Functional equations for Appell's F_1 arising from transformations of elliptic curves

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Abstract

We give a functional equation for Appell's hypergeometric function F_1, which arises from transformations of elliptic curves. As an application, we give an efficient algorithm for computing incomplete elliptic integrals of the first kind. We also give a reduction formula that simplifies Lauricella's hypergeometric function F_D of five variables to F_1.

Key Words and Phrases. Hypergeometric Functions, Transformation Formulas, Incomplete Elliptic Integrals of the First Kind.

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1 Introduction

It is classically known that Gauss' hypergeometric function $F(\alpha, \beta, \gamma; z)$ satisfies the transformation formula

$$(1+2z)^{2p} \cdot F\left(p, p-q+\frac{1}{2}, q+\frac{1}{2}; z^2\right) = F\left(p, q, 2q; \frac{4z}{(1+z)^2}\right).$$

By this transformation formula and an Euler-type integral representation of $F(\alpha, \beta, \gamma; z)$, we can express the arithmetic-geometric mean of $(a, b) \in (\mathbb{R}_{>0})^2$ by a complete elliptic integral of the first kind, where $\mathbb{R}_{>0}$ is the set of positive real numbers. Transformation formulas for other hypergeometric functions are also applied to the study of iterations of several means of several terms. For example, in [4] it is shown that a transformation formula for Appell's hypergeometric function F_1 implies three means of three terms and that the triple of sequences defined by the iteration of these means converges and has a common limit expressed by an incomplete elliptic integral of the first kind.

In this paper, we find a new transformation formula for Appell's hypergeometric function F_1 by considering transformations of elliptic curves. Our main theorem (Theorem 3.1) is as follows:

$$F_1\left(1,\frac{1}{2},p,p+1;1-z_1^2,1-z_2^2\right)=\frac{1}{z_1}F_1\left(1,p,p,p+1;1-w_1,1-w_2\right),$$
$$w_1=\frac{z_1+z_2^2+\sqrt{(1-z_2^2)(z_1^2-z_2^2)}}{2z_1},\quad w_2=\frac{z_1+z_2^2-\sqrt{(1-z_2^2)(z_1^2-z_2^2)}}{2z_1}$$

We prove this formula by using the integration by substitution that corresponds to the isogeny map. We apply our theorem to the computation of incomplete elliptic integrals. By our transformation formula, we define a map $(\mathbb{R}_{>0})^3 \to (\mathbb{R}_{>0})^3$, the iteration of which implies a triple $(a_n, b_n, c_n)_{n\in\mathbb{N}}$ of sequences. It turns out that the sequences converge and satisfy

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \neq \lim_{n\to\infty} c_n$$

for general initial terms. An incomplete elliptic integral of the first kind can be expressed in terms of these limits. Since their convergence is quadratic, we thus obtain an efficient algorithm for computing incomplete elliptic integrals. As has been mentioned above, there are several extensions and analogies of the arithmetic-geometric mean; each of them is based on a common limit of a multiple of sequences. This example suggests to us the application of the iterations of a mapping, even if the resulting sequences do not have a common limit.

The contents of this paper are as follows. First, we describe transformations of elliptic curves in terms of the theta functions by using the results in [3], and we give expressions for the isogeny and the doubling map, which are convenient for our study. Next, we prove the main theorem by using the expression of the isogeny, and we explain the algorithm for computing incomplete elliptic integrals of the first kind. Finally, we consider a triple of sequences given by the transformation formula in [4]:

$$\begin{aligned}&\frac{z_1+z_2}{2}F_1\left(1,\frac{1}{2},\frac{1}{2},\frac{3}{2};1-z_1^2,1-z_2^2\right)\\&=F_1\left(1,\frac{1}{2},\frac{1}{2},\frac{3}{2};1-\frac{z_1(1+z_2)}{z_1+z_2},1-\frac{z_2(1+z_1)}{z_1+z_2}\right)\end{aligned}$$

(equivalent to Proposition 5.3). By calculating an elliptic integral by using a substitution that arises from the doubling map, we give another proof of this formula and also a reduction formula of Lauricella's hypergeometric function F_D of five variables to Appell's hypergeometric function F_1.

2 Elliptic curves and complex tori

We begin by reviewing some results in [3].

2.1 Abel–Jacobi map

We consider the elliptic curves

$$C(\lambda): y^2 = x(1-x)(1-\lambda x),\ \lambda \in \mathbb{C} - \{0,1\},$$

which are double coverings of the complex projective line $\mathbb{P}^1$. We choose a symplectic basis $A,\ B \in H_1(C(\lambda), \mathbb{Z})$ so that $A \cdot A = B \cdot B = 0,\ B \cdot A = 1$, and

$$\int_A \frac{dx}{y} = 2\int_1^{\frac{1}{\lambda}} \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}} \in i\,\mathbb{R}_{>0},$$
$$\int_B \frac{dx}{y} = 2\int_0^{1} \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}} \in \mathbb{R}_{>0},$$

when λ is in the open interval $(0,1)$. We set

$$\tau_A := \int_A \frac{dx}{y},\ \tau_B := \int_B \frac{dx}{y},\ \tau := \frac{\tau_A}{\tau_B};$$

note that τ belongs to the upper half-plane $\mathbb{H}$. Let $L(\tau)$ be the lattice $\mathbb{Z}\tau + \mathbb{Z}$; then the complex torus $E(\tau) := \mathbb{C}/L(\tau)$ is isomorphic to $C(\lambda)$ by the Abel–Jacobi map

$$\Phi : C(\lambda) \longrightarrow E(\tau);\ P \mapsto \frac{1}{\tau_B}\int_{P_\infty}^{P} \frac{dx}{y} \mod L(\tau),$$

where P_∞ is the point at infinity in $C(\lambda)$. We represent the inverse map of Φ by the theta functions with half-integral characteristics. For $a, b \in \{0,1\}$, the theta function is defined by

$$\vartheta_{ab}(z,\tau) := \sum_{n\in\mathbb{Z}} \exp\left(\pi i\left(n+\frac{a}{2}\right)^2 \tau + 2\pi i\left(n+\frac{a}{2}\right)\left(z+\frac{b}{2}\right)\right),$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$. We denote $\vartheta_{ab}(0,\tau)$ by $\vartheta_{ab}(\tau)$.

Proposition 2.1 ([3]). *The inverse map of Φ is expressed as follows:*

$$\Phi^{-1}([z]) = \left(\frac{\vartheta_{00}(\tau)^2\,\vartheta_{01}(z,\tau)^2}{\vartheta_{10}(\tau)^2\,\vartheta_{11}(z,\tau)^2}, \frac{\vartheta_{01}(\tau)^2\,\vartheta_{00}(z,\tau)\vartheta_{01}(z,\tau)\vartheta_{10}(z,\tau)}{\vartheta_{10}(\tau)^2\,\vartheta_{11}(z,\tau)^3}\right),$$

where $[z]$ *means the element of* $E(\tau)$ *represented by* $z \in \mathbb{C}$. *Further the parameter* λ *of the elliptic curve* $C(\lambda)$ *is expressed as*

$$\lambda = \frac{\vartheta_{10}(\tau)^4}{\vartheta_{00}(\tau)^4} = 1 - \frac{\vartheta_{01}(\tau)^4}{\vartheta_{00}(\tau)^4}.$$

2.2 Maps between elliptic curves

We use the following formulas from [2] and [5].

Facts 2.2.

(1) $\vartheta_{00}(\tau)^3\vartheta_{00}(2z,\tau) = \vartheta_{00}(z,\tau)^4 + \vartheta_{11}(z,\tau)^4$,

(2) $\vartheta_{01}(\tau)^3\vartheta_{01}(2z,\tau) = \vartheta_{01}(z,\tau)^4 - \vartheta_{11}(z,\tau)^4$,

(3) $\vartheta_{10}(\tau)^3\vartheta_{10}(2z,\tau) = \vartheta_{10}(z,\tau)^4 - \vartheta_{11}(z,\tau)^4$,

(4) $\vartheta_{00}(\tau)^2\vartheta_{01}(\tau)\vartheta_{01}(2z,\tau) = \vartheta_{00}(z,\tau)^2\vartheta_{01}(z,\tau)^2 + \vartheta_{10}(z,\tau)^2\vartheta_{11}(z,\tau)^2$,

(5) $\vartheta_{00}(\tau)\vartheta_{01}(\tau)\vartheta_{10}(\tau)\vartheta_{11}(2z,\tau) = 2\vartheta_{00}(z,\tau)\vartheta_{01}(z,\tau)\vartheta_{10}(z,\tau)\vartheta_{11}(z,\tau)$,

(6) $2\vartheta_{00}(2\tau)\vartheta_{00}(2z,2\tau) = \vartheta_{00}(z,\tau)^2 + \vartheta_{01}(z,\tau)^2$,

(7) $\vartheta_{01}(2\tau)\vartheta_{01}(2z,2\tau) = \vartheta_{00}(z,\tau)\vartheta_{01}(z,\tau)$.

We consider the isogeny and the translation by $\frac{\tau}{2}$:

$$pr : E(2\tau) \longrightarrow E(\tau);\ z \bmod L(2\tau) \mapsto z \bmod L(\tau),$$
$$T_{\frac{\tau}{2}} : E(\tau) \longrightarrow E(\tau);\ z \bmod L(\tau) \mapsto z + \frac{\tau}{2} \bmod L(\tau).$$

By Proposition 2.1, $E(2\tau)$ is isomorphic to the elliptic curve $C(\lambda')$ with

$$\lambda' = 1 - \frac{\vartheta_{01}(2\tau)^4}{\vartheta_{00}(2\tau)^4} = 1 - \frac{\vartheta_{00}(\tau)^2\vartheta_{01}(\tau)^2}{\left(\frac{\vartheta_{00}(\tau)^2+\vartheta_{01}(\tau)^2}{2}\right)^2} = \left(\frac{\vartheta_{00}(\tau)^2 - \vartheta_{01}(\tau)^2}{\vartheta_{00}(\tau)^2 + \vartheta_{01}(\tau)^2}\right)^2,$$

where the second equality follows from (6) and (7). Via the Abel–Jacobi maps, pr and $T_{\frac{\tau}{2}}$ induce $\widetilde{pr} : C(\lambda') \to C(\lambda)$ and $\widetilde{T_{\frac{\tau}{2}}} : C(\lambda) \to C(\lambda)$, respectively.

Proposition 2.3 ([3]). *We have*

(i) $\widetilde{pr}(x', y') = \left(\frac{(\sqrt{\lambda'}x'+1)^2}{4\sqrt{\lambda'}x'}, \frac{\sqrt{\lambda'}(1+\sqrt{1-\lambda})}{8}\left(1 - \frac{1}{\lambda' x'^2}\right)y'\right)$, *where*

$$\sqrt{\lambda'} = \frac{\vartheta_{10}(2\tau)^2}{\vartheta_{00}(2\tau)^2} = \frac{\vartheta_{00}(2\tau)^2 - \vartheta_{01}(2\tau)^2}{\vartheta_{00}(2\tau)^2 + \vartheta_{01}(2\tau)^2},\quad \sqrt{1-\lambda} = \frac{\vartheta_{01}(\tau)^2}{\vartheta_{00}(\tau)^2},$$

(ii) $\widetilde{T_{\frac{\tau}{2}}}(x,y) = \left(\frac{1}{\lambda x}, -\frac{y}{\lambda x^2}\right)$,

(iii) $\widetilde{T_{\frac{\tau}{2}}} \circ \widetilde{pr}(x',y') = \left(\frac{4\sqrt{\lambda'}x'}{\lambda(\sqrt{\lambda'}x'+1)^2}, \frac{-2\sqrt{\lambda'}(\sqrt{\lambda'}x'-1)y'}{(1-\sqrt{1-\lambda})(\sqrt{\lambda'}x'+1)^3}\right)$.

We consider the map $\psi : C(\lambda) \longrightarrow C(\lambda)$ induced from

$$E(\tau) \longrightarrow E(\tau);\ z \bmod L(\tau) \longmapsto 2z \bmod L(\tau)$$

via the Abel–Jacobi map Φ. The following proposition appears in some textbooks on elliptic curves (e.g., [6]). However, we give our proof using the theta functions, because this representation of x is key to the study of section 5.

Proposition 2.4. *The map $\psi : C(\lambda) \longrightarrow C(\lambda)$ is represented as follows:*

$$\psi(x',y') = \left(\frac{(1-\lambda x'^2)^2}{4\lambda x'(1-x')(1-\lambda x')}, \frac{(\lambda x'^2-1)(\lambda x'^2-2x'+1)(\lambda x'^2-2\lambda x'+1)}{8\lambda y'^3}\right).$$

Proof. Letting $\psi(x',y') = (x,y)$, we then have

$$\begin{aligned} x &= \frac{\vartheta_{00}(\tau)^2}{\vartheta_{10}(\tau)^2}\frac{\vartheta_{01}(2z,\tau)^2}{\vartheta_{11}(2z,\tau)^2} \\ &= \frac{1}{4}\left(\frac{\vartheta_{00}(z,\tau)^2\vartheta_{01}(z,\tau)^2}{\vartheta_{10}(z,\tau)^2\vartheta_{11}(z,\tau)^2} + 2 + \frac{\vartheta_{10}(z,\tau)^2\vartheta_{11}(z,\tau)^2}{\vartheta_{00}(z,\tau)^2\vartheta_{01}(z,\tau)^2}\right) \\ &= \frac{1}{4}\left(\frac{\lambda x'(1-x')}{1-\lambda x'} + 2 + \frac{1-\lambda x'}{\lambda x'(1-x')}\right) = \frac{1}{4}\cdot\frac{(1-\lambda x'^2)^2}{\lambda x'(1-x')(1-\lambda x')}, \end{aligned}$$

by (4) and (5). Similarly, we obtain the expression of y by applying (1), (2), and (3). □

3 Transformation formula for Appell's hypergeometric function F_1

Appell's hypergeometric function F_1 of two variables z_1, z_2 with parameters $\alpha, \beta_1, \beta_2, \gamma$ is defined as

$$F_1(\alpha,\beta_1,\beta_2,\gamma;z_1,z_2) = \sum_{n_1,n_2=0}^{\infty} \frac{(\alpha,n_1+n_2)(\beta_1,n_1)(\beta_2,n_2)}{(\gamma,n_1+n_2)(1,n_1)(1,n_2)} z_1^{n_1} z_2^{n_2},$$

where z_j's satisfy $|z_j| < 1$, $\gamma \neq 0, -1, -2, \ldots$, and $(\alpha, n) = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$. This function admits an Euler-type integral representation:

$$F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2)$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^\alpha (1-t)^{\gamma-\alpha} (1-z_1 t)^{-\beta_1} (1-z_2 t)^{-\beta_2} \frac{dt}{t(1-t)}.$$

Theorem 3.1. *We have a transformation formula for F_1:*

$$(8) \quad F_1\left(1, \frac{1}{2}, p, p+1; 1-z_1^2, 1-z_2^2\right) = \frac{1}{z_1} F_1\left(1, p, p, p+1; 1-w_1, 1-w_2\right),$$
$$w_1 = \frac{z_1+z_2^2+\sqrt{(1-z_2^2)(z_1^2-z_2^2)}}{2z_1}, \quad w_2 = \frac{z_1+z_2^2-\sqrt{(1-z_2^2)(z_1^2-z_2^2)}}{2z_1},$$

where (z_1, z_2) is in a small neighborhood of $(1,1)$.

Remark 3.2. *If we choose another branch of $\sqrt{(1-z_2^2)(z_1^2-z_2^2)}$, then w_1 and w_2 are interchanged. By $F_1(\alpha, \beta, \beta, \gamma; z_1, z_2) = F_1(\alpha, \beta, \beta, \gamma; z_2, z_1)$, the right-hand side of (8) is independent of the choice of the branch of the square root.*

Proof of Theorem 3.1. Replace $1-z_1^2$ and $1-z_2^2$ with z_1 and z_2, respectively, and use the integral representation for F_1. Then it is sufficient to show that

$$(9) \qquad \int_0^1 (1-t)^{p-1}(1-z_1 t)^{-\frac{1}{2}}(1-z_2 t)^{-p} dt$$
$$= \frac{1}{\sqrt{1-z_1}} \int_0^1 (1-t')^{p-1}(1-w_1 t')^{-p}(1-w_2 t')^{-p} dt'$$

for $z_1, z_2 \in \mathbb{R}$ satisfying $0 < z_1 < z_2 < 1$, where

$$w_1 = 1 - \frac{\sqrt{1-z_1} + 1 - z_2 + \sqrt{z_2(z_2-z_1)}}{2\sqrt{1-z_1}},$$
$$w_2 = 1 - \frac{\sqrt{1-z_1} + 1 - z_2 - \sqrt{z_2(z_2-z_1)}}{2\sqrt{1-z_1}}.$$

To prove the identity (9), we use three kinds of substitutions. By the first substitution

$$t = \frac{1-z_2}{z_2} x + 1,$$

we have

$$\frac{dt}{dx} = \frac{1-z_2}{z_2},\ 1-t = -\frac{1-z_2}{z_2}x,$$

$$1-z_1t = (1-z_1)\left(1-\frac{(1-z_2)z_1}{(1-z_1)z_2}x\right),\ 1-z_2t = (1-z_2)(1-x),$$

$$(10)\qquad \int_0^1 (1-t)^{p-1}(1-z_1t)^{-\frac{1}{2}}(1-z_2t)^{-p}dt = \frac{(1-z_1)^{-\frac{1}{2}}}{z_2\cdot(-z_2)^{p-1}}\int_{R_1}^0 x^{p-1}(1-x)^{-p}(1-\lambda x)^{-\frac{1}{2}}dx,$$

where

$$\lambda = \frac{(1-z_2)z_1}{(1-z_1)z_2} = 1-\frac{z_2-z_1}{(1-z_1)z_2},\ R_1 = -\frac{z_2}{1-z_2}.$$

We set

$$\lambda' = \left(\frac{1-\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}}\right)^2 = \left(\frac{1-\sqrt{\frac{z_2-z_1}{(1-z_1)z_2}}}{1+\sqrt{\frac{z_2-z_1}{(1-z_1)z_2}}}\right)^2$$

and consider the integral in the right-hand side of (10) by the second substitution

$$x = \frac{4\sqrt{\lambda'}x'}{\lambda(\sqrt{\lambda'}x'+1)^2}$$

in Proposition 2.3 (iii). If $x = 0$, then $x' = 0$. On the other hand, the equation

$$R_1 = \frac{4\sqrt{\lambda'}x'}{\lambda(\sqrt{\lambda'}x'+1)^2}$$

has two solutions

$$x' = R_2^{\pm} := \frac{-\lambda R_1 + 2(1\pm\sqrt{1-\lambda R_1})}{\lambda\sqrt{\lambda'}R_1}.$$

Since $R_1 < 0$, the inequality $R_2^+ < R_2^- < 0$ holds. Hence the integral interval

$[R_1, 0]$ for x is changed to the integral interval $[R_2^-, 0]$ for x'. We have

$$\begin{aligned}
\frac{dx}{dx'} &= \frac{4\sqrt{\lambda'}(1-\sqrt{\lambda'}x')}{\lambda(\sqrt{\lambda'}x'+1)^3}, \\
1-x &= \frac{\lambda(\sqrt{\lambda'}x'+1)^2 - 4\sqrt{\lambda'}x'}{\lambda(\sqrt{\lambda'}x'+1)^2} \\
&= \frac{1}{(\sqrt{\lambda'}x'+1)^2}\left(1+\frac{2(\lambda-2)}{(1+\sqrt{1-\lambda})^2}x' + \lambda' x'^2\right) \\
&= \frac{1}{(\sqrt{\lambda'}x'+1)^2}(1-x')(1-\lambda'x'), \\
1-\lambda x &= \frac{\lambda(\sqrt{\lambda'}x'+1)^2 - 4\lambda\sqrt{\lambda'}x'}{\lambda(\sqrt{\lambda'}x'+1)^2} = \left(\frac{\sqrt{\lambda'}x'-1}{\sqrt{\lambda'}x'+1}\right)^2
\end{aligned}$$

Note that if $R_2^- < x' < 0$, then $\sqrt{\lambda'}x'+1>0$ and $\sqrt{\lambda'}x'-1<0$. Thus the identity (10) is equivalent to

$$\begin{aligned}
(11)\quad & \int_0^1 (1-t)^{p-1}(1-z_1t)^{-\frac{1}{2}}(1-z_2t)^{-p}dt \\
&= \frac{(1-z_1)^{-\frac{1}{2}}}{z_2\cdot(-z_2)^{p-1}} \frac{2^{2p}}{\left(1+\sqrt{\frac{z_2-z_1}{(1-z_1)z_2}}\right)^{2p}} \int_{R_2}^0 x'^{p-1}(1-x')^{-p}(1-\lambda'x')^{-p}dx'.
\end{aligned}$$

Finally, we consider the integral in the right-hand side of (11) by the third substitution

$$x' = -R_2^- t' + R_2^-.$$

Then it follows that

$$\begin{aligned}
& \frac{dx'}{dt'} = -R_2^-,\ x' = R_2^-(1-t'),\ 1-x' = (1-R_2^-)\left(1-\frac{-R_2^-}{1-R_2^-}t'\right), \\
& 1-\lambda'x' = (1-\lambda'R_2^-)\left(1-\frac{-\lambda'R_2^-}{1-\lambda'R_2^-}t'\right).
\end{aligned}$$

Using $\lambda R_1 = -\frac{z_1}{1-z_1}$, we calculate $\sqrt{\lambda'}$ and R_2^-:

$$\begin{aligned}
\sqrt{\lambda'} &= \frac{\sqrt{(1-z_1)z_2}-\sqrt{z_2-z_1}}{\sqrt{(1-z_1)z_2}+\sqrt{z_2-z_1}} = \frac{-z_1+2z_2-z_1z_2-2\sqrt{z_2(1-z_1)(z_2-z_1)}}{(1-z_2)z_1}, \\
R_2^- &= \frac{1}{\sqrt{\lambda'}}\left(-1+2\cdot\frac{1-\frac{1}{\sqrt{1-z_1}}}{-\frac{z_1}{1-z_1}}\right) = \frac{-z_1-2(1-z_1-\sqrt{1-z_1})}{\sqrt{\lambda'}z_1} = \frac{z_1-2+2\sqrt{1-z_1}}{\sqrt{\lambda'}z_1}.
\end{aligned}$$

This implies that

$$
\begin{aligned}
1-\frac{-R_2^-}{1-R_2^-} &= \frac{1}{1-R_2^-} = \frac{\sqrt{\lambda'}z_1}{\sqrt{\lambda'}z_1 - z_1 + 2 - 2\sqrt{1-z_1}} \\
&= \frac{1}{2\sqrt{1-z_1}} \cdot \frac{(\sqrt{1-z_1} - \sqrt{z_2(z_2-z_1)})^2 - (1-z_2)^2}{\sqrt{1-z_1} - \sqrt{z_2(z_2-z_1)} - (1-z_2)} \\
&= \frac{\sqrt{1-z_1} - \sqrt{z_2(z_2-z_1)} + 1 - z_2}{2\sqrt{1-z_1}} = 1 - w_2, \\
1-\frac{-\lambda' R_2^-}{1-\lambda' R_2^-} &= \frac{1}{1-\lambda' R_2^-} = \frac{z_1}{z_1 - \sqrt{\lambda'}(z_1 - 2 + 2\sqrt{1-z_1})} \\
&= \frac{\sqrt{1-z_1} + \sqrt{z_2(z_2-z_1)} + 1 - z_2}{2\sqrt{1-z_1}} = 1 - w_1.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\int_0^1 (1-t)^{p-1}(1-z_1t)^{-\frac{1}{2}}(1-z_2t)^{-p}dt \\
&= \frac{(1-z_1)^{-\frac{1}{2}}}{z_2 \cdot (-z_2)^{p-1}} \frac{2^{2p}}{\left(1+\sqrt{\frac{z_2-z_1}{(1-z_1)z_2}}\right)^{2p}} (-R_2^-)(R_2^-)^{p-1} \\
&\quad \cdot (1-R_2^-)^{-p}(1-\lambda' R_2^-)^{-p} \int_0^1 (1-t')^{p-1}(1-w_1t')^{-p}(1-w_2t')^{-p}dt',
\end{aligned}
$$

and hence, to conclude the identity (9), it is sufficient to show the following:

$$
(12)\qquad z_2^{-p}\left(\frac{2}{1+\sqrt{\frac{z_2-z_1}{(1-z_1)z_2}}}\right)^{2p}(-R_2^-)^p(1-R_2^-)^{-p}(1-\lambda' R_2^-)^{-p} = 1.
$$

By these calculations, we obtain

$$
\begin{aligned}
&\frac{-R_2^-}{(1-R_2^-)(1-\lambda' R_2^-)} \\
&= \frac{2 - z_1 - 2\sqrt{1-z_1}}{\sqrt{\lambda'}z_1} \cdot \frac{(\sqrt{1-z_1} + 1 - z_2)^2 - z_2(z_2-z_1)}{4(1-z_1)} \\
&= \frac{(1-z_2)z_1}{4(1-z_1)\sqrt{\lambda'}} = \frac{(1-z_2)z_1}{4(1-z_1)} \frac{\sqrt{(1-z_1)z_2} + \sqrt{z_2-z_1}}{\sqrt{(1-z_1)z_2} - \sqrt{z_2-z_1}} \\
&= \left(\frac{\sqrt{(1-z_1)z_2} + \sqrt{z_2-z_1}}{2\sqrt{1-z_1}}\right)^2 = z_2\left(\frac{1+\sqrt{\frac{z_2-z_1}{(1-z_1)z_2}}}{2}\right)^2,
\end{aligned}
$$

which implies (12). □

4 Triple of sequences and its application to computing elliptic integrals

We now apply Theorem 3.1 to produce an efficient algorithm for computing incomplete elliptic integrals of the first kind. We consider a triple of sequences (a_n, b_n, c_n) where

$$
\begin{aligned}
&(a_0, b_0, c_0) = (a, b, c), \quad a \geq b \geq c > 0, \\
&a_{n+1} := \sqrt{a_n b_n}, \\
&b_{n+1} := \frac{c_n + \sqrt{a_n b_n} + \sqrt{(a_n - c_n)(b_n - c_n)}}{2}, \\
&c_{n+1} := \frac{c_n + \sqrt{a_n b_n} - \sqrt{(a_n - c_n)(b_n - c_n)}}{2}.
\end{aligned} \tag{13}
$$

Lemma 4.1. (i) *The sequences* $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, *and* $\{c_n\}_{n\in\mathbb{N}}$ *converge.*

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n \iff b = c$.

(iv) *If* $b > c$, *then* $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, *and* $\{c_n\}_{n\in\mathbb{N}}$ *converge quadratically.*

Proof. If we assume $a_n \geq b_n \geq c_n > 0$, then we have

$$
\begin{aligned}
a_n - a_{n+1} &= \sqrt{a_n}(\sqrt{a_n} - \sqrt{b_n}) \geq 0, \\
a_{n+1} - b_{n+1} &= \frac{\sqrt{a_n b_n} - c_n - \sqrt{(a_n - c_n)(b_n - c_n)}}{2} \\
&= \frac{c_n\left(a_n + b_n - c_n - \sqrt{a_n b_n} - \sqrt{(a_n - c_n)(b_n - c_n)}\right)}{2(\sqrt{a_n b_n} + \sqrt{(a_n - c_n)(b_n - c_n)})} \\
&\geq \frac{c_n\left(a_n + b_n - c_n - \frac{a_n + b_n}{2} - \frac{a_n - c_n + b_n - c_n}{2}\right)}{2(\sqrt{a_n b_n} + \sqrt{(a_n - c_n)(b_n - c_n)})} = 0, \\
b_{n+1} - b_n &= \frac{\sqrt{b_n}(\sqrt{a_n} - \sqrt{b_n}) + \sqrt{b_n - c_n}(\sqrt{a_n - c_n} - \sqrt{b_n - c_n})}{2} \geq 0, \\
c_{n+1} - c_n &= a_{n+1} - b_{n+1} \geq 0.
\end{aligned}
$$

It follows that

$$
a \geq a_n \geq a_{n+1} \geq b_{n+1} \geq b_n \geq c_n \geq c_{n-1} \geq c \quad (n \geq 1),
$$

which implies (i). By $a_{n+1} = \sqrt{a_n b_n}$, we have (ii). Inequalities

$$b_{n+1} - c_{n+1} = \sqrt{(a_n - c_n)(b_n - c_n)} \geq b_n - c_n \geq b - c \quad (n \in \mathbb{N})$$

show (iii). Since (iii) and

$$\begin{aligned}
a_{n+1} - b_{n+1} &= c_{n+1} - c_n \\
&= \frac{\left(\sqrt{(\sqrt{a_n}-\sqrt{c_n})(\sqrt{b_n}+\sqrt{c_n})} - \sqrt{(\sqrt{a_n}+\sqrt{c_n})(\sqrt{b_n}-\sqrt{c_n})}\right)^2}{4} \\
&= (a_n - b_n)^2 \cdot \frac{c_n}{(\sqrt{a_n}+\sqrt{b_n})^2} \\
&\cdot \left(\sqrt{(\sqrt{a_n}-\sqrt{c_n})(\sqrt{b_n}+\sqrt{c_n})} + \sqrt{(\sqrt{a_n}+\sqrt{c_n})(\sqrt{b_n}-\sqrt{c_n})}\right)^{-2},
\end{aligned}$$

there exists $M > 0$ such that

$$a_{n+1} - b_{n+1} \leq M(a_n - b_n)^2, \quad c_{n+1} - c_n \leq M(c_n - c_{n-1})^2.$$

These inequalities mean (iv). □

Example 4.2. *Let* (a, b, c) *be* $(1,\ 0.5,\ 0.3)$*. The values of* (a_n, b_n, c_n) *and* $[-\log_{10}(a_n - b_n)]$*, computed using Maple version 14, are shown in Table 1, where* $[d]$ *means the largest integer not greater than* d*. Note that the rate of growth of* $[-\log_{10}(a_n - b_n)]$ *means the rapidity of the convergence, because* a_n *and* b_n *are in agreement until the* $[-\log_{10}(a_n - b_n)]$*-th decimal place. Comparing Table 1, below, to Table 2 in section 5, we notice that this triple of sequences converges much faster.*

n	a_n	b_n	c_n
0	1.00000000000000000	0.50000000000000000	0.30000000000000000
1	0.70710678118654752	0.69063625993197083	0.31647052125457669
2	0.69882299814131164	0.69880291625039502	0.31649060314549330
3	0.69881295712371630	0.69881295709385839	0.31649060317535121
4	0.69881295710878734	0.69881295710878734	0.31649060317535121
5	0.69881295710878734	0.69881295710878734	0.31649060317535121

n	1	2	3	4	5	6	7	8	9	10
$[-\log_{10}(a_n - b_n)]$	1	4	10	22	45	92	185	371	744	1490

Table 1: Fast convergence

Theorem 4.3. *For $0 < z_1 < z_2 < 1$, we consider the triple of sequences (a_n, b_n, c_n) with $(a, b, c) = (1, 1 - z_1, 1 - z_2)$ and set*

$$\alpha := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n, \quad \gamma := \lim_{n\to\infty} c_n.$$

Then we have

$$\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}} = \frac{a}{\alpha\sqrt{1-\frac{\gamma}{\alpha}}}\left(\log\left(\frac{\gamma}{\alpha}\right) - 2\log\left(1-\sqrt{1-\frac{\gamma}{\alpha}}\right)\right).$$

Proof. We set $z_1 = \sqrt{\dfrac{b_n}{a_n}}$, $z_2 = \sqrt{\dfrac{c_n}{a_n}}$ and $p = \dfrac{1}{2}$ in Theorem 3.1; then we have

$$\int_0^1 \frac{dt}{\sqrt{(1-t)(1-B_1 t)(1-C_1 t)}} = \sqrt{\frac{a_n}{b_n}}\int_0^1 \frac{dt'}{\sqrt{(1-t')(1-B_2 t')(1-C_2 t')}},$$

$$B_1 = 1 - \frac{b_n}{a_n}, \quad C_1 = 1 - \frac{c_n}{a_n},$$

$$B_2 = 1 - \frac{c_n + \sqrt{a_n b_n} + \sqrt{(a_n - c_n)(b_n - c_n)}}{2\sqrt{a_n b_n}} = 1 - \frac{b_{n+1}}{a_{n+1}},$$

$$C_2 = 1 - \frac{c_n + \sqrt{a_n b_n} - \sqrt{(a_n - c_n)(b_n - c_n)}}{2\sqrt{a_n b_n}} = 1 - \frac{c_{n+1}}{a_{n+1}}.$$

This implies that the function

$$\mu(p, q, r) := \frac{p}{\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t)(1-(1-q/p)t)(1-(1-r/p)t)}}}$$

satisfies $\mu(a_n, b_n, c_n) = \mu(a_{n+1}, b_{n+1}, c_{n+1})$ for all $n \in \mathbb{N}$. Then we obtain

$$\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}} = \int_0^1 \frac{dt}{\sqrt{(1-t)(1-(1-\frac{b}{a})t)(1-(1-\frac{c}{a})t)}}$$

$$= \frac{a}{\mu(a,b,c)} = \frac{a}{\mu(\alpha,\alpha,\gamma)} = \frac{a}{\alpha}\int_0^1 \frac{dt}{\sqrt{(1-t)(1-(1-\frac{\gamma}{\alpha})t)}}$$

$$= \frac{a}{\alpha\sqrt{1-\frac{\gamma}{\alpha}}}\left(\log\left(\frac{\gamma}{\alpha}\right) - 2\log\left(1-\sqrt{1-\frac{\gamma}{\alpha}}\right)\right).$$

□

Theorem 4.3 and Lemma 4.1 (iv) imply an efficient algorithm for computing incomplete elliptic integrals of the first kind:

Algorithm 4.4. *To approximate*

$$\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1t)(1-z_2t)}} \quad (0 < z_1 < z_2 < 1), \tag{14}$$

we evaluate (a_N, b_N, c_N) *in Theorem 4.3 by the recurrence relation (13), where* N *is sufficiently large. Thus* a_N *and* c_N *approximate* α *and* γ*, respectively, and hence an approximation of the integral (14) is evaluated as*

$$\frac{a}{a_N\sqrt{1-\frac{c_N}{a_N}}}\left(\log\left(\frac{c_N}{a_N}\right) - 2\log\left(1-\sqrt{1-\frac{c_N}{a_N}}\right)\right).$$

Remark 4.5. *Note that* N *does not have to be very large, since the convergence of* (a_n, b_n, c_n) *is quadratic by Lemma 4.1 (iv). For example, to evaluate the integral (14) for* $z_1 = 0.5$, $z_2 = 0.7$, *we approximate* α *and* γ *as* a_{10} *and* c_{10}*, respectively, then* $|a_{10} - \alpha|$, $|c_{10} - \gamma| < 10^{-1000}$ *by Example 4.2.*

5 Triple of sequences in [4]

5.1 Triple of sequences and their common limit

We define a triple of sequences (a_n, b_n, c_n) by

$$(a_0, b_0, c_0) = (a, b, c), \; a \geq b \geq c > 0,$$

$$(a_{n+1}, b_{n+1}, c_{n+1}) = \left(\frac{\sqrt{a_n}(\sqrt{b_n}+\sqrt{c_n})}{2}, \frac{\sqrt{b_n}(\sqrt{a_n}+\sqrt{c_n})}{2}, \frac{\sqrt{c_n}(\sqrt{a_n}+\sqrt{b_n})}{2}\right)$$

Fact 5.1 ([4]). *The sequences* $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, *and* $\{c_n\}_{n\in\mathbb{N}}$ *converge and satisfy*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

This common limit of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ is denoted by $m_*^\infty(a, b, c)$.

Theorem 5.2 ([4]). *The common limit of the triple of sequences can be expressed as*

$$m_*^\infty(a, b, c) = \frac{2a}{\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1t)(1-z_2t)}}}, \tag{15}$$

where $z_1 = 1 - \frac{b}{a}$, $z_2 = 1 - \frac{c}{a}$.

To prove this theorem, we use the following proposition which we prove in the next subsection.

Proposition 5.3 ([4]). *If $a \geq b \geq c > 0$, then we have*

$$\int_0^1 \frac{dt'}{\sqrt{(1-t')(1-w_1t')(1-w_2t')}} = \frac{\sqrt{ab}+\sqrt{ac}}{2a} \int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1t)(1-z_2t)}},$$

where

$$z_1 = 1-\frac{b}{a}, \quad z_2 = 1-\frac{c}{a}, \quad w_1 = 1-\frac{\sqrt{ab}+\sqrt{bc}}{\sqrt{ab}+\sqrt{ac}}, \quad w_2 = 1-\frac{\sqrt{ac}+\sqrt{bc}}{\sqrt{ab}+\sqrt{ac}}.$$

Proof of Theorem 5.2 (Refer to [4]). Let $\mu(a,b,c)$ be the right-hand side of (15). Proposition 5.3 implies that

$$\mu(a_n, b_n, c_n) = \mu(a_{n+1}, b_{n+1}, c_{n+1})$$

for all $n \in \mathbb{N}$. Hence we have

$$\begin{aligned}
\mu(a,b,c) &= \lim_{n\to\infty} \mu(a_n, b_n, c_n) = \mu(m_*^\infty(a,b,c), m_*^\infty(a,b,c), m_*^\infty(a,b,c)) \\
&= 2m_*^\infty(a,b,c) \Big/ \int_0^1 \frac{dt}{\sqrt{1-t}} = m_*^\infty(a,b,c).
\end{aligned}$$

□

Remark 5.4. *By this triple of sequences, we can also compute an incomplete elliptic integral of the first kind. However, the convergence of (a_n, b_n, c_n) is not rapid. For example, the values of (a_n, b_n, c_n) and $[-\log_{10}(a_n - b_n)]$ with $(a,b,c) := (1,\ 0.5,\ 0.3)$ are computed by Maple version 14 and are shown in Table 2.*

5.2 Another proof of Proposition 5.3

In [4], Proposition 5.3 is proved as a consequence of the transformation formula for Appell's hypergeometric function F_1, which is obtained by the calculation of connection matrices of integrable Pfaffian systems. Here we give our proof using integration by substitution.

We consider two elliptic curves

$$\begin{aligned}
C &: s^2 = (1-t)(1-z_1t)(1-z_2t), \\
C' &: s'^2 = (1-t')(1-w_1t')(1-w_2t'),
\end{aligned}$$

n	a_n	b_n	c_n
0	1.000000000000000	0.500000000000000	0.300000000000000
1	0.627414669345856	0.547202557903644	0.467510446062953
2	0.563765287089548	0.545863514844305	0.523691167084954
3	0.549050549905967	0.544702305679079	0.539010662167320
4	0.545439775683462	0.544360558450349	0.542928227999162
5	0.544541226396508	0.544271910695070	0.543913237208964
$\vdots$		$\vdots$	
20	0.544242076130621	0.544242076130370	0.544242076130036

n	1	2	3	4	5	6	7	8	9	10	$\cdots$	20
$[-\log_{10}(a_n - b_n)]$	1	1	2	2	3	4	4	5	5	6	$\cdots$	12

Table 2: Slow convergence

where z_1, z_2, w_1, and w_2 are as in Proposition 5.3. Both of these curves are isomorphic to

$$C(\lambda) : y^2 = x(1-x)(1-\lambda x), \ \lambda = \frac{(1-z_2)z_1}{(1-z_1)z_2} = \frac{(1-w_2)w_1}{(1-w_1)w_2} = \frac{(a-b)c}{(a-c)b}.$$

Then there is an isomorphism

$$C \ni (t,s) \longmapsto \left(\frac{(1-w_1)z_1}{(1-z_1)w_1} t + \frac{w_1 - z_1}{(1-z_1)w_1}, \frac{1-w_1}{1-z_1} \sqrt{\frac{(1-w_2)z_1}{(1-z_2)w_1}} s \right) \in C',$$

which maps the branched points 1, $1/z_1$, and $1/z_2$ of $C \to \mathbb{P}^1$ to 1, $1/w_1$, and $1/w_2$ of $C' \to \mathbb{P}^1$, respectively. We calculate the integral

$$\int_0^1 \frac{dt'}{\sqrt{(1-t')(1-w_1 t')(1-w_2 t')}}$$

by the substitution

$$t' = \frac{(1-w_1)z_1}{(1-z_1)w_1} t + \frac{w_1 - z_1}{(1-z_1)w_1}.$$

Then we have

$$\begin{aligned} &\int_0^1 \frac{dt'}{\sqrt{(1-t')(1-w_1 t')(1-w_2 t')}} \\ &= \frac{\sqrt{ab} + \sqrt{ac}}{a} \int_{t_0}^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}}, \quad t_0 = \frac{z_1 - w_1}{(1-w_1)z_1}. \end{aligned}$$

Comparing to Proposition 5.3, we have to show that

$$(16)\quad \int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1t)(1-z_2t)}} = 2\int_{t_0}^1 \frac{dt}{\sqrt{(1-t)(1-z_1t)(1-z_2t)}}.$$

Claim 5.5. *The equation (16) corresponds to the doubling map via the Abel–Jacobi map that sends $(1,0) \in C$ to the origin of the complex torus. More precisely, $(t_0, \sqrt{(1-t_0)(1-z_1t_0)(1-z_2t_0)}) \in C$ multiplied by 2 is $(0,1) \in C$.*

We should thus make a different substitution that uses the doubling map. We define an isomorphism by

$$\begin{aligned}\rho : C &\longrightarrow C(\lambda);\\ (t,s) &\longmapsto \left(\frac{1-z_1}{z_1}\frac{1}{t-1}, \frac{1}{z_1}\sqrt{\frac{1-z_1}{-z_2}}\frac{s}{(t-1)^2}\right),\end{aligned}$$

which maps $(1,0) \in C$ to the point at infinity of $C(\lambda)$ (the isomorphism $\rho' : C' \longrightarrow C(\lambda)$ is given in a similar way). Via ρ and the Abel Jacobi map for $C(\lambda)$, $(0,1) \in C$ corresponds to the origin of the complex torus $E(\tau)$. If we let ψ be as in Proposition 2.4 and (t,s) be $\rho^{-1} \circ \psi \circ \rho'(t',s')$, then we obtain

$$(17)\qquad t = 1 - 4 \cdot \frac{(1-z_1)(1-w_2)w_1(1-t')(1-w_1t')(1-w_2t')}{z_1(w_1w_2t'^2 - 2w_1w_2t' + w_1 + w_2 - 1)^2}$$

Proof of Proposition 5.3. We prove Proposition 5.3 by making the substitution (17). Then we have

$$\begin{aligned}\frac{dt}{dt'} &= -4 \cdot \frac{(1-z_1)(1-w_2)w_1}{z_1}\\ &\quad \cdot \frac{(w_1w_2t'^2 - 2w_1t' + w_1 - w_2 + 1)(w_1w_2t'^2 - 2w_2t' - w_1 + w_2 + 1)}{(w_1w_2t'^2 - 2w_1w_2t' + w_1 + w_2 - 1)^3}.\end{aligned}$$

For simplicity, we set

$$f_1(t') = w_1w_2t'^2 - 2w_1t' + w_1 - w_2 + 1,\quad f_2(t') = w_1w_2t'^2 - 2w_2t' - w_1 + w_2 + 1,$$
$$f_3(t') = w_1w_2t'^2 - 2w_1w_2t' + w_1 + w_2 - 1.$$

It is easy to show that if $0 \le t' \le 1$, then $f_1(t') > 0$, $f_2(t') > 0$, and $f_3(t') < 0$. This implies that $\frac{dt}{dt'} > 0$ when $0 \le t' \le 1$. Since

$$\begin{aligned}f_1(t')^2 - f_3(t')^2 &= (2w_1w_2t'^2 - 2w_1(1+w_2)t' + 2w_1)(2w_1(w_2-1)t' - 2w_2 + 2)\\ &= 4w_1(1-w_2)(1-t')(1-w_1t')(1-w_2t'),\end{aligned}$$

we obtain

$$1 - z_1 t = (1 - z_1)\frac{f_1(t')^2}{f_3(t')^2}, \quad 1 - z_2 t = (1 - z_2)\frac{f_2(t')^2}{f_3(t')^2}$$

(the latter is followed similarly by $\frac{(1-z_1)(1-w_2)w_1}{z_1} = \frac{(1-z_2)(1-w_1)w_2}{z_2}$). Therefore we conclude

$$\begin{aligned}
&\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}} \\
&= \int_0^1 \sqrt{\frac{z_1}{4(1-z_1)(1-w_2)w_1}} \frac{-f_3(t')}{\sqrt{(1-t')(1-w_1 t')(1-w_2 t')}} \\
&\quad \cdot \frac{1}{\sqrt{1-z_1}} \frac{-f_3(t')}{f_1(t')} \frac{1}{\sqrt{1-z_2}} \frac{-f_3(t')}{f_2(t')} \left(-4 \cdot \frac{(1-z_1)(1-w_2)w_1}{z_1} \frac{f_1(t')f_2(t')}{f_3(t')^3}\right) dt' \\
&= 2\sqrt{\frac{(1-w_2)w_1}{(1-z_2)z_1}} \int_0^1 \frac{dt'}{\sqrt{(1-t')(1-w_1 t')(1-w_2 t')}}.
\end{aligned}$$

This completes our proof of Proposition 5.3, since

$$\frac{(1-w_2)w_1}{(1-z_2)z_1} = \frac{a^2}{(\sqrt{ab}+\sqrt{ac})^2} \frac{(\sqrt{ac}+\sqrt{bc})(\sqrt{ac}-\sqrt{bc})}{c(a-b)} = \left(\frac{a}{\sqrt{ab}+\sqrt{ac}}\right)^2.$$

□

5.3 Reduction formula

Using the substitution (17), we now obtain a reduction formula from Lauricella's F_D of five variables to Appell's F_1. Lauricella's hypergeometric function F_D of m-variables $z_1, \ldots, z_m$ with parameters α, $(\beta_j) = (\beta_1, \ldots, \beta_m)$, γ is defined as

$$F_D(\alpha, (\beta_j), \gamma; z_1, \ldots, z_m) = \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{(\alpha, \sum_{j=1}^m n_j) \prod_{j=1}^m (\beta_j, n_j)}{(\gamma, \sum_{j=1}^m n_j) \prod_{j=1}^m (1, n_j)} \prod_{j=1}^m z_j^{n_j},$$

where z_j's satisfy $|z_j| < 1$, $\gamma \neq 0, -1, -2, \ldots$. Note that if we set $m = 2$, then $F_D(\alpha, (\beta_1, \beta_2), \gamma; z_1, z_2) = F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2)$. The function F_D admits an integral representation:

$$\begin{aligned}
&F_D(\alpha, (\beta_j), \gamma; z_1, \ldots, z_m) \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha}(1-t)^{\gamma-\alpha} \left(\prod_{j=1}^m (1 - z_j t)^{-\beta_j}\right) \frac{dt}{t(1-t)}.
\end{aligned}$$

We consider the integral representation for F_1 with the substitution (17). Replacing z_1 and z_2 with $1-z_1^2$ and $1-z_2^2$, respectively, we then have

$$w_1 = 1 - \frac{z_1(1+z_2)}{z_1+z_2}, \quad w_2 = 1 - \frac{z_2(1+z_1)}{z_1+z_2}.$$

Since calculations in section 5.2 are valid after replacing them, and we can simplify the right-hand side of (17) as

$$t = \frac{8z_1^2z_2^2}{(z_1+z_2)^3 f_3(t')^2}t'\left(1-\frac{1-z_1}{2}t'\right)\left(1-\frac{1-z_2}{2}t'\right)\left(1+\frac{(1-z_1)(1-z_2)}{2(z_1+z_2)}t'\right),$$

the following theorem is obtained.

Theorem 5.6. *We have*

$$\begin{aligned}
&\left(\frac{z_1+z_2}{2}\right)^p F_1\left(p,\frac{1}{2},\frac{1}{2},\frac{3}{2};1-z_1^2,1-z_2^2\right) \\
&= F_D\left(p,\left(p-\frac{1}{2},p-\frac{1}{2},1-p,1-p,1-p\right),\frac{3}{2};w_1,w_2,w_3,w_4,w_5\right), \\
&(w_1,w_2,w_3,w_4,w_5) \\
&= \left(1-\frac{z_1(1+z_2)}{z_1+z_2},1-\frac{z_2(1+z_1)}{z_1+z_2},\frac{1-z_1}{2},\frac{1-z_2}{2},-\frac{(1-z_1)(1-z_2)}{2(z_1+z_2)}\right),
\end{aligned}$$

where (z_1,z_2) *is in a small neighborhood of* $(1,1)$.

This theorem is a generalization of Proposition 5.3, which is different from Theorem 1.1 in [4]. Indeed, if we let $p=1$, $z_1=\sqrt{\frac{b}{a}}$, and $z_2=\sqrt{\frac{c}{a}}$, then we obtain Proposition 5.3.

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