Minimum Augmentation of Local Edge-Connectivity between Vertices and Vertex Subsets in Undirected Graphs ⋆

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Abstract

Given an undirected multigraph $G = (V, E)$, a family $\mathcal{W}$ of sets $W \subseteq V$ of vertices (areas), and a requirement function $r : \mathcal{W} \to Z^+$ (where Z^+ is the set of nonnegative integers), we consider the problem of augmenting G by the smallest number of new edges so that the resulting graph has at least $r(W)$ edge-disjoint paths between v and W for every pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$. So far this problem was shown to be NP-hard in the uniform case of $r(W) = 1$ for each $W \in \mathcal{W}$, and polynomially solvable in the uniform case of $r(W) = r \geq 2$ for each $W \in \mathcal{W}$. In this paper, we show that the problem can be solved in $O(m + pn^4\ (r^* + \log n))$ time, even if $r(W) \geq 2$ holds for each $W \in \mathcal{W}$, where $n = |V|$, $m = |\{\{u, v\} | (u, v) \in E\}|$, $p = |\mathcal{W}|$, and $r^* = \max\{r(W) \mid W \in \mathcal{W}\}$.

Key words: undirected graph, connectivity augmentation problem, local edge-connectivity, node-to-area connectivity, polynomial time deterministic algorithm

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1 Introduction

In a communication network, graph connectivity is a fundamental measure of its robustness. An undirected graph $G = (V, E)$ is k-edge-connected if the deletion of any $k-1$ or fewer edges leaves a connected graph; equivalently, there exist at least k pairwise edge-disjoint paths between every two vertices. The connectivity augmentation problem asks to add to a given graph the smallest number of new edges such that the connectivity of the graph increases up to a specified value k. The problem has important applications such as the network design problem [5], and so on (see [4,14] for surveys).

Most of all those researches have dealt with connectivity between two vertices in a graph. However, in many real-world networks, the connectivity between every two vertices is not necessarily required. For example, in a multimedia network, some vertices of the network may have functions of offering several types of services for users. For a set W of vertices offering certain service i, a user at a vertex v can use service i by communicating with one vertex $w \in W$ through a path between w and v. In such networks, it is desirable that the network has some pairwise disjoint paths from the vertex v to *at least one* of vertices in W. This means that the measure of reliability is the connectivity between a vertex and a set of vertices rather than that between two vertices. From this point of view, Ito et al. considered the node to area connectivity (NA-connectivity, for short) as a concept that represents the connectivity between vertices and sets of vertices (areas) in a graph [7,9]. As related problems, the problem of locating a set W of vertices offering service with requirements measured by connectivity has been also studied [1,8,9,15].

In this paper, given a graph $G = (V, E)$ with a family $\mathcal{W}$ of sets W of vertices (areas), and a requirement function $r : \mathcal{W} \to Z^+$, we consider the problem of asking to augment G by adding the smallest number of new edges so that the resulting graph has at least $r(W)$ pairwise edge-disjoint paths between v and W for every pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$. We call this problem *r-NA-edge-connectivity augmentation problem* (for short, *r-NA-ECAP*).

Figure 1 gives an instance of r-NA-ECAP with $r(W_1) = 2$, $r(W_2) = 3$, and $r(W_3) = 4$. In the graph G in (i), some pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$ (say, v_7 and W_3) cannot have $r(W)$ edge-disjoint paths between them, and r-NA-ECAP asks to add the minimum number of new edges to G to construct a graph like (ii) in which there are at least $r(W)$ edge-disjoint paths between every pair of $v \in V$ and $W \in \mathcal{W}$. So far k-NA-ECAP in the uniform case that $r(W) = k$ holds for every area $W \in \mathcal{W}$ has been studied, and several algorithms for solving k-NA-ECAP have been proposed. Miwa and Ito [10] showed that 1-NA-ECAP is NP-hard and that 2-NA-ECAP is polynomially solvable. Recently, Ishii et al. [6] proposed a polynomial time algorithm for

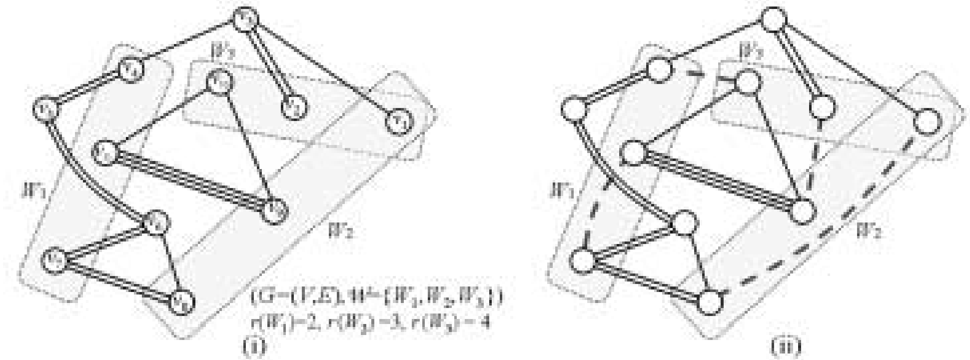

Fig. 1. Illustration of an instance of r-NA-ECAP. (i) An initial graph $G = (V, E)$ with a family $\mathcal{W} = \{W_1 = \{v_4, v_7, v_{11}\}, W_2 = \{v_1, v_8, v_9\}, W_3 = \{v_1, v_2, v_{10}\}\}$ of areas, where a requirement function $r : \mathcal{W} \to Z^+$ satisfies $r(W_1) = 2$, $r(W_2) = 3$, and $r(W_3) = 4$. (ii) An r-NA-edge-connected graph obtained from G by adding a set of edges drawn as broken lines; there are at least $r(W)$ edge-disjoint paths between every pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$.

solving k-NA-ECAP in the case of $k \geq 3$. However, it was still open whether the problem in general requirements $r \geq 2$ is polynomially solvable or not. In this paper, we show that if $r(W) \geq 2$ holds for each $W \in \mathcal{W}$, then r-NA-ECAP can be solved in $O(m+ \; pn^4 \; (r^* + \log n))$ time, where $n = |V|$, $m = |\{\{u, v\}|(u, v) \in E\}|$, $p = |\mathcal{W}|$, and $r^* = \max\{r(W) \mid W \in \mathcal{W}\}$.

The paper is organized as follows. In Section 2, we define r-NA-ECAP, after introducing some basic notations. In Section 3, we derive lower bounds on the optimal value $opt(G, \mathcal{W}, r)$ to r-NA-ECAP, and state our main result that a min-max formula to the r-NA-ECAP with $r \geq 2$ is established and that r-NA-ECAP is polynomially solvable for $r \geq 2$. We give an algorithm, called r-NAEC-AUG, which finds a solution E' with $|E'| = opt(G, \mathcal{W}, r)$ in Section 4. In Sections 5 and 6, we prove the correctness of algorithm r-NAEC-AUG. In Section 7, we give concluding remarks.

2 Problem Definition

Let $G = (V, E)$ stand for an undirected graph with a set V of *vertices* and a set E of *edges*. An edge with end vertices u and v is denoted by (u, v). We denote $|V|$ by n and $|\{\{u, v\}|(u, v) \in E\}|$ by m. A singleton set $\{x\}$ may be simply written as x, and " $\subset$ " implies proper inclusion while " $\subseteq$ " means " $\subset$ " or " $=$ ". In $G = (V, E)$, its vertex set V and edge set E may be denoted by $V(G)$ and $E(G)$, respectively. For a subset $V' \subseteq V$ in G, $G[V']$ denotes the subgraph induced by V'. For an edge set E' with $E' \cap E = \emptyset$, we denote the augmented graph $(V, E \cup E')$ by $G + E'$. For an edge set E', we denote by $V[E']$ the set of all end vertices of edges in E'.

An *area graph* is defined as a graph $G = (V, E)$ with a family $\mathcal{W}$ of vertex subsets $W \subseteq V$ which are called *areas* (see Figure 1). We denote an area graph G with $\mathcal{W}$ by $(G, \mathcal{W})$. In the sequel, we may denote $(G, \mathcal{W})$ by G simply if no confusion arises. For two disjoint subsets $X,\ Y \subset V$ of vertices, we denote by $E_G(X, Y)$ the set of edges $e = (x, y)$ such that $x \in X$ and $y \in Y$, and also denote $|E_G(X, Y)|$ by $d_G(X, Y)$. In particular, $E_G(u, v)$ is the set of edges with end vertices u and v. A *cut* is defined as the subset X of V with $\emptyset \neq X \neq V$, and the *size* of a cut X is defined by $d_G(X, V - X)$, which may also be written as $d_G(X)$. Moreover, we define $d(\emptyset) = 0$. For two cuts $X, Y \subset V$ with $X \cap Y = \emptyset$ in G, we denote by $\lambda_G(X, Y)$ the minimum size of cuts which separate X and Y, i.e., $\lambda_G(X, Y) = \min\{d_G(S) | S \supseteq X, S \subseteq V - Y\}$. For two cuts $X, Y \subset V$ with $X \cap Y \neq \emptyset$ in G, we define $\lambda_G(X, Y) = \infty$. The *edge-connectivity* of G, denoted by $\lambda(G)$, is defined as $\min_{X \subset V, Y \subset V} \lambda_G(X, Y)$. For a vertex $v \in V$ and a set $W \subseteq V$ of vertices, the *node-to-area edge-connectivity* (*NA-edge-connectivity*, for short) between v and W is defined as $\lambda_G(v, W)$. Note that $\lambda_G(v, W) = \infty$ holds for $v \in W$. We say that a vertex v and an area W is k-NA-edge-connected if $\lambda_G(v, W) \geq k$ holds for an integer k. For an area graph $(G, \mathcal{W})$ and a function $r : \mathcal{W} \to Z^+$, we say that $(G, \mathcal{W})$ is *r-NA-edge-connected* if $\lambda(v, W) \geq r(W)$ holds for every pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$. Note that the area graph $(G, \mathcal{W})$ in Figure 1(ii) is r-NA-edge-connected, where $r(W_1) = 2$, $r(W_2) = 3$, and $r(W_3) = 4$.

In this paper, we consider the following problem, called r-NA-ECAP.

Problem 1 (r-NA-edge-connectivity augmentation problem, r-NA-ECAP)
Input: *An area graph* $(G = (V, E), \mathcal{W})$ *and a requirement function* $r : \mathcal{W} \to Z^+$.
Output: *A set* E^* *of new edges with the minimum cardinality such that* $G + E^*$ *is* r*-NA-edge-connected.* □

3 Lower Bound on the Optimal Value

For an area graph $(G, \mathcal{W})$ and a fixed function $r : \mathcal{W} \to Z^+$, let $opt(G, \mathcal{W}, r)$ denote the optimal value to r-NA-ECAP in $(G, \mathcal{W})$, i.e., the minimum size $|E^*|$ of a set E^* of new edges such that $G + E^*$ is r-NA-edge-connected. In this section, we derive lower bounds on $opt(G, \mathcal{W}, r)$ to r-NA-ECAP with $(G, \mathcal{W})$. In the sequel, let $\mathcal{W} = \{W_1, W_2, \ldots, W_p\}$.

A family $\mathcal{X} = \{X_1, \ldots, X_t\}$ of cuts in G is called a *partition of* V, if every two cuts $X_i, X_j \in \mathcal{X}$ satisfy $X_i \cap X_j = \emptyset$ and $\cup_{X_i \in \mathcal{X}} X_i = V$ holds. For a subset $X \subseteq V$ of vertices, a partition of X is called a *subpartition of* V. For an area graph $(G, \mathcal{W})$ and an area $W_i \in \mathcal{W}$, let $\mathcal{A}_i$ denote the family of cuts X with $X \cap W_i = \emptyset$ and $\mathcal{B}_i$ denote the family of cuts X with $X \supseteq W_i$ (note that a cut

X of $\mathcal{B}_i$ satisfies $X \neq V$ by the definition of a cut). We easily see the following property.

Lemma 2 *An area graph $(G, \mathcal{W})$ is r-NA-edge-connected if and only if all cuts $X \in \mathcal{A}_i \cup \mathcal{B}_i$ satisfy $d_G(X) \geq r(W_i)$ for each area $W_i \in \mathcal{W}$.* $\square$

Let X be a cut in $(G, \mathcal{W})$. If X is a cut of $\mathcal{A}_i \cup \mathcal{B}_i$ with $d_G(X) < r(W_i)$ for some area $W_i \in \mathcal{W}$, then it is necessary to add at least $r(W_i) - d_G(X)$ edges between X and $V - X$. It follows since if X belongs to $\mathcal{A}_i$ (resp., $\mathcal{B}_i$), then the NA-edge-connectivity between a vertex in X (resp., $V - X$) and an area $W_i \in \mathcal{W}$ with $W_i \cap X = \emptyset$ (resp., $W_i \subseteq X$) need be augmented to at least $r(W_i)$. Here we define $\alpha_{G,\mathcal{W},r}(X)$ as follows, which indicates the number of necessary edges incident to X.

Definition 3 *For each cut $X \in \mathcal{A}_j \cup \mathcal{B}_j$ for some W_j, we define i_X as an index i satisfying $r(W_i) = \max\{r(W) \mid W \in \mathcal{W},\ X \cap W = \emptyset, \text{ or } X \supseteq W\}$, and define $\alpha_{G,\mathcal{W},r}(X) = \max\{0, r(W_{i_X}) - d_G(X)\}$. For any other cut X, $X = \emptyset$, or $X = V$, define $\alpha_{G,\mathcal{W},r}(X) = 0$.* $\square$

Lemma 4 *It is necessary to add at least $\alpha_{G,\mathcal{W},r}(X)$ edges between X and $V - X$.* $\square$

Let

$$\alpha(G, \mathcal{W}, r) = \max_{\mathcal{X}} \Big\{ \sum_{X \in \mathcal{X}} \alpha_{G,\mathcal{W},r}(X) \Big\}, \tag{1}$$

where the maximization is taken over all subpartitions of V. Then any feasible solution to r-NA-ECAP with $(G, \mathcal{W})$ must contain an edge which joins two vertices from a cut X with $\alpha_{G,\mathcal{W},r}(X) > 0$ and the cut $V - X$. Therefore we see the following lemma.

Lemma 5 *$opt(G, \mathcal{W}, r) \geq \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$ holds.* $\square$

The area graph $(G, \mathcal{W})$ in Figure 1(i) satisfies $\alpha(G, \mathcal{W}, r) = 8$. We have $\sum_{X \in \mathcal{X}} \alpha_{G,\mathcal{W},r}(X) = 8$ for the subpartition $\mathcal{X} = \{\{v_1\}, \{v_2\}, \{v_4\}, \{v_6, v_7, v_8\}, \{v_9, v_{11}\}, \{v_{10}\}\}$ of V.

We remark that there is an area graph $(G, \mathcal{W})$ with $opt(G, \mathcal{W}, r) > \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$. Figure 2 gives an instance for $r = r(W_1) = r(W_2) = r(W_3) = 2$. Each cut $\{v_i\}$, $i = 1, 2, 4, 5$ belongs to $\mathcal{A}_3$, $r - d_G(v_i) = 1$ holds for $i = 1, 2, 5$, and $r - d_G(v_4) = 2$ holds. The cut $\{v_3\}$ belongs to $\mathcal{A}_1$ and satisfies $r - d_G(v_3) = 1$. It is not hard to see that in (1) the minimum is achieved for the subpartition $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$ and $\lceil \alpha(G, \mathcal{W}, r)/2 \rceil = 3$. In order to make $(G, \mathcal{W})$ r-NA-edge-connected by adding three new edges, we must add $E' = \{(v_1, v_2), (v_3, v_4), (v_4, v_5)\}$ or $E' = \{(v_1, v_4), (v_2, v_4), (v_3, v_5)\}$

without loss of generality. In both cases, $G + E'$ is not r-NA-edge-connected by $\lambda_{G+E'}(v_1, W_3) = 1$. We will show that all such instances can be completely

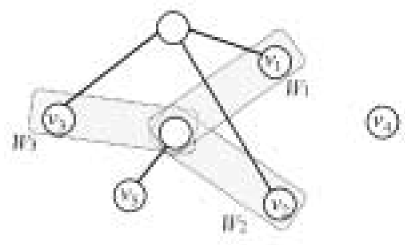

Fig. 2. Illustration of an area graph $(G, \mathcal{W})$ with $opt(G, \mathcal{W}, r) = \lceil \frac{\alpha(G,\mathcal{W},r)}{2} \rceil + 1$.

characterized as follows.

Definition 6 *We say that an area graph $(G, \mathcal{W})$ has property (P) if $\alpha(G, \mathcal{W}, r)$ is even and there is a subpartition $\mathcal{X}$ of V with $\sum_{X \in \mathcal{X}} \alpha_{G,\mathcal{W},r}(X) = \alpha(G, \mathcal{W}, r)$ satisfying the following conditions $(P1)$–$(P3)$:*

$(P1)$ Each cut $X \in \mathcal{X}$ belongs to $\mathcal{A}_i$ for some $W_i \in \mathcal{W}$.

$(P2)$ There is a cut $X^* \in \mathcal{X}$ with $\alpha_{G,\mathcal{W},r}(X^*) = 1$.

$(P3)$ Let $\mathcal{X}_1$ denotes the family of cuts $X \in \mathcal{X}$ with $d_G(X) = 0$ and $\alpha_{G,\mathcal{W},r}(X) = 2$. For each $X \in \mathcal{X} - \mathcal{X}_1 - \{X^*\}$, there is a cut $Y_X \in \mathcal{B}_j$ for some $W_j \in \mathcal{W}$ such that the following (i)–(iv) hold: (i) $X \cup X^* \subseteq Y_X$, (ii) $V - Y_X - (\cup_{X'' \in \mathcal{X}_1} X'') \neq \emptyset$, (iii) $\sum_{X' \in \mathcal{X}, X' \subset Y_X} \alpha_{G,\mathcal{W},r}(X') \leq (r(W_j) + 1) - d_G(Y_X)$, and (iv) every cut $X' \in \mathcal{X}$ satisfies $X' \subset Y_X$ or $X' \cap Y_X = \emptyset$. □

Note that $(G, \mathcal{W})$ in Figure 2 has property (P) because $\alpha(G, \mathcal{W}, r) = 6$ holds and the subpartition $\mathcal{X} = \{X^* = \{v_5\}, X_1 = \{v_1\}, X_2 = \{v_2\}, X_3 = \{v_3\}, X_4 = \{v_4\}\}$ of V satisfies $\mathcal{X}_1 = \{X_4\}$, $Y_{X_1} = C_1 \cup \{v_1\}$, $Y_{X_2} = C_1 \cup \{v_2\}$, and $Y_{X_3} = C_1 \cup \{v_3\}$ for the component C_1 of G containing v_5.

Lemma 7 *If $(G, \mathcal{W})$ has property* (P)*, then $opt(G, \mathcal{W}, r) \geq \lceil \alpha(G, \mathcal{W}, r)/2 \rceil + 1$ holds.*

PROOF. Assume by contradiction that $(G, \mathcal{W})$ has property (P) and there is an edge set E^* with $|E^*| = \alpha(G, \mathcal{W}, r)/2$ such that $G + E^*$ is r-NA-edge-connected (note that $\alpha(G, \mathcal{W}, r)$ is even). Let $\mathcal{X} = \{X_1, \ldots, X_t\}$ denote a subpartition of V satisfying $\sum_{X \in \mathcal{X}} \alpha_{G,\mathcal{W},r}(X) = \alpha(G, \mathcal{W}, r)$ and the above (P1)–(P3). Since $|E^*| = \alpha(G, \mathcal{W}, r)/2$ holds, each cut $X \in \mathcal{X}$ satisfies $d_{G+E^*}(X) = r(W_{i_X})$, and hence $d_{G'}(X) = r(W_{i_X}) - d_G(X) = \alpha_{G,\mathcal{W},r}(X)$, where $G' = (V, E^*)$. Therefore, any edge $(x, x') \in E^*$ satisfies $x \in X$ and $x' \in X'$ for some two cuts $X, X' \in \mathcal{X}$ with $X \neq X'$. Hence $\sum_{v \in X''} d_{G'}(v) =$

$d_{G'}(X'')$ for $X'' \in \{X, X'\}$. From this, there exists a cut $X_1 \in \mathcal{X} - \{X^*\}$ with $E_{G'}(X^*, X_1) \neq \emptyset$. Now note that $\mathcal{X} - \mathcal{X}_1 - \{X^*\} \neq \emptyset$ holds since otherwise $\alpha(G, \mathcal{W}, r) = 2|\mathcal{X}_1| + 1$ by the properties (P2) and (P3), which contradicts that $\alpha(G, \mathcal{W}, r)$ is even.

Assume that $X_1 \in \mathcal{X} - \mathcal{X}_1$ holds. Since $(G, \mathcal{W})$ satisfies property (P), there is a cut $Y_{X_1} \in \mathcal{B}_j$ which satisfies (P3), and hence $\sum_{v \in Y_{X_1}} d_{G'}(v) = \sum_{X' \in \mathcal{X}, X' \subset Y_{X_1}} d_{G'}(X') = \sum_{X' \in \mathcal{X}, X' \subset Y_{X_1}} \alpha_{G,\mathcal{W},r}(X') \leq (r(W_j) + 1) - d_G(Y_{X_1})$. Since $G'[Y_{X_1}]$ contains one edge in $E_{G'}(X_1, X^*)$, we have $d_{G'}(Y_{X_1}) \leq (r(W_j) - 1) - d_G(Y_{X_1})$, which implies that $d_{G+E^*}(Y_{X_1}) = d_G(Y_{X_1}) + d_{G'}(Y_{X_1}) \leq r(W_j) - 1$. Hence a vertex $v \in V - Y_{X_1}$ satisfies $\lambda_{G+E^*}(v, W_j) \leq r(W_j) - 1$, contradicting that $G + E^*$ is r-NA-edge-connected (note that $Y_{X_1} \in \mathcal{B}_j$ holds and hence we have $W_j \subseteq Y_{X_1}$).

Assume that $X_1 \in \mathcal{X}_1$ holds. From the properties (P2) and (P3), we have $d_{G'}(X^* \cup X_1) = 1$, and this implies that there exists an edge $e \in E^*$ connecting X_1 and some cut in $\mathcal{X} - \{X^*, X_1\}$. Let $\mathcal{X}'_1 = \{X^*, X_1, X_2, \ldots, X_{t'}, X_{t'+1}\}$ be a family of cuts in $\mathcal{X}$ such that we have $X_i \in \mathcal{X}_1$ for each $i = 1, 2, \ldots, t'$ and $X_{t'+1} \in \mathcal{X} - \mathcal{X}_1$ and $E_{G'}(X_i, X_{i+1}) \neq \emptyset$ for each $i = 1, \ldots, t'$ (note that such $X_{t'+1}$ exists by $\mathcal{X} - \mathcal{X}_1 - \{X^*\} \neq \emptyset$). Note that such $\mathcal{X}'_1$ is determined uniquely by

$$d_{G'}(X^*) = 1 \text{ and } d_{G'}(X) = 2 \text{ for each } X \in \mathcal{X}_1. \tag{2}$$

From the definition of property (P), there is a cut $Y_{X_{t'+1}} \in \mathcal{B}_j$ for some $W_j \in \mathcal{W}$ satisfying (P3) for $X_{t'+1}$. Let $Y_{t'+1} = Y_{X_{t'+1}} \cup (\cup_{X \in \mathcal{X}'_1} X)$. Note that $d_G(Y_{t'+1}) = d_G(Y_{X_{t'+1}})$ holds by $d_G(X) = 0$ for each $X \in \mathcal{X}_1$. We have

$$\sum_{v \in Y_{t'+1}} d_{G'}(v) \leq (r(W_j) + 1) - d_G(Y_{t'+1}) + 2t' \tag{3}$$

by $\sum_{v \in Y_{X_{t'+1}}} d_{G'}(v) \leq (r(W_j)+1) - d_G(Y_{X_{t'+1}})$, (2), and $d_G(Y_{t'+1}) = d_G(Y_{X_{t'+1}})$. Also by (2), we can observe that each edge in E^* incident to $(\cup_{X \in \mathcal{X}'_1 - \{X_{t'+1}\}} X)$ is contained in $E(G'[Y_{t'+1}])$; $E(G'[Y_{t'+1}])$ contains at least $t' + 1$ edges in E^*. From (3) and this, we have $d_{G'}(Y_{t'+1}) \leq (r(W_j) + 1) - d_G(Y_{t'+1}) + 2t' - 2(t' + 1) = r(W_j) - 1 - d_G(Y_{t'+1})$, which implies that $d_{G+E^*}(Y_{t'+1}) \leq r(W_j) - 1$ holds. Since we have $W_j \subseteq Y_{X_{t'+1}}$ and $V - Y_{X_{t'+1}} - (\cup_{X \in \mathcal{X}_1} X) \neq \emptyset$ from the property (P3), it follows that $Y_{t'+1} \in \mathcal{B}_j$, contradicting that $G + E^*$ is r-NA-edge-connected. □

In this paper, we prove that r-NA-ECAP enjoys the following min-max theorem and is polynomially solvable.

Theorem 8 *For r-NA-ECAP with $r(W) \geq 2$ for each area $W \in \mathcal{W}$, opt$(G, \mathcal{W}, r) = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$ holds if $(G, \mathcal{W})$ does not have property (P), and opt$(G, \mathcal{W}, r) = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil + 1$ holds otherwise. Moreover, a solution E^* with $|E^*| =$ opt$(G, \mathcal{W}, r)$ can be obtained in $O(m + pn^4 (r^* + \log n))$ time, where $n = |V|$, $m = |\{\{u,v\}|(u,v) \in E\}|$, $p = |\mathcal{W}|$, and $r^* = \max\{r(W) \mid W \in \mathcal{W}\}$.* □

4 Algorithm

Based on the lower bounds in the previous section, we give an algorithm, called r-NAEC-AUG, which finds a feasible solution E' to r-NA-ECAP with $|E'| = opt(G, \mathcal{W}, r)$, for a given area graph $(G, \mathcal{W})$ and a requirement function $r : \mathcal{W} \rightarrow Z^+ - \{1\}$. It finds a feasible solution E' with $|E'| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil + 1$ if $(G, \mathcal{W})$ has property (P), $|E'| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$ otherwise.

To find a minimum set E' of new edges, we do not immediately add some new edges to G. Instead we first try to find the set of vertices in G that are end vertices of such an E'. For this, we create a new vertex s outside of G and add new edges between s and G.

For a graph $H = (V \cup \{s\}, E)$ and a designated vertex $s \notin V$, an operation called *edge-splitting* (*at* s) is defined as deleting two edges $(s,u), (s,v) \in E$ and adding one new edge (u,v). That is, the graph $H' = (V \cup \{s\}, (E - \{(s,u),(s,v)\}) \cup \{(u,v)\})$ is obtained from such edge-splitting operation. Then we say that H' is obtained from H by *splitting* a pair of edges (s,u) and (s,v) (or by splitting (s,u) and (s,v)). A sequence of splittings is *complete* if the resulting graph H' does not have any neighbor of s. The edge-splitting operation is known to be a useful tool for solving connectivity augmentation problems [3].

We here give an outline of algorithm r-NAEC-AUG. In the first step, we add to a given graph $(G, \mathcal{W})$ a new vertex s and a set F_1 of new edges between s and V with $|F_1| = \alpha(G, \mathcal{W}, r)$ such that the resulting graph $H = (V \cup \{s\}, E \cup F_1)$ satisfies $\lambda_H(v, W_i) \geq r(W_i)$ for every pair of $v \in V$ and $W_i \in \mathcal{W}$. (The vertex s will be discarded upon the completion of the algorithm.) If F_1 is odd, then we add an arbitrary one edge to F_1. Then we can check if G has Property (P) or not. In the next step, we repeat edge-splittings at s while preserving $r(W_i)$-NA-edge-connectivity between every pair of $v \in V$ and $W_i \in \mathcal{W}$. If $(G, \mathcal{W})$ does not have property (P), then the algorithm finds such a complete splitting, and hence the set E^* of added edges satisfies $|E^*| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$ and $\lambda_{G+E^*}(v, W_i) \geq r(W_i)$ for every pair of a vertex $v \in V$ and an area $W_i \in \mathcal{W}$. If $(G, \mathcal{W})$ has property (P), then the algorithm finds such a complete splitting by adding one extra edge to G, and hence the obtained edge set E^*

satisfies $|E^*| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil + 1$. In both cases, E^* is optimal by Lemmas 5 and 7.

More precisely, we describe the algorithm below, and introduce three theorems necessary to justify the algorithm, which will be proved in the subsequent sections. An example of computational process of r-NAEC-AUG is shown in Figure 3.

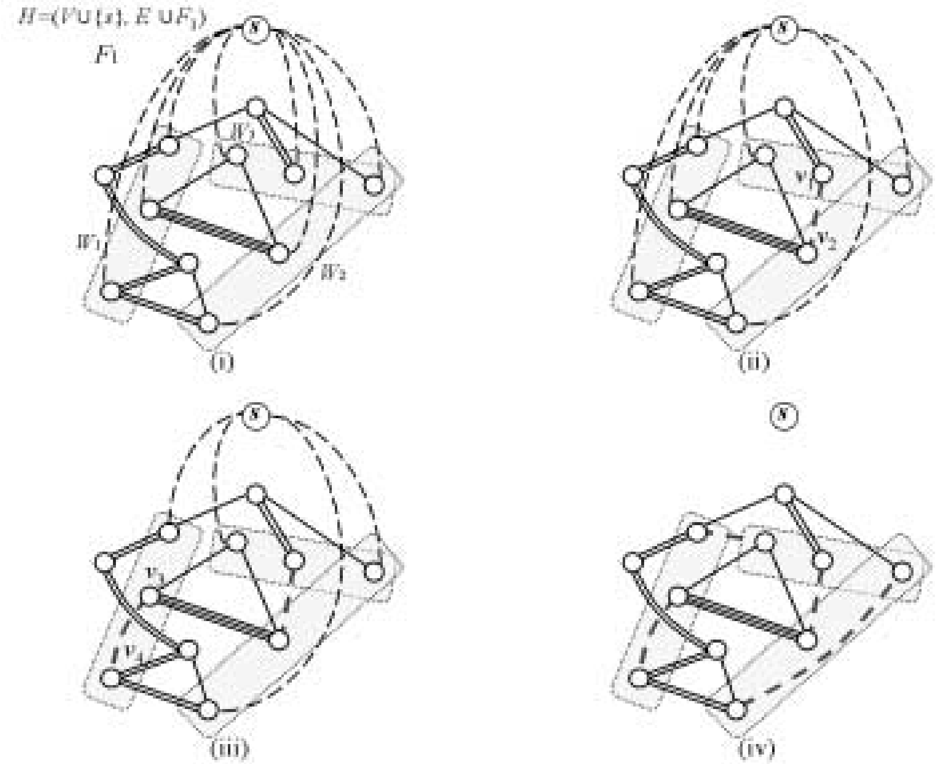

Fig. 3. Computational process of algorithm r-NAEC-AUG applied to the area graph $(G, \mathcal{W})$ in Figure 1 and $(r(W_1), r(W_2), r(W_3)) = (2, 3, 4)$. The lower bound in Section 3 is $\lceil \alpha(G, \mathcal{W})/2 \rceil = 4$. (i) $H = (V \cup \{s\}, E \cup F_1)$ obtained by Step 1. Edges in F_1 are drawn as broken lines. Then $\lambda_H(v, W) \geq r(W)$ holds for every pair of $v \in V$ and $W \in \mathcal{W}$. (ii) $H_1 = (H - \{(s, v_1), (s, v_2)\}) \cup \{(v_1, v_2)\}$ obtained from H by the admissible splitting of (s, v_1) and (s, v_2). (iii) $H_2 = (H_1 - \{(s, v_3), (s, v_4)\}) \cup \{(v_3, v_4)\}$ obtained from H_1 by the admissible splitting of (s, v_3) and (s, v_4). (iv) H_3 obtained from H_2 by a complete admissible splitting at s. The graph $G_3 = H_3 - s$ is r-NA-edge-connected.

Algorithm r-NAEC-AUG

Input: An area graph $(G = (V, E), \mathcal{W})$ and a requirement function $r : \mathcal{W} \to Z^+ - \{1\}$.

Output: A set E^* of new edges with $|E^*| = opt(G, \mathcal{W}, r)$ such that $G + E^*$ is r-NA-edge-connected.

Step 1: We add a new vertex s and a set F_1 of new edges between s and V such that in the resulting graph $H = (V \cup \{s\}, E \cup F_1)$,

$$\text{all cuts } X \subset V \text{ of } \mathcal{A}_i \cup \mathcal{B}_i \text{ satisfy } d_H(X) \geq r(W_i) \text{ for each } W_i \in \mathcal{W}, \quad (4)$$

and no $F \subset F_1$ satisfies this property (as will be shown, $|F_1| = \alpha(G, \mathcal{W}, r)$ holds). If $d_H(s)$ is odd, then we add to F_1 one extra edge between s and V.

Step 2: We split two edges incident to s while preserving (4) (such splitting pair is called *admissible*).

If at least one of the following conditions (I)–(III) does not hold, then find a complete admissible splitting at s in H after replacing at most one edge f_1 in F_1 with another edge f_2 incident to s. Output the set E^* of all split edges, where $|E^*| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$ holds. If all conditions (I)–(III) hold, then we can prove that G has property (P). By adding one new edge e^* to G, find a complete admissible splitting at s in $H + \{e^*\}$. Output the edge set $E^* := E_3 \cup \{e^*\}$, where E_3 denotes the set of all split edges and $|E^*| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil + 1$ holds. (The procedures of finding a complete splitting at s and finding edges f_1, f_2, and e^* are complicated, and hence the details will be described in Section 6 later.)

(I) G has exactly one component C^* with $d_H(s, C^*) = 1$.

(II) For the edge (s, u^*) with $\{(s, u^*)\} = E_H(s, C^*)$, u^* is contained in a cut $X \subseteq C^*$ with $X \in \mathcal{A}_j$ and $d_H(X) = r(W_j)$ for some area $W_j \in \mathcal{W}$.

(III) Let $\mathcal{C}_1$ be the family of all components C of G which satisfies $d_H(C) = d_H(s, C) = 2$ and belongs to $\mathcal{A}_i$ for some area $W_i \in \mathcal{W}$. $\{(s, u^*), e\}$ is not admissible in H for any edge $e \in E_H(s, V - \cup_{C \in \mathcal{C}_1} C)$. □

To justify the algorithm r-NAEC-AUG, it suffices to show the following Theorems 9–11.

Theorem 9 *Let $(G = (V, E), \mathcal{W})$ be an area graph, and $r(W) \geq 0$ for each $W \in \mathcal{W}$. Let $H = (V \cup \{s\}, E \cup F_1)$ be a graph with $s \notin V$ and $F_1 = E_H(s, V)$ such that H satisfies* (4) *and no $F \subset F_1$ satisfies this property. Then $|F_1| = \alpha(G, \mathcal{W}, r)$ holds.* □

In the sequel, we shall often consider an area graph $(G = (V, E), \mathcal{W})$, and a graph $H = (V \cup \{s\}, E \cup F)$ with a designated vertex $s \notin V$ and $F =$

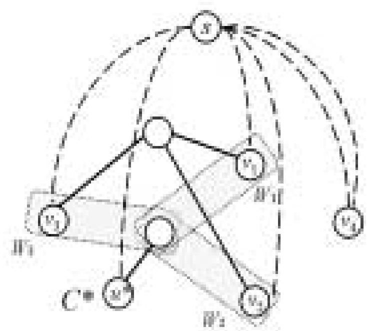

Fig. 4. Illustration of a graph $H = (V \cup \{s\}, E \cup F)$ satisfying the statements (I) – (III) in Theorem 10. The graph H is constructed from the graph G in Figure 2 by adding a designated vertex s and a set F of edges between s and V so that H satisfies (4). Observe that the component C^* corresponds to the C^* in the statement (I), the cut $\{u^*\} \subseteq C^*$ satisfies $\{u^*\} \in \mathcal{A}_1$ and $d_H(\{u^*\}) = r(W_1) = 2$, $\{(s, u^*), (s, v_j)\}$, $j = 1, 2, 3$ is not admissible in H, and $\mathcal{C}_1 = \{\{v_4\}\}$ holds.

$E_H(s, V) \neq \emptyset$ satisfying the following (a) – (c):

$$\begin{array}{ll} \text{(a) } |F| = d_H(s, V) \text{ is even,} & \\ \text{(b) } r(W) \geq 2 \text{ holds for each area } W \in \mathcal{W}, & \qquad (5) \\ \text{(c) } H \text{ satisfies (4).} & \end{array}$$

Theorem 10 *Let $G = (V, E)$, $H = (V \cup \{s\}, E \cup F)$, and r satisfy (5). If H satisfies the following conditions (I)–(III), then G has property (P) (see Figure 4). Otherwise H has a complete admissible splitting at s after replacing at most one edge in F with a new edge incident to s.*

(I) G has exactly one component C^* with $d_H(s, C^*) = 1$.

(II) For the edge $(s, u^*) \in F$ with $E_H(s, C^*) = \{(s, u^*)\}$, u^* is contained in a cut $X \subseteq C^*$ with $X \in \mathcal{A}_i$ and $d_H(X) = r(W_i)$ for some area $W_i \in \mathcal{W}$.

(III) Let $\mathcal{C}_1$ be the family of all components C' of G such that $d_H(s, C') = 2$ and $C' \in \mathcal{A}_j$ for some W_j. For any edge $e \in E_H(s, V - \cup_{C' \in \mathcal{C}_1} C')$, $\{(s, u^*), e\}$ is not admissible in H. □

Theorem 11 *Let $G = (V, E)$, $H = (V \cup \{s\}, E \cup F)$, and r satisfy (5). Then there is a graph $H' = H + \{e\}$ obtained from H by adding some edge e to G such that H' has a complete admissible splitting at s.* □

By Theorems 10 and 11, for the set E^* of edges obtained by algorithm r-NAEC-AUG, the graph $H^* = (V \cup \{s\}, E \cup E^*)$ satisfies (4), i.e., all cuts $X \subset V$ of $\mathcal{A}_i \cup \mathcal{B}_i$ satisfy $d_{H^*}(X) \geq r(W_i)$ for each area $W_i \in \mathcal{W}$. By $d_{H^*}(s) = 0$, all cuts $X \subset V$ satisfy $d_{G+E^*}(X) = d_{H^*}(X)$. From Lemma 2, it follows that

$G + E^*$ is r-NA-edge-connected. Theorem 9 indicates that $|F_1| = \alpha(G, \mathcal{W}, r)$ holds. Again by Theorems 10 and 11, we have $|E^*| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil + 1$ in the cases where an initial area graph $(G, \mathcal{W})$ has property (P), $|E^*| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$ otherwise. By Lemmas 5 and 7, we have $|E^*| = \mathit{opt}(G, \mathcal{W}, r)$.

5 Proof of Theorem 9

In the subsequent sections, for a graph $H = (V \cup \{s\}, E \cup F)$, let $s \notin V$, $F = E_H(s, V)$, and the graph $H - s$ be the area graph $(G, \mathcal{W})$, if no confusion occurs.

For two cuts $X, Y \subseteq V$ in a graph $G = (V, E)$, we say that X and Y *cross* each other in G or X crosses with Y if none of $X \cap Y$, $X - Y$, $Y - X$, and $V - (X \cup Y)$ is empty. For a family $\mathcal{X}$ of subsets of V and a vertex set $Y \subseteq V$, $\mathcal{X}$ *covers* Y if $Y \subseteq \cup_{X \in \mathcal{X}} X$ holds. For a graph $G = (V, E)$, every two cuts $X, Y \subset V$ satisfy the following equalities.

$$d_G(X) + d_G(Y) = d_G(X - Y) + d_G(Y - X) + 2d_G(X \cap Y, V - (X \cup Y)) \quad (6)$$
$$d_G(X) + d_G(Y) = d_G(X \cup Y) + d_G(X \cap Y) + 2d_G(X - Y, Y - X). \quad (7)$$

For a graph $G = (V, E)$, every three cuts X, Y, and Z satisfy the following inequality.

$$\begin{aligned} d_G(X) + d_G(Y) + d_G(Z) \geq\ & d_G(X - Y - Z) + d_G(Y - X - Z) \\ & + d_G(Z - X - Y) + d_G(X \cap Y \cap Z) \\ & + 2d_G(X \cap Y \cap Z, V - (X \cup Y \cup Z)). \end{aligned} \quad (8)$$

In a graph $H = (V \cup \{s\}, E \cup F)$ satisfying (4), for each area $W_i \in \mathcal{W}$, the following properties hold:

If a cut $X \subset V$ belongs to $\mathcal{A}_i$, then every cut $X' \subseteq X$ also belongs to $\mathcal{A}_i$ and hence satisfies $d_H(X') \geq r(W_i)$. (9)

If a cut $X \subset V$ belongs to $\mathcal{B}_i$, then every cut $X' \supseteq X$ with $X' \neq V$ also belongs to $\mathcal{B}_i$ and hence satisfies $d_H(X') \geq r(W_i)$. (10)

Theorem 9 can be proved from the theory of polymatroids as follows. Let V be a finite ground set and let $p : 2^V \to Z \cup \{-\infty\}$ be an integer-valued function with $p(\emptyset) = 0$. A set function p is called *skew-supermodular* if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ or $p(X) + p(Y) \leq p(X - Y) + p(Y - X)$ hold for every two subsets X and Y of V. A set function p is called *symmetric* if $p(X) = p(V - X)$ holds for all $X \subseteq V$. In [3], it was shown that given a symmetric skew-supermodular integer-valued function $p : 2^V \rightarrow Z \cup \{-\infty\}$, a vector $z : V \rightarrow Z^+$ such that $\sum_{v \in V} z(v)$ is the minimum and $z(X) \geq p(X)$ holds for every $X \subseteq V$ can be found by a greedy algorithm. Now it is not difficult to see from (6), (7), (9), and (10) that $\alpha_{G,\mathcal{W},r}$ is a symmetric skew-supermodular integer-valued function. Note that $H = (V \cup \{s\}, E \cup F)$ satisfies (4) if and only if a vector $z : V \rightarrow Z^+$ with $z(v) = d_H(s, v)$ satisfies $z(X) \geq \alpha_{G,\mathcal{W},r}(X)$ for every $X \subseteq V$. This observation proves Theorem 9.

Here we also give a graph theoretical proof of Theorem 9. We first show several properties of a graph $H = (V \cup \{s\}, E \cup F)$ satisfying (4).

Lemma 12 *Let $G = (V, E)$, $H = (V \cup \{s\}, E \cup F)$, and r satisfy (4), and two cuts $X, Y \subset V$ with $d_H(X) = r(W_i)$ and $d_H(Y) = r(W_j)$ cross each other in H ($r(W_i) = r(W_j)$ may hold). Assume that one of the following (i)-(iii) holds. Then we have $d_H(X \cap Y, V \cup \{s\} - (X \cup Y)) = 0$. Moreover, if (i) holds, then we have $X - Y \in \mathcal{A}_i$, $d_H(X - Y) = r(W_i)$, and $d_H(Y - X) = r(W_j)$. If (ii) or (iii) hold, then we have $X - Y \in \mathcal{A}_j$, $d_H(X - Y) = r(W_j)$, and $d_H(Y - X) = r(W_i)$.*
(i) $X \in \mathcal{A}_i$ and $Y \in \mathcal{A}_j$.
(ii) $X \in \mathcal{B}_i$ and $Y \in \mathcal{B}_j$.
(iii) $X \in \mathcal{A}_i$, $Y \in \mathcal{B}_j$, and $V = X \cup Y$.

PROOF. If (i) holds (resp., (ii) or (iii) hold), then $X - Y$ belongs to $\mathcal{A}_i$ (resp., $\mathcal{A}_j$) by (9) (resp., $(X - Y) \cap W_j = \emptyset$). It suffices to show that if (i) holds (resp., (ii) or (iii) hold), then $d_H(X - Y) \geq r(W_i)$ and $d_H(Y - X) \geq r(W_j)$ (resp., $d_H(X - Y) \geq r(W_j)$ and $d_H(Y - X) \geq r(W_i)$) hold. This follows since if $d_H(X - Y) \geq r(W_i)$ and $d_H(Y - X) \geq r(W_j)$ (resp., $d_H(X - Y) \geq r(W_j)$ and $d_H(Y - X) \geq r(W_i)$) hold, then by (6), it follows that $r(W_i) + r(W_j) = d_H(X) + d_H(Y) = d_H(X - Y) + d_H(Y - X) + 2d_H(X \cap Y, V \cup \{s\} - (X \cup Y)) \geq r(W_i) + r(W_j)$, which proves the lemma.

(i) (9) says $d_H(X - Y) \geq r(W_i)$ and $d_H(Y - X) \geq r(W_j)$.

(ii) There are two areas $W_i, W_j \in \mathcal{W}$ with $W_i \subseteq X$ and $W_j \subseteq Y$. $(Y - X) \cap W_i = \emptyset$ says that $Y - X \in \mathcal{A}_i$ holds. From (4), it follows that $d_H(X - Y) \geq r(W_j)$ and $d_H(Y - X) \geq r(W_i)$.

(iii) There are two areas $W_i, W_j \in \mathcal{W}$ with $W_i \subseteq V - X$ and $W_j \subseteq Y$. Since $X - Y \in \mathcal{A}_j$ holds, we have $d_H(X - Y) \geq r(W_j)$ by (4). Moreover, from $X \cup Y = V$, it follows that $W_i \subseteq Y - X$ holds. Hence the cut $Y - X$ belongs to $\mathcal{B}_i$ and satisfies $d_H(Y - X) \geq r(W_i)$ by (4). $\square$

Lemma 13 *Let $G = (V, E)$, $H = (V \cup \{s\}, E \cup F)$, and r satisfy* (4) *and two cuts $X, Y \subset V$ with $d_H(X) = r(W_i)$ and $d_H(Y) = r(W_j)$ cross each other in H ($r(W_i) = r(W_j)$ may hold). If $X \in \mathcal{A}_i$, $Y \in \mathcal{B}_j$, and $V \neq X \cup Y$ hold, then the cut $X \cup Y$ belongs to $\mathcal{B}_j$ and satisfies $d_H(X \cup Y) = r(W_j)$.*

PROOF. We have $d_H(X \cap Y) \geq r(W_i)$ by (9) and $d_H(X \cup Y) \geq r(W_j)$ by (10). By (7), we have $r(W_i) + r(W_j) = d_H(X) + d_H(Y) \geq d_H(X \cap Y) + d_H(X \cup Y) \geq r(W_i) + r(W_j)$, which proves the lemma. □

PROOF of Theorem 9. We first show $|F_1| \geq \alpha(G, \mathcal{W}, r)$. Let $\mathcal{X}^*$ be a subpartition of V such that every cut $X^* \in \mathcal{X}^*$ satisfies $\alpha_{G,\mathcal{W},r}(X^*) > 0$ and $\sum_{X^* \in \mathcal{X}^*} \alpha_{G,\mathcal{W},r}(X^*) = \alpha(G, \mathcal{W}, r)$ holds. For any graph $H' = (V \cup \{s\}, E \cup F')$ satisfying (4), we have $d_{H'}(s, X^*) \geq r(W_{i_{X^*}}) - d_G(X^*)$ for all cuts $X^* \in \mathcal{X}^*$. This means $|F'| \geq \alpha(G, \mathcal{W}, r)$. Therefore we have $|F_1| \geq \alpha(G, \mathcal{W}, r)$.

We next show $|F_1| \leq \alpha(G, \mathcal{W}, r)$. From the minimality of F_1, there is a cut $X_v \subset V$ of $\mathcal{A}_i \cup \mathcal{B}_i$ with some area $W_i \in \mathcal{W}$, satisfying $v \in X_v$ and $d_H(X_v) = r(W_i)$ for every edge $(s, v) \in F_1$. We call such cut X_v a *critical cut with respect to v*. Let $\mathcal{X}$ be a family of critical cuts X_v, $v \in V[F_1] - s$, such that $\mathcal{X}$ covers $V[F_1] - s$ and $\cup_{X \in \mathcal{X}} |X|$ is the minimum (note that such $\mathcal{X}$ exists from the minimality of F_1). We claim that $\mathcal{X}$ is a subpartition of V. If $\mathcal{X}$ is a subpartition of V, then we have $|F_1| = \sum_{X \in \mathcal{X}} (r(W_{i_X}) - d_G(X)) \leq \alpha(G, \mathcal{W}, r)$ by the maximality of $\alpha(G, \mathcal{W}, r)$ (note that each critical cut $X \subset V$ satisfies $d_H(s, X) = r(W_{i_X}) - d_G(X) = \alpha_{G,\mathcal{W},r}(X) > 0$ by the definition of critical cuts and (4)).

Assume by contradiction that $\mathcal{X}$ is not a subpartition of V. Then there are two cuts $X_1, X_2 \in \mathcal{X}$ which cross each other in H (note that from the minimality of $\cup_{X \in \mathcal{X}} |X|$, no cut $X' \in \mathcal{X}$ satisfies $X' \subset X$ for some cut $X \in \mathcal{X}$). There are the following four possible cases: (I) $X_1 \in \mathcal{A}_i$ and $X_2 \in \mathcal{A}_j$ hold for some $W_i, W_j \in \mathcal{W}$, (II) $X_1 \in \mathcal{B}_i$ and $X_2 \in \mathcal{B}_j$ hold for some $W_i, W_j \in \mathcal{W}$, (III) $X_1 \in \mathcal{A}_i$ and $X_2 \in \mathcal{B}_j$ hold for some $W_i, W_j \in \mathcal{W}$ and $X_1 \cup X_2 = V$, and (IV) $X_1 \in \mathcal{A}_i$ and $X_2 \in \mathcal{B}_j$ hold for some $W_i, W_j \in \mathcal{W}$ and $X_1 \cup X_2 \neq V$.

In the cases of (I) (resp., (II) or (III)), from Lemma 12 (i) (resp., (ii)(iii)), it follows that $X_1 - X_2$ is a critical cut of $\mathcal{A}_i$ (resp., $\mathcal{A}_j$). Hence, the new family $\mathcal{X}' = (\mathcal{X} - \{X_1\}) \cup \{X_1 - X_2\}$ of critical cuts covers $V[F_1] - s$ and contradicts the minimality of $\cup_{X \in \mathcal{X}} |X|$. In the case of (IV), from Lemma 13, it follows that $X_1 \cup X_2$ is a critical cut of $\mathcal{B}_j$. Hence, the new family $\mathcal{X}' = (\mathcal{X} - \{X_1, X_2\}) \cup \{X_1 \cup X_2\}$ of critical cuts covers $V[F_1] - s$ and contradicts the minimality of $\cup_{X \in \mathcal{X}} |X|$. □

6 Proofs of Theorems 10 and 11

In this section, we give proofs of Theorems 10 and 11. In Section 6.1, we first show several properties about edge-splitting operations and give a proof of Theorem 11. Based on this, we prove Theorem 10; we show in Section 6.2 that if at least one of conditions (I)–(III) in Theorem 10 does not hold, then there is a complete splitting at s, and show in Section 6.3 that if all conditions (I)–(III) hold, then G has property (P).

Through this section, let $\mathcal{C}_1$ be the family of all components C of G such that $d_H(C) = d_H(s, C) = 2$ and $C \in \mathcal{A}_i$ for some $W_i \in \mathcal{W}$, and $V_1 = \cup_{C\in\mathcal{C}_1} C$. Let $\mathcal{C}_2$ be the family of all components C of G such that $C \notin \mathcal{C}_1$ and $d_H(s, C) > 0$, and $V_2 = \cup_{C\in\mathcal{C}_2} C$.

6.1 Edge-splitting operations

In this section, we show the following theorem and lemmas, which are keys for splitting operations in algorithm r-NAEC-AUG.

Lemma 14 *Let G, H, and r satisfy* (5). *If*

$$d_H(s, C) \leq 2 \text{ holds for all components } C \text{ of } G, \tag{11}$$

then we can continue admissible edge-splittings at s until isolating s.

Lemma 15 *Let G, H, and r satisfy* (5). *If*

$$d_H(s, C) \text{ is even for all components } C \text{ of } G, \tag{12}$$

then we can continue admissible edge-splittings at s until isolating s.

Theorem 16 *Let G, H, and r satisfy* (5). *Assume that no pair of two edges in $E_H(s, V_2)$ is admissible and that neither* (11) *nor* (12) *holds. Then $G[V_2]$ has exactly two components C_1 and C_2. Moreover, C_1 and C_2 satisfy the followings*:
(a) *$d_H(s, C_1) \geq 3$ holds, every cut $X \subseteq C_1$ satisfies $d_H(X) \geq 2$, and every cut $X \subseteq C_1$ with $d_H(X) = 2$ belongs to $\mathcal{A}_i$ for some $W_i \in \mathcal{W}$.*
(b) *$d_H(s, C_2) = 1$ holds.* □

Remark: The second and third properties in Theorem 16 (a) will be used for further analysis about complete splittings in Section 6.3.

From Lemma 15 and Theorem 16, Corollary 17 follows.

Corollary 17 *Let C_1 and C_2 be two components of $G[V_2]$ in Theorem 16. Then, in the graph $H + e^*$ obtained by adding one arbitrary new edge e^* to $E_G(C_1, C_2)$, we can continue admissible edge-splittings at s until isolating s.*

PROOF. In the graph $H' = H + e^*$ obtained by adding one arbitrary new edge e^* to $E_G(C_1, C_2)$, $d_{H'}(s, C)$ is even for every component C of $H'[V]$ (note that $d_H(s, C_1 \cup C_2)$ is even since $d_H(s, V)$ and $d_H(s, V_1)$ are both even and $d_H(s, C_1 \cup C_2) = d_H(s, V) - d_H(s, V_1)$ holds). Lemma 15 proves this corollary. □

It is not difficult to observe that this corollary proves Theorem 11. Let H' denote the resulting graph obtained from H by continuing admissible splittings of two edges in $E_H(s, V_2)$ as possible. If H' satisfies (11) or (12), then it follows from Lemmas 14 and 15 that H' has a complete splitting at s. Otherwise H' satisfies the assumption of Theorem 16 (note that the case of $E_{H'}(s, V_2) = \emptyset$ implies that (12) holds by the definition of $\mathcal{C}_1$). In this case, it follows from Corollary 17 that we can obtain a complete admissible splitting in H' after adding one extra edge.

Before proving these theorem and lemmas, we introduce several preparatory properties about splittings. For a graph $H = (V \cup \{s\}, E \cup F)$ with satisfying (4), a pair $\{(s, u), (s, v)\} \subseteq F$ of two edges is not admissible if there is a cut $Y \subset V$ of $\mathcal{A}_i \cup \mathcal{B}_i$ for some i with $\{u, v\} \subseteq Y$ and $d_H(Y) \leq r(W_i) + 1$. Such cut Y is called a *dangerous cut*. Conversely, a pair $\{(s, u), (s, v)\}$ is not admissible only if there is a dangerous cut $Y \subset V$ with $\{u, v\} \subseteq Y$.

The following two lemmas are used for seeking an admissible pair of two edges while avoiding dangerous cuts. Lemma 18 says that any dangerous cut cannot cover $V[F] - s$.

Lemma 18 *Let $G = (V, E)$, $H = (V \cup \{s\}, E \cup F)$, and r satisfy* (4) *and $Y \subset V$ be a dangerous cut. Then we have $d_H(s, V - Y) \geq d_H(s, Y) - 1 > 0$.*

PROOF. Assume that Y is a dangerous cut of $\mathcal{A}_i \cup \mathcal{B}_i$ for an area $W_i \in \mathcal{W}$. Since Y is a dangerous cut, we have $d_H(Y) = d_H(s, Y) + d_H(Y, V - Y) \leq r(W_i) + 1$. Moreover, $Y \in \mathcal{A}_i \cup \mathcal{B}_i$ holds, and hence so does $V - Y$, which implies $d_H(V - Y) = d_H(s, V - Y) + d_H(Y, V - Y) \geq r(W_i)$ by (4). Hence we have $d_H(s, V - Y) \geq r(W_i) - d_H(Y, V - Y) \geq d_H(s, Y) - 1$. From the definition of dangerous cuts, it follows that $d_H(s, Y) \geq 2$ holds. □

The next lemma says that any two dangerous cuts containing a common vertex in V cannot cover $V[F] - s$.

Lemma 19 *Let $G = (V, E)$, $H = (V \cup \{s\}, E \cup F)$, and r satisfy* (4) *and an even $d_H(s)$. Assume that there are two dangerous cuts $Y_1, Y_2 \subset V$ with $d_H(s, Y_1 - Y_2) > 0$, $d_H(s, Y_2 - Y_1) > 0$, and $d_H(s, Y_1 \cap Y_2) > 0$. Then we have $d_H(s, V - Y_1 - Y_2) > 0$.*

PROOF. Assume that Y_i (resp., Y_j) is a dangerous cut of $\mathcal{A}_i \cup \mathcal{B}_i$ (resp., $\mathcal{A}_j \cup \mathcal{B}_j$) for an area $W_i \in \mathcal{W}$ (resp., $W_j \in \mathcal{W}$). Assume $d_H(s, Y_1 - Y_2) \geq d_H(s, Y_2 - Y_1)$ without loss of generality. By Lemma 18, we have $d_H(s, Y_2 - Y_1) + d_H(s, V - Y_1 - Y_2) = d_H(s, V - Y_1) \geq d_H(s, Y_1) - 1 = d_H(s, Y_1 - Y_2) + d_H(s, Y_1 \cap Y_2) - 1$ $\geq d_H(s, Y_2 - Y_1) + d_H(s, Y_1 \cap Y_2) - 1$. Hence $d_H(s, V - Y_1 - Y_2) = 0$ would imply that the above inequalities hold by equality since $d_H(s, Y_1 \cap Y_2) \geq 1$ holds. This means $d_H(s, Y_1 - Y_2) = d_H(s, Y_2 - Y_1)$, which implies $d_H(s) = 2d_H(s, Y_1 - Y_2) + 1$, contradicting that $d_H(s)$ is even. $\square$

The next two lemmas show properties for cuts $Y \in \mathcal{A}_i \cup \mathcal{B}_i$ with some i satisfying $d_H(Y) \leq r(W_i) + 1$ (note that Y is not necessarily dangerous). We will be often referred to the next Lemma 20 in the subsequent arguments, when we observe that a dangerous cut of $\mathcal{A}_i$ induces a connected component, or that a dangerous cut which does not a connected component belongs to $\mathcal{B}_j$.

Lemma 20 *Let $G = (V, E)$, $H = (V \cup \{s\}, E \cup F)$, and r satisfy* (4)*. For every cut $Y \subset V$ of $\mathcal{A}_i$ with $r(W_i) \geq 2$ and $d_H(Y) \leq r(W_i) + 1$, $\lambda(G[Y]) \geq r(W_i) - \lfloor \frac{d_H(Y)}{2} \rfloor$ (≥ 1) holds.*

PROOF. By (9), for any partition $\{Y_1, Y_2\}$ of Y, we have $d_H(Y_j) \geq r(W_i)$ for $j = 1, 2$. Hence, we have $d_H(Y_1, Y_2) = \frac{1}{2}(d_H(Y_1) + d_H(Y_2)) - \frac{d_H(Y)}{2} \geq r(W_i) - \frac{d_H(Y)}{2} > 0$ by $r(W_i) \geq 2$. $\square$

The next lemma is often used under a situation where two crossing dangerous cuts Y_1, Y_2 satisfy $d_H(s, Y_1 \cap Y_2) > 0$.

Lemma 21 *Let $G = (V, E)$, $H = (V \cup \{s\}, E \cup F)$, and r satisfy* (4)*, and Y_1 and Y_2 be two cuts with $d_H(Y_1) \leq r(W_i) + 1$, $d_H(Y_2) \leq r(W_j) + 1$, and $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) > 0$ such that Y_1 and Y_2 satisfy one of the following* (i) *or* (ii)*. Assume that Y_1 and Y_2 cross each other in H. Then we have $d_H(Y_1) = r(W_i) + 1$, $d_H(Y_2) = r(W_j) + 1$, and $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) = 1$. Moreover, if* (i) *holds, then $Y_1 - Y_2$ is a cut of $\mathcal{A}_i$ with $d_H(Y_1 - Y_2) = r(W_i)$ and $Y_2 - Y_1$ is a cut of $\mathcal{A}_j$ with $d_H(Y_2 - Y_1) = r(W_j)$. If* (ii) *holds, then $Y_1 - Y_2$ is a cut of $\mathcal{A}_j$ with $d_H(Y_1 - Y_2) = r(W_j)$ and $Y_2 - Y_1$ is a cut of $\mathcal{A}_i$ with $d_H(Y_2 - Y_1) = r(W_i)$.*
(i) *$Y_1 \in \mathcal{A}_i$ and $Y_2 \in \mathcal{A}_j$.*
(ii) *$Y_1 \in \mathcal{B}_i$ and $Y_2 \in \mathcal{B}_j$.*

PROOF. If (i) (resp., (ii)) holds, then the cuts $Y_1 - Y_2$ belongs to $\mathcal{A}_i$ (resp., $\mathcal{A}_j$) and $Y_2 - Y_1$ belongs to $\mathcal{A}_j$ (resp., $\mathcal{A}_i$) by (9) (resp., by $(Y_1 - Y_2) \cap W_j = \emptyset$ $= (Y_2 - Y_1) \cap W_i$). Hence, from (4), it follows that $d_H(Y_1 - Y_2) \geq r(W_i)$ (resp., $d_H(Y_1 - Y_2) \geq r(W_j)$) and $d_H(Y_2 - Y_1) \geq r(W_j)$ (resp., $d_H(Y_2 - Y_1) \geq r(W_i)$). By this, $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) > 0$, and (6), we have $r(W_i) + 1 + r(W_j) + 1 \geq d_H(Y_1) + d_H(Y_2) = d_H(Y_1 - Y_2) + d_H(Y_2 - Y_1)$ $+2d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) \geq r(W_i) + r(W_j) + 2$. This proves the lemma. $\square$

Based on these, we give proofs of Lemmas 14 and 15. In the subsequent arguments, we will be often referred to the following two conditions (11) and (12) as evidences that we can continue admissible edge-splittings in H until isolating s.

PROOF of Lemma 14. Assume that $|F| \geq 4$ holds, since otherwise $|F| = 2$ holds and Lemma 18 implies that the pair of two edges in F is admissible. Hence G has at least two components $C \in \mathcal{C}$, where $\mathcal{C}$ denotes the family of all components C of G with $d_H(s, C) > 0$. We prove the lemma by showing that there is a pair of two edges in F which is admissible in H (note that the resulting graph obtained by an admissible splitting at s also satisfies the assumption of this lemma).

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$, $t \geq 2$ satisfy $d_H(s, C_1) \geq d_H(s, C_2) \geq \cdots \geq d_H(s, C_t)$. Let $(s, u_j) \in F$ be an edge with $u_j \in C_j \in \mathcal{C}$ for $j = 1, 2$. It suffices to show that $\{(s, u_1), (s, u_2)\}$ is admissible in H. Assume by contradiction that there exists a dangerous cut Y_1 with $\{u_1, u_2\} \subseteq Y_1$. Then Lemma 20 implies that $Y_1 \in \mathcal{B}_i$ holds for some area $W_i \in \mathcal{W}$.

We claim that $W_i \cap C \neq \emptyset$ holds for each $C \in \mathcal{C} - \{C_1, C_2\}$. Assume that some $C' \in \mathcal{C}$ satisfies $C' \cap W_i = \emptyset$. Then $C' \in \mathcal{A}_i$ holds and hence $d_H(C') = d_H(s, C') \geq r(W_i) \geq 2$ holds by (4). It follows that $d_H(s, C') = 2$, $r(W_i) = 2$, and $d_H(Y_1) \leq 3$. From the choice of u_1, u_2 and $d_H(s, C') = 2$, we have $d_H(s, C_1) = d_H(s, C_2) = 2$, and hence $d_H(C_1 \cap Y_1) \geq 2$ and $d_H(C_2 \cap Y_1) \geq 2$ hold. It follows that $d_H(Y_1) \geq d_H(C_1 \cap Y_1) + d_H(C_2 \cap Y_1) \geq 4$, contradicting $d_H(Y_1) \leq 3$.

Lemma 18 says that there is an edge $(s, v_1) \in E_H(s, V - Y_1)$. Then let $v_1 \in C_1 \cup C_2$ if $d_H(s, C_1 \cup C_2 - Y_1) > 0$ holds. Let C' be the component in $\mathcal{C}$ with $v_1 \in C'$. From $(C' - Y_1) \cap W_i = \emptyset$ and (4), it follows that $d_H(C' - Y_1) \geq r(W_i)$. Hence $d_G(C' - Y_1) = d_G(C' - Y_1, C' \cap Y_1) \geq r(W_i) - d_H(s, C' - Y_1)$. From this, it follows that $d_H(Y_1) \geq d_H(s, Y_1) + d_G(C' - Y_1, C' \cap Y_1) \geq r(W_i) + d_H(s, Y_1) - d_H(s, C' - Y_1)$. Note that $1 \leq d_H(s, C' - Y_1) \leq 2$ holds; there are two possible cases (I) $d_H(s, C' - Y_1) = 1$ and (II) $d_H(s, C' - Y_1) = 2$.

We consider the case of (I). By $d_H(s, Y_1) \geq 2$ and $d_H(Y_1) \leq r(W_i)+1$, we have $d_H(s, Y_1) = 2$, $d_H(Y_1) = r(W_i) + 1$, and $d_G(Y_1) = d_G(C' - Y_1, C' \cap Y_1)$. Since every component $C \in \mathcal{C}$ satisfies $C \cap Y_1 \neq \emptyset$ and $d_G(Y_1) = d_G(C' - Y_1, C' \cap Y_1)$ holds, every component $C \in \mathcal{C} - \{C'\}$ satisfies $C \subseteq Y_1$. From $d_H(s, Y_1) = 2$, it follows that (a) $\mathcal{C} = \{C_1, C_2, C'\}$ and $d_H(s, C_1) = d_H(s, C_2) = d_H(s, C') = 1$ hold, or (b) $\mathcal{C} = \{C_1 = C', C_2\}$ and $d_H(s, C_1) = 2$ and $d_H(s, C_2) = 1$ hold. This contradicts that $|F|$ is even.

We consider the case of (II). By $d_H(Y_1) \leq r(W_i) + 1$, we have $d_H(s, Y_1) \leq 3$. $d_H(s, C' - Y_1) = 2$ implies that $C' \in \mathcal{C} - \{C_1, C_2\}$. From the choice of u_1 and u_2, $d_H(s, C_1) = d_H(s, C_2) \geq d_H(s, C') = 2$ hold. From the choice of v_1, $d_H(s, C_1 \cup C_2 - Y_1) = 0$ holds, from which $d_H(s, Y_1) \geq 4$, a contradiction to $d_H(s, Y_1) \leq 3$. □

PROOF of Lemma 15. We prove the lemma by showing that there is a pair of two edges in F which is admissible in H (note that the resulting graph obtained by an admissible splitting at s also satisfies the assumption of this lemma). Let $(s, u) \in F$. Assume by contradiction that there is no edge $(s, v) \in F$ such that $\{(s, u), (s, v)\}$ is admissible in H. Then we claim that there are three dangerous cuts Y_1, Y_2, and Y_3 with $u \in Y_1 \cap Y_2 \cap Y_3$, $d_H(s, Y_1 - Y_2 - Y_3) > 0$, $d_H(s, Y_2 - Y_3 - Y_1) > 0$, and $d_H(s, Y_3 - Y_1 - Y_2) > 0$. Assume by contradiction that the claim does not hold. Then there is a dangerous cut Y with $F = E_H(s, Y)$ or two dangerous cuts Y_1 and Y_2 with $F = E_H(s, Y_1 \cup Y_2)$, $u \in Y_1 \cap Y_2$, $d_H(s, Y_1 - Y_2) > 0$, and $d_H(s, Y_2 - Y_1) > 0$. The former case (resp., the latter case) would contradict Lemma 18 (resp., Lemma 19).

Then there are the following four possible cases.

(Case-1) $Y_1 \in \mathcal{A}_i, Y_2 \in \mathcal{A}_j$, and $Y_3 \in \mathcal{A}_k$.

(Case-2) $Y_1 \in \mathcal{B}_i, Y_2 \in \mathcal{B}_j$, and $Y_3 \in \mathcal{B}_k$.

(Case-3) $Y_1 \in \mathcal{A}_i, Y_2 \in \mathcal{B}_j$, and $Y_3 \in \mathcal{B}_k$.

(Case-4) $Y_1 \in \mathcal{A}_i, Y_2 \in \mathcal{A}_j$, and $Y_3 \in \mathcal{B}_k$.

Note that in each case, every cut $Y_\ell \in \mathcal{A}_h \cup \mathcal{B}_h$ satisfies $d_H(Y_\ell) \leq r(W_h) + 1$ for some W_h and $d_H(Y_1 \cap Y_2 \cap Y_3, V \cup \{s\} - (Y_1 \cup Y_2 \cup Y_3)) \geq d_H(s, Y_1 \cap Y_2 \cap Y_3)$ $\geq d_H(s, u) > 0$ holds. Also note that we have $Y_1 - Y_2 - Y_3 \neq \emptyset$, $Y_2 - Y_3 - Y_1 \neq \emptyset$, and $Y_3 - Y_1 - Y_2 \neq \emptyset$.

(Case-1) By (9), we have $d_H(Y_1 - Y_2 - Y_3) \geq r(W_i)$, $d_H(Y_2 - Y_3 - Y_1) \geq r(W_j)$, $d_H(Y_3 - Y_1 - Y_2) \geq r(W_k)$, and $d_H(Y_1 \cap Y_2 \cap Y_3) \geq r(W_i)$. From (8), it follows that $r(W_i) + r(W_j) + r(W_k) + 3 \geq \sum_{i=1}^{3} d_H(Y_i) \geq d_H(Y_1 - Y_2 - Y_3) + d_H(Y_2 - Y_3 - Y_1)$ $+ d_H(Y_3 - Y_1 - Y_2) + d_H(Y_1 \cap Y_2 \cap Y_3)$ $+ 2d_H(Y_1 \cap Y_2 \cap Y_3, V \cup \{s\} - (Y_1 \cup Y_2 \cup Y_3))$

$\geq 2r(W_i)+r(W_j)+r(W_k)+2$, contradicting $r(W_i) \geq 2$. This case cannot occur.

(Case-2) Without loss of generality, let $r(W_i) \leq r(W_j) \leq r(W_k)$. By $(Y_1 - Y_2 - Y_3) \cap W_k = \emptyset$, the cut $Y_1 - Y_2 - Y_3$ belongs to $\mathcal{A}_k$ and hence satisfies $d_H(Y_1 - Y_2 - Y_3) \geq r(W_k)$ by (4). Similarly, we have $d_H(Y_2 - Y_3 - Y_1) \geq r(W_k)$ and $d_H(Y_3 - Y_1 - Y_2) \geq r(W_j)$. We have $d_H(Y_1 \cap Y_2 \cap Y_3) \geq d_H(s, Y_1 \cap Y_2 \cap Y_3) \geq 1$. From (8), it follows that $r(W_i) + r(W_j) + r(W_k) + 3 \geq \sum_{i=1}^{3} d_H(Y_i) \geq d_H(Y_1 - Y_2 - Y_3) + d_H(Y_2 - Y_3 - Y_1) + d_H(Y_3 - Y_1 - Y_2) + d_H(Y_1 \cap Y_2 \cap Y_3)$ $+2d_H(Y_1 \cap Y_2 \cap Y_3, V \cup \{s\} - (Y_1 \cup Y_2 \cup Y_3)) \geq r(W_j) + 2r(W_k) + 3$. Hence we have $r(W_i) \geq r(W_k)$. This and $r(W_i) \leq r(W_j) \leq r(W_k)$ imply that $r(W_i) = r(W_j) = r(W_k)$ holds and every inequality turns out to be an equality. Hence, $d_H(Y_1 \cap Y_2 \cap Y_3) = d_H(s, Y_1 \cap Y_2 \cap Y_3) = d_H(s, u) = 1$ holds, from which u is contained in a component C' of G with $d_H(s, C') = 1$. This contradicts the assumption of H.

(Case-3) By (9), we have $d_H(Y_1 - Y_2 - Y_3) \geq r(W_i)$ and $d_H(Y_1 \cap Y_2 \cap Y_3) \geq r(W_i)$. By $(Y_2 - Y_3 - Y_1) \cap W_k = \emptyset$ and (4), we have $d_H(Y_2 - Y_3 - Y_1) \geq r(W_k)$. Similarly, $d_H(Y_3 - Y_1 - Y_2) \geq r(W_j)$ holds. Similarly to Case-1, by $r(W_i) \geq 2$ and (8), this case cannot occur.

(Case-4) There are the following three possible cases (a)–(c) without loss of generality. (a) $r(W_i) \leq r(W_j) \leq r(W_k)$. (b) $r(W_i) \leq r(W_k) \leq r(W_j)$. (c) $r(W_k) \leq r(W_i) \leq r(W_j)$. We show that in each case of (a)–(c), we have $d_H(Y_3 - Y_1 - Y_2) = d_H(s, Y_3 - Y_1 - Y_2) = 1$; $Y_3 - Y_2 - Y_1$ contains a component C'' of G with $d_H(s, C'') = 1$, a contradiction to the assumption of H.

(a) By (9), we have $d_H(Y_1 \cap Y_2 \cap Y_3) \geq r(W_j)$. By $(Y_1 - Y_2 - Y_3) \cap W_k = \emptyset = (Y_2 - Y_3 - Y_1) \cap W_k$, we have $d_H(Y_1 - Y_2 - Y_3) \geq r(W_k)$ and $d_H(Y_2 - Y_3 - Y_1) \geq r(W_k)$. We have $d_H(Y_3 - Y_1 - Y_2) \geq d_H(s, Y_3 - Y_1 - Y_2) \geq 1$. Similarly to Case-2, we have $r(W_i) + r(W_j) + r(W_k) + 3 \geq 2r(W_k) + r(W_j) + 3$ by (8). Hence $r(W_i) \geq r(W_k)$ holds. From this and $r(W_i) \leq r(W_j) \leq r(W_k)$, we have $r(W_i) = r(W_j) = r(W_k)$ and we see that every inequality turns out to be an equality, from which $d_H(Y_3 - Y_1 - Y_2) = d_H(s, Y_3 - Y_1 - Y_2) = 1$ holds.

(b) By (9), we have $d_H(Y_2 - Y_3 - Y_1) \geq r(W_j)$, and $d_H(Y_1 \cap Y_2 \cap Y_3) \geq r(W_j)$. By $(Y_1 - Y_2 - Y_3) \cap W_k = \emptyset$, we have $d_H(Y_1 - Y_2 - Y_3) \geq r(W_k)$. By (8) we have $r(W_i) + r(W_j) + r(W_k) + 3 \geq 2r(W_j) + r(W_k) + 3$, from which $r(W_i) \geq r(W_j)$ holds. Similarly to (a), we have $r(W_i) = r(W_j) = r(W_k)$ and $d_H(Y_3 - Y_1 - Y_2) = d_H(s, Y_3 - Y_1 - Y_2) = 1$.

(c) By (9), we have $d_H(Y_1 - Y_2 - Y_3) \geq r(W_i)$, $d_H(Y_2 - Y_3 - Y_1) \geq r(W_j)$, and $d_H(Y_1 \cap Y_2 \cap Y_3) \geq r(W_j)$. By (8) we have $r(W_i) + r(W_j) + r(W_k) + 3 \geq r(W_i) + 2r(W_j) + 3$, from which $r(W_k) \geq r(W_j)$ holds. Similarly to (a), we have $r(W_i) = r(W_j) = r(W_k)$ and $d_H(Y_3 - Y_1 - Y_2) = d_H(s, Y_3 - Y_1 - Y_2) = 1$. □

In the rest of this section, we will give a proof of Theorem 16 via the following Lemma 22 and Theorem 23. Lemma 22 shows a property of dangerous cuts containing u, v for $\{(s,u),(s,v)\} \subseteq E_H(s,V_2)$. Theorem 23 shows a situation where for an edge $(s,u) \in E_H(s,V_2)$, $\{(s,u),e\}$ is not admissible in H for any edge $e \in E_H(s,V_2)$.

Lemma 22 *Let G, H, and r satisfy* (5). *Let $\{(s,u),(s,v)\}$ be a pair of edges in $E_H(s,V_2)$ which is not admissible in H, and $Y \subset V$ be a dangerous cut with $u, v \in Y$. Then if $Y \cap C \neq \emptyset$ for some $C \in \mathcal{C}_1$ and $V \neq Y \cup C$ holds, then $Y \cup V_1$ is also dangerous.*

PROOF. First we claim that $Y \cup C$ is dangerous. Lemma 20 implies that $Y \in \mathcal{B}_j$ holds for some $W_j \in \mathcal{W}$ because $G[Y]$ is not connected. Assume that $C - Y \neq \emptyset$ holds. Since $C \in \mathcal{A}_i$ holds for some $W_i \in \mathcal{W}$ from the definition of $\mathcal{C}_1$, every $X \subseteq C$ satisfies $d_H(X) \geq r(W_i) \geq 2$. This indicates that $d_H(Y) = d_H(Y \cap C) + d_H(Y - C) \geq 2 + d_H(Y - C) = d_H(Y \cup C)$ holds. From $V \neq Y \cup C$, it follows that $Y \cup C$ is a dangerous cut of $\mathcal{B}_j$.

Let $Y' = Y \cup C$. Note that $d_H(Y') \geq d_H(s,Y') \geq d_H(s,\{u,v\}) + d_H(s,C) \geq 4$ holds. From the definition of dangerous cuts and Lemma 18, it follows that $r(W_j) \geq 3$ and $d_H(s, V - Y') \geq d_H(s,Y') - 1 \geq 3$. $r(W_j) \geq 3$ indicates that any component $C' \in \mathcal{C}_1 - \{C\}$ satisfies $C' \cap Y' \neq \emptyset$, since if $C' \cap Y' = \emptyset$, then $C' \in \mathcal{A}_j$ and $d_H(C') = 2 < r(W_j)$ hold, contradicting that H satisfies (4). Moreover, $d_H(s, V - Y') \geq 3$ implies that $Y' \cup C' \neq V$ holds. Hence, by applying the above claim, we can observe that for each $C' \in \mathcal{C}_1$, $Y' \cup C'$ is also a dangerous cut of $\mathcal{B}_j$ and satisfies $d_H(s, Y' \cup C') \geq 4$.

By repeating those arguments, it is not hard to show that $Y \cup V_1$ is a dangerous cut of $\mathcal{B}_j$. □

Theorem 23 *Let G, H, and r satisfy* (5) *such that neither* (11) *nor* (12) *holds. Let $e_1 = (s,u_1) \in E_H(s,V_2)$ such that u_1 is contained in a component C_1 with $d_H(s,C_1) \geq 2$ (note that such e_1 exists since H does not satisfy* (11)*). Assume that there is no edge $e' \in E_H(s,V_2)$ such that $\{e_1, e'\}$ is admissible in H. Then one of the following statements* (i) *and* (ii) *holds*:

(i) (11) *or* (12) *hold after splitting one admissible pair of edges in $E_H(s,V_2)$.*

(ii) *Exactly one component C of $G[V_2]$ other than C_1 satisfies $d_H(s,C) > 0$ (we denote the component by C_2). Then $d_H(s,C_2) = 1$ holds. C_1 satisfies one of the following (a) and (b) :*

(a) *C_1 is a dangerous cut of $\mathcal{A}_i$ for some $W_i \in \mathcal{W}$.*
(b) *There are two dangerous cuts $Y_1 \in \mathcal{A}_i$ and $Y_2 \in \mathcal{A}_j$ for some $W_i, W_j \in$*

$\mathcal{W}$ such that $u_1 \in Y_1 \cap Y_2$, $d_H(s, Y_1 - Y_2) > 0$, $d_H(s, Y_2 - Y_1) > 0$, $r(W_i) = r(W_j)$, and $C_1 = Y_1 \cup Y_2$.

PROOF. Note that $d_H(s, V_2) = |F| - d_H(s, V_1)$ is even. We can assume that $d_H(s, V_2) \geq 4$ holds, since if $d_H(s, V_2) = 2$ holds, then H satisfies (11) or (12), contradicting the assumption of the theorem. There are the following three possible cases (I)–(III).

(I) There is a dangerous cut Y_1 with $E_H(s, Y_1) \supseteq E_H(s, V_2)$.

(II) (I) does not hold. There are two dangerous cuts Y_1 and Y_2

satisfying $u_1 \in Y_1 \cap Y_2$, $E_H(s, Y_1 \cup Y_2) \supseteq E_H(s, V_2)$,

$d_H(s, V_2 \cap (Y_1 - Y_2)) > 0$, and $d_H(s, V_2 \cap (Y_2 - Y_1)) > 0$.

(III) Neither (I) nor (II) holds.

(I) If $Y_1 \in \mathcal{A}_i$ holds for some $W_i \in \mathcal{W}$, then Lemma 20 implies that V_2 is a component of G and H satisfies (12), a contradiction. So $Y_1 \in \mathcal{B}_i$ holds for some $W_i \in \mathcal{W}$. From $d_H(Y_1) \geq d_H(s, Y_1) \geq d_H(s, V_2) \geq 4$, it follows that $r(W_i) \geq 3$. Hence, $Y_1 \cap C \neq \emptyset$ holds for each $C \in \mathcal{C}_1$ since we have $C \in \mathcal{A}_j$ for some W_j and $d_H(C) = 2 < r(W_i)$. Now Lemma 18 says that $d_H(s, V - Y_1) \geq d_H(s, Y_1) - 1 \geq 3$ holds, from which $V \neq Y_1 \cup C$ holds. Lemma 22 implies that $Y_1 \cup V_1$ is also dangerous, contradicting $d_H(s, V - Y_1 - V_1) = 0$ and Lemma 18.

(II) If $Y_1 \in \mathcal{A}_i$ and $Y_2 \in \mathcal{A}_j$ hold for some $W_i, W_j \in \mathcal{W}$, respectively, then Lemma 20 implies that $V_2 \supseteq Y_1 \cup Y_2$ is a component of G and H satisfies (12), a contradiction. Let Y_1 belong to $\mathcal{B}_i$ for some W_i without loss of generality. Now $V_1 \neq \emptyset$ holds since Lemma 19 indicates that $d_H(s, V - Y_1 - Y_2) > 0$. We claim that $Y_1 \cap V_1 = \emptyset$ holds. This follows since if $C \cap Y_1 \neq \emptyset$ holds for some $C \in \mathcal{C}_1$, then $V_2 - Y_1 \neq \emptyset$ and Lemma 22 indicate that $Y_1 \cup V_1$ is also dangerous, contradicting Lemma 19 and $F = E_H(s, Y_1 \cup Y_2 \cup V_1)$. From $Y_1 \cap V_1 = \emptyset$ and (4), it follows that $r(W_i) = 2$ (note that each $C \in \mathcal{C}_1$ belongs to $\mathcal{A}_\ell$ for some W_ℓ and satisfies $d_H(C) = 2$).

Assume that $Y_2 \in \mathcal{B}_j$ holds for some W_j. Similarly, $r(W_j) = 2$ and $Y_2 \cap V_1 = \emptyset$ hold. From Lemma 21 and $(s, u_1) \in E_H(s, Y_1 \cap Y_2)$, it follows that $d_H(Y_1) = d_H(Y_2) = d_H(Y_1 - Y_2) + 1 = d_H(Y_2 - Y_1) + 1 = 3$, $d_H(s, Y_1 \cap Y_2) = d_H(s, u_1) = 1$, $Y_1 - Y_2 \in \mathcal{A}_j$, and $Y_2 - Y_1 \in \mathcal{A}_i$. By $Y_1 \cup Y_2 \subseteq V - V_1$, neither $Y_1 - Y_2$ nor $Y_2 - Y_1$ belongs to $\mathcal{C}_1$ and hence $d_G(Y_1 - Y_2) > 0$ and $d_G(Y_2 - Y_1) > 0$ hold. This implies that $d_H(s, Y_1 - Y_2) = d_H(s, Y_2 - Y_1) = 1$ hold by $d_H(s, Y_1 - Y_2) > 0$ and $d_H(s, Y_2 - Y_1) > 0$. From this and $d_H(s, Y_1 \cap Y_2) = 1$, $d_H(s, V_2) = d_H(s, Y_1 \cup Y_2) = 3$ holds, contradicting $d_H(s, Y_1 \cup Y_2) \geq 4$.

Assume that $Y_2 \in \mathcal{A}_j$ holds for some W_j. Lemma 20 implies that $Y_2 \cap V_1 = \emptyset$ holds. Note that since (I) does not hold, $Y_1 \cup Y_2$ is not dangerous and so we have $d_H(Y_1 \cup Y_2) \geq 4$ by $r(W_i) = 2$ (note that $Y_1 \cup Y_2 \neq V$ from $V_1 \neq \emptyset$). By (7), we have $3 + r(W_j) + 1 \geq d_H(Y_1) + d_H(Y_2) = d_H(Y_1 \cap Y_2) + d_H(Y_1 \cup Y_2)$ $+2d_H(Y_1 - Y_2, Y_2 - Y_1) \geq 4 + r(W_j)$. It follows that $d_H(Y_1 \cup Y_2) = 4$, $d_H(Y_1) = 3$, $d_H(Y_1 \cap Y_2) = r(W_j)$, and $d_H(Y_1 - Y_2, Y_2 - Y_1) = 0$. $d_H(s, Y_1 \cup Y_2) \geq 4$ implies that $d_G(Y_1 \cup Y_2) = 0$ and $d_H(s, Y_1 \cup Y_2) = 4$ hold. By the connectedness of $G[Y_2]$, we have $d_G(Y_1, Y_2 - Y_1) > 0$. From $d_H(Y_1) = 3$, $d_H(s, Y_1 - Y_2) \geq 1$, and $d_H(s, Y_1 \cap Y_2) \geq 1$, it follows that $d_H(s, Y_1 - Y_2) = d_H(s, Y_1 \cap Y_2) = 1$, $d_G(Y_1) = d_G(Y_1, Y_2 - Y_1) = 1$, and $d_H(s, Y_2 - Y_1) = d_H(s, Y_1 \cup Y_2) - d_H(s, Y_1) = 2$ hold. Now $V[F] \cap V_2$ is not contained in one component of G, since H does not satisfy (12). This implies that $G[V_2]$ contains two components C_1 and C_2 with $u_1 \in C_1$, $Y_2 \subseteq C_1$, $C_2 \subseteq Y_1 - Y_2$, $d_H(s, C_1) = 3$, $d_H(s, C_2) = 1$, $E_H(s, C_1) = E_H(s, Y_2)$, and $E_H(s, C_2) = E_H(s, Y_1 - Y_2)$.

If there is an admissible pair of two edges in $E_H(s, V_2)$, then the resulting graph satisfies (11) or (12), which indicates the statement (i) of the theorem. Assume that no pair of two edges in $E_H(s, V_2)$ is admissible. We then show that $C_1 = Y_2$ holds, which indicates the statement (ii)(a) of the theorem. Assume by contradiction that $Z = C_1 - Y_2 \neq \emptyset$ holds. From $d_G(Y_1 \cup Y_2) = d_G(Y_1 - Y_2, Y_2 - Y_1) = 0$, it follows that $Z \subset Y_1 - Y_2$ and $E_G(Z, Y_2) \subseteq E_G(Y_1 \cap Y_2)$ hold. Note that $d_H(Y_2 - Y_1) = 3$ implies that $r(W_j) \in \{2, 3\}$ holds since $Y_2 - Y_1 \in \mathcal{A}_j$ holds. If $r(W_j) = 2$ holds, then it follows from $d_H(s, Y_2) = 3 = r(W_j) + 1$ that $d_G(Y_2) = 0$ and $Y_2 = C_1$, which would contradict $Z \neq \emptyset$. Hence we have $r(W_j) = 3$ and $d_G(Z) = d_G(Z, Y_1 \cap Y_2) = 1$. Let $(s, u_2) \in E_H(s, Y_2 - Y_1)$ and $\{(s, u_3)\} = E_H(s, C_2)$. Now $\{(s, u_2), (s, u_3)\}$ is not admissible from the assumption. Let Y_3 be a dangerous cut with $\{u_2, u_3\} \subseteq Y_3$. We have $Y_3 \in \mathcal{B}_k$ for some $W_k \in \mathcal{W}$ by Lemma 20. Since (I) does not hold, $d_H(s, Y_2 - Y_3) > 0$ holds. By $E_H(Y_2 \cup Y_3) \supseteq E_H(s, V_2)$, we have $Y_3 \subseteq V - V_1$, $d_H(Y_3) = 3$, $E_H(s, Y_2 \cap Y_3) = \{(s, u_2)\}$, and $d_G(Y_3) = d_G(Y_3, Y_2 - Y_3) = 1$ by a similar argument about Y_1 and Y_2. Now it follows from the connectedness of $G[Y_2]$ and $d_G(Y_2 - Y_1, Y_1 \cap Y_2) = 1$ that $G[Y_2 - Y_1]$ and $G[Y_1 \cap Y_2]$ are both connected. So from $u_2 \in Y_2 - Y_1$, $d_G(Y_3) = d_G(Y_3, Y_2 - Y_3) = 1$, and $E_H(s, Y_2 \cap Y_3) = \{(s, u_2)\}$, it follows that $E_G(Y_3) \subseteq E(G[Y_2 - Y_1])$ and $C_1 \cap Y_3 \subseteq Y_2 - Y_1$ hold. Therefore $Z \cap Y_3 = \emptyset = Z \cap W_k$ holds. From this, it follows that $d_H(Z) = d_G(Z) = 1$ contradicts (4).

(III) Let $\mathcal{Y}$ be the family of all dangerous cuts Y with $u_1 \in Y$ and $E_H(s, Y \cap (V_2 - \{u_1\})) \neq \emptyset$. Since neither (I) nor (II) holds, then there are three dangerous cuts Y_1, Y_2, and Y_3 with $u_1 \in Y_1 \cap Y_2 \cap Y_3$, $d_H(s, (Y_1 - Y_2 - Y_3) \cap V_2) > 0$, $d_H(s, (Y_2 - Y_3 - Y_1) \cap V_2) > 0$, and $d_H(s, (Y_3 - Y_1 - Y_2) \cap V_2) > 0$. Here we choose such three cuts Y_1, Y_2, and Y_3 in $\mathcal{Y}$ satisfying the property that

$d_H(s, Y_1 \cup Y_2 \cup Y_3)$ is the maximum. We have the following four possible cases.

(Case-1) $Y_1 \in \mathcal{A}_i, Y_2 \in \mathcal{A}_j$, and $Y_3 \in \mathcal{A}_k$.

(Case-2) $Y_1 \in \mathcal{B}_i, Y_2 \in \mathcal{B}_j$, and $Y_3 \in \mathcal{B}_k$.

(Case-3) $Y_1 \in \mathcal{A}_i$, $Y_2 \in \mathcal{B}_j$, and $Y_3 \in \mathcal{B}_k$.

(Case-4) $Y_1 \in \mathcal{A}_i$, $Y_2 \in \mathcal{A}_j$, and $Y_3 \in \mathcal{B}_k$.

Similarly to the proof of Lemma 15, observe that neither Case-1 nor Case-3 can occur, and that in both of Case-2 and Case-4, every inequality obtained from (8) by substituting three cuts Y_1, Y_2, and Y_3 turns out to be an equality. In Case-2, we have $d_H(Y_1 \cap Y_2 \cap Y_3) = d_H(s, Y_1 \cap Y_2 \cap Y_3) = d_H(s, u_1) = 1$. This indicates that $d_G(Y_1 \cap Y_2 \cap Y_3) = 0$ holds and the component C' of G containing u_1 satisfies $d_H(s, C') = 1$, contradicting the choice of (s, u_1).

In Case-4, we have $d_H(Y_1 - Y_2 - Y_3) = d_H(Y_2 - Y_3 - Y_1) = d_H(Y_1 \cap Y_2 \cap Y_3) = r(W_i) = r(W_j) = r(W_k)$ and $d_H(Y_3 - Y_1 - Y_2) = d_H(s, Y_3 - Y_1 - Y_2) = 1$. Lemma 20 implies that $Y_1 \cup Y_2$ induces a connected component in G and $Y_1 \cup Y_2 \subseteq V - V_1$ holds. From $d_H(Y_3 - Y_1 - Y_2) = d_H(s, Y_3 - Y_1 - Y_2) = 1$, it follows that $d_G(Y_3 - Y_1 - Y_2) = 0$ and $Y_3 - Y_1 - Y_2 \subseteq V - V_1$ hold, and there is a component C_2 of $G[V_2]$ with $d_H(s, C_2) = 1$. Hence, for proving that the statement (ii)(b) of the theorem holds, it suffices to show that $V - V_1 = Y_1 \cup Y_2 \cup Y_3$ holds (note that $Y_1 \cup Y_2$ corresponds to C_1).

The maximality of $d_H(s, Y_1 \cup Y_2 \cup Y_3)$ and $d_H(s, Y_3 - Y_1 - Y_2) = 1$ means that $d_H(s, Y_1 \cup Y_2)$ is the maximum among all two dangerous cuts Y' and Y'' with $u_1 \in Y' \cap Y''$, $d_H(s, V_2 \cap (Y' - Y'')) > 0$, and $d_H(s, V_2 \cap (Y'' - Y')) > 0$ (note that neither (I) nor (II) holds). Now we can see the following claim.

Claim 24 *For three cuts Y_1, Y_2, $Y_3 \in \mathcal{Y}$ such that $d_H(s, Y_3 - Y_1 - Y_2) > 0$ and $d_H(s, Y_1 \cup Y_2) \geq \max\{d_H(s, Y_1 \cup Y_3), d_H(s, Y_2 \cup Y_3)\}$, we have*

$$d_H(s, Y_1 - Y_2 - Y_3) > 0, \; d_H(s, Y_2 - Y_1 - Y_3) > 0, \; d_H(s, Y_3 - Y_1 - Y_2) > 0. \tag{13}$$

□

Assume by contradiction that $V - V_1 - Y_1 - Y_2 - Y_3 \neq \emptyset$ holds. We first show the following claim.

Claim 25 $E_H(s, V - V_1 - (Y_1 \cup Y_2 \cup Y_3)) = \emptyset$ *holds.*

PROOF. Assume by contradiction that there is an edge $(s, v_4) \in E_H(s, V - V_1 - (Y_1 \cup Y_2 \cup Y_3))$. Let Y_4 be the corresponding dangerous cut in $\mathcal{Y}$ with

$\{u_1, v_4\} \subseteq Y_4$. Assume that $Y_4 \in \mathcal{A}_\ell$ holds for some ℓ. From the maximality of $d_H(s, Y_1 \cup Y_2)$ and Claim 24, the cuts Y_1, Y_2, and Y_4 satisfy Case-1, which cannot occur. Assume that $Y_4 \in \mathcal{B}_\ell$ holds for some W_ℓ. From the maximality of $d_H(s, Y_1 \cup Y_2)$ and Claim 24, the cuts Y_1, Y_2, and Y_4 satisfy Case-4. Then we have $d_G(Y_4 - Y_1 - Y_2) = 0$ and $d_H(Y_4 - Y_1 - Y_2) = 1$, implying that $Z = Y_4 - Y_1 - Y_2 - (Y_3 - Y_1 - Y_2)$ satisfies $Z \cap W_k = \emptyset$ but $d_H(Z) = 1$, a contradiction to (4) and $r(W_k) \geq 2$. Therefore we have $E_H(s, V - V_1 - (Y_1 \cup Y_2 \cup Y_3)) = \emptyset$. □

From $V - V_1 - Y_1 - Y_2 - Y_3 \neq \emptyset$, $d_G(Y_3 - Y_1 - Y_2) = 0$, and $d_G(V_1) = 0$, it follows that $d_H(V - V_1 - Y_1 - Y_2 - Y_3) = d_G(V - V_1 - Y_1 - Y_2 - Y_3) = d_G(Y_1 \cup Y_2)$. By $(V - V_1 - Y_1 - Y_2 - Y_3) \cap W_k = \emptyset$ and (4), $d_H(V - V_1 - Y_1 - Y_2 - Y_3) \geq r(W_k)$ holds, from which $d_G(Y_1 \cup Y_2) \geq r(W_k) = r(W_i)$ holds (note that $r(W_i) = r(W_j) = r(W_k)$). By $d_H(s, Y_1 - Y_2) > 0$, $d_H(s, Y_2 - Y_1) > 0$, and $d_H(s, Y_1 \cap Y_2) > 0$, we have $d_H(s, Y_1 \cup Y_2) \geq 3$, from which $d_H(Y_1 \cup Y_2) \geq r(W_i) + 3$ holds. (9) implies $d_H(Y_1 \cap Y_2) \geq r(W_i)$ (note that $Y_1 \in \mathcal{A}_i$ and $Y_2 \in \mathcal{A}_j$ hold). By (7), we have $2(r(W_i)+1) \geq d_H(Y_1) + d_H(Y_2) \geq d_H(Y_1 \cap Y_2) + d_H(Y_1 \cup Y_2) \geq r(W_i) + r(W_i) + 3$, a contradiction. □

PROOF of Theorem 16. Let $e_1 = (s, u_1) \in E_H(s, V_2)$ be an arbitrary edge such that u_1 is contained in a component C_1 of $G[V_2]$ with $d_H(s, C_1) \geq 2$ (note that such e_1 exists since H does not satisfy (11)). Since no pair of two edges in $E_H(s, V_2)$ is admissible, it follows that the statement (ii) of Theorem 23 holds for the edge (s, u_1). Hence, it follows that there are exactly two components C_1 and C_2 with $d_H(s, C_i) > 0$, and that the statement (b) of the theorem holds.

We show that the statement (a) of the theorem holds. We can observe directly from the proof of Theorem 23 that $d_H(s, C_1) \geq 3$ holds.

We next show that every cut $X \subseteq C_1$ satisfies $d_H(X) \geq 2$. Assume by contradiction that there is a cut $X \subseteq C_1$ with $d_H(X) = 1$. Since each $W \in \mathcal{W}$ satisfies $r(W) \geq 2$, $X \notin \mathcal{A}_k \cup \mathcal{B}_k$ holds for any W_k. Hence, it follows that C_1 satisfies the statement in Theorem 23 (ii)(b); $C_1 = Y_1 \cup Y_2$ where Y_1, Y_2 denote two dangerous cuts with $Y_1 \in \mathcal{A}_j$ and $Y_2 \in \mathcal{A}_k$ for some W_j, W_k. Moreover, $X - Y_1 \neq \emptyset \neq X - Y_2$ holds. Now by applying Lemma 21 to Y_1 and Y_2, we have $d_H(Y_1 - Y_2) = r(W_j)$ and $d_H(Y_2 - Y_1) = r(W_k)$, $Y_1 - Y_2 \in \mathcal{A}_j$, and $Y_2 - Y_1 \in \mathcal{A}_k$. Lemma 20 implies that $G[Y_1 - Y_2]$ and $G[Y_2 - Y_1]$ are both connected. Then it is not difficult to see that $d_H(X) = 1$ would contradict the connectedness of $G[Y_1 - Y_2]$ and $G[Y_2 - Y_1]$.

We finally show that every cut $X \subseteq C_1$ with $d_H(X) = 2$ belongs to $\mathcal{A}_i$ for some $W_i \in \mathcal{W}$. Assume by contradiction that $X \subseteq C_1$ does not belong to $\mathcal{A}_i$ for any $W_i \in \mathcal{W}$. Hence, it follows that C_1 satisfies the statement in Theorem 23 (ii)(b); $C_1 = Y_1 \cup Y_2$ where Y_1 and Y_2 denote two dangerous cuts of $\mathcal{A}_j$ and $\mathcal{A}_k$ for some W_j and W_k. Moreover, we have $X - Y_1 \neq \emptyset \neq X - Y_2$. By $d_H(X) = 2$ and $d_H(C_1) \geq 3$, we have $C_1 - X \neq \emptyset$. Since $G[C_1]$ is connected, it follows that $d_G(X) \geq 1$ holds, from which $d_H(s, X) \leq 1$ holds. This implies that X and Y_1 cross each other in H. From (6) and $X - Y_1 \subseteq Y_2$, we have $r(W_j) + 1 + 2 \geq d_H(Y_1) + d_H(X) = d_H(Y_1 - X) + d_H(X - Y_1) + 2d_H(X \cap Y_1, V \cup \{s\} - X - Y_1) \geq r(W_j) + r(W_k)$. From $r(W_k) \geq 2$ and $d_H(X - Y_1) \geq r(W_k) = r(W_j)$, it follows that $d_H(X \cap Y_1, V \cup \{s\} - X - Y_1) = 0$ and $d_H(Y_1 - X) \leq 3$. Hence $Y_1 - Y_2 - X \neq \emptyset \neq (Y_1 \cap Y_2) - X$ holds by $d_H(s, Y_1 - Y_2) > 0$ and $d_H(s, Y_1 \cap Y_2) > 0$. By these and $X \cap (Y_1 - Y_2) \neq \emptyset$, $Y_1 - X$ and $Y_1 - Y_2$ cross each other in H. From (6) and $d_H(Y_1 - X) \leq 3$, it follows that $d_H(Y_1 - Y_2) + 3 \geq d_H(Y_1 - Y_2) + d_H(Y_1 - X) \geq d_H((Y_1 - Y_2) \cap X) + d_H(Y_1 \cap Y_2 - X) + 2d_H(s, Y_1 - Y_2 - X) \geq r(W_j) + r(W_k) + 2$. We have $d_H(Y_1 - Y_2) \geq r(W_j) + 1$ by $r(W_k) \geq 2$. Now by applying Lemma 21 to Y_1 and Y_2, we have $d_H(Y_1 - Y_2) = r(W_j)$, a contradiction. □

6.2 Hooking up operations

We show via the following two lemmas that if at least one of conditions (I)–(III) in Theorem 10 does not hold, then there is a complete admissible splitting at s.

Lemma 26 *Let G, H, and r satisfy* (5)*. If one of the following* (*i*)*–*(*iii*) *holds, then we can continue admissible edge-splittings at s until isolating s.*
(*i*) *Every component C of $G[V_2]$ satisfies $d_H(s, C) \geq 2$.*
(*ii*) *There is exactly one component C' of $G[V_2]$ with $d_H(s, C') = 1$ where $\{(s, u)\} = E_H(s, C')$ holds. $\{(s, u), (s, v)\}$ is admissible for some $(s, v) \in E_H(s, V_2) - \{(s, u)\}$.*
(*iii*) *There are at least two components C of $G[V_2]$ with $d_H(s, C) = 1$.* □

Lemma 27 *Let G, H, and r satisfy* (5) *such that there is exactly one component C' of $G[V_2]$ with $d_H(s, C') = 1$ where $\{(s, u)\} = E_H(s, C')$ holds. If u is contained in no critical cut of $\mathcal{A}_i$ for any area W_i in H, then after replacing the edge (s, u) with a new edge (s, x) for some vertex $x \in V_1 \cup V_2 - C'$, we can continue admissible edge-splittings until isolating s.* □

For proving these lemmas, we first consider a situation where no admissible pair exists after a sequence of greedy admissible splittings for a given G and H. Then we consider *hooking up* some split edge and resplitting some pair of edges in order to attain a complete admissible splitting. We say that H' is obtained from H by *hooking up* an edge $(u, v) \in E(H - s)$ at s, if we construct H' by replacing the edge (u, v) with two edges (s, u) and (s, v) in H. Even in the case of $opt(G, \mathcal{W}, r) = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$, a greedy splitting in Step 2 of algorithm r-NAEC-AUG may not construct an optimal solution unless hooking up operations are used (see Figure 5). Let $B(G)$ denote the set of

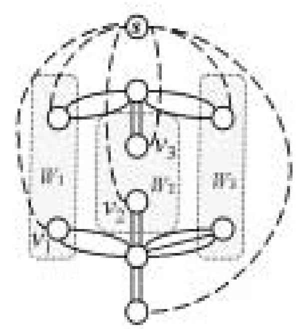

Fig. 5. Illustration of a graph $H = (V \cup \{s\}, E \cup F_1)$ satisfying (4) for $r(W_1) = r(W_2) = r(W_3) = 3$, where edges in F_1 are drawn by broken lines. If we first execute the admissible splitting of (s, v_2) and (s, v_3), then a complete splitting can be found. However, the resulting graph $H_1 = (H - \{(s, v_1), (s, v_2)\}) \cup \{(v_1, v_2)\}$ obtained from H by the admissible splitting of (s, v_1) and (s, v_2) has no admissible splitting pair at s.

bridges in a graph G.

Consider a situation where some pairs of two edges in F have been split and those split edges can be hooked up, defined as follows. Let G, H, and r satisfy (5). Let $\mathcal{H}_1$ be the family of all graphs H^* obtained from H by a sequence of admissible splittings of two edges in $E_H(s, V_2)$ such that no pair of two edge in $E_{H^*}(s, V_2)$ is admissible in H^* and H^* satisfies neither (11) nor (12). Note that $d_{H^*}(s, V_2) > 0$ holds since H^* does not satisfy (12). Let $F(H^*) = E_{H^*}(s, V)$ and $E_1(H^*)$ be the set of all split edges in $H^* \in \mathcal{H}_1$. By Theorem 16, $H^*[V_2]$ has exactly two components C with $d_{H^*}(s, C) > 0$. Let $C_1(H^*)$ (resp., $C_2(H^*)$) denote the component of $H^*[V_2]$ with $d_{H^*}(s, C_1(H^*)) \geq 3$ (resp., $d_{H^*}(s, C_1(H^*)) = 1$), corresponding to C_1 (resp., C_2) in the statement of Theorem 16.

The next two Lemmas 28 and 29 show situations where re-splittings are available in $H^* \in \mathcal{H}_1$. In particular, Lemma 29 shows cases where we can find a complete admissible splitting at s in $H^* \in \mathcal{H}_1$ after hooking up one split edge; H has a complete admissible splitting.

Lemma 28 *For a graph $H^* \in \mathcal{H}_1$, let $\{(s, u_2)\} = E_{H^*}(s, C_2(H^*))$. Assume that $H^*[V - V_1 - C_1(H^*)]$ has a split edge (v_1, v_2). Then in the graph H' obtained from H^* by hooking up the edge (v_1, v_2), there is an admissible pair $\{(s, u_1), (s, z)\}$ in H' for some $(s, u_1) \in E_{H^*}(s, C_1(H^*))$ and some $(s, z) \in \{(s, u_2), (s, v_1), (s, v_2)\}$.*

PROOF. From the definition of $\mathcal{H}_1$, no pair of two edges in $E_{H^*}(s, V - V_1)$ is admissible. It follows that the statement (ii) in Theorem 23 holds for any edge $(s, u) \in E_{H^*}(s, C_1(H^*))$; for any pair $\{(s, u), (s, v)\} \subseteq E_{H^*}(s, C_1(H^*))$ of two edges, there is a dangerous cut $Y \subseteq C_1(H^*)$ with $\{u, v\} \subseteq Y$.

Such Y is dangerous also in H', and it follows that no pair $\{(s, u), (s, v)\} \subseteq E_{H'}(s, C_1(H^*)) = E_{H^*}(s, C_1(H^*))$ is admissible also in H'. Assume by contradiction that this lemma does not hold; H' has no admissible pair of two edges in $E_{H'}(s, V - V_1)$, except $\{(s, u_2), (s, v_1)\}$, $\{(s, v_1), (s, v_2)\}$, and $\{(s, v_2), (s, u_2)\}$. Hence, an edge $(s, u_1) \in E_{H^*}(s,\ C_1(H^*))$ satisfies the assumption in Theorem 23. On the other hand, from $d_{H'}(s, V - V_1 - C_1(H^*)) = 3$, it follows that the statement (ii) in Theorem 23 for the edge (s, u_1) does not hold. Now $d_{H'}(s, C_1(H^*)) \geq 3$ holds and $d_{H'}(s, C_1(H^*))$ is odd. From these and the assumption, it is not difficult to see that the statement (i) in Theorem 23 for the edge (s, u_1) does not hold, a contradiction. $\square$

Lemma 29 *For a graph $H^* \in \mathcal{H}_1$, let $C_2(H^*)$ contain a split edge in $E_1(H^*)$. Then we can continue admissible edge-splittings at s until isolating s, after hooking up one split edge in $E_1(H^*)$.*

PROOF. Let $C_1 = C_1(H^*)$, $C_2 = C_2(H^*)$, and E_2 be the set of all split edges in $H^*[C_2]$. Let $\{(s, v_2)\} = E_{H^*}(s, C_2)$. There are the following two possible cases (i) and (ii). (i) $E_2 - B(H^*[C_2]) \neq \emptyset$ holds. (ii) $E_2 \subseteq B(H^*[C_2])$ holds.

(i) Let $e_1 = (u_1, u_2) \in E_2 - B(H^*[C_2])$, and H_1 be the graph obtained from H^* by hooking up the edge e_1. Lemma 28 implies that some pair $\{e, e'\}$ is admissible for $e \in E_{H_1}(s, C_1)$ and $e' \in E_{H_1}(s, C_2)$. Let H_2 be the graph obtained from H_1 by splitting such two edges e and e'. Since $H_1[C_2]$ is connected from the choice of e_1, $H_2[C_1 \cup C_2]$ is connected. Now $d_{H_2}(s, C_1 \cup C_2) = d_{H_2}(s, V) - d_{H_2}(s, V_1)$ is even. So Lemma 15 proves the lemma.

(ii) Let $e_1 = (u_1, u_2) \in E_2$ be an edge. Let $\{X_1, X_2\}$ be the partition of C_2 such that $E_{H^*}(X_1, X_2) = \{(u_1, u_2)\}$, $u_1 \in X_1$ and $\{u_2, v_2\} \subseteq X_2$. Let H_1 be the graph obtained from H^* by hooking up the edge e_1. Lemma 28 implies that some pair $\{e, e'\}$ is admissible for $e = (s, v_1) \in E_{H_1}(s, C_1)$ and $e' = (s, z) \in E_{H_1}(s, C_2)$. Let H_2 be the graph obtained from H_1 by splitting two edges (s, v_1) and (s, z). If $z = u_1$ holds, then H_2 satisfies (12), proving the lemma.

We claim that also in the case of $z \in \{u_2, v_2\}$, we can continue admissible edge-splittings until isolating s, which proves the lemma. Assume by contradiction that $z \in \{u_2, v_2\}$ holds and we cannot obtain a graph satisfying (11) or (12) by a sequence of admissible splittings of two edges in $E_{H_2}(s, V - V_1)$. Hence, by a sequence of admissible splittings of two edges in $E_{H_2}(s, V - V_1)$, we obtain a graph $H_3 \in \mathcal{H}_1$. Then $d_{H_3}(X_2) = 2$, $X_2 \subseteq C_1(H_3)$, and $E_{H_3}(X_2) \subseteq F(H_3) \cup E_1(H_3)$ hold. By Theorem 16, we can observe from $d_{H_3}(X_2) = 2$ that X_2 belongs to $\mathcal{A}_i$ for some W_i.

If X_2 contains no split edge, then it follows from $E_{H_3}(X_2) \subseteq F(H_3) \cup E_1(H_3)$ that X_2 is the component of G with $d_H(s, X_2) = d_H(X_2) = 2$ and $X_2 \in \mathcal{C}_1$, which contradicts the construction of H_3.

Consider the case where X_2 contains a split edge. Note that each of such split edges belongs to $B(H^*[C_2])$ from the assumption. On the other hand, by Theorem 16, $d_{H_3}(X) \geq 2$ holds for each cut $X \subseteq C_1(H_3)$. Therefore, it is not difficult to see that there is a cut $X' \subset X_2$ such that $E_{H_3}(X') \subseteq F(H_3) \cup E_1(H_3)$ and $d_{H_3}(X') = 2$ hold and $H_3[X']$ has no split edge. Hence, X' is the component of G with $d_H(s, X') = d_H(X') = 2$. By $X_2 \in \mathcal{A}_i$, it follows that $X' \in \mathcal{A}_i$. Therefore, it follows that $X' \in \mathcal{C}_1$ holds, contradicting the construction of H_3. $\Box$

Based on these observation, we give proofs of Lemmas 26 and 27.

PROOF of Lemma 26. (i) Assume that we obtain a graph $H_1 \in \mathcal{H}_1$ from H by a sequence of admissible splittings of two edges in $E_H(s, V_2)$ (otherwise the resulting graph H' obtained from H by splitting all edges in $E_H(s, V_2)$ satisfies (11) and has a complete admissible splitting at s). Then $C_2(H_1)$ contains a split edge since every component C of $G[V_2]$ satisfies $d_H(s, C) \geq 2$. By Lemma 29, we can continue admissible splittings at s until isolating s after hooking up one split edge.

(ii) Let H_1 be the graph obtained from H by splitting edges (s, u) and (s, v). Assume that we obtain a graph $H_2 \in \mathcal{H}_1$ from H_1 by a sequence of admissible splittings of two edges in $E_{H_1}(s, V - V_1)$ (otherwise the lemma is proved, similarly to (i)). Also in this case, $C_2(H_2)$ contains a split edge, and Lemma 29 indicates that the lemma is proved.

(iii) Assume that we obtain a graph $H_1 \in \mathcal{H}_1$ from H by a sequence of admissible splittings of two edges in $E_H(s, V_2)$ (otherwise the lemma is proved). If $C_2(H_1)$ contains a split edge, then Lemma 29 indicates that the lemma is proved.

Assume that $C_2(H_1)$ contains no split edge; $C_2(H_1)$ is the component of $G[V_2]$ satisfying $d_H(s, C_2(H_1)) = 1$. Let C' and C'' be two distinct components of $G[V_2]$ with $C' = C_2(H_1)$ and $d_H(s, C') = d_H(s, C'') = 1$, where $\{(s, u_1)\} = E_H(s, C')$ and $\{(s, u_2)\} = E_H(s, C'')$. Then it follows from $d_{H_1}(s, C_1(H_1)) \geq 3$ that u_2 is an end vertex of some split edge $e_2 = (u_2, v_2)$ in H_1. Note that $u_2 \notin C_1(H_1)$ holds, since otherwise $C'' \subset C_1(H_1)$ and $d_{H_1}(C'') = 1$ hold by

$d_H(s, C'') = 1$, contradicting the statement (a) in Theorem 16. So in the graph H_2 obtained from H_1 by hooking up the edge e_2, there are three components C', C'', and C_3 of $H_2[V - V_1]$ with $d_{H_2}(s, C') = d_{H_2}(s, C'') = d_{H_2}(s, C_3) = 1$ and $E_{H_2}(s, C_3) = \{(s, v_2)\}$. Lemma 28 implies that $\{(s, v'), (s, z)\}$ is admissible for some $v' \in C_1(H_1)$ and some $z \in \{u_1, u_2, v_2\}$. Let H_3 be the graph obtained from H_2 by splitting two edges (s, v') and (s, z).

Finally we claim that we can continue admissible edge-splittings until isolating s in H_3, which proves the lemma. Assume by contradiction that by a sequence of admissible splitting of two edges in $E_{H_3}(s, V - V_1)$, we obtain a graph $H_4 \in \mathcal{H}_1$. Then it follows that $C' \subseteq C_1(H_4)$, $C'' \subseteq C_1(H_4)$, or $C_3 \subseteq C_1(H_4)$ hold. In each case, $d_{H_4}(C') = d_{H_4}(C'') = d_{H_4}(C_3) = 1$ would contradict the statement (a) in Theorem 16. □

PROOF of Lemma 27. Assume that u is contained in no critical cut of $\mathcal{A}_i$ for any area $W_i \in \mathcal{W}$ in H. Let X_u denote a critical cut of $\mathcal{B}_j$ for an area $W_j \in \mathcal{W}$ satisfying $u \in X_u \subset V$ such that no cut $X' \subset X_u$ with $u \in X'$ is critical of $\mathcal{B}_h$ for any h if exists, $X_u = V$ otherwise. Then $X_u \cap (V_1 \cup V_2 - C') \neq \emptyset$ holds since otherwise $(V - V_1 - V_2) \cup C'$ belongs to $\mathcal{B}_j$ and hence $d_H((V - V_1 - V_2) \cup C') \geq r(W_j) \geq 2$ holds by (4), contradicting $d_H((V - V_1 - V_2) \cup C') = d_H(s, C') = 1$. Let $H_1 = (H - \{(s, u)\}) \cup \{(s, x)\}$ be a graph obtained from H by replacing the edge (s, u) with (s, x) with some $x \in X_u \cap (V_1 \cup V_2 - C')$ in H.

We claim that H_1 also satisfies (4). Assume by contradiction that H_1 violates (4). Then H has a critical cut $X' \subset V$ with $u \in X' \cap X_u$ and $x \in X_u - X'$. Note that $X' \in \mathcal{B}_\ell$ holds for an area W_ℓ from the assumption of u. We have $X' - X_u \neq \emptyset$ from the minimality of X_u and hence X_u and X' cross each other in H. Lemma 12 says $d_H(s, X_u \cap X') = 0$, contradicting $u \in X_u \cap X'$.

Let $C'' \subseteq V_1 \cup V_2 - C'$ be the component of G with $x \in C''$. By the assumption, $d_H(s, C'') \geq 2$ holds and hence $d_{H_1}(s, C'') \geq 3$ holds. Since H_1 satisfies (i) in Lemma 26, the lemma is proved. □

6.3 Property (P)

In this section, we prove that G has property (P) if all statements (I) – (III) of Theorem 10 hold. For this, we show that if $H = (V \cup \{s\}, E \cup F)$ with $F = E_H(s, V)$ belongs to the family $\mathcal{H}_2$ of graphs defined as follows, then $H - s = (V, E)$ has property (P). Let $\mathcal{H}_2$ be the family of all graphs H such that G, H, and r satisfy (5) and the following (I)–(III).

(I) There is exactly one component C^* of $G[V_2]$ with $d_H(s, C^*) = 1$ where $E_H(s, C^*) = \{(s, u^*)\}$.

(II) The vertex u^* is contained in a critical cut of $\mathcal{A}_i$ for some area $W_i \in \mathcal{W}$.

(III) $\{(s, u^*), e\}$ is not admissible in H for any edge $e \in E_H(s, V_2)$.

By (III), for each $(s, v) \in E_H(s, V_2 - C^*)$ there is a dangerous cut Y with $\{u^*, v\} \subseteq Y$, which will play a role as a cut Y_X in Definition 6 in the subsequent arguments. We first show properties of such dangerous cuts in Lemma 30, and show by Lemma 31 that for $H \in \mathcal{H}_2$, G has property (P).

Lemma 30 *For a graph $H \in \mathcal{H}_2$, let $(s, v) \in E_H(s, V_2 - C^*)$ and Y_v be a dangerous cut with $\{u^*, v\} \subseteq Y_v$ (such Y_v exists by the property (III) of $\mathcal{H}_2$). Then*
(i) $d_H(s, V_2 - Y_v) \geq 1$ holds.
(ii) For some $(s, w) \in E_H(s, V_2 - C^*) - \{(s, v)\}$, Y_v and Y_w cross each other in H, where Y_w denotes a dangerous cut with $\{u^*, w\} \subseteq Y_w$ in H. Moreover, $v \in Y_v - Y_w$ and $Y_v \subset V - V_1$ hold and $Y_v - Y_w$ is a critical cut of $\mathcal{A}_i$ for some $W_i \in \mathcal{W}$.
(iii) $Y_v \cup C^*$ is also dangerous.

PROOF. Note that $Y_v \in \mathcal{B}_i$ holds for some i by Lemma 20. Also note that $d_H(s, V_2) \geq 4$ holds since $|F|$ and $d_H(s, V_1)$ are even and the property (I) of $\mathcal{H}_2$ holds.

(i) Assume that $d_H(s, V_2 - Y_v) = 0$ holds. Hence $d_H(Y_v) \geq d_H(s, Y_v) \geq d_H(s, V_2) \geq 4$ holds. So we have $r(W_i) \geq 3$. Hence, each $C \in \mathcal{C}_1$ satisfies $C \cap Y_v \neq \emptyset$ since $d_H(C) = 2 < r(W_i)$ holds (note that C belongs to $\mathcal{A}_\ell$ for some W_ℓ). Moreover, each $C \in \mathcal{C}_1$ satisfies $C \cup Y_v \neq V$, since otherwise $d_H(s, Y_v) \geq 4$ and $d_H(s, C - Y_v) \leq 2$ would contradict Lemma 18. Lemma 22 says that $Y_v \cup V_1$ is also dangerous. It follows that $d_H(s, V - (Y_v \cup V_1)) = d_H(s, V_2 - Y_v) = 0$, contradicting Lemma 18.

(ii) Let Y'_v be a dangerous cut with $\{u^*, v\} \subseteq Y'_v$ and $Y'_v \supseteq Y_v$ such that no $Y \supset Y'_v$ is dangerous in H. By (i), $d_H(s, V_2 - Y'_v) > 0$ holds. Let $w \in V_2 - Y'_v$ be a vertex with $d_H(s, w) > 0$ and Y_w be a dangerous cut with $\{u^*, w\} \subseteq Y_w$. Then Y'_v and Y_w cross each other in H since we have $u^* \in Y'_v \cap Y_w$, $w \in Y_w - Y'_v$, and $Y'_v - Y_w \neq \emptyset$ by the maximality of Y'_v. Note that $Y_w \in \mathcal{B}_j$ holds for some $W_j \in \mathcal{W}$. Lemma 21 implies that $d_H(s, Y'_v \cap Y_w) = 1$ holds, and it follows from $u^* \in Y'_v \cap Y_w$ that $v \in Y_v - Y_w$. This implies that Y_v and Y_w also cross each other in H.

Again by Lemma 21, $Y_v - Y_w$ is a critical cut of $\mathcal{A}_j$ and $G[Y_v - Y_w]$ is connected by Lemma 20. Similarly, $G[Y_w - Y_v]$ is connected, from which $(Y_v - Y_w) \cup (Y_w - Y_v) \subseteq V_2$ holds. Finally, we prove that $Y_v \cap Y_w \cap V_1 = \emptyset$ holds in order to show that $Y_v \subset V - V_1$ holds (note that $V - V_1 - Y_v \neq \emptyset$ holds by $d_H(s, V_2 - Y_v) > 0$). Assume by contradiction that $Y_v \cap Y_w \cap C \neq \emptyset$ holds for some $C \in \mathcal{C}_1$. From

$d_H(s, V_2 - Y_v) > 0$, $d_H(s, V_2 - Y_w) > 0$, and Lemma 22, it follows that $Y_v \cup V_1$ and $Y_w \cup V_1$ are dangerous cuts of $\mathcal{B}_i$ and $\mathcal{B}_j$, respectively, and cross each other in H. $d_H(s, (Y_v \cap Y_w) \cup V_1) \geq 3$ would contradict Lemma 21.

(iii) Let $Y_v'' = Y_v \cup C^*$. By (i) and $u^* \in Y_v$, we have $d_H(s, V - Y_v'') \geq 1$. Hence $V - Y_v'' \neq \emptyset$ implies that Y_v'' also belongs to $\mathcal{B}_i$. By $E_H(s, C^*) \subseteq E_H(s, Y_v)$ and $d_H(s, C^*) = d_H(C^*)$, we have $d_H(Y_v'') \leq d_H(Y_v)$, which proves the lemma. □

Lemma 31 *For each graph $H \in \mathcal{H}_2$, G has property (P).*

PROOF. Lemma 30 implies that for each $v \in V[F] - V_1 - \{s, u^*\}$, there are two cuts $X_v \subset V - V_1$ and $Y_v \subset V - V_1$ with $v \in X_v \subseteq Y_v$ satisfying the following (a) and (b).

(a) X_v is a critical cut of $\mathcal{A}_i$ for some area $W_i \in \mathcal{W}$, and no cut $X' \subset X_v$ with $v \in X'$ satisfies this property.

(b) Y_v satisfies $u^* \in Y_v$ and $C^* \subseteq Y_v \subset V - V_1$ (by (ii)(iii) in Lemma 30) and is a dangerous cut of $\mathcal{B}_k$ for some area $W_k \in \mathcal{W}$.

Let X_{u^*} be a critical cut of $\mathcal{A}_i$ for some W_i with $u^* \in X_{u^*}$ such that no cut $X' \subset X_{u^*}$ satisfies this property (such X_{u^*} exists from the property (II) of $\mathcal{H}_2$). Note that X_{u^*} induces a connected component by Lemma 20, and it follows that we have $X_{u^*} \subseteq C^*$ and $X_{u^*} \cap X_v = \emptyset$ for any $v \in V[F] - V_1 - \{s, u^*\}$. Let $\mathcal{X}$ be the family of all cuts X_v, $v \in V[F] - \{s\} - V_1$ such that $\mathcal{X}$ covers $V[F] - \{s\} - V_1$ and $X_v \in \mathcal{X}$ does not satisfy $X_v \subset X$ for any $X \in \mathcal{X}$, and $\mathcal{Y}$ be the family of the corresponding cuts Y_v. We will show that $\alpha(G, \mathcal{W}, r)$ is even and the family $\mathcal{X} \cup \mathcal{C}_1$ is a subpartition of V satisfying $\sum_{X \in \mathcal{X} \cup \mathcal{C}_1} (\alpha_{G,\mathcal{W},r}(X)) = \alpha(G, \mathcal{W}, r)$ and (P1)–(P3), which proves the lemma.

We claim that

$$\mathcal{X} \text{ is a subpartition of } V - V_1. \tag{14}$$

Assume by contradiction that there are two cuts $X_u, X_v \in \mathcal{X}$ which cross each other in H. By Lemma 12(i), we have $d_H(X_u - X_v) = r(W_j)$, $d_H(X_v - X_u) = r(W_k)$, and $d_H(X_u \cap X_v, (V \cup \{s\}) - X_u - X_v) = 0$, where $X_u \in \mathcal{A}_j$ and $X_v \in \mathcal{A}_k$ hold. Hence $u \in X_u - X_v$ holds and $X_u - X_v$ is also a critical cut of $\mathcal{A}_j$, contradicting the minimality of X_u.

Now each $C \in \mathcal{C}_1$ is a critical cut of $\mathcal{A}_i$ for some $W_i \in \mathcal{W}$, since it follows from $d_H(C) = 2$, (4), and $r(W) \geq 2$ for each $W \in \mathcal{W}$ that $C \in \mathcal{A}_i$ holds for some $W_i \in \mathcal{W}$ with $r(W_i) = 2$. Hence, by (14), $\mathcal{X} \cup \mathcal{C}_1$ is a subpartition of V and a family of critical cuts which covers $V[F] - \{s\}$. It follows that

$\sum_{X\in\mathcal{X}\cup\mathcal{C}_1}\alpha_{G,\mathcal{W},r}(X)=|F|=\alpha(G,\mathcal{W},r)$. Since $|F|$ is even, $\alpha(G,\mathcal{W},r)$ is even. Moreover, $\mathcal{X}\cup\mathcal{C}_1$ is a subpartition of V satisfying (P1) and (P2) by taking $X^*=X_{u^*}$. Now for every dangerous cut $Y\in\mathcal{Y}$ of $\mathcal{B}_j$ which does not cross with any $X\in\mathcal{X}$ in H, we have $\sum_{X'\in\mathcal{X},X'\subseteq Y}\alpha_{G,\mathcal{W},r}(X')\le(r(W_j)+1)-d_G(Y)$. Moreover, note that each $Y\in\mathcal{Y}$ is disjoint with any cut $C\in\mathcal{C}_1$ and satisfies $V-V_1-Y\neq\emptyset$. Therefore, by regarding $\mathcal{C}_1$ as $\mathcal{X}_1$ in Definition 6, in order to show that $\mathcal{X}\cup\mathcal{C}_1$ satisfies (P3), it suffices to prove that for any $X_u\in\mathcal{X}$ with $u\neq u^*$, there is a cut $Y_w\in\mathcal{Y}$ with $X_u\subseteq Y_w$ such that for any cut $X\in\mathcal{X}$, Y_w and X do not cross each other in H. For this, we show that

$$\begin{aligned}&\text{if there is a cut } Y_u\in\mathcal{Y} \text{ which crosses with some } X_v\in\mathcal{X} \text{ in } H,\\&\text{then } v\neq u^* \text{ and } Y_u\subseteq Y_v \text{ hold.}\end{aligned}\tag{15}$$

Since each $Y\in\mathcal{Y}$ satisfies $X_{u^*}\subseteq C^*\subseteq Y$, $v\neq u^*$ holds. Assume by contradiction that $Y_u-Y_v\neq\emptyset$ holds. Let $Y_u\in\mathcal{B}_j$, $Y_v\in\mathcal{B}_k$, and $X_v\in\mathcal{A}_\ell$. By $X_v-Y_u\neq\emptyset\neq X_v\cap Y_u$, Y_u and Y_v cross each other in H. From Lemma 21, it follows that $Y_v-Y_u\in\mathcal{A}_j$, $d_H(Y_u-Y_v)=r(W_k)$, $d_H(Y_v-Y_u)=r(W_j)$, and $d_H(s,u^*)=d_H(Y_u\cap Y_v,V\cup\{s\}-Y_u-Y_v)=1$. Hence we have $v\in X_v-Y_u$, from which $X_v\cap(Y_v-Y_u)\neq\emptyset$ holds. Note that $X_v-(Y_v-Y_u)\neq\emptyset$ holds since X_v and Y_u cross each other in H. Moreover, $(Y_v-Y_u)-X_v\neq\emptyset$ holds since if $Y_v-Y_u\subseteq X_v$ holds, then the cut Y_v-Y_u contradicts the minimality of X_v by $d_H(Y_v-Y_u)=r(W_j)$, $v\in Y_v-Y_u$, and $X_v-(Y_v-Y_u)\neq\emptyset$. This means that X_v and Y_v-Y_u cross each other in H. Now $d_H(X_v\cap(Y_v-Y_u),V\cup\{s\}-X_v-(Y_v-Y_u))>0$ holds by $v\in X_v-Y_u$. By applying Lemma 21, we have $d_H(X_v)=r(W_\ell)+1$, contradicting $d_H(X_v)=r(W_\ell)$ (note that $X_v\in\mathcal{A}_\ell$ and $Y_v-Y_u\in\mathcal{A}_j$ hold). Hence (15) holds. □

Before closing this section, we will analyze the time complexity of algorithm r-NAEC-AUG, after describing the details of Step 2 in the algorithm. Step 2 is described as follows.

Step 2:
2-1: Check whether $H\in\mathcal{H}_2$ holds or not.
2-2: If $H\in\mathcal{H}_2$ holds, then execute 2-2-1 and 2-2-2.
 2-2-1: Repeat splitting an admissible pair of two edges in $E_H(s,V_2)$
  as possible. Denote the resulting graph by H_1. /** $H_1\in\mathcal{H}_1$ holds. **/
 2-2-2: After adding one extra edge (x,y) to $E_{H_1}(C_1(H_1),C_2(H_1))$, find a
  complete admissible splitting at s (according to Corollary 17).
  Output the set $E^*=E_2\cup\{(x,y)\}$ of edges, where E_2 is the set of all
  split edges and $|E^*|=\lceil\alpha(G,\mathcal{W},r)/2\rceil+1$ holds.
2-3: If $H\notin\mathcal{H}_2$ holds, then execute 2-3-1 – 2-3-9.
 2-3-1: If $G[V_2]$ has exactly one component C' with $d_H(s,C')=1$ where
  $\{(s,u')\}=E_H(s,C')$ holds and u' is contained in no critical cut of $\mathcal{A}_i$

for any area W_i in H, then we replace the edge (s, u') with a new edge (s, x) for some $x \in V_1 \cup V_2 - C'$ while preserving (4) (according to Lemma 27). Redenote the resulting graph by H.

2-3-1-1: If the component C'' of G with $x \in C''$ belongs to $\mathcal{C}_1$, then let $\mathcal{C}_1 := \mathcal{C}_1 - \{C''\}$ and $\mathcal{C}_2 := \mathcal{C}_2 \cup \{C''\}$.

/** H satisfies one of the statements (i)–(iii) in Lemma 26. **/

2-3-2: If H satisfies (ii) in Lemma 26, we first split the pair $\{(s, u), (s, v)\} \subseteq E_H(s, V_2)$ described in Lemma 26 (ii). Redenote the resulting graph by H.

2-3-3: Repeat splitting an admissible pair of two edges in $E_H(s, V_2)$ as possible. Denote the resulting graph by H_1.

2-3-4: If H_1 satisfies (11) or (12), then find a complete admissible splitting in H_1, according to Lemmas 14 or 15. Output the set E^* of all split edges, where $|E^*| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$.

/** In the sequel, H_1 satisfies neither (11) nor (12); $H_1 \in \mathcal{H}_1$ holds. **/

/** $E(H_1[V - V_1 - C_1(H_1)])$ contains a split edge from the proof of Lemma 26. **/

2-3-5: If $E(H_1[C_2(H_1)])$ contains a split edge, then find a split edge $e_1 = (v_1, v_2)$ in $C_2(H_1)$ such that if there are at least one split edge in $E(H_1[C_2(H_1)]) - B(H_1[C_2(H_1)])$, then $e_1 \in E(H_1[C_2(H_1)]) - B(H_1[C_2(H_1)])$, according to Lemma 29.

2-3-6: If $E(H_1[C_2(H_1)])$ contains no split edge, then find a split edge $e_1 = (v_1, v_2) \in E(H_1[V - V_1 - C_1(H_1)])$, one of whose end vertices, say v_1 is contained in a component C' of G with $E_H(s, C') = \{(s, v_1)\}$, according to the proof of Lemma 26 (iii).

2-3-7: After hooking up the edge e_1, split an admissible pair $\{(s, u), (s, v)\}$ with some $(s, u) \in E_{H_1}(s, C_1(H_1))$ and some $v \in \{u_2, v_1, v_2\}$, where $\{(s, u_2)\} = E_{H_1}(s, C_2(H_1))$ holds, according to Lemma 28. Denote the resulting graph by H_2.

2-3-8: In H_2, repeat splitting an admissible pair of two edges in $E_{H_2}(s, V_2)$ until all edges in $E_{H_2}(s, V_2)$ are split off. Denote the resulting graph by H_3.

/** Lemma 26 says that this is possible. H_3 satisfies (11). **/

2-3-9: Find a complete splitting at s in H_3, and output the set E^* of all split edges, where $|E^*| = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$ holds. $\square$

Finally, we analyze the time complexity of algorithm r-NAEC-AUG. We first show that it can be checked in $O(p(mn + n^2 \log n))$ time whether H satisfies (4) or not. Note that for an area $W_i \in \mathcal{W}$, we have $\lambda_H(v, W_i) \geq r(W_i)$ for each vertex $v \in V$ if and only if the graph $H(i)$ obtained from H by contracting W_i satisfies $d_{H(i)}(X) \geq r(W_i)$ for each cut $X \subset V(H(i)) - \{s\}$. By computing a family of cuts called *extreme sets*, in $H(i) - s$, we can check if $H(i)$ satisfies $d_{H(i)}(X) \geq r(W_i)$ for each cut $X \subset V(H(i)) - \{s\}$. In a graph $G = (V, E)$, a cut $X \subset V$ is called *extreme* if $d_G(X') > d_G(X)$ holds for any subset $\emptyset \neq X' \subset X$, and it is known [2,11] that the family $\mathcal{X}(G)$ of all extreme sets of G enjoys the following property:

For a graph $H = (V \cup \{s\}, E \cup F)$ and an integer k, $d_H(Y) \geq k$ for all cuts $Y \subset V$ if and only if $d_H(X) \geq k$ for all cuts $X \in \mathcal{X}(G)$.

Moreover, it was shown that $\mathcal{X}(G)$ can be found in $O(mn+n^2 \log n)$ time [11]. Hence, by computing the family of extreme sets in $H(i) - s$ for each W_i, we can check in $O(p(mn + n^2 \log n))$ time if H satisfies (4) or not.

In Step 1, for each vertex $v \in V$, after deleting all edges between s and v, we check whether the resulting graph H' satisfies (4) or not. If (4) is violated, then we add $\max_{x \in V, W_i \in \mathcal{W}}\{r(W_i) - \lambda_{H'}(x, W_i)\}$ edges between s and v in H'. So Step 1 takes $O(np\ (mn + n^2 \log n))$ time.

In Step 2, we first remark that for each pair $\{u, v\} \subseteq V$ of two vertices, we can check in $O(p(mn+n^2 \log n))$ time how many pairs of $\{(s, u), (s, v)\}$ can be split. This can be done by checking whether the resulting graph H' satisfies (4) after splitting $\min\{d_H(s, u), d_H(s, v)\}$ pairs $\{(s, u), (s, v)\}$. If (4) is violated, then we hook up $\lceil \frac{1}{2} \max_{x \in V, W_i \in \mathcal{W}}\{r(W_i) - \lambda_{H'}(x, W_i)\}\rceil$ pairs in H'. Hence, it takes $O(p(mn + n^2 \log n))$ time. Moreover, we can observe that it takes $O(|V'|^2 p(mn + n^2 \log n))$ time to execute admissible splittings of two edges in $E_H(s, V')$ as possible for a vertex set $V' \subseteq V$.

Step 2 first needs to check whether $H \in \mathcal{H}_2$ or not. It is not difficult to see that checking the statement (I) in the definition of $\mathcal{H}_2$ takes linear time. For the statement (II), we need to compute minimal critical cuts containing the vertex u', where u' is in the component C' of G with $E_H(s, C') = \{(s, u')\}$ found by the checking of (I). This can be found in $O(p\ (mn + n^2 \log n))$ time by computing the family of all extreme sets in $H(i) - s$ for each $W_i \in \mathcal{W}$. For the statement (III), we need splitting $O(n)$ pairs. Hence it takes $O(np$ $(mn + n^2 \log n))$ time to check whether $H \in \mathcal{H}_2$ or not.

We next claim that Step 2 contains at most one hooking up operations. If $H \in \mathcal{H}_2$, then we can obtain an optimal solution without hooking up operations, according to Corollary 17 (see step 2-2 in the above description of Step 2). We show that the step 2-3 contains at most one hooking up operations. Since $H \notin \mathcal{H}_2$, H violates at least one of (I)–(III) in the definition of $\mathcal{H}_2$. If H violates (I) or (III), then H satisfies at least one of conditions (i)–(iii) in Lemma 26. Then according to the proof of Lemma 26 and the choice of split edges in steps 2-3-5 and 2-3-6, we can continue admissible splitting at s while at most one hooking up operations is executed (see steps 2-3-2 – 2-3-9). If H satisfies (I) and (III) but violates (II), then replacing one edge in $E_H(s, V)$ can convert H to H' satisfying (i) in Lemma 26 without violating (4), according to the proof of Lemma 27 (see step 2-3-1). It follows that the claim is proved.

Note that the above observation about hooking operations indicates that at most one replacing operations occurs. The time complexity of replacing operations depends on minimal critical cut containing u_2, which is the same as that of checking the statement (II) in the definition of $\mathcal{H}_2$. Also note that finding a split edge in steps 2-3-5 or 2-3-6 takes linear time. Consequently, it is not difficult to see that the time complexity of Step 2 depends on that of splitting $O(n^2)$ pairs. It follows that Step 2 can be implemented to run in $O(n^2 p\ (mn + n^2 \log n))$ time.

As a result, the total complexity of the algorithm is $O(n^2 p\ (mn + n^2 \log n))$, which can be reduced to $O(m+\ pn^4\ (r^* + \log n))$ time by applying the procedure to a sparse spanning subgraph of G' with $O(r^* n)$ edges, where such sparsification takes $O(m + n \log n)$ time [12,13].

Lemma 32 *Algorithm NAEC-AUG can be implemented to run in $O(m+\ pn^4\ (r^* + \log n))$ time.* □

Summarizing the argument given so far, Theorem 8 is now established.

7 Concluding Remarks

In this paper, given an area graph $(G = (V, E), \mathcal{W})$ and a requirement function $r : \mathcal{W} \to Z^+$, we considered the problem of asking to augment $(G = (V, E), \mathcal{W})$ by adding the minimum number of new edges such that the resulting graph becomes r-NA-edge-connected. We first gave a polynomial time algorithm for the problem in the case where each area $W \in \mathcal{W}$ satisfies $r(W) \geq 2$. The time complexity of our algorithm is $O(m+\ pn^4\ (r^* + \log n))$, where $n = |V|$, $m = |\{\{u, v\}|(u, v) \in E\}|$, $p = |\mathcal{W}|$, $r^* = \max\{r(W) \mid W \in \mathcal{W}\}$. It is a future work to consider generalized problems in such a way that the connectivity requirement is general for each pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$.

We finally introduce a problem of augmenting a symmetric skew-supermodular integer-valued function by a multigraph G' as another generalization of r-NA-ECAP. r-NA-ECAP asks to augment a symmetric skew-supermodular integer-valued function $\alpha_{G,\mathcal{W},r}$ by a multigraph G' with the minimum number of edges, as observed in Section 5. In [16], Szigeti showed the following Theorem 33 and that the problem of augmenting an integer-valued symmetric skew-supermodular function by a hypergraph H' with the minimum $\sum_{Y \in \mathcal{E}(H')} |Y|$ is polynomially solvable, where $\mathcal{E}(H')$ denotes the family of hyperedges in H'.

Theorem 33 [16] *Let p be a symmetric skew-supermodular integer-valued function on the ground set V. Then*

$$\min\{\sum_{Y \in \mathcal{E}(H')} |Y| \mid d_{H'}(X) \geq p(X) \textit{ for all } X \subseteq V\} = \max\{\sum p(V_i)\}$$

where the maximization is taken over all subpartitions $\{V_1, \ldots, V_\ell\}$ *of* V. □

However, Figure 2 indicates that r-NA-ECAP does not enjoy Theorem 33.

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