW_t^Q - \sigma^{(w)} dt$ is the $Q^{(w)}$-Brownian increment. Define $Q_t^{(w)}$ as the $\mathcal{F}_t$-conditional probability under $Q^{(w)}$. Now that $w_s = w_t R_{t,s}^{-1} Z_{t,s} \eta_{t,s}^{(w)}$ from (5), the second integral can be expressed as $\bar{h}w_t \int_t^T Z_{t,s} Q_t^{(w)}(H_s \leq B_s) ds$. The desired expression $Q_t^{(w)}(H_s \leq B_s) =$

$N(-d_{t,s}^{(+)}(\bar{h}w_t, \beta_s))$ is immediate if we recognize

$$H_s = \bar{h}w_t \exp\left((r - \zeta + (\sigma^{(w)})^2/2)(s-t) + \sigma^{(w)}(W_s^{(w)} - W_t^{(w)})\right) A_{s,T}(\zeta) \quad under \quad Q^{(w)}.$$

6.4 Proof of Proposition 5

The key trick for that purpose is to introduce a fictitious time-dependent strike price process K_t and $Y_t := H_t/K_t = \bar{h}w_t A_{t,T}(\zeta)/K_t$ so that the discounted put payoff function is $R_{t,T}K_t(1-Y_t)^+$. Using the integration by parts, the Tanaka-Mayer formula and the martingale nature of the discounted human capital process $(R_{t,s}H_s)_{s\in[t,T]}$ under Q, as in Carr, Jarrow and Myneni (1992, p.102), we have

$$p(t,H) = (K_0 - H_0)^+ + \int_t^T R_{t,s}K_s E_t^Q[d\Lambda_Y(1,s)] + \int_t^T E_t^Q[1_{\{H_t<K_t\}}]d(R_{t,T}K_t) \qquad (23)$$

where $\Lambda_Y(1,t)$ is the local time of $(Y_s)_{s\in[t,T]}$ in the vicinity of one. By the Ito's lemma,

$$d\ln Y_t = \mu_t^{(Y)}dt + \sigma^{(w)}dW_t^Q$$

where $\mu_t^{(Y)} = r - \zeta - (\sigma^{(w)})^2/2 + \alpha_t - \kappa_t$, $\alpha_t = \partial \ln A_{t,T}(\zeta)/\partial t$ and $\kappa_t = \partial \ln K_t/\partial t$. Then, with a bit of calculation as in Carr, Jarrow and Myneni (1992, p.102-103), we can compute $E_t^Q[d\Lambda_Y(1,s)]$ as follows:

$$\frac{(\sigma^{(w)})^2/2}{\sigma^{(w)}\sqrt{s-t}} \cdot n\left(-\left\{\frac{\ln(Y_t) + \int_t^s \mu_v^{(Y)}dv}{\sigma\sqrt{s-t}}\right\}\right)ds = \frac{\sigma^{(w)}}{2\sqrt{s-t}} \cdot n\left(-d_{t,s}^{(-)}(\bar{h}w_t A_{s,T}(\zeta), K_t)\right)ds.$$

Substituting $K_t = K$ and using the identity

$$KR_{t,T} \cdot n\left(-d_{0,t}^{(-)}(\bar{h}w_t A_{s,T}(\zeta), K)\right) = \bar{h}w_t A_{s,T}(\zeta) Z_{t,T} \cdot n\left(-d_{0,t}^{(+)}(\bar{h}w_t A_{s,T}(\zeta), K)\right),$$

the second integral in (23) now gives the first component in the first integral of $D_{t,T}$. It is easy to evaluate the second integral in (23), which is similar to the argument in the proof of Proposition 3. It gives the first component in the second integral of $D_{t,T}$.

Table 1: Values of Parameters.

Risk-free rate	$r = .02$	The total endowment of time	$\bar{h} = 1$
Market price of risk	$\theta = .3$	Average wage growth	$\mu^{(w)} = .01$
Stock-market volatility	$\sigma = .2$	Wage volatility	$\sigma^{(w)} = .03$
The total horizon	$T = 60$	Working-Age weight	$f = 1.784$
Retirement timing	$T_r = 40$	Discount Rate (Working)	$g = .0227$
Subjective discount rate	$b = 0$	Discount Rate (Retirement)	$\bar{g} = .0234$
Relative weight	$\eta = 2/3$	Relative Risk Aversion	$R = 4$

$T_r = 40$ means that there is no retirement period in this configuration. The simulation result in Figure 1 relies on the parameter configurations in the first three rows and $T = T_r = 40$ without retirement period: the end of working life is also the end of life.

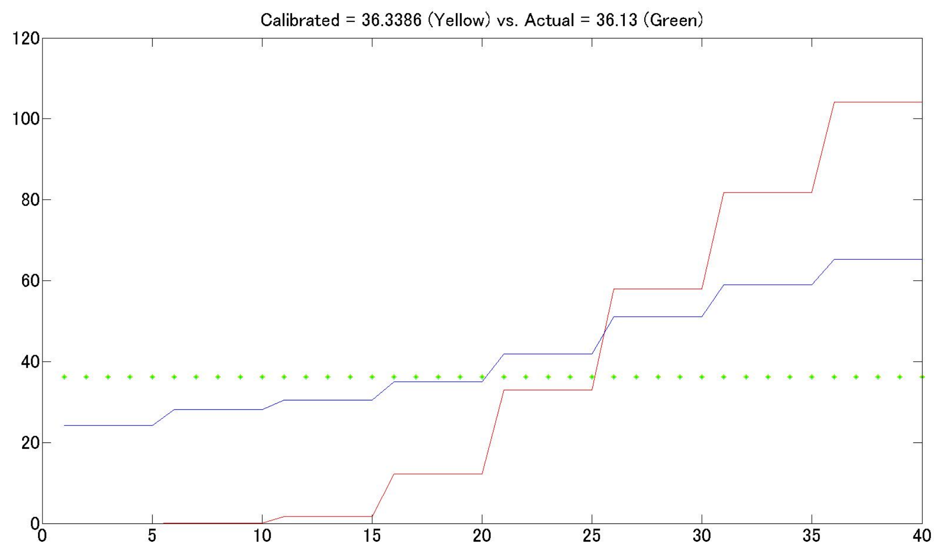

Figure 1: The suicide rates per 100 thousands: simulated (red), its average (yellow), actual (blue) and its average (green). The vertical axis is the number of suicides againt the the working year and not the actual age in the horizontal axis. If this representative worker starts working at the age of 20, she/he is keep working by the age of $T = T_r = 60$, then exits from the labor market. We do not model the retirement period. The average simulated and actual values of suicide rate are very similar. The simulated pattern of suicide is under-stating the number of suicides in the younger ages and over-stating in the older ages.

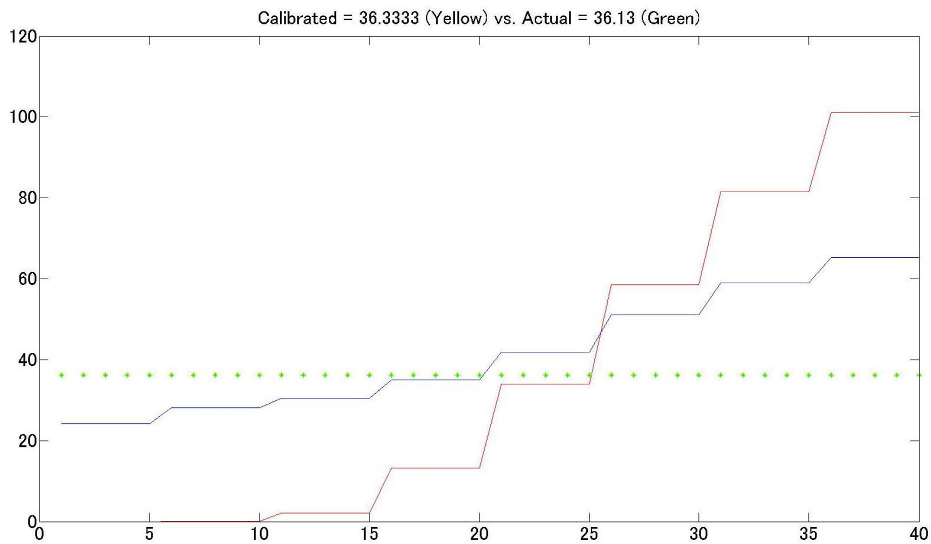

Figure 2: The suicide rates per 100 thousands: simulated (red), its average (yellow), actual (blue) and its average (green). The wealth process implied from the optimal choice of consumption and leisure as the underlying asset of suicide option. The vertical axis is the number of suicides againt the the working year and not the actual age in the horizontal axis. If this representative worker starded working at the age of 20, she/he is keep working by the age of 60, then exit from the labor market. We do not model the retirement period. The simulated pattern of suicide is under-stating the number of suicides in the younger ages and over-stating in the older ages. Overall, the pattern is similar to the previous one in Figure 1.