TWISTED PERIOD RELATIONS FOR LAURICELLA'S HYPERGEOMETRIC FUNCTION F_A

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ABSTRACT. We study Lauricella's hypergeometric function F_A of m variables and the system E_A of differential equations annihilating F_A, by using twisted (co)homology groups. We construct twisted cycles with respect to an integral representation of Euler type of F_A. These cycles correspond to 2^m linearly independent solutions to E_A, which are expressed by hypergeometric series F_A. Using intersection forms of twisted (co)homology groups, we obtain twisted period relations which give quadratic relations for Lauricella's F_A.

1. INTRODUCTION

Lauricella's hypergeometric series F_A of m variables $x_1, \dots, x_m$ with complex parameters $a, b_1, \dots, b_m, c_1, \dots, c_m$ is defined by

$$F_A(a,b,c;x) = \sum_{n_1,\dots,n_m=0}^{\infty} \frac{(a, n_1+\cdots+n_m)(b_1,n_1)\cdots(b_m,n_m)}{(c_1,n_1)\cdots(c_m,n_m)n_1!\cdots n_m!} x_1^{n_1}\cdots x_m^{n_m},$$

where $x=(x_1,\dots,x_m)$, $b=(b_1,\dots,b_m)$, $c=(c_1,\dots,c_m)$, $c_1,\dots,c_m \notin \{0,-1,-2,\dots\}$ and $(c_1,n_1)=\Gamma(c_1+n_1)/\Gamma(c_1)$. This series converges in the domain

$$D_A := \left\{ (x_1,\dots,x_m) \in \mathbb{C}^m \;\middle|\; \sum_{k=1}^{m} |x_k| < 1 \right\},$$

and admits the integral representation (3). The system $E_A(a,b,c)$ of differential equations annihilating $F_A(a,b,c;x)$ is a holonomic system of rank 2^m with the singular locus S given in (1). There is a fundamental system of solutions to $E_A(a,b,c)$ in a simply connected domain in $D_A - S$, which is given in terms of Lauricella's hypergeometric series F_A with different parameters, see (2) for their expressions.

In this paper, we construct 2^m twisted cycles which represent elements of the m-th twisted homology group concerning with the integral representation (3). They imply integral representations of the solutions (2) expressed by the series F_A. We evaluate the intersection numbers of these 2^m twisted cycles. Further, by using the intersection matrix of a basis of the twisted cohomology group in [9], we give twisted period relations for two fundamental systems of E_A with different parameters.

In the study of twisted homology groups, twisted cycles given by bounded chambers are useful. For Lauricella's F_A, twisted cycles defined by 2^m bounded chambers are studied in [10]. Though the integrals on these cycles are solutions to E_A, they do not give integral representations of the solutions (2), except for one cycle. We construct other twisted cycles from these 2^m bounded chambers by using a method introduced in [5]. For a subset $\{i_1,\dots,i_r\}$ of $\{1,\dots,m\}$ of cardinality r, we construct a twisted cycle $\Delta_{i_1\cdots i_r}$ from the direct product of an r-simplex and $(m-r)$ intervals, by a similar manner to [5]. See Section 4, for details. Our first main theorem states that this twisted cycle corresponds to the solution (2) expressed by the power function $\prod_{p=1}^{r} x_{i_p}^{1-c_{i_p}}$ and the series F_A. Our construction has a

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simple combinatorial structure, and enables us to evaluate the intersection matrix formally. Once the intersection matrix for bases of twisted homology groups and that of twisted cohomology groups are evaluated, then we obtain twisted period relations which are originally identities among the integrals given by the pairings of elements of twisted homology and cohomology groups. Our first main theorem transforms these identities into quadratic relations among hypergeometric series F_A's. Our second main theorem states these formulas in Section 6.

As is in [2], the irreducibility condition of the system $E_A(a,b,c)$ is known to be

$$b_1,\dots,b_m,\ c_1-b_1,\dots,c_m-b_m,\ a-\sum_{p=1}^{r} c_{i_r}\notin\mathbb{Z}$$

for any subset $\{i_1,\dots,i_r\}$ of $\{1,\dots,m\}$. Since our interest is in the property of solutions to $E_A(a,b,c)$ expressed in terms of the hypergeometric series F_A, we assume throughout this paper that the parameters a, $b=(b_1,\dots,b_m)$ and $c=(c_1,\dots,c_m)$ satisfy the condition above and $c_1,\dots,c_m\notin\mathbb{Z}$.

2. Differential equations and integral representations

In this section, we collect some facts about Lauricella's F_A and the system E_A of hypergeometric differential equations annihilating it.

Notation 2.1. Throughout this paper, the letter k always stands for an index running from 1 to m. If no confusion is possible, $\sum_{k=1}^{m}$ and $\prod_{k=1}^{m}$ are often simply denoted by $\sum$ (or $\sum_k$) and $\prod$ (or $\prod_k$), respectively. For example, under this convention $F_A(a,b,c;x)$ is expressed as

$$F_A(a,b,c;x)=\sum_{n_1,\dots,n_m=0}^{\infty}\frac{(a,\sum n_k)\prod(b_k,n_k)}{\prod(c_k,n_k)\cdot\prod n_k!}\prod x_k^{n_k}.$$

Let ∂_k $(k=1,\dots,m)$ be the partial differential operator with respect to x_k. Lauricella's $F_A(a,b,c;x)$ satisfies hypergeometric differential equations

$$\Big[x_k(1-x_k)\partial_k^2-x_k\sum_{\substack{1\le i\le m\\ i\neq k}}x_i\partial_k\partial_i$$
$$+(c_k-(a+b_k+1)x_k)\partial_k-b_k\sum_{\substack{1\le i\le m\\ i\neq k}}x_i\partial_i-ab_k\Big]f(x)=0,$$

for $k=1,\dots,m$. The system generated by them is called Lauricella's system $E_A(a,b,c)$ of hypergeometric differential equations.

Proposition 2.2 ([8], [11]). *The system $E_A(a,b,c)$ is a holonomic system of rank 2^m with the singular locus*

$$S:=\left(\prod_{k=1}^{m}x_k\cdot\prod_{\{i_1,\dots,i_r\}\subset\{1,\dots,m\}}\left(1-\sum_{p=1}^{r}x_{i_p}\right)=0\right)\subset\mathbb{C}^m. \tag{1}$$

If $c_1,\dots,c_m\notin\mathbb{Z}$, then the vector space of solutions to $E_A(a,b,c)$ in a simply connected domain in D_A-S is spanned by the following 2^m elements:

$$f_{i_1\cdots i_r}:=\left(\prod_{p=1}^{r}x_{i_p}^{1-c_{i_p}}\right)\cdot F_A\left(a+r-\sum_{p=1}^{r}c_{i_p},b^{i_1\cdots i_r},c^{i_1\cdots i_r};x\right). \tag{2}$$

Here r runs from 0 to m, indices $i_1, \dots, i_r$ satisfy $1 \le i_1 < \cdots < i_r \le m$, and the row vectors $b^{i_1\cdots i_r}$ and $c^{i_1\cdots i_r}$ are defined by

$$b^{i_1\cdots i_r} := b + \sum_{p=1}^{r}(1 - c_{i_p})e_{i_p}, \quad c^{i_1\cdots i_r} := c + 2\sum_{p=1}^{r}(1 - c_{i_p})e_{i_p},$$

where e_i is the i-th unit row vector of $\mathbb{C}^m$.

For the above $i_1, \dots, i_r$, we take $j_1, \dots, j_{m-r}$ so that $1 \le j_1 < \cdots < j_{m-r} \le m$ and $\{i_1, \dots, i_r, j_1, \dots, j_{m-r}\} = \{1, \dots, m\}$. It is easy to see that the i_p-th entries of $b^{i_1\cdots i_r}$ and $c^{i_1\cdots i_r}$ are $b_{i_p} - c_{i_p} + 1$ and $2 - c_{i_p}$ $(1 \le p \le r)$ and the j_q-th entries are b_{j_q} and c_{j_q} $(1 \le q \le m - r)$, respectively.

We denote the multi-index "$i_1 \cdots i_r$" by a letter I expressing the set $\{i_1, \dots, i_r\}$. Note that the solution (2) for $r = 0$ is $f(= f_\emptyset) = F_A(a, b, c; x)$.

Proposition 2.3 (Integral representation of Euler type, [8]). *For sufficiently small positive real numbers $x_1, \dots, x_m$, if $\mathrm{Re}(c_k) > \mathrm{Re}(b_k) > 0$ $(k = 1, \dots, m)$, then $F_A(a, b, c; x)$ admits the following integral representation:*

(3)
$$F_A(a,b,c;x) = \prod \frac{\Gamma(c_k)}{\Gamma(b_k)\Gamma(c_k - b_k)} \cdot \int_{(0,1)^m} \prod \left(t_k^{b_k-1} \cdot (1 - t_k)^{c_k-b_k-1}\right) \cdot \left(1 - \sum x_k t_k\right)^{-a} dt_1 \wedge \cdots \wedge dt_m.$$

3. Twisted homology groups

We review twisted homology groups and the intersection form between twisted homology groups in general situations, by referring to Chapter 2 of [1] and Chapters IV, VIII of [12].

For polynomials $P_j(t) = P_j(t_1, \dots, t_m)$ $(1 \le j \le n)$, we set $D_j := \{t \mid P_j(t) = 0\} \subset \mathbb{C}^m$ and $M := \mathbb{C}^m - (D_1 \cup \cdots \cup D_n)$. We consider a multi-valued function $u(t)$ on M defined as

$$u(t) := \prod_{j=1}^{n} P_j(t)^{\lambda_j}, \ \lambda_j \in \mathbb{C} - \mathbb{Z} \ (1 \le j \le n).$$

For a k-simplex σ in M, we define a loaded k-simplex $\sigma \otimes u$ by σ loading a branch of u on it. We denote the $\mathbb{C}$-vector space of finite sums of loaded k-simplexes by $\mathcal{C}_k(M, u)$, called the k-th twisted chain group. An element of $\mathcal{C}_k(M, u)$ is called a twisted k-chain. For a loaded k-simplex $\sigma \otimes u$ and a smooth k-form φ on M, the integral $\int_{\sigma\otimes u} u \cdot \varphi$ is defined by

$$\int_{\sigma\otimes u} u \cdot \varphi := \int_{\sigma} [\text{the fixed branch of } u \text{ on } \sigma] \cdot \varphi.$$

By the linear extension of this, we define the integral on a twisted k-chain.

We define the boundary operator $\partial^u : \mathcal{C}_k(M, u) \to \mathcal{C}_{k-1}(M, u)$ by

$$\partial^u(\sigma \otimes u) := \partial(\sigma) \otimes u|_{\partial(\sigma)},$$

where ∂ is the usual boundary operator and $u|_{\partial(\sigma)}$ is the restriction of u to $\partial(\sigma)$. It is easy to see that $\partial^u \circ \partial^u = 0$. Thus we have a complex

$$\mathcal{C}_\bullet(M, u) : \cdots \xrightarrow{\partial^u} \mathcal{C}_k(M, u) \xrightarrow{\partial^u} \mathcal{C}_{k-1}(M, u) \xrightarrow{\partial^u} \cdots,$$

and its k-th homology group $H_k(\mathcal{C}_\bullet(M, u))$. It is called the k-th twisted homology group. An element of $\ker \partial^u$ is called a twisted cycle.

By considering $u^{-1} = 1/u$ instead of u, we have $H_k(\mathcal{C}_\bullet(M, u^{-1}))$. There is the intersection pairing I_h between $H_m(\mathcal{C}_\bullet(M, u))$ and $H_m(\mathcal{C}_\bullet(M, u^{-1}))$ (in fact, the intersection pairing is defined between $H_k(\mathcal{C}_\bullet(M,u))$ and $H_{2m-k}(\mathcal{C}_\bullet(M,u^{-1}))$, however we do not consider the cases $k \neq m$). Let Δ and Δ' be elements of $H_m(\mathcal{C}_\bullet(M,u))$ and $H_m(\mathcal{C}_\bullet(M,u^{-1}))$ given by twisted cycles $\sum_i \alpha_i \cdot \sigma_i \otimes u_i$ and $\sum_j \alpha'_j \cdot \sigma'_j \otimes u_j^{-1}$ respectively, where u_i (resp. u_j^{-1}) is a branch of u (resp. u^{-1}) on σ_i (resp. σ'_j). Then their intersection number is defined by

$$I_h(\Delta,\Delta') := \sum_{i,j} \sum_{s\in\sigma_i\cap\sigma'_j} \alpha_i\alpha'_j \cdot (\sigma_i\cdot\sigma'_j)_s \cdot \frac{u_i(s)}{u_j(s)},$$

where $(\sigma_i\cdot\sigma'_j)_s$ is the topological intersection number of m-simplexes σ_i and σ'_j at s.

In this paper, we mainly consider

$$M := \mathbb{C}^m - \left(\bigcup_k (t_k = 0) \cup \bigcup_k (1-t_k=0) \cup (v=0)\right),$$

where $v := 1-\sum x_k t_k$. We consider the twisted homology group on M with respect to the multi-valued function

$$u := \prod_{k=1}^m t_k^{b_k}(1-t_k)^{c_k-b_k-1} \cdot v^{-a}.$$

Let Δ be the regularization of $(0,1)^m \otimes u$, which gives an element in $H_m(\mathcal{C}_\bullet(M,u))$. For the construction of regularizations, refer to Sections 3.2.4 and 3.2.5 of [1]. Proposition 2.3 means that the integral

$$\int_\Delta u\varphi, \quad \varphi := \frac{dt_1\wedge\cdots\wedge dt_m}{t_1\cdots t_m}$$

represents $F_A(a,b,c;x)$ modulo Gamma factors.

4. Twisted cycles corresponding to local solutions $f_{i_1\cdots i_r}$

In this section, we construct 2^m twisted cycles in M corresponding to the solutions (2) to $E_A(a,b,c)$.

Let $0\le r\le m$ and subsets $\{i_1,\dots,i_r\}$ and $\{j_1,\dots,j_{m-r}\}$ of $\{1,\dots,m\}$ satisfy $i_1<\cdots<i_r$, $j_1<\cdots<j_{m-r}$ and $\{i_1,\dots,i_r,j_1,\dots,j_{m-r}\}=\{1,\dots,m\}$.

NOTATION 4.1. From now on, the letter p (resp. q) is always stands for an index running from 1 to r (resp. from 1 to $m-r$). We use the abbreviations $\sum$, $\prod$ for the indices p,q as are mentioned in Notation 2.1.

We set

$$:= \mathbb{C}^m - \left(\bigcup_k (s_k=0) \cup \bigcup_p (s_{i_p}-x_{i_p}=0) \cup \bigcup_q (1-s_{j_q}=0) \cup (v_{i_1\cdots i_r}=0)\right),$$

where

$$v_{i_1\cdots i_r} := 1-\sum_p s_{i_p} - \sum_q x_{j_q}s_{j_q}.$$

Let $u_{i_1\cdots i_r}$ and $\varphi_{i_1\cdots i_r}$ be a multi-valued function and an m-form on $M_{i_1\cdots i_r}$ defined as

$$u_{i_1\cdots i_r} := \prod_{p=1}^{r} s_{i_p}^{b_{i_p}}\left(s_{i_p}-x_{i_p}\right)^{c_{i_p}-b_{i_p}-1}\cdot\prod_{q=1}^{m-r} s_{j_q}^{b_{j_q}}(1-s_{j_q})^{c_{j_q}-b_{j_q}-1}\cdot v_{i_1\cdots i_r}^{-a},$$
$$\varphi_{i_1\cdots i_r} := \frac{ds_1\wedge\cdots\wedge ds_m}{s_1\cdots s_m}.$$

We construct a twisted cycle $\tilde{\Delta}_{i_1\cdots i_r}$ in $M_{i_1\cdots i_r}$ with respect to $u_{i_1\cdots i_r}$. Note that if $\{i_1,\dots,i_r\}=\emptyset$, then these settings coincide with those in the end of Section 3. We choose positive real numbers $\varepsilon_1,\dots,\varepsilon_m$ and ε so that $\varepsilon<1-\sum_k\varepsilon_k$ and $\varepsilon_k<\frac{1}{4}$. And let $x_1,\dots,x_m$ be small positive real numbers satisfying

$$x_k<\varepsilon_k,\quad \sum_k x_k(1+\varepsilon_k)<\varepsilon$$

(for example, if

$$\varepsilon_k=\varepsilon=\frac{1}{5m},\quad 0<x_k<\frac{1}{6m^2},$$

these conditions hold). Thus the closed subset

$$\sigma_{i_1\cdots i_r} := \left\{(s_1,\dots,s_m)\in\mathbb{R}^m \;\middle|\; \begin{array}{l} s_{i_p}\geq\varepsilon_{i_p},\ 1-\sum s_{i_p}\geq\varepsilon, \\ s_{j_q}\geq\varepsilon_{j_q},\ 1-s_{j_q}\geq\varepsilon_{j_q}\end{array}\right\}$$

is nonempty, since we have $(\varepsilon_1+\frac{\delta}{2m},\dots,\varepsilon_m+\frac{\delta}{2m})\in\sigma_{i_1\cdots i_r}$, where $\delta:=1-\sum\varepsilon_k-\varepsilon>0$. Further, $\sigma_{i_1\cdots i_r}$ is contained in the bounded domain

$$\left\{(s_1,\dots,s_m)\in\mathbb{R}^m \;\middle|\; \begin{array}{l} s_{i_p}-x_{i_p}>0, \\ 0<s_{j_q}<1,\end{array}\ 1-\sum s_{i_p}-\sum x_{j_q}s_{j_q}>0\right\}\subset(0,1)^m.$$

and is a direct product of an r-simplex and $(m-r)$ intervals. Indeed, $(s_1,\dots,s_m)\in\sigma_{i_1\cdots i_r}$ satisfies

$$s_{i_p}-x_{i_p}>s_{i_p}-\varepsilon_{i_p}>0,$$
$$1-\sum s_{i_p}-\sum x_{j_q}s_{j_q}>\varepsilon-\sum x_{j_q}>\varepsilon-\sum x_k>0.$$

The orientation of $\sigma_{i_1\cdots i_r}$ is induced from the natural embedding $\mathbb{R}^m\subset\mathbb{C}^m$. We construct a twisted cycle from $\sigma_{i_1\cdots i_r}\otimes u_{i_1\cdots i_r}$, where the branch of $u_{i_1\cdots i_r}$ on $\sigma_{i_1\cdots i_r}$ is defined by the principal value. We may assume that $\varepsilon_k=\varepsilon$ (the above example satisfies this condition), and denote them by ε. Set $L_1:=(s_1=0)$, ..., $L_m:=(s_m=0)$, $L_{m+1}:=(1-s_{j_1}=0),\dots,L_{2m-r}:=(1-s_{j_{m-r}})$, $L_{2m-r+1}:=(1-\sum s_{i_p}=0)$, and let $U(\subset\mathbb{R}^m)$ be a bounded chamber surrounded by $L_1,\dots,L_{2m-r+1}$, then $\sigma_{i_1\cdots i_r}$ is contained in U. Note that we do not consider the hyperplane L_{2m-r+1} (resp. the hyperplanes $L_{m+1},\dots,L_{2m-r}$), when $r=0$ (resp. $r=m$). For $J\subset\{1\dots,2m-r+1\}$, we consider $L_J:=\cap_{j\in J}L_j$, $U_J:=\overline{U}\cap L_J$ and $T_J:=\varepsilon$-neighborhood of U_J. Then we have

$$\sigma_{i_1\cdots i_r}=U-\bigcup_J T_J.$$

Using these neighborhoods T_J, we can construct a twisted cycle $\tilde{\Delta}_{i_1\cdots i_r}$ in the same manner as Section 3.2.4 of [1] (notations L and U correspond to H and Δ in [1], respectively). Note that we have to consider contribution of branches of $s_{i_p}^{b_{i_p}}\left(s_{i_p}-x_{i_p}\right)^{c_{i_p}-b_{i_p}-1}$, when we deal with the circle associated to L_{i_p} ($p=1,\dots,r$), because of $x_{i_p}<\varepsilon$. Thus the exponent about this contribution is

$$b_{i_p}+(c_{i_p}-b_{i_p}-1)=c_{i_p}-1.$$

The exponents about the contributions of the circles associated to L_{j_q}, L_{m+q}, L_{2m-r+1} are simply

$$b_{j_q},\ c_{j_q}-b_{j_q}-1,\ -a,$$

respectively. We briefly explain the expression of $\tilde{\Delta}_{i_1\cdots i_r}$. For $j=1,\dots,2m-r+1$, let l_j be the $(m-1)$-face of $\sigma_{i_1\cdots i_r}$ given by $\sigma_{i_1\cdots i_r}\cap T_j$, and let S_j be a positively oriented circle with radius ε in the orthogonal complement of L_j starting from the projection of l_j to this space and surrounding L_j. Then $\tilde{\Delta}_{i_1\cdots i_r}$ is written as

$$\sigma_{i_1\cdots i_r}\otimes u_{i_1\cdots i_r}+\sum_{\emptyset\neq J\subset\{1,\dots,2m-r+1\}}\left(\prod_{j\in J}\frac{1}{d_j}\right)\cdot\left(\left(\bigcap_{j\in J}l_j\right)\times\prod_{j\in J}S_j\right)\otimes u_{i_1\cdots i_r},$$

where

$$d_{i_p}:=\gamma_{i_p}-1,\ d_{j_q}:=\beta_{j_q}-1,\ d_{m+q}:=\gamma_{j_q}\beta_{j_q}^{-1}-1,\ d_{2m-r+1}:=\alpha^{-1}-1,$$

and $\alpha:=e^{2\pi\sqrt{-1}a}$, $\beta_k:=e^{2\pi\sqrt{-1}b_k}$, $\gamma_k:=e^{2\pi\sqrt{-1}c_k}$. The branch of $u_{i_1\cdots i_r}$ on $(\cap_{j\in J}l_j)\times\prod_{j\in J}S_j$ is defined by the analytic continuation of that on $\sigma_{i_1\cdots i_r}$. Note that we define an appropriate orientation for each $(\cap_{j\in J}l_j)\times\prod_{j\in J}S_j$, see Section 3.2.4 of [1] for details.

EXAMPLE 4.2. We give explicit forms of $\tilde{\Delta}$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_{12}$, for $m=2$.

(i) In the case of $I=\emptyset$, $\tilde{\Delta}$ is the usual regularization of $(0,1)^m\otimes u$.

(ii) In the case of $I=\{1\}$, we have

$$\begin{aligned}\tilde{\Delta}_1=&\sigma_1\otimes u_1+\frac{(S_1\times l_1)\otimes u_1}{1-\gamma_1}+\frac{(S_2\times l_2)\otimes u_1}{1-\beta_2}+\frac{(S_4\times l_4)\otimes u_1}{1-\alpha^{-1}}+\frac{(S_3\times l_3)\otimes u_1}{1-\gamma_2\beta_2^{-1}}\\&+\frac{(S_1\times S_2)\otimes u_1}{(1-\gamma_1)(1-\beta_2)}+\frac{(S_2\times S_4)\otimes u_1}{(1-\beta_2)(1-\alpha^{-1})}\\&+\frac{(S_4\times S_3)\otimes u_1}{(1-\alpha^{-1})(1-\gamma_2\beta_2^{-1})}+\frac{(S_3\times S_1)\otimes u_1}{(1-\gamma_2\beta_2^{-1})(1-\gamma_1)},\end{aligned}$$

where the 1-chains l_j satisfy $\partial\sigma=\sum_{j=1}^4 l_j$ (see Figure 1), and the orientation of each direct product is induced from those of its components.

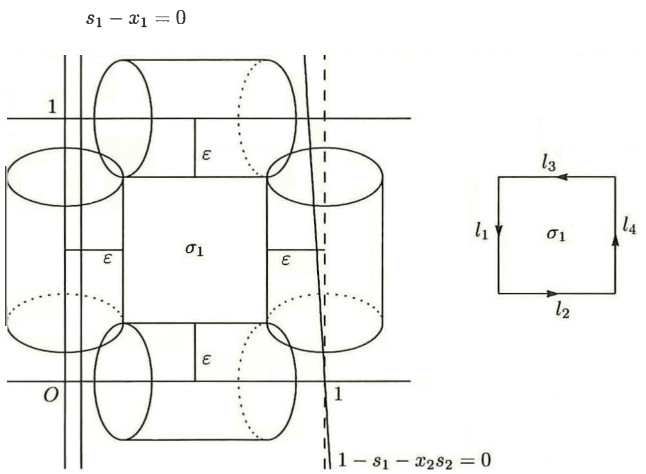

FIGURE 1. $\tilde{\Delta}_1$ for $m=2$.

(iii) In the case of $I = \{1,2\}$, we have

$$\tilde{\Delta}_{12} = \sigma_{12} \otimes u_{12} + \frac{(S_1 \times l_1) \otimes u_{12}}{1-\gamma_1} + \frac{(S_2 \times l_2) \otimes u_{12}}{1-\gamma_2} + \frac{(S_3 \times l_3) \otimes u_{12}}{1-\alpha^{-1}}$$
$$+ \frac{(S_1 \times S_2) \otimes u_{12}}{(1-\gamma_1)(1-\gamma_2)} + \frac{(S_2 \times S_3) \otimes u_{12}}{(1-\gamma_2)(1-\alpha^{-1})} + \frac{(S_3 \times S_1) \otimes u_{12}}{(1-\alpha^{-1})(1-\gamma_1)},$$

where the 1-chains l_j satisfy $\partial\sigma = l_1 + l_2 + l_3$ (see Figure 2), and the orientation of each direct product is induced from those of its components.

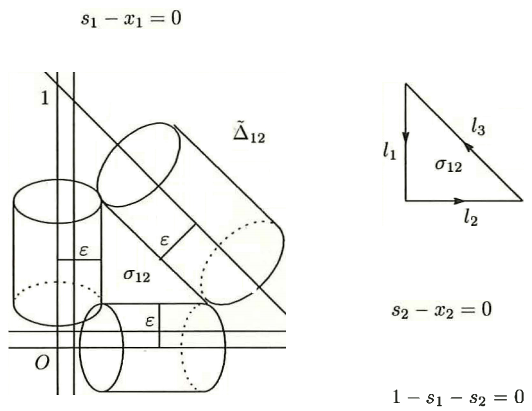

FIGURE 2. $\tilde{\Delta}_{12}$ for $m = 2$.

We consider the following integrals:

$$F_{i_1\cdots i_r} := \int_{\tilde{\Delta}_{i_1\cdots i_r}} u_{i_1\cdots i_r}\varphi_{i_1\cdots i_r}$$
$$= \int_{\tilde{\Delta}_{i_1\cdots i_r}} \prod_{p=1}^{r} s_{i_p}^{c_{i_p}-2}\left(1-\frac{x_{i_p}}{s_{i_p}}\right)^{c_{i_p}-b_{i_p}-1} \cdot \prod_{q=1}^{m-r} s_{j_q}^{b_{j_q}-1}(1-s_{j_q})^{c_{j_q}-b_{j_q}-1}$$
$$\cdot \left(1-\sum_{p=1}^{r} s_{i_p} - \sum_{q=1}^{m-r} x_{j_q}s_{j_q}\right)^{-a} ds_1 \wedge \cdots \wedge ds_m.$$

Proposition 4.3.

$$F_{i_1\cdots i_r} = \prod_{p=1}^{r}\Gamma(c_{i_p}-1) \cdot \prod_{q=1}^{m-r}\frac{\Gamma(b_{j_q})\Gamma(c_{j_q}-b_{j_q})}{\Gamma(c_{j_q})} \cdot \frac{\Gamma(1-a)}{\Gamma(\sum c_{i_p} - a - r + 1)}$$
$$\cdot F_A(a + r - \sum_{p=1}^{r} c_{i_p}, b^{i_1\cdots i_r}, c^{i_1\cdots i_r}; x).$$

Proof. We compare the power series expansions of the both sides. Note that the coefficient of $x_1^{n_1}\cdots x_m^{n_m}$ in the series expression of $F_A(a+r-\sum_{p=1}^{r} c_{i_p}, b^{i_1\cdots i_r}, c^{i_1\cdots i_r}; x)$ is

$$A_{n_1\ldots n_m} := \frac{\Gamma(a+r-\sum_p c_{i_p} + \sum_k n_k)}{\Gamma(a+r-\sum_p c_{i_p})}\prod_p \frac{\Gamma(b_{i_p}+1-c_{i_p}+n_{i_p})}{\Gamma(b_{i_p}+1-c_{i_p})}\prod_q \frac{\Gamma(b_{j_q}+n_{j_q})}{\Gamma(b_{j_q})}$$
$$\cdot \prod_p \frac{\Gamma(2-c_{i_p})}{\Gamma(2-c_{i_p}+n_{i_p})} \cdot \prod_q \frac{\Gamma(c_{j_q})}{\Gamma(c_{j_q}+n_{j_q})} \cdot \prod_k \frac{1}{n_k!}.$$

On the other hand, we have

$$\left(1-\frac{x_{i_p}}{s_{i_p}}\right)^{c_{i_p}-b_{i_p}-1}=\sum_{n_{i_p}}\frac{\Gamma(b_{i_p}-c_{i_p}+1+n_{i_p})}{\Gamma(b_{i_p}-c_{i_p}+1)\cdot n_{i_p}!}s_{i_p}^{-n_{i_p}}x_{i_p}^{n_{i_p}}$$

and

$$\left(1-\sum_{p=1}^{r}s_{i_p}-\sum_{q=1}^{m-r}x_{j_q}s_{j_q}\right)^{-a}$$
$$\sum_{n_{j_1},\dots,n_{j_{m-r}}}\frac{\Gamma(a+\sum n_{j_q})}{\Gamma(a)\cdot\prod n_{j_q}!}(1-\sum s_{i_p})^{-a-\sum n_{j_q}}\cdot\prod s_{j_q}^{n_{j_q}}x_{j_q}^{n_{j_q}}.$$

When $r=0$ (resp. $r=m$), we do not need the first (resp. second) expansion. The convergences of these power series expansions are verified as follows. By the construction of $\tilde{\Delta}_{i_1\cdots i_r}$, we have

$$0<x_k<\varepsilon_k,\ \varepsilon_{i_p}\leq|s_{i_p}|,\ |s_{j_q}|\leq 1+\varepsilon_{j_q},\ \left|1-\sum s_{i_p}\right|\geq\varepsilon.$$

Thus the uniform convergences on $\tilde{\Delta}_{i_1\cdots i_r}$ follow from

$$\left|\frac{x_{i_p}}{s_{i_p}}\right|<\frac{\varepsilon_{i_p}}{\varepsilon_{i_p}}=1,$$
$$\left|\frac{1}{1-\sum s_{i_p}}\cdot\sum x_{j_q}s_{j_q}\right|\leq\frac{1}{|1-\sum s_{i_p}|}\cdot\sum x_{j_q}|s_{j_q}|\leq\frac{1}{\varepsilon}\cdot\sum x_{j_q}(1+\varepsilon_{j_q})<\frac{\varepsilon}{\varepsilon}=1.$$

Since $\tilde{\Delta}_{i_1\cdots i_r}$ is constructed as a finite sum of loaded (compact) simplexes, we can exchange the sum and the integral in the expression of $F_{i_1\cdots i_r}$. Then the coefficient of $x_1^{n_1}\cdots x_m^{n_m}$ in the series expansion of $F_{i_1\cdots i_r}$ is

(4)
$$B_{n_1\dots n_m}:=\prod_p\frac{\Gamma(b_{i_p}-c_{i_p}+1+n_{i_p})}{\Gamma(b_{i_p}-c_{i_p}+1)}\cdot\frac{\Gamma(a+\sum n_{j_q})}{\Gamma(a)}\cdot\prod_k\frac{1}{n_k!}$$
$$\cdot\int_{\tilde{\Delta}_{i_1\cdots i_r}}\prod_p s_{i_p}^{c_{i_p}-2-n_{i_p}}\cdot(1-\sum s_{i_p})^{-a-\sum n_{j_q}}\cdot\prod_q s_{j_q}^{b_{j_q}-1-n_{j_q}}(1-s_{j_q})^{c_{j_q}-b_{j_q}-1}ds.$$

By the construction, the twisted cycle $\tilde{\Delta}_{i_1\cdots i_r}$ of this integral is identified with the usual regularization of the loaded domain

$$\left\{(s_1,\dots,s_m)\in\mathbb{R}^m\ \mid s_{i_p}>0,\ 1-\sum s_{i_p}>0,\ 0<s_{j_q}<1\right\}$$

for the multi-valued function

$$\prod_p s_{i_p}^{c_{i_p}-1-n_{i_p}}(1-\sum s_{i_p})^{-a-\sum n_{j_q}}\cdot\prod_q s_{j_q}^{b_{j_q}-n_{j_q}}(1-s_{j_q})^{c_{j_q}-b_{j_q}-1}$$

on $\mathbb{C}^m-\left(\bigcup_k(s_k=0)\cup\bigcup_q(1-s_{j_q}=0)\cup(1-\sum s_{i_p}=0)\right)$. Hence the integral in (4) is equal to

$$\frac{\prod_p\Gamma(c_{i_p}-n_{i_p}-1)\cdot\Gamma(-a-\sum n_{j_q}+1)}{\Gamma(\sum c_{i_p}-a-\sum n_k-r+1)}\cdot\prod_q\frac{\Gamma(b_{j_q}+n_{j_q})\Gamma(c_{j_q}-b_{j_q})}{\Gamma(c_{j_q}+n_{j_q})}.$$

Using the formula

(5) $$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)},$$

we thus have

$$\frac{B_{n_1\dots n_m}}{A_{n_1\dots n_m}} = \prod_p \Gamma(c_{i_p}-1)\cdot\prod_q \frac{\Gamma(b_{j_q})\Gamma(c_{j_q}-b_{j_q})}{\Gamma(c_{j_q})}\frac{\Gamma(1-a)}{\Gamma(\sum c_{i_p}-a-r+1)},$$

which implies the proposition. □

We define a bijection $\iota_{i_1\cdots i_r} : M_{i_1\cdots i_r} \to M$ by

$$\iota_{i_1\cdots i_r}(s_1,\dots,s_m) := (t_1,\dots,t_m);\ t_{i_p} = \frac{s_{i_p}}{x_{i_p}},\ t_{j_q} = s_{j_q}.$$

For example, $\iota(=\iota_\emptyset)$ is the identity map on $M = M_\emptyset$.

We also define branches of the multi-valued function u on real bounded chambers in M. On the domain

$$D_{i_1\cdots i_r} := \{(t_1,\dots,t_r)\in\mathbb{R}^m \mid t_k>0,\ 1-\sum x_k t_k>0,\ 1-t_{i_p}<0,\ 1-t_{j_q}>0\},$$

the arguments of t_k, $1-\sum x_k t_k$, $1-t_{i_p}$ and $1-t_{j_q}$ are given as follows.

t_k	$1-\sum x_k t_k$	$1-t_{i_p}$	$1-t_{j_q}$
0	0	$-\pi$	0

We state our first main theorem.

Theorem 4.4. *We define a twisted cycle $\Delta_{i_1\cdots i_r}$ in M by*

$$\Delta_{i_1\cdots i_r} := (\iota_{i_1\cdots i_r})_*(\tilde{\Delta}_{i_1\cdots i_r}).$$

Then we have

$$\int_{\Delta_{i_1\cdots i_r}} \prod\left(t_k^{b_k-1}\cdot(1-t_k)^{c_k-b_k-1}\right)\cdot\left(1-\sum x_k t_k\right)^{-a} dt_1\wedge\cdots\wedge dt_m$$

$$\left(=\int_{\Delta_{i_1\cdots i_r}} u\varphi\right) = e^{\pi\sqrt{-1}(\sum b_{i_p}-\sum c_{i_p}+r)}\prod_{p=1}^{r} x_{i_p}^{1-c_{i_p}}\cdot F_{i_1\cdots i_r},$$

and hence this integral corresponds to the local solution $f_{i_1\cdots i_r}$ to $E_A(a,b,c)$ given in Proposition 2.2.

Proof. Since $\iota_{i_1\cdots i_r}(\sigma_{i_1\cdots i_r})\subset D_{i_1\cdots i_r}$, the left hand side is equal to

$$e^{\pi\sqrt{-1}(\sum b_{i_p}-\sum c_{i_p}+r)}\cdot\int_{\Delta_{i_1\cdots i_r}}\prod_p\left(t_{i_p}^{b_{i_p}-1}\cdot(t_{i_p}-1)^{c_{i_p}-b_{i_p}-1}\right)$$
$$\cdot\prod_q\left(t_{j_q}^{b_{j_q}-1}\cdot(1-t_{j_q})^{c_{j_q}-b_{j_q}-1}\right)\cdot\left(1-\sum x_k t_k\right)^{-a} dt_1\wedge\cdots\wedge dt_m,$$

where the branch of the integrand is determined naturally. Pulling back this integral by $\iota_{i_1\cdots i_r}$ leads the first claim. This and Proposition 4.3 imply the second claim. □

REMARK 4.5. Except in the case of $\{i_1,\dots,i_r\}=\emptyset$, the twisted cycle $\Delta_{i_1\cdots i_r}$ is different from the regularization of $D_{i_1\cdots i_r}\otimes u$ as elements in $H_m(\mathcal{C}_\bullet(M,u))$.

The replacement $u\mapsto u^{-1}=1/u$ and the construction same as $\Delta_{i_1\cdots i_r}$ give the twisted cycle $\Delta^\vee_{i_1\cdots i_r}$ which represents an element in $H_m(\mathcal{C}_\bullet(M,u^{-1}))$. We obtain the intersection numbers of the twisted cycles $\{\Delta_{i_1\cdots i_r}\}$ and $\{\Delta^\vee_{i_1\cdots i_r}\}$.

Theorem 4.6. *(i) For $I,J\subset\{1,\dots,m\}$ such that $I\neq J$, we have $I_h(\Delta_I,\Delta^\vee_J)=0$.*

(ii) The self-intersection number of $\Delta_{i_1\cdots i_r}$ is

$$I_h(\Delta_{i_1\cdots i_r},\Delta^\vee_{i_1\cdots i_r}) = \frac{\alpha-\prod_p\gamma_{i_p}}{(\alpha-1)\prod_p(1-\gamma_{i_p})}\cdot\prod_q\frac{\beta_{j_q}(1-\gamma_{j_q})}{(1-\beta_{j_q})(\beta_{j_q}-\gamma_{j_q})}.$$

Proof. (i) Since $\Delta_{i_1\cdots i_r}$'s represent local solutions (2) to $F_A(a,b,c)$ by Theorem 4.4, this claim is followed from similar arguments to the proof of Lemma 4.1 in [6].
(ii) By $\iota_{i_1\cdots i_r}$, the self-intersection number of $\Delta_{i_1\cdots i_r}$ is equal to that of $\tilde{\Delta}_{i_1\cdots i_r}$ with respect to the multi-valued function $u_{i_1\cdots i_r}$. To calculate this, we apply results in [7]. Since we construct the twisted cycle $\tilde{\Delta}_{i_1\cdots i_r}$ from the direct product of an r-simplex and $(m-r)$ intervals, the self intersection number of $\tilde{\Delta}_{i_1\cdots i_r}$ is obtained as the product of those of the loaded simplex and the loaded intervals. Thus we have

$$I_h(\Delta_{i_1\cdots i_r}, \Delta^{\vee}_{i_1\cdots i_r}) = \frac{1-\prod_p \gamma_{i_p}\cdot\alpha^{-1}}{\prod_p(1-\gamma_{i_p})\cdot(1-\alpha^{-1})}\cdot\prod_q \frac{1-\gamma_{j_q}}{(1-\beta_{j_q})(1-\gamma_{j_q}\beta_{j_q}^{-1})}.$$

□

5. Intersection numbers of twisted cohomology groups

In this section, we review twisted cohomology groups and the intersection form between twisted cohomology groups in our situation, and collect some results of [9] in which intersection numbers of twisted cocycles are evaluated.

Recall that

$$M = \mathbb{C}^m - \left(\bigcup_k (t_k=0)\cup\bigcup_k(1-t_k=0)\cup(v=0)\right),$$

$$u = \prod t_k^{b_k}(1-t_k)^{c_k-b_k-1}\cdot v^{-a}.$$

We consider the logarithmic 1-form

$$\omega := d\log u = \frac{du}{u}.$$

We denote the $\mathbb{C}$-vector space of smooth k-forms on M by $\mathcal{E}^k(M)$. We define the covariant differential operator $\nabla_\omega : \mathcal{E}^k(M)\to\mathcal{E}^{k+1}(M)$ by

$$\nabla_\omega(\psi) := d\psi + \omega\wedge\psi,\quad \psi\in\mathcal{E}^k(M).$$

Because of $\nabla_\omega\circ\nabla_\omega = 0$, we have a complex

$$\mathcal{E}^\bullet(M) : \cdots \xrightarrow{\nabla_\omega} \mathcal{E}^k(M) \xrightarrow{\nabla_\omega} \mathcal{E}^{k+1}(M) \xrightarrow{\nabla_\omega} \cdots$$

and its k-th cohomology group $H^k(M,\nabla_\omega)$. It is called the k-th twisted de Rham cohomology group. An element of $\ker\nabla_\omega$ is called a twisted cocycle. By replacing $\mathcal{E}^k(M)$ with the $\mathbb{C}$-vector space $\mathcal{E}^k_c(M)$ of smooth k-forms on M with compact support, we obtain the twisted de Rham cohomology group $H^k_c(M,\nabla_\omega)$ with compact support. By [3], we have $H^k(M,\nabla_\omega)=0$ for all $k\neq m$. Further, by Lemma 2.9 in [1], there is a canonical isomorphism

$$\jmath : H^m(M,\nabla_\omega)\to H^m_c(M,\nabla_\omega).$$

By considering $u^{-1}=1/u$ instead of u, we have the covariant differential operator $\nabla_{-\omega}$ and the twisted de Rham cohomology group $H^k(M,\nabla_{-\omega})$. The intersection form I_c between $H^m(M,\nabla_\omega)$ and $H^m(M,\nabla_{-\omega})$ is defined by

$$I_c(\psi,\psi') := \int_M \jmath(\psi)\wedge\psi',\quad \psi\in H^m(M,\nabla_\omega),\ \psi'\in H^m(M,\nabla_{-\omega}),$$

which converges because of the compactness of the support of $\jmath(\psi)$.

Remark 5.1. By Lemma 2.8 and Theorem 2.2 in [1], we have

$$\dim H_k(\mathcal{C}_\bullet(M,u)) = 0\quad (k\neq m),$$
$$\dim H_m(\mathcal{C}_\bullet(M,u)) = \dim H^m(M,\nabla_\omega) = (-1)^m\chi(M) = 2^m,$$

where $\chi(M)$ is the Euler characteristic of M. Under our assumption for the parameters a, b and c (see Section 1), since the determinant of the intersection matrix $(I_h(\Delta_I, \Delta^\vee_J))$ is not zero by Theorem 4.6, the twisted cycles $\{\Delta_I\}_I$ form a basis of $H_m(\mathcal{C}_\bullet(M,u))$.

The intersection numbers of some twisted cocycles are evaluated in [9]. We use a part of these results. We consider m-forms

$$\varphi^{i_1\cdots i_r} := \frac{dt_1\wedge\cdots\wedge dt_m}{\prod_p (t_{i_p}-1)\cdot\prod_q t_{j_q}}$$

on M, which is denoted by $\varphi_{x,(v_1,\dots,v_m)}$ with $v_{i_p}=1$, $v_{j_q}=0$ in [9]. Note that $\varphi=\varphi^{\emptyset}$ is equal to $\varphi=\varphi_\emptyset$ defined in Section 3 (and 4). We put

$$A_{i_1\cdots i_r} = A_I := \sum_{\{I^{(l)}\}}\prod_{l=1}^{r}\frac{1}{a-\sum c_{i_p^{(l)}}+l},$$

where $\{I^{(l)}\}$ runs sequences of subsets of $I=\{i_1,\dots,i_r\}$, which satisfy

$$I = I^{(r)} \supsetneq I^{(r-1)} \supsetneq \cdots \supsetneq I^{(2)} \supsetneq I^{(1)} \neq \emptyset,$$

and we write $I^{(l)} = \{i_1^{(l)},\dots,i_l^{(l)}\}$.

Proposition 5.2 ([9]). *We have*

$$I_c(\varphi^I,\varphi^{I'}) = (2\pi\sqrt{-1})^m\cdot\sum_{N\subset\{1,\dots,m\}}\left(A_N\prod_{n\notin N}\frac{\delta_{I,I'}(n)}{\tilde{b}_I(n)}\right),$$

where

$$\delta_{I,I'}(n) := \begin{cases} 1 & (n\in(I\cap I')\cup(I^c\cap I'^c)) \\ 0 & (\text{otherwise}),\end{cases}$$

$$\tilde{b}_I(n) := \begin{cases} c_n-b_n-1 & (n\in I) \\ b_n & (n\in I^c).\end{cases}$$

Under our assumptions for the parameters, $\{\varphi^I\}_I$ form a basis of $H^m(M,\nabla_\omega)$.

6. Twisted period relations

Because of the compatibility of intersection forms and pairings obtained by integrations (see [4]), we have the following theorem.

Theorem 6.1 (Twisted period relations, [4]). *We have*

$$I_c(\varphi^I,\varphi^{I'}) = \sum_{N\subset\{1,\dots,m\}}\frac{1}{I_h(\Delta_N,\Delta^\vee_N)}\cdot g_{I,N}\cdot g^\vee_{I',N}, \tag{6}$$

where

$$g_{I,N} = \int_{\Delta_N} u\varphi^I,\ g^\vee_{I',N} = \int_{\Delta^\vee_N} u^{-1}\varphi^{I'}.$$

By the results in Sections 4 and 5, twisted period relations (6) can be reduced to quadratic relations among F_A's. We write out two of them as a corollary.

Corollary 6.2. *We use the notations*

$$b^{i_1\cdots i_r} = b+\sum(1-c_{i_p})e_{i_p},\ c^{i_1\cdots i_r} = c+2\sum(1-c_{i_p})e_{i_p}\quad\text{(see Proposition 2.2)},$$

$$a_{i_1\cdots i_r} := a+r-\sum c_{i_p},$$

$$\tilde{b}^{i_1\cdots i_r} := (1,\dots,1)-b^{i_1\cdots i_r},\ \tilde{c}^{i_1\cdots i_r} := (2,\dots,2)-c^{i_1\cdots i_r}.$$

(i) The equality (6) for $I = I' = \emptyset$ is reduced to

$$\frac{\prod(c_k - 1)}{a} \cdot \sum_I \left(A_I \prod_{j \notin I} \frac{1}{b_j} \right)$$
$$= \sum_I \left[\prod_q \frac{c_{j_q} - b_{j_q} - 1}{b_{j_q}} \cdot \frac{1}{a_{i_1 \cdots i_r}} \cdot F_A(a_{i_1 \cdots i_r}, b^{i_1 \cdots i_r}, c^{i_1 \cdots i_r}; x) \right.$$
$$\left. \cdot F_A(-a_{i_1 \cdots i_r}, -b^{i_1 \cdots i_r}, \tilde{c}^{i_1 \cdots i_r}; x) \right].$$

(ii) The equality (6) for $I = \emptyset$, $I' = \{1, \ldots, m\}$ is reduced to

$$\frac{\prod(1 - c_k)}{} \cdot A_{1 \cdots m}$$
$$= \sum_I \frac{(-1)^r}{a_{i_1 \cdots i_r}} \cdot F_A(a_{i_1 \cdots i_r}, b^{i_1 \cdots i_r}, c^{i_1 \cdots i_r}; x) \cdot F_A(-a_{i_1 \cdots i_r}, \tilde{b}^{i_1 \cdots i_r}, \tilde{c}^{i_1 \cdots i_r}; x).$$

Proof. We prove (i). By Proposition 4.3 and Theorem 4.4, we have

$$g_{i_1 \cdots i_r} = e^{\pi\sqrt{-1}(\sum b_{i_p} - \sum c_{i_p} + r)} \cdot \prod_{p=1}^{r} \Gamma(c_{i_p} - 1) \cdot \prod_{q=1}^{m-r} \frac{\Gamma(b_{j_q})\Gamma(c_{j_q} - b_{j_q})}{\Gamma(c_{j_q})}$$
$$\cdot \frac{\Gamma(1-a)}{\Gamma(\sum c_{i_p} - a - r + 1)} \cdot \prod_{p=1}^{r} x_{i_p}^{1 - c_{i_p}} \cdot F_A(a + r - \sum c_{i_p}, b^{i_1 \cdots i_r}, c^{i_1 \cdots i_r}; x).$$

On the other hand, we can express $g^{\vee}_{i_1 \cdots i_r}$ like this by the replacement

$$(a, b, c) \longmapsto (-a, -b, (2, \ldots, 2) - c),$$

since $u^{-1}\varphi$ is written as

$$u^{-1}\varphi = \prod t_k^{-b_k - 1}(1 - t_k)^{-c_k + b_k + 1} \cdot (1 - \sum x_k t_k)^a dt_1 \wedge \cdots \wedge dt_m.$$

Thus we obtain

$$g^{\vee}_{i_1 \cdots i_r} = e^{\pi\sqrt{-1}(-\sum b_{i_p} + \sum c_{i_p} - r)}$$
$$\cdot \prod_{p=1}^{r} \Gamma(1 - c_{i_p}) \cdot \prod_{q=1}^{m-r} \frac{\Gamma(-b_{j_q})\Gamma(2 - c_{j_q} + b_{j_q})}{\Gamma(2 - c_{j_q})} \frac{\Gamma(1+a)}{\Gamma(-\sum c_{i_p} + a + r + 1)}$$
$$\cdot \prod_{p=1} x_{i_p}^{c_{i_p} - 1} \cdot F_A(-a - r + \sum c_{i_p}, -b^{i_1 \cdots i_r}, (2, \ldots, 2) - c^{i_1 \cdots i_r}; x).$$

By the formula (5) and Theorem 4.6, we have

$$\frac{\Gamma(1-a)\Gamma(1+a)}{\Gamma(\sum c_{i_p} - a - r + 1)\Gamma(-\sum c_{i_p} + a + r + 1)} \cdot \prod_p \Gamma(c_{i_p} - 1)\Gamma(1 - c_{i_p})$$
$$\cdot \prod_q \frac{\Gamma(b_{j_q})\Gamma(-b_{j_q}) \cdot \Gamma(c_{j_q} - b_{j_q})\Gamma(1 - c_{j_q} + b_{j_q})}{\Gamma(c_{j_q})\Gamma(2 - c_{j_q})}$$
$$= (2\pi\sqrt{-1})^m \cdot \prod_k \frac{1}{c_k - 1} \cdot \prod_q \frac{c_{j_q} - b_{j_q} - 1}{b_{j_q}} \cdot \frac{a}{a + r - \sum c_{i_p}} \cdot I_h(\Delta_{i_1 \cdots i_r}, \Delta^{\vee}_{i_1 \cdots i_r}).$$

Hence, we obtain (i) by Proposition 5.2. A similar calculation shows (ii). □

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