THE FORMULA OF HIGHER ORDER DERIVATIVES OF IMPLICIT FUNCTIONS

By

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We denote that

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}, \quad N_k = {\mathbb{N}_0}^k \backslash \{(\overbrace{0, \dots, 0}^{k})\} \quad (k \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

$$\begin{aligned} \mathfrak{P}_l(n_1, \dots, n_k) = \{u \in {\mathbb{N}_0}^{N_k}; \#\{(i_1, \dots, i_k) \in N_k; u(i_1, \dots, i_k) \neq 0\} < \aleph_0, \\ \sum_{(i_1, \dots, i_k) \in N_k} u(i_1, \dots, i_k) = l, \\ \sum_{(i_1, \dots, i_k) \in N_k} i_m u(i_1, \dots, i_k) = n_m (1 \leqslant m \leqslant k)\} \end{aligned}$$

and

$$\mathfrak{P}(n_1, \dots, n_k) = \bigcup_{l=0}^{\infty} \mathfrak{P}_l(n_1, \dots, n_k),$$

where $l, n_1, \dots, n_k$ are elements of $\mathbb{N}_0$.

An element u of $\mathfrak{P}_l(n_1, \dots, n_k)$ expresses a vector partition of $\begin{pmatrix} n_1 \\ \vdots \\ n_k \end{pmatrix}$ into l parts. For example,

$$u = (u(i, j))_{(i,j) \in N_2} = \begin{pmatrix} & u(0,1) & u(0,2) \\ u(1,0) & u(1,1) & \\ u(2,0) & & \cdots \end{pmatrix} = \begin{pmatrix} & 2 & 1 \\ 1 & 3 & \\ 0 & & O \end{pmatrix} \in \mathfrak{P}_7(4, 7)$$

expresses a vector partition:

$$\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Let $p(n_1, \dots, n_k)$ be the cardinal of the set $\mathfrak{P}(n_1, \dots, n_k)$. Then the generating function of the arithmetical function $p(n_1, \ldots, n_k)$ is given by

$$\sum_{(n_1,\ldots,n_k)\in\mathbb{N}_0{}^k} p(n_1, \ldots, n_k){x_1}^{n_1} \ldots {x_k}^{n_k} = \prod_{(m_1,\ldots,m_k)\in N_k} (1-{x_1}^{m_1} \ldots {x_k}^{m_k})^{-1}$$

(see [1]).

Suppose that a function $f(x_1, \ldots, x_k)$ $(k \geqslant 2)$ is of class $C^r(1 \leqslant n \leqslant r \leqslant \infty$ or $r = \omega)$ in a domain $G \subset \mathbb{R}^k$ and that $f(a_1, \ldots a_k) = 0, f_{x_k}(a_1, \ldots a_k) \neq 0$ at a point $(a_1, \ldots a_k)$ in G. Let $x_k = g(x_1, \ldots, x_{k-1})$ be the implicit function determined by $f = 0$.

Theorem. Let $n_1, \ldots, n_{k-1} \in \mathbb{N}_0$ and $n = n_1 + \cdots + n_{k-1} \geqslant 1$. Then in a neighborhood of $(a_1, \ldots a_k)$,

(1) $$g^{(n_1,\ldots,n_{k-1})}(x_1, \ldots, x_{k-1}) = -\frac{1}{{f_{x_k}}^{2n-1}} \sum_{u\in\mathfrak{P}_{2n-1}(n_1,\ldots,n_{k-1},2n-2)} \nu(u) f(u; x_1, \ldots, x_k),$$

where $$\nu(u) = \frac{(-1)^{u(e_k)} u(e_k)!\,(2n - u(e_k) - 2)!\, n_1! \cdots n_{k-1}!}{\prod_{(i_1,\ldots,i_k)\in N_k} (i_1! \cdots i_k!)^{u(i_1,\ldots,i_k)} u(i_1, \ldots, i_k)!},$$

$$f(u; x_1, \ldots, x_k) = \prod_{(i_1,\ldots,i_k)\in N_k} \{f^{(i_1,\ldots,i_k)}(x_1, \ldots, x_k)\}^{u(i_1,\ldots,i_k)}$$

and $e_m = (0, \ldots, \overset{m}{\breve{1}}, \ldots, 0)$ $(1 \leqslant m \leqslant k)$ are fundamental vectors.

For the proof of Theorem, we prepare following two lemmata.

Lemma 1. Suppose that $k \geqslant 2$; $n_1, \ldots, n_{k-1} \in \mathbb{N}_0$, $n = n_1 + \cdots + n_{k-1} \geqslant 1$ and $u \in \mathfrak{P}_{2n-1}(n_1, \ldots, n_{k-1}, 2n-2)$. If $u(e_k) = 0$, then

$$u(i_1, \ldots, i_k) = \begin{cases} n_m & \text{, if } (i_1, \ldots, i_k) = e_m \text{ for all } m\,(1 \leqslant m \leqslant k-1), \\ n-1 & \text{, if } (i_1, \ldots, i_k) = 2e_k, \\ 0 & \text{, otherwise.} \end{cases}$$

LEMMA 2. Suppose that $k \geqslant 2$; $n_1, \ldots, n_{k-1} \in \mathbb{N}_0$, $n = n_1 + \cdots + n_{k-1} \geqslant 1$ and $u \in \mathfrak{P}_{2n-1}(n_1, \ldots, n_{k-1}, 2n-2)$. If $u(e_k) = 1$, then u coincides with either of the following two cases u.

(i) $n \geqslant 2$ and for some h such that $1 \leqslant h \leqslant k-1$,

$$u(i_1, \ldots, i_k) = \begin{cases} n_h - 1 & , \text{ if } (i_1, \ldots, i_k) = e_h, \\ n_m & , \text{ if } (i_1, \ldots, i_k) = e_m \text{ for all } m\,(1 \leqslant m \leqslant k-1, m \neq h), \\ 1 & , \text{ if } (i_1, \ldots, i_k) = e_k \text{ or } e_h + e_k, \\ n-2 & , \text{ if } (i_1, \ldots, i_k) = 2e_k, \\ 0 & , \text{ otherwise.} \end{cases}$$

(ii) $n \geqslant 3$ and

$$u(i_1, \ldots, i_k) = \begin{cases} n_m & , \text{ if } (i_1, \ldots, i_k) = e_m \text{ for all } m\,(1 \leqslant m \leqslant k-1), \\ 1 & , \text{ if } (i_1, \ldots, i_k) = e_k \text{ or } 3e_k, \\ n-3 & , \text{ if } (i_1, \ldots, i_k) = 2e_k, \\ 0 & , \text{ otherwise.} \end{cases}$$

Proof of Lemma 1.

For any $u \in \mathfrak{P}_{2n-1}(n_1, \ldots, n_{k-1}, 2n-2)$, we set that

$$s(i) = \sum_{i_1 + \cdots + i_k = i} u(i_1, \ldots, i_k) \quad (i \in \mathbb{N}).$$

Then

$$\sum_{i=1}^{\infty} s(i) = \sum_{(i_1, \ldots, i_k) \in N_k} u(i_1, \ldots, i_k) = 2n - 1$$

and

$$\begin{aligned} \sum_{i=1}^{\infty} i\, s(i) &= \sum_{(i_1, \ldots, i_k) \in N_k} (i_1 + \cdots + i_k)\, u(i_1, \ldots, i_k) \\ &= n_1 + \cdots + n_{k-1} + 2n - 2 = 3n - 2. \end{aligned}$$

Therefore $s \in \mathfrak{P}_{2n-1}(3n-2)$. Let $s^*(i) = s(i+1)$ $(i \in \mathbb{N})$. Then

$$\sum_{i=1}^{\infty} i\, s^*(i) = \sum_{i=1}^{\infty} i\, s(i+1) = \sum_{i=1}^{\infty}(i-1)s(i)$$
$$= (3n-2) - (2n-1) = n-1.$$

We have $s^* \in \mathfrak{P}(n-1)$. Let λ be $\sum_{i=1}^{\infty} s^*(i)$. Then $\lambda \leqslant n-1$ and

$$s(1) + \lambda = \sum_{i=1}^{\infty} s(i) = 2n-1.$$

So that $s(1) \geqslant n$. From the assumption $u(e_k) = 0$, we have

(2) $$s(1) = u(e_1) + \cdots + u(e_{k-1}) \geqslant n.$$

On the other hand,

(3) $$u(e_1) + \cdots + u(e_{k-1}) = \sum_{(i_1,\ldots,i_k)\in\{e_1,\ldots,e_{k-1}\}} (i_1 + \cdots + i_{k-1})\, u(i_1,\ldots,i_k)$$
$$\leqslant \sum_{(i_1,\ldots,i_k)\in N_k} (i_1 + \cdots + i_{k-1})\, u(i_1,\ldots,i_k)$$
$$= n_1 + \cdots + n_{k-1} = n.$$

From (2) and (3), we have

(4) $$s(1) = u(e_1) + \cdots + u(e_{k-1}) = n.$$

For all m such that $1 \leqslant m \leqslant k-1$,we have also

(5) $$u(e_m) \leqslant \sum_{(i_1,\ldots,i_k)\in N_k} i_m u(i_1,\ldots,i_k) = n_m.$$

By (4) and (5), we get

$$u(e_m) = n_m (1 \leqslant m \leqslant k-1).$$

From that $s(1) + \lambda = \sum s(i) = 2n-1$ and (4),

$$\lambda = \sum s^*(i) = n-1.$$

Thus $s^* \in \mathfrak{P}_{n-1}(n-1)$ expresses a partition of $n-1$ into $n-1$ parts. So that

$$s^*(i) = \begin{cases} n-1 & \text{, if } i = 1, \\ 0 & \text{, if } i > 1. \end{cases}$$

Namely,

$$s(2) = n-1,\ s(3) = s(4) = \cdots = 0.$$

We have

$$s(2) = \sum_{i_1+\cdots+i_k=2} u(i_1,\ldots,i_k) = \sum_{1\leqslant p\leqslant q\leqslant k} u(e_p+e_q).$$

If there exists a pair (p,q) such that

$$1 \leqslant p \leqslant q \leqslant k,\ p < k,\ u(e_p+e_q) > 0,$$

then $\sum i_p u(i_1,\ldots,i_k) > n_p$. This is contradiction. So that

$$\sum_{1\leqslant p\leqslant q\leqslant k,\, p<k} u(e_p+e_q) = 0.$$

This leads $s(2) = u(2e_k) = n-1$. q.e.d.

Proof of Lemma 2.

We use the same setting of s, s^*, λ as used in the proof Lemma 1. We have $s \in \mathfrak{P}_{2n-1}(3n-2)$, $s^* \in \mathfrak{P}(n-1)$ and

$$s(1) \geqslant n,\ \ \lambda = 2n-1-s(1) \leqslant n-1.$$

From the assumption $u(e_k)=1$, we have

$$u(e_1)+\cdots+u(e_{k-1}) \geqslant n-1$$

because that $s(1) \geqslant n$. On the other hand,

$$u(e_1)+\cdots+u(e_{k-1}) \leqslant \sum (i_1+\cdots+i_{k-1})\, u(i_1,\ldots,i_k) = n.$$

Hence

$$u(e_1)+\cdots+u(e_{k-1}) = n-1 \text{ or } n.$$

(i) Case of $u(e_1)+\cdots+u(e_{k-1}) = n-1$.

In this case, from that $u(e_m) \leqslant n_m$ for all $m (1 \leqslant m \leqslant k-1)$, there exists $h (1 \leqslant h \leqslant k-1)$ such that

$$u(e_h) = n_h - 1$$

and for any $m(1 \leqslant m \leqslant k-1,\ m \neq h)$,

$$u(e_m) = n_m.$$

In this case, we have

$$s(1) = n,\ \ \lambda = n-1.$$

By the same way as in the proof of Lemma 1,

$$s^*(i) = \begin{cases} n-1 & \text{, if } i = 1, \\ 0 & \text{, if } i > 1. \end{cases}$$

Namely,

$$s(2) = n-1,\ s(3) = s(4) = \cdots = 0.$$

From $\sum i_m u(i_1, \ldots, i_k) = n_m$ $(1 \leqslant m \leqslant k-1)$, we get

$$n \geqslant 2,\ u(e_h + e_k) = 1,\ u(2e_k) = n-2,$$
$$u(e_p + e_q) = 0\ (\,1 \leqslant p \leqslant q \leqslant k,\ (p,q) \notin \{(h,k),(k,k)\}\,).$$

This consequence coincides with case(i) of Lemma 2.

(ii) Case of $u(e_1) + \cdots + u(e_{k-1}) = n$.
In this case, from that $u(e_m) \leqslant n_m$ for all $m(1 \leqslant m \leqslant k-1)$,

$$u(e_m) = n_m \text{ for all } m\,(1 \leqslant m \leqslant k-1)$$

Hence

$$s(1) = n+1,\ \lambda = n-2.$$

Therefore $s^* \in \mathfrak{P}_{n-2}(n-1)$. This leads

$$s^*(i) = \begin{cases} n-3 & \text{, if } i = 1, \\ 1 & \text{, if } i = 2, \\ 0 & \text{, if } i > 2. \end{cases}$$

Namely,

$$s(2) = n-3,\ s(3) = 1,\ s(4) = s(5) = \cdots = 0.$$

From $\sum i_m u(i_1, \ldots, i_k) = n_m$ $(1 \leqslant m \leqslant k-1)$, we get

$$n \geqslant 3,\ u(2e_k) = n-3,\ u(3e_k) = 1,$$
$$u(i_1, \ldots, i_k) = 0\ (\,(i_1, \ldots, i_k) \notin \{e_1, \ldots, e_k, 2e_k, 3e_k\}\,).$$

This consequence coincides with case(ii) of Lemma 2. q.e.d.

Proof of the theorem.

We denote the differential operator $\dfrac{\partial}{\partial x_m}$, by ∂_m $(1 \leqslant m \leqslant k)$, without regarding that x_k is the function $g(x_1, \ldots, x_{k-1})$ of $x_1, \ldots, x_{k-1}$. And denote $\dfrac{\partial}{\partial x_m}$ $(1 \leqslant m \leqslant k-1)$ by $\dot{\partial}_m$, when we regard that

$$x_k = g(x_1, \ldots, x_{k-1}).$$

Then

$$\begin{aligned}\dot{\partial}_m &= \partial_m + \dot{\partial}_m[x_k]\,\partial_k \\ &= \partial_m - \frac{f_{x_m}}{f_{x_k}}\,\partial_k \quad (1 \leqslant m \leqslant k-1).\end{aligned}$$

If $n = 1$, then that for some j $(1 \leqslant j \leqslant k-1)$,

$$\mathfrak{P}_{2n-1}(n_1, \ldots, n_{k-1}, 2n-2) = \mathfrak{P}_1(e_j) = \{u_j\},$$

where

$$u_j(i_1, \ldots, i_k) = \begin{cases} 1 & \text{, if } (i_1, \ldots, i_k) = e_j \\ 0 & \text{, otherwise.} \end{cases}$$

Then

$$\nu(u_j) = 1,\ f(u_j; x_1, \ldots, x_k) = f_{x_j}.$$

Hence the theorem holds, if $n = 1$.

Suppose that the theorem folds for $n-1$ $(n \geqslant 2)$. In the case for n, there exists j $(1 \leqslant j \leqslant k-1)$ such that

$$n_j > 0.$$

By the hypothesis of induction, we have

$$\begin{aligned}(6)\quad g^{(n_1,\ldots,n_{k-1})} &= \dot{\partial}_j[g^{(n_1,\ldots,n_j-1,\ldots,n_{k-1})}] = \dot{\partial}_j\left[-\frac{1}{{f_{x_k}}^{2n-3}}\sum_{u\in\mathfrak{P}}\nu(u)f(u)\right] \\ &= -\frac{1}{{f_{x_k}}^{2n-1}}\left\{{f_{x_k}}^2\sum_{u\in\mathfrak{P}}\nu(u)\left(\partial_j - \frac{f_{x_j}}{f_{x_k}}\partial_k\right)[f(u)]\right. \\ &\qquad\left.-(2n-3)f_{x_k}\left(\partial_j - \frac{f_{x_j}}{f_{x_k}}\partial_k\right)[f_{x_k}]\sum_{u\in\mathfrak{P}}\nu(u)f(u)\right\} \\ &= -\frac{1}{{f_{x_k}}^{2n-1}}\sum_{u\in\mathfrak{P}}\nu(u)\{\,{f_{x_k}}^2\partial_j[f(u)] - f_{x_j}f_{x_k}\partial_k[f(u)] \\ &\qquad -(2n-3)f_{x_k}f_{x_jx_k}f(u) + (2n-3)f_{x_j}f_{x_kx_k}f(u)\,\},\end{aligned}$$

where

$$\mathfrak{P} = \mathfrak{P}_{2n-3}(n_1, \ldots, n_j - 1, \ldots, n_{k-1}, 2n-4)$$

and

$$f(u) = f(u; x_1, \ldots, x_k).$$

Concerning $\partial_m[f(u)]$ $(1 \leqslant m \leqslant k)$, we have

(7) $$\partial_m[f(u)] = \partial_m\left[\prod_{(i_1,\ldots,i_k)\in N_k}\{f^{(i_1,\ldots,i_k)}\}^{u(i_1,\ldots,i_k)}\right]$$
$$= \sum_{\substack{(q_1,\ldots,q_k)\in N_k\\ u(q_1,\ldots,q_k)\neq 0}} u(q_1,\ldots,q_k)\,\{f^{(q_1,\ldots,q_k)}\}^{u(q_1,\ldots,q_k)-1} f^{(q_1,\ldots,q_m+1,\ldots,q_k)} \cdot \prod_{\substack{(i_1,\ldots,i_k)\in N_k\\ (i_1,\ldots,i_k)\neq(q_1,\ldots,q_k)}}\{f^{(i_1,\ldots,i_k)}\}^{u(i_1,\ldots,i_k)}.$$

Suppose that

$$u \in \mathfrak{P},\ q = (q_1,\ldots,q_k) \in N_k \text{ and } u(q) \neq 0.$$

We define $u_q^{j,1} \in {\mathbb{N}_0}^{N_k}$, by the following:

(i) If $q \neq e_k$,

$$u_q^{j,1}(i_1,\ldots,i_k) = \begin{cases} u(i_1,\ldots,i_k)+2 & \text{, if } (i_1,\ldots,i_k) = e_k, \\ u(i_1,\ldots,i_k)-1 & \text{, if } (i_1,\ldots,i_k) = q, \\ u(i_1,\ldots,i_k)+1 & \text{, if } (i_1,\ldots,i_k) = q+e_j, \\ u(i_1,\ldots,i_k) & \text{, otherwise.} \end{cases}$$

(ii) If $q = e_k$,

$$u_q^{j,1}(i_1,\ldots,i_k) = \begin{cases} u(i_1,\ldots,i_k)+1 & \text{, if } (i_1,\ldots,i_k) \in \{e_k, e_j+e_k\}, \\ u(i_1,\ldots,i_k) & \text{, otherwise.} \end{cases}$$

Then

$$u_q^{j,1} \in \mathfrak{P}' = \mathfrak{P}_{2n-1}(n_1,\ldots,n_{k-1},2n-2)$$

and by (7), we get

(8) $$\sum_{u\in\mathfrak{P}} \nu(u) {f_{x_k}}^2\, \partial_j[f(u)] = \sum_{u\in\mathfrak{P}} \nu(u) \sum_{\substack{q\in N_k\\ u(q)\neq 0}} u(q) f(u_q^{j,1}).$$

We define $u_q^{j,2} \in {\mathbb{N}_0}^{N_k}$, by the following:

(i) If $q \notin \{e_j, e_k\}$,

$$u_q^{j,2}(i_1,\ldots,i_k) = \begin{cases} u(i_1,\ldots,i_k)+1 & \text{, if } (i_1,\ldots,i_k) \in \{e_j, e_k, q+e_k\}, \\ u(i_1,\ldots,i_k)-1 & \text{, if } (i_1,\ldots,i_k) = q, \\ u(i_1,\ldots,i_k) & \text{, otherwise.} \end{cases}$$

(ii) If $q = e_j$,

$$u_q^{j,2}(i_1,\ldots,i_k) = \begin{cases} u(i_1,\ldots,i_k)+1 & \text{, if } (i_1,\ldots,i_k) \in \{e_k, e_j+e_k\}, \\ u(i_1,\ldots,i_k) & \text{, otherwise.} \end{cases}$$

(iii) If $q = e_k$,

$$u_q^{j,2}(i_1,\dots,i_k)=\begin{cases} u(i_1,\dots,i_k)+1 & \text{, if } (i_1,\dots,i_k)\in\{e_j,2e_k\}, \\ u(i_1,\dots,i_k) & \text{, otherwise.}\end{cases}$$

Then

$$u_q^{j,2} \in \mathfrak{P}'$$

and by (7), we get

$$\sum_{u\in\mathfrak{P}} \nu(u) f_{x_j} f_{x_k}\, \partial_k[f(u)] = \sum_{u\in\mathfrak{P}} \nu(u) \sum_{\substack{q\in N_k\\ u(q)\neq 0}} u(q) f(u_q^{j,2}). \tag{9}$$

For any $u \in \mathfrak{P}$, we define $u^{j,3}$, $u^{j,4}$ by the following:

$$u^{j,3}(i_1,\dots,i_k)=\begin{cases} u(i_1,\dots,i_k)+1 & \text{, if } (i_1,\dots,i_k)\in\{e_k,e_j+e_k\}, \\ u(i_1,\dots,i_k) & \text{, otherwise,}\end{cases}$$

$$u^{j,4}(i_1,\dots,i_k)=\begin{cases} u(i_1,\dots,i_k)+1 & \text{, if } (i_1,\dots,i_k)\in\{e_j,2e_k\}, \\ u(i_1,\dots,i_k) & \text{, otherwise.}\end{cases}$$

Then we have

$$u^{j,3}, u^{j,4} \in \mathfrak{P}',$$

$$\sum_{u\in\mathfrak{P}} \nu(u) f_{x_k} f_{x_j x_k}\, f(u) = \sum_{u\in\mathfrak{P}} \nu(u) f(u^{j,3}) \tag{10}$$

and

$$\sum_{u\in\mathfrak{P}} \nu(u) f_{x_j} f_{x_k x_k}\, f(u) = \sum_{u\in\mathfrak{P}} \nu(u) f(u^{j,4}). \tag{11}$$

From the observation that

$$u_{e_k}^{j,1} = u^{j,3},$$

by (8),

$$\sum_{u\in\mathfrak{P}} \nu(u) {f_{x_k}}^2\, \partial_j[f(u)] = \sum_{u\in\mathfrak{P}} \nu(u) \left\{ u(e_k) f(u^{j,3}) + \sum_{\substack{q\in N_k\setminus\{e_k\}\\ u(q)\neq 0}} u(q) f(u_q^{j,1}) \right\}. \tag{12}$$

Remarking that

$$u_{e_j}^{j,2} = u^{j,3},\ u_{e_k}^{j,2} = u^{j,4};$$

by (9),

$$\sum_{u\in\mathfrak{P}} \nu(u) f_{x_j} f_{x_k} \partial_k[f(u)] = \sum_{u\in\mathfrak{P}} \nu(u) \left\{ u(e_j) f(u^{j,3}) + u(e_k) f(u^{j,4}) + \sum_{\substack{q\in N_k\setminus\{e_j,e_k\} \\ u(q)\neq 0}} u(q) f(u_q^{j,2}) \right\}. \tag{13}$$

From (6) and (10)–(13), we have

$$g^{(n_1,\ldots,n_{k-1})} = -\frac{1}{{f_{x_k}}^{2n-1}} \sum_{u\in\mathfrak{P}} \nu(u) \Bigg[\{u(e_k) - u(e_j) - (2n-3)\} f(u^{j,3}) + \{2n-3-u(e_k)\} f(u^{j,4}) + \sum_{\substack{q\in N_k\setminus\{e_k\} \\ u(q)\neq 0}} u(q) f(u_q^{j,1}) - \sum_{\substack{q\in N_k\setminus\{e_j,e_k\} \\ u(q)\neq 0}} u(q) f(u_q^{j,2}) \Bigg]. \tag{14}$$

Concerning the first term of (14),

$$\sum_{u\in\mathfrak{P}} \nu(u)\{u(e_k) - u(e_j) - (2n-3)\} f(u^{j,3}) = \sum_{v\in\mathfrak{P}'} f(v) \sum_{\substack{u\in\mathfrak{P} \\ v=u^{j,3}}} \nu(u)\{u(e_k) - u(e_j) - (2n-3)\}.$$

We denote that

$$A(v) = \sum_{\substack{u\in\mathfrak{P} \\ v=u^{j,3}}} \nu(u)\{u(e_k) - u(e_j) - (2n-3)\}. \tag{15}$$

Then from

$$u^{j,3}(e_k) > 0,\ u^{j,3}(e_j + e_k) > 0,\ u^{j,3}(e_j) = u(e_j),\ u^{j,3}(e_k) = u(e_k) + 1\,;$$

$$A(v)=\begin{cases} 0 & \text{, if } v(e_k)\,v(e_j+e_k)=0, \\ -\{2n+v(e_j)-v(e_k)-2\}\,\nu(v-\alpha_1) & \text{, if } v(e_k)\,v(e_j+e_k)>0, \end{cases} \tag{16}$$

where $\alpha_1 \in {\mathbb{N}_0}^{N_k}$ and

$$\alpha_1(i_1,\ldots,i_k)=\begin{cases} 1 \text{ , if } (i_1,\ldots,i_k)\in\{e_k, e_j+e_k\}, \\ 0 \text{ , otherwise.} \end{cases}$$

Let $B(v)$ be a function of $v \in \mathfrak{P}'$ such that

$$B(v) = \sum_{\substack{u\in\mathfrak{P} \\ v=u^{j,4}}} \nu(u)\{2n-3-u(e_k)\}. \tag{17}$$

Then from

$$u^{j,4}(e_j)>0,\ u^{j,4}(2e_k)>0,\ u^{j,4}(e_k)=u(e_k)\,;$$

$$B(v)=\begin{cases} 0 & \text{, if } v(e_j)\,v(2e_k)=0, \\ \{2n-v(e_k)-3\}\,\nu(v-\alpha_2) & \text{, if } v(e_j)\,v(2e_k)>0, \end{cases} \tag{18}$$

where $\alpha_2 \in {\mathbb{N}_0}^{N_k}$ and

$$\alpha_2(i_1,\ldots,i_k)=\begin{cases} 1 \text{ , if } (i_1,\ldots,i_k)\in\{e_j, 2e_k\}, \\ 0 \text{ , otherwise.} \end{cases}$$

Let $C(v)$ be a function of $v \in \mathfrak{P}'$ such that

$$C(v) = \sum_{\substack{u\in\mathfrak{P} \\ v=u_q^{j,1}}} \nu(u) \sum_{\substack{q\in N_k\setminus\{e_k\} \\ u(q)\neq 0}} u(q). \tag{19}$$

If $v = u_q^{j,1},\ q \neq e_k$, we set $p = q + e_j$ and

$$\alpha_p^{j,1}(i_1,\ldots,i_k)=\begin{cases} 2 & \text{, if } (i_1,\ldots,i_k)=e_k, \\ -1 & \text{, if } (i_1,\ldots,i_k)=p-e_j, \\ 1 & \text{, if } (i_1,\ldots,i_k)=p, \\ 0 & \text{, otherwise.} \end{cases}$$

Then we have

$$u = v-\alpha_p^{j,1},\ q=p-e_j,\ u(q)=u(p-e_j)=v(p-e_j)+1$$

and

$$(20)\qquad C(v)=\begin{cases} 0 & \text{, if } v(e_k)<2,\\ \displaystyle\sum_{\substack{v(p)>0,\, p\neq e_j+e_k\\ p_j>0,\, \sum p_i\geqq 2}} \{v(p-e_j)+1\}\,\nu(v-\alpha_p^{j,1}) & \text{, if } v(e_k)\geqq 2\,,\end{cases}$$

where $p=(p_1,\dots,p_k)\in N_k$.

Let $D(v)$ be a function of $v\in\mathfrak{P}'$ such that

$$(21)\qquad D(v)=-\sum_{\substack{u\in\mathfrak{P}\\ v=u_q^{j,2}}}\nu(u)\sum_{\substack{q\in N_k\setminus\{e_j,e_k\}\\ u(q)\neq 0}}u(q).$$

If $v=u_q^{j,2}$, $q\notin\{e_j,e_k\}$, we set $p=q+e_j$ and

$$\alpha_p^{j,2}(i_1,\dots,i_k)=\begin{cases} 1 & \text{, if } (i_1,\dots,i_k)\in\{e_j,e_k,p\},\\ -1 & \text{, if } (i_1,\dots,i_k)=p-e_k,\\ 0 & \text{, otherwise.}\end{cases}$$

Then we have

$$u=v-\alpha_p^{j,2},\ q=p-e_k,\ u(q)=u(p-e_k)=v(p-e_k)+1$$

and

$$(22)\qquad D(v)=\begin{cases} 0 & \text{, if } v(e_j)v(e_k)=0,\\ \displaystyle -\sum_{\substack{v(p)>0\\ p\notin\{e_j+e_k,2e_k\}\\ p_k>0,\, \sum p_i\geqq 2}} \{v(p-e_k)+1\}\,\nu(v-\alpha_p^{j,2}) & \text{, if } v(e_j)v(e_k)>0\,.\end{cases}$$

From (14), (15), (17), (19) and (21), we have

$$g^{(n_1,\dots,n_{k-1})}=-\frac{1}{{f_{x_k}}^{2n-1}}\sum_{v\in\mathfrak{P}'}f(v)\{A(v)+B(v)+C(v)+D(v)\}.$$

Therefore, for the proof of the theorem, it is sufficient to show that

$$(23)\qquad A(v)+B(v)+C(v)+D(v)=\nu(v).$$

We denote

$$\prod_{(i_1,\dots,i_k)\in N_k} (i_1!\cdots i_k!)^{v(i_1,\dots,i_k)} v(i_1,\dots,i_k)!$$

by $P(v)$. If $v(e_k)v(e_j+e_k)>0$, then

$$\nu(v-\alpha_1) = \frac{(-1)^{v(e_k)-1}\{v(e_k)-1\}!\,\{2n-4-v(e_k)+1\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!}{P(v)/\{v(e_k)v(e_j+e_k)\}}.$$

So that, by (16),

$$A(v) = (-1)^{v(e_k)} v(e_k)!\,\{2n-v(e_k)-3\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}! \cdot \frac{1}{P(v)}\{2n+v(e_j)-v(e_k)-2\}\,v(e_j+e_k),$$

So we have

$$A(v)=\begin{cases} 0 & \text{, if } v(e_k)=0,\\ Q(v)\,v(e_j+e_k)\{2n+v(e_j)-v(e_k)-2\} & \text{, if } v(e_k)>0, \end{cases} \tag{24}$$

where

$$Q(v) = \frac{(-1)^{v(e_k)} v(e_k)!\,\{2n-v(e_k)-3\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!}{P(v)}.$$

If $v(e_j)v(2e_k)>0$, then

$$\nu(v-\alpha_2) = \frac{(-1)^{v(e_k)} v(e_k)!\,\{2n-v(e_k)-4\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!}{P(v)/\{v(e_j)\cdot 2v(2e_k)\}}.$$

So that, by (18),

$$\begin{aligned} B(v) &= \{2n-v(e_k)-3\}\nu(v-\alpha_2)\\ &= (-1)^{v(e_k)} v(e_k)!\,\{2n-v(e_k)-3\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!\\ &\quad\cdot \frac{1}{P(v)}\,2v(e_j)v(2e_k). \end{aligned}$$

So we have

$$B(v) = Q(v)2v(e_j)v(2e_k). \tag{25}$$

If $v(e_k) \geqslant 2$, then by (20),

$$C(v) = \sum_{\substack{v(p)>0,\, p\neq e_j+e_k \\ p_j>0,\, \sum p_i \geqslant 2}} \frac{(-1)^{v(e_k)}\{v(e_k)-2\}!\,\{2n-v(e_k)-2\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!}{P(v)/[v(e_k)\{v(e_k)-1\}p_j v(p)]}$$

$$= (-1)^{v(e_k)} v(e_k)!\,\{2n-v(e_k)-2\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!$$

$$\cdot \frac{1}{P(v)} \sum_{\substack{v(p)>0,\, p\neq e_j+e_k \\ p_j>0,\, \sum p_i \geqslant 2}} p_j v(p)$$

$$= (-1)^{v(e_k)} v(e_k)!\,\{2n-v(e_k)-2\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!$$

$$\cdot \frac{1}{P(v)} \{n_j - v(e_j) - v(e_j+e_k)\}.$$

So we have

$$(26) \quad C(v) = \begin{cases} 0 & \text{, if } v(e_k) < 2, \\ Q(v)\{2n-v(e_k)-2\}\,\{n_j - v(e_j) - v(e_j+e_k)\} & \text{, if } v(e_k) \geqslant 2. \end{cases}$$

If $v(e_j)v(e_k) > 0$, then by (22),

$$D(v)$$

$$= -\sum_{\substack{v(p)>0 \\ p\notin\{e_j+e_k,2e_k\} \\ p_k>0,\, \sum p_i \geqslant 2}} \frac{(-1)^{v(e_k)-1}\{v(e_k)-1\}!\,\{2n-v(e_k)-3\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!}{P(v)/\{v(e_j)v(e_k)p_k v(p)\}}$$

$$= (-1)^{v(e_k)} v(e_k)!\,\{2n-v(e_k)-3\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!$$

$$\cdot \frac{1}{P(v)} v(e_j) \sum_{\substack{v(p)>0 \\ p\notin\{e_j+e_k,2e_k\} \\ p_k>0,\, \sum p_i \geqslant 2}} p_k v(p)$$

$$= (-1)^{v(e_k)} v(e_k)!\,\{2n-v(e_k)-3\}!\,n_1!\cdots(n_j-1)!\cdots n_{k-1}!$$

$$\cdot \frac{1}{P(v)} v(e_j)\{2n-2-v(e_k)-v(e_j+e_k)-2v(2e_k)\}.$$

So we have

$$D(v)=\begin{cases} 0 & \text{, if } v(e_k)=0, \\ Q(v)v(e_j)\{2n-v(e_k)-v(e_j+e_k)-2v(2e_k)-2\} & \text{, if } v(e_k)>0. \end{cases} \tag{27}$$

1. Case of $v(e_k) = 0$.

In this case, we get

$$\begin{aligned} A(v)+B(v)+C(v)+D(v) &= B(v) \\ &= Q(v)\cdot 2v(e_j)v(2e_k) \\ &= Q(v)\cdot 2n_j(n-1) \\ &= \nu(v), \end{aligned}$$

by Lemma 1.

2. Case of $v(e_k) = 1$.

In this case, we have

$$C(v) = 0.$$

So that

$$\begin{aligned} &A(v)+B(v)+C(v)+D(v) = A(v)+B(v)+D(v) \\ &= Q(v)\,[\,v(e_j+e_k)\,\{2n+v(e_j)-v(e_k)-2\}+2v(e_j)v(2e_k) \\ &\qquad +v(e_j)\,\{2n-v(e_k)-v(e_j+e_k)-2v(2e_k)-2\}\,] \\ &= Q(v)\,\{2n-v(e_k)-2\}\{v(e_j)+v(e_j+e_k)\}. \end{aligned}$$

For the proof of (23), it is sufficient to show that

$$v(e_j)+v(e_j+e_k)=n_j. \tag{28}$$

If $v \in \mathfrak{P}'$ coincides with type(i) of Lemma 2, then

$$(v(e_j), v(e_j+e_k)) = (n_j-1, 1) \text{ or } (n_j, 0),$$

according to $h = j$ or $h \neq j$, respectively. In this case, (28) holds.

If $v \in \mathfrak{P}'$ coincides with type(ii) of Lemma 2. then

$$(v(e_j), v(e_j+e_k)) = (n_j, 0),$$

In this case, also (28) holds.

3. Case of $v(e_k) \geqslant 2$.

From (24)–(27), we get

$$
\begin{aligned}
&A(v)+B(v)+C(v)+D(v)\\
&\quad = Q(v)\,\big[\,v(e_j+e_k)\,\{2n+v(e_j)-v(e_k)-2\}+2v(e_j)v(2e_k)\\
&\qquad\qquad +\{2n-v(e_k)-2\}\,\{n_j-v(e_j)-v(e_j+e_k)\}\\
&\qquad\qquad +v(e_j)\,\{2n-v(e_k)-v(e_j+e_k)-2v(2e_k)-2\}\,\big]\\
&\quad = Q(v)\,\{2n-v(e_k)-2\}\,n_j\\
&\quad = \nu(v).
\end{aligned}
$$

q.e.d.

Reference

[1] Andrews, G.E., The Theory of Partitions, Addison-Wesley Publishing Co., 1976.

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APPENDIX

Table of the case $k = 2$ in the theorem

$$[\ f(x,y) = 0,\ y = g(x)\ ]$$

$$g^{(n)}(x) = -\frac{1}{{f_y}^{2n-1}} \sum_{u \in \mathfrak{P}_{2n-1}(n,2n-2)} \nu(u) f(u; x, y)$$

• $n = 1$:

$$\begin{array}{rccc} \mathfrak{P}_1(1,0) \ni u : & \begin{pmatrix} & 0 \\ 1 & \end{pmatrix} \\ \nu(u) : & 1 \\ f(u;x,y) : & f_x \end{array} \qquad g'(x) = -\frac{f_x}{f_y}$$

• $n = 2$:

$$\begin{array}{rccc} \mathfrak{P}_3(2,2) \ni u : & \begin{pmatrix} & 2 & 0 \\ 0 & 0 & \\ 1 & & \end{pmatrix} & \begin{pmatrix} & 1 & 0 \\ 1 & 1 & \\ 0 & & \end{pmatrix} & \begin{pmatrix} & 0 & 1 \\ 2 & 0 & \\ 0 & & \end{pmatrix} \\ \nu(u) : & 1 & -2 & 1 \\ f(u;x,y) : & {f_y}^2 f_{xx} & f_x f_y f_{xy} & {f_x}^2 f_{yy} \end{array}$$

$$g''(x) = -\frac{1}{{f_x}^3} \left({f_y}^2 f_{xx} - 2 f_x f_y f_{xy} + {f_x}^2 f_{yy} \right)$$

• $n = 3$: $[\, u \in \mathfrak{P}_5(3,4)\,]$

$$\begin{array}{rccccc} \nu(u) : & 1 & -3 & 3 & -1 & \\ f(u;x,y) : & {f_y}^4 f_{xxx} & f_x {f_y}^3 f_{xxy} & {f_x}^2 {f_y}^2 f_{xyy} & {f_x}^3 f_y f_{yyy} & \\ \nu(u) : & -3 & 6 & 3 & -9 & 3 \\ f(u;x,y) : & {f_y}^3 f_{xx} f_{xy} & f_x {f_y}^2 {f_{xy}}^2 & f_x {f_y}^2 f_{xx} f_{yy} & {f_x}^2 f_y f_{xy} f_{yy} & {f_x}^3 {f_{yy}}^2 \end{array}$$

$$g^{(3)}(x) = -\frac{1}{{f_y}^5} \sum_{u \in \mathfrak{P}_5(3,4)} \nu(u) f(u; x, y)$$

• $n = 4 : [\, u \in \mathfrak{P}_7(4,6)\,]$

$\nu(u)$:	1	-4	6	-4	1
$f(u)$:	$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}4\\0\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}3\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}2\\2\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}1\\3\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\4\end{smallmatrix}\right|$ ¶

-4	-6	4	24	12
$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}3\\0\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}2\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}3\\0\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}2\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\2\end{smallmatrix}\right|$

-18	-36	-6	24	16	-10
$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}2\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\2\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\3\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\2\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\3\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\3\end{smallmatrix}\right|$

3	12	-24	-36
$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|^{2}$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|^{3}$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|$

18	72	-60	15
$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|^{2}$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|^{2}$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|^{3}$

• $n = 5 : [\, u \in \mathfrak{P}_9(5,8)\,]$

$\nu(u)$:	1	-5	10	-10	5	-1
$f(u)$:	$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{8}\left.\begin{smallmatrix}5\\0\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{7}\left.\begin{smallmatrix}4\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}3\\2\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}2\\3\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}1\\4\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\5\end{smallmatrix}\right|$

-5	-10	5	40	30
$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{7}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}4\\0\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{7}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}3\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}4\\0\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}3\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}2\\2\end{smallmatrix}\right|$

-30	-90	-30	60	80	10
$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}3\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}2\\2\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\3\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}2\\2\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\3\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\4\end{smallmatrix}\right|$

-50	-25	15
$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\3\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{4}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{3}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\4\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}1\\0\end{smallmatrix}\right|^{5}\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\4\end{smallmatrix}\right|$

20	10	60	15
$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}3\\0\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}0\\2\end{smallmatrix}\right|\left.\begin{smallmatrix}3\\0\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|\left.\begin{smallmatrix}1\\1\end{smallmatrix}\right|\left.\begin{smallmatrix}2\\1\end{smallmatrix}\right|$	$\left.\begin{smallmatrix}0\\1\end{smallmatrix}\right|^{6}\left.\begin{smallmatrix}2\\0\end{smallmatrix}\right|^{2}\left.\begin{smallmatrix}1\\2\end{smallmatrix}\right|$

¶ $\left.\begin{smallmatrix}\nu\\\mu\end{smallmatrix}\right| = f^{(\nu,\mu)} = \frac{\partial^{\nu}}{\partial x^{\nu}}\frac{\partial^{\mu}}{\partial y^{\mu}} f$

$$\begin{matrix}-60\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{5}\left.{1\atop 1}\right|\left.{0\atop 2}\right|\left.{3\atop 0}\right|\end{matrix}\qquad \begin{matrix}-180\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{5}\left.{1\atop 1}\right|^{2}\left.{2\atop 1}\right|\end{matrix}\qquad \begin{matrix}-90\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{5}\left.{2\atop 0}\right|\left.{0\atop 2}\right|\left.{2\atop 1}\right|\end{matrix}\qquad \begin{matrix}-180\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{5}\left.{2\atop 0}\right|\left.{1\atop 1}\right|\left.{1\atop 2}\right|\end{matrix}\qquad \begin{matrix}-15\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{5}\left.{2\atop 0}\right|^{2}\left.{0\atop 3}\right|\end{matrix}$$

$$\begin{matrix}30\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{4}\left.{0\atop 2}\right|^{2}\left.{3\atop 0}\right|\end{matrix}\qquad \begin{matrix}360\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{4}\left.{1\atop 1}\right|\left.{0\atop 2}\right|\left.{2\atop 1}\right|\end{matrix}\qquad \begin{matrix}360\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{4}\left.{1\atop 1}\right|^{2}\left.{1\atop 2}\right|\end{matrix}\qquad \begin{matrix}180\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{4}\left.{2\atop 0}\right|\left.{0\atop 2}\right|\left.{1\atop 2}\right|\end{matrix}\qquad \begin{matrix}120\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{4}\left.{2\atop 0}\right|\left.{1\atop 1}\right|\left.{0\atop 3}\right|\end{matrix}$$

$$\begin{matrix}-150\\ \left.{1\atop 0}\right|^{3}\left.{0\atop 1}\right|^{3}\left.{0\atop 2}\right|^{2}\left.{2\atop 1}\right|\end{matrix}\qquad \begin{matrix}-600\\ \left.{1\atop 0}\right|^{3}\left.{0\atop 1}\right|^{3}\left.{1\atop 1}\right|\left.{0\atop 2}\right|\left.{1\atop 2}\right|\end{matrix}\qquad \begin{matrix}-200\\ \left.{1\atop 0}\right|^{3}\left.{0\atop 1}\right|^{3}\left.{1\atop 1}\right|^{2}\left.{0\atop 3}\right|\end{matrix}\qquad \begin{matrix}-100\\ \left.{1\atop 0}\right|^{3}\left.{0\atop 1}\right|^{3}\left.{1\atop 1}\right|^{2}\left.{0\atop 3}\right|\end{matrix}$$

$$\begin{matrix}225\\ \left.{1\atop 0}\right|^{4}\left.{0\atop 1}\right|^{2}\left.{0\atop 2}\right|^{2}\left.{1\atop 2}\right|\end{matrix}\qquad \begin{matrix}300\\ \left.{1\atop 0}\right|^{4}\left.{0\atop 1}\right|^{2}\left.{1\atop 1}\right|\left.{0\atop 2}\right|\left.{0\atop 3}\right|\end{matrix}\qquad \begin{matrix}-150\\ \left.{1\atop 0}\right|^{5}\left.{0\atop 1}\right|\left.{0\atop 2}\right|^{2}\left.{0\atop 3}\right|\end{matrix}$$

$$\begin{matrix}-10\\ \left.{0\atop 1}\right|^{7}\left.{3\atop 0}\right|\left.{2\atop 1}\right|\end{matrix}\qquad \begin{matrix}30\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{6}\left.{2\atop 1}\right|^{2}\end{matrix}\qquad \begin{matrix}20\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{6}\left.{3\atop 0}\right|\left.{1\atop 2}\right|\end{matrix}\qquad \begin{matrix}-90\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{5}\left.{2\atop 1}\right|\left.{1\atop 2}\right|\end{matrix}\qquad \begin{matrix}-10\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{5}\left.{3\atop 0}\right|\left.{0\atop 3}\right|\end{matrix}$$

$$\begin{matrix}60\\ \left.{1\atop 0}\right|^{3}\left.{0\atop 1}\right|^{4}\left.{1\atop 2}\right|^{2}\end{matrix}\qquad \begin{matrix}40\\ \left.{1\atop 0}\right|^{3}\left.{0\atop 1}\right|^{4}\left.{2\atop 1}\right|\left.{0\atop 3}\right|\end{matrix}\qquad \begin{matrix}-50\\ \left.{1\atop 0}\right|^{4}\left.{0\atop 1}\right|^{3}\left.{1\atop 2}\right|\left.{0\atop 3}\right|\end{matrix}\qquad \begin{matrix}10\\ \left.{1\atop 0}\right|^{5}\left.{0\atop 1}\right|^{2}\left.{0\atop 3}\right|^{2}\end{matrix}$$

$$\begin{matrix}-60\\ \left.{0\atop 1}\right|^{5}\left.{2\atop 0}\right|\left.{1\atop 1}\right|^{3}\end{matrix}\qquad \begin{matrix}-45\\ \left.{0\atop 1}\right|^{5}\left.{2\atop 0}\right|^{2}\left.{1\atop 1}\right|\left.{0\atop 2}\right|\end{matrix}\qquad \begin{matrix}120\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{4}\left.{1\atop 1}\right|^{4}\end{matrix}\qquad \begin{matrix}360\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{4}\left.{2\atop 0}\right|\left.{1\atop 1}\right|^{2}\left.{0\atop 2}\right|\end{matrix}\qquad \begin{matrix}45\\ \left.{1\atop 0}\right|\left.{0\atop 1}\right|^{4}\left.{2\atop 0}\right|^{2}\left.{0\atop 2}\right|^{2}\end{matrix}$$

$$\begin{matrix}-600\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{3}\left.{1\atop 1}\right|^{3}\left.{0\atop 2}\right|\end{matrix}\qquad \begin{matrix}-450\\ \left.{1\atop 0}\right|^{2}\left.{0\atop 1}\right|^{3}\left.{2\atop 0}\right|\left.{1\atop 1}\right|\left.{0\atop 2}\right|^{2}\end{matrix}\qquad \begin{matrix}900\\ \left.{1\atop 0}\right|^{3}\left.{0\atop 1}\right|^{2}\left.{1\atop 1}\right|^{2}\left.{0\atop 2}\right|^{2}\end{matrix}\qquad \begin{matrix}150\\ \left.{1\atop 0}\right|^{3}\left.{0\atop 1}\right|^{2}\left.{2\atop 0}\right|\left.{0\atop 2}\right|^{3}\end{matrix}$$

$$\begin{matrix}-525\\ \left.{1\atop 0}\right|^{4}\left.{0\atop 1}\right|\left.{1\atop 1}\right|\left.{0\atop 2}\right|^{3}\end{matrix}\qquad \begin{matrix}105\\ \left.{1\atop 0}\right|^{5}\left.{0\atop 2}\right|^{4}\end{matrix}$$