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INTERSECTION NUMBERS AND TWISTED PERIOD RELATIONS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION ${}_{m+1}F_m$

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Abstract. We consider the generalized hypergeometric function ${}_{m+1}F_m$ and the differential equation ${}_{m+1}E_m$ that it satisfies. We use the twisted (co)homology groups associated with an Euler-type integral representation. We evaluate the intersection numbers of the twisted cocycles that are defined as the mth exterior products of logarithmic 1-forms. We also provide the twisted cycles corresponding to the local solutions to ${}_{m+1}E_m$ around the origin, and we evaluate their intersection numbers. The intersection numbers of the twisted (co)cycles lead to the twisted period relations between two fundamental systems of solutions to ${}_{m+1}E_m$.

1. Introduction

The generalized hypergeometric function ${}_{m+1}F_m$ of a variable x with complex parameters $a_0, \dots, a_m, b_1, \dots, b_m$ is defined by

$${}_{m+1}F_m\left(\begin{matrix} a_0, \dots, a_m \\ b_1, \dots, b_m \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_0, n) \cdots (a_m, n)}{(b_1, n) \cdots (b_m, n) n!} x^n,$$

where $b_1, \dots, b_m \notin \{0, -1, -2, \dots\}$ and $(c, n) = \Gamma(c+n)/\Gamma(c)$. This series converges in the unit disk $|x| < 1$, and satisfies the generalized hypergeometric differential equation

$${}_{m+1}E_m = {}_{m+1}E_m\left(\begin{matrix} a_0, \dots, a_m \\ b_1, \dots, b_m \end{matrix}\right) : \left[\theta \prod_{i=1}^{m} (\theta + b_i - 1) - x \prod_{j=0}^{m} (\theta + a_j)\right] f(x) = 0,$$

where $\theta = x(d/dx)$. The linear differential equation ${}_{m+1}E_m$ is of rank $m+1$ with regular singular points $x = 0, 1, \infty$. We put $b_0 := 0$ (although b_0 is usually defined to be 1, we use this setting for our convenience). If $b_i - b_j \notin \mathbb{Z}$ $(0 \le i < j \le m)$, a fundamental system of solutions to ${}_{m+1}E_m$ around $x = 0$ is given by the following $m+1$ functions:

$$\begin{aligned} f_0 &:= {}_{m+1}F_m\left(\begin{matrix} a_0, \dots, a_m \\ b_1, \dots, b_m \end{matrix}; x\right), \\ f_r &:= x^{1-b_r} \cdot {}_{m+1}F_m\left(\begin{matrix} a_0 - b_r + 1, \dots, a_m - b_r + 1 \\ b_1 - b_r + 1, \dots, 2 - b_r, \dots, b_m - b_r + 1 \end{matrix}; x\right), \end{aligned} \tag{1}$$

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where $1 \leq r \leq m$. It is known that ${}_{m+1}F_m$ admits an Euler-type integral representation:

$$
\begin{aligned}
&{}_{m+1}F_m\left(\begin{matrix} a_0, \ldots, a_m \\ b_1, \ldots, b_m \end{matrix}; x\right) \\
&= \prod_{i=1}^{m} \frac{\Gamma(b_i)}{\Gamma(a_i)\Gamma(b_i - a_i)} \int_D \prod_{j=1}^{m-1} \left(t_j^{a_j - b_{j+1}} (t_j - t_{j+1})^{b_{j+1} - a_{j+1} - 1} \right) \\
&\qquad \cdot t_m^{a_m - 1} (1 - t_1)^{b_1 - a_1 - 1} (1 - x t_m)^{-a_0}\, dt_1 \wedge \cdots \wedge dt_m, \qquad (2)
\end{aligned}
$$

where $D := \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid 0 < t_m < t_{m-1} < \cdots < t_1 < 1\}$. The branch of the integrand is defined by the principal value for x near to 0.

In this paper, we consider the twisted (co)homology groups associated with the integral representation (2). Note that the singular locus of the integrand of (2) is not normally crossing. In such a case, as is studied in [**7**], the resolution of singularities is an effective way to study the intersections of the twisted (co)homology groups. We resolute the singularities and evaluate the intersection numbers of the twisted cocycles. We provide formulas for the intersection numbers of the mth exterior products of the logarithmic 1-forms, which span the twisted cohomology group. To describe the formulas, we use the combinatorial structure of the divisors. For the study of the twisted homology group, we introduce a method that does not require the resolution of the singularities. We use the method given in [**5**] and [**6**] to construct twisted cycles corresponding to the $m+1$ solutions (1) to ${}_{m+1}E_m$. These are made by the bounded chambers, and their boundaries are not regularized in the usual way. It is an advantage of our construction that we can evaluate their intersection numbers by the formula in [**11**] for a normally crossing singular locus. Intersection numbers of twisted homology and cohomology groups lead to twisted period relations between two fundamental systems of solutions to ${}_{m+1}E_m$ that have different parameters. We transform these relations into quadratic relations between the hypergeometric series ${}_{m+1}F_m$. Since our intersection matrices are diagonal, it is easy to reduce the twisted period relations to quadratic relations between the ${}_{m+1}F_m$.

In [**9**], twisted cycles corresponding to the solutions (1) to ${}_{m+1}E_m$ are obtained as the usual regularizations of real (unbounded) chambers, and their intersection numbers are evaluated. This evaluation is based on the method in [**7**], and it requires that the singularities be resolved. Note that we do not need this resolution for the study of the twisted homology group. Our method is much simpler than that in [**9**].

In [**10**], twisted period relations for ${}_{m+1}F_m$ are obtained from the study of the intersection forms of the (co)homology groups with coefficients in the local system of rank m. Another integral representation of ${}_{m+1}F_m$ and its inductive structure are used. Our (co)homology groups have coefficients in the local system of rank 1 for the general m variable case. The twisted period relations in [**10**] are transformed into quadratic relations between the ${}_{m+1}F_m$ and their derivatives. Since our quadratic relations consist of only the ${}_{m+1}F_m$, the structures of twisted period relations in [**10**] and that of ours are completely different.

There have been few studies of the intersection forms of the twisted cohomology groups associated with (2). We give two systems of twisted cocycles that have some orthogonality with respect to the intersection form. This is the first time the intersection numbers of such cocycles have been evaluated.

As shown in [2], the irreducibility condition of the differential equation ${}_{m+1}E_m$ is known to be $a_i - b_j \notin \mathbb{Z}$ $(0 \le i,\ j \le m)$. Since we use the fundamental system (1) of solutions to ${}_{m+1}E_m$, we assume throughout this paper that the parameters $a_i,\ b_j$ satisfy the condition

$$a_i - b_j \notin \mathbb{Z} \quad (0 \le i,\ j \le m), \qquad b_i - b_j \notin \mathbb{Z} \quad (0 \le i < j \le m). \tag{3}$$

2. Twisted (co)homology groups associated with the integral representation (2)

For twisted homology groups, twisted cohomology groups, and the intersection forms, refer to [1, 11], or [5]. We use the same notation as in [5] and [6].

In this paper, we primarily consider the twisted (co)homology group for

$$M := \mathbb{C}^m - \left(\bigcup_{j=1}^{m} (t_j = 0) \cup \bigcup_{j=2}^{m} (t_{j-1} - t_j = 0) \cup (1 - t_1 = 0) \cup (1 - x t_m = 0) \right)$$

and the multivalued function

$$u := \prod_{j=1}^{m} t_j^{a_j - b_{j+1}} \cdot \prod_{j=2}^{m} (t_{j-1} - t_j)^{b_j - a_j} \cdot (1 - t_1)^{b_1 - a_1} \cdot (1 - x t_m)^{-a_0}.$$

We put $\omega := d \log u$, where d is the exterior derivative with respect to the variables $t_1, \ldots, t_m$ (note that this is not with respect to x, which is regarded as a parameter). The twisted cohomology group, the twisted cohomology group with compact support, and the twisted homology group are denoted by $H^k(M, \nabla_\omega)$, $H^k_c(M, \nabla_\omega)$, and $H_k(\mathcal{C}_\bullet(M, u))$, respectively. Here, ∇_ω is the covariant differential operator defined as $\nabla_\omega := d + \omega\wedge$. The expression (2) means that the integral

$$\int_{D \otimes u} u \varphi_0, \qquad \varphi_0 := \frac{dt_1 \wedge \cdots \wedge dt_m}{t_m (1 - t_1)(t_1 - t_2) \cdots (t_{m-1} - t_m)}$$

represents ${}_{m+1}F_m$ modulo Gamma factors. By [1] and [3], we have $H^k(M, \nabla_\omega) = 0$ $(k \neq m)$, $\dim H^m(M, \nabla_\omega) = m + 1$, and there is a canonical isomorphism

$$\jmath : H^m(M, \nabla_\omega) \to H^m_c(M, \nabla_\omega).$$

By the Poincaré duality, we have

$$\dim H_k(\mathcal{C}_\bullet(M, u)) = \dim H^k(M, \nabla_\omega) = 0 \quad (k \neq m),$$
$$\dim H_m(\mathcal{C}_\bullet(M, u)) = \dim H^m(M, \nabla_\omega) = m + 1.$$

The intersection form I_h on the twisted homology groups is the pairing between $H_m(\mathcal{C}_\bullet(M, u))$ and $H_m(\mathcal{C}_\bullet(M, u^{-1}))$. The intersection form I_c on the twisted cohomology groups is the pairing between $H^m_c(M, \nabla_\omega)$ and $H^m(M, \nabla_{-\omega})$. By using $\jmath$, we can regard the intersection form I_c as the pairing between $H^m(M, \nabla_\omega)$ and $H^m(M, \nabla_{-\omega})$, i.e.

$$I_c(\psi, \psi') := \int_M \jmath(\psi) \wedge \psi', \qquad \psi \in H^m(M, \nabla_\omega), \qquad \psi' \in H^m(M, \nabla_{-\omega}).$$

3. Twisted cohomology groups and intersection numbers

In this section, we give two systems of twisted cocycles and evaluate their intersection numbers.

We embed M into the projective space $\mathbb{P}^m$, that is, we regard M as an open subset of $\mathbb{P}^m$:

$$M = \mathbb{P}^m - \left(\bigcup_{j=0}^{m} L_j \cup \bigcup_{j=0}^{m} H_j\right) \subset \mathbb{C}^m \subset \mathbb{P}^m,$$

where

$$L_j := (T_j = 0) \quad (0 \le j \le m),$$
$$H_j := (T_{j-1} - T_j = 0) \quad (1 \le j \le m), \qquad H_0 := (T_0 - xT_m = 0).$$

In terms of the homogeneous coordinates $T_0, \dots, T_m$, the multivalued function u is expressed as

$$u = T_0^{\lambda_0}(T_0 - xT_m)^{\mu_0} \cdot \prod_{j=1}^{m} T_j^{\lambda_j}(T_{j-1} - T_j)^{\mu_j},$$

where

$$\lambda_j := a_j - b_{j+1} \quad (1 \le j \le m-1), \qquad \lambda_m := a_m,$$
$$\mu_j := b_j - a_j \quad (1 \le j \le m), \qquad \mu_0 := -a_0,$$
$$\lambda_0 := -\left(\sum_{j=1}^{m} \lambda_j + \sum_{j=0}^{m} \mu_j\right) = a_0 - b_1.$$

Note that $L_0 = (T_0 = 0)$ is the hyperplane at infinity, i.e. $M \subset \mathbb{C}^m = \mathbb{P}^m - L_0 \subset \mathbb{P}^m$, and the coordinates $t_1, \dots, t_m$ on $\mathbb{C}^m$ are given as $t_j = T_j/T_0$. Hereafter, we regard subscripts as elements in $\mathbb{Z}/(m+1)\mathbb{Z}$. For example, we have $a_{m+1} = a_0$, $b_{m+1} = b_0 = 0$, and

$$\lambda_j = a_j - b_{j+1}, \quad \mu_j = b_j - a_j \quad (0 \le j \le m).$$

Let ℓ_k and h_k $(0 \le k \le m)$ be the defining linear forms of L_k and H_k, respectively. We define an m-form on M by

$$\phi(f_0, \dots, f_m) := d\log\left(\frac{f_0}{f_1}\right) \wedge d\log\left(\frac{f_1}{f_2}\right) \wedge \cdots \wedge d\log\left(\frac{f_{m-1}}{f_m}\right)$$

for $f_0, \dots, f_m \in \{\ell_0, \dots, \ell_m, h_0, \dots, h_m\}$. We consider two systems $\{\varphi_k\}_{k=0}^m$ and $\{\psi_k\}_{k=0}^m$ given as

$$\varphi_k := \phi(h_0, \dots, h_{k-1}, \ell_{k-1}, h_{k+1}, \dots h_m),$$
$$\psi_k := \phi(h_0, \dots, h_{k-1}, \ell_k, h_{k+1}, \dots h_m).$$

Using the coordinates $t_j = T_j/T_0$ $(1 \le j \le m)$ of $\mathbb{C}^m = \mathbb{P}^m - L_0$, we have

$$\varphi_0 = \frac{dt_1 \wedge \cdots \wedge dt_m}{t_m(1-t_1)(t_1-t_2)\cdots(t_{m-1}-t_m)}$$

$$\psi_0 = \frac{dt_1 \wedge \cdots \wedge dt_m}{(1-t_1)(t_1-t_2)\cdots(t_{m-1}-t_m)}$$

$$\varphi_r = \frac{x\, dt_1 \wedge \cdots \wedge dt_m}{t_{r-1}(1-xt_m)(1-t_1)(t_1-t_2)\cdots\widehat{(t_{r-1}-t_r)}\cdots(t_{m-1}-t_m)}$$

$$\psi_r = \frac{dt_1 \wedge \cdots \wedge dt_m}{t_r(1-xt_m)(1-t_1)(t_1-t_2)\cdots\widehat{(t_{r-1}-t_r)}\cdots(t_{m-1}-t_m)}$$

where $1 \le r \le m$. Note that the m-form φ_0 coincides with that defined in Section 2.

THEOREM 3.1. *We have*

$$I_c(\varphi_i, \varphi_j) = I_c(\psi_i, \psi_j) = 0 \quad (i \ne j), \tag{4}$$

$$I_c(\varphi_k, \varphi_k) = (2\pi\sqrt{-1})^m \prod_{0<l<m} \frac{b_l - b_k}{(a_l - b_k)(b_l - a_l)}, \tag{5}$$

$$I_c(\psi_k, \psi_k) = (2\pi\sqrt{-1})^m \prod_{\substack{0 \le l \le m \\ l \ne k}} \frac{a_l - a_k}{(b_l - a_k)(b_l - a_l)}, \tag{6}$$

$$I_c(\varphi_i, \psi_j) = I_c(\psi_j, \varphi_i) = \varepsilon_{ij}(2\pi\sqrt{-1})^m \frac{(b_i - a_i)(b_j - a_j)}{(b_i - a_j)} \prod_{l=0}^{m} \frac{1}{b_l - a_l}, \tag{7}$$

where

$$\varepsilon_{ij} := \begin{cases} -1 & i \ne j \text{ and } (i = 0 \text{ or } j = 0), \\ 1 & \text{otherwise.} \end{cases}$$

The following corollary follows immediately from this theorem.

COROLLARY 3.2. *Under the condition (3), the m-forms $\varphi_0, \ldots, \varphi_m$ form a basis of $H^m(M, \nabla_\omega)$.*

Proof. We set $C := (I_c(\varphi_i, \varphi_j))_{i,j=0,\ldots,m}$; this is called the intersection matrix. Then we have

$$\det(C) = (2\pi\sqrt{-1})^{m(m+1)} \prod_{l=0}^{m} \frac{1}{(b_l - a_l)^m} \prod_{0 \le i \ne j \le m} \frac{b_i - b_j}{a_i - b_j},$$

which does not vanish under the condition (3). □

In the remainder of this section, we prove Theorem 3.1. According to **[8]**, we can evaluate intersection numbers by blowing up $\mathbb{P}^m$ so that the pole divisor of the pullback of $\omega = d \log u$ is a normal crossing divisor. We need information about the m-forms around the points at which the m components of the pole divisor intersect.

For $i \ne j,\ j+1$, let $L_{j,j+1,\ldots,i-1}$ be the exceptional divisor obtained by blowing up along $L_j \cap L_{j+1} \cap \cdots \cap L_{i-1} = (T_j = T_{j+1} = \cdots = T_{i-1} = 0)$. The residue of the pullback of ω along $L_{j,j+1,\dots,i-1}$ is

$$\lambda_{j,j+1,\dots,i-1} = \sum_{l=j}^{i-1} \lambda_l + \sum_{l=j+1}^{i-1} \mu_l = \sum_{l=j}^{i-1} (a_l - b_{l+1}) + \sum_{l=j+1}^{i-1} (b_l - a_l) = a_j - b_i$$

(recall that the indices are regarded as elements in $\mathbb{Z}/(m+1)\mathbb{Z}$). Note, for example, that L_{12} is an exceptional divisor, but L_1 is not.

First, we describe the intersections of $L_1, \dots, L_m, H_0, \dots, H_m$ and the exceptional divisors $L_{j,j+1,\dots,m}$ $(1 \le j \le m-1)$ in $\mathbb{C}^m = \mathbb{P}^m - L_0$. It is easy to show the following lemma.

LEMMA 3.3. *We blow up* $\mathbb{P}^m - L_0$ *along*

$$L_j \cap L_{j+1} \cap \dots \cap L_m \quad (1 \le j \le m-1).$$

In $\{\varphi_0, \dots, \varphi_m, \psi_0, \dots, \psi_m\}$, *only* φ_0 *and* ψ_j *have* $L_{j,j+1,\dots,m}$ *as a component of the pole divisor. Further, we have*

$$H_k \cap L_{j,j+1,\dots,m} = \emptyset \Longleftrightarrow k = 0 \text{ or } k = j.$$

Second, we describe all of the intersections of $L_0, \dots, L_m, H_0, \dots, H_m$ and the exceptional divisors.

LEMMA 3.4. *After blowing up along all* $L_j \cap L_{j+1} \cap \dots \cap L_{i-1}$, *the pole divisor of the pullback of* ω *is a normal crossing divisor. Let* Φ_k *(respectively* Ψ_k*) be a set consisting of the components of the pole divisor of the pullback of* φ_k *(respectively* ψ_k*). Then we have*

$$\Phi_k = \{H_{k+1}, H_{k+2}, \dots, H_{k-1}, L_{k+1,k+2,\dots,k-1}, L_{k+2,\dots,k-1}, \dots, L_{k-2,k-1}, L_{k-1}\},$$

$$\Psi_k = \{H_{k+1}, H_{k+2}, \dots, H_{k-1}, L_k, L_{k,k+1}, \dots, L_{k,k+1,\dots,k-3}, L_{k,k+1\dots,k-3,k-2}\}.$$

Moreover, we have

$$H_k \cap L_{j,j+1,\dots,i-1} = \emptyset \Longleftrightarrow k = i \text{ or } k = j.$$

Proof. Recall that u is expressed as

$$u = t_1^{\lambda_1} \cdots t_m^{\lambda_m} \cdot (1 - xt_m)^{\mu_0} (1 - t_1)^{\mu_1} \cdot (t_1 - t_2)^{\mu_2} \cdots (t_{m-1} - t_m)^{\mu_m}$$

on $\mathbb{C}^m = \mathbb{P}^m - L_0$ (with the coordinates $t_j = T_j/T_0,\ 1 \le j \le m$). On the other hand, on $\mathbb{P}^m - L_k$, it is expressed as

$$\begin{aligned} u &= s_0^{\lambda_0} \cdots s_{k-1}^{\lambda_{k-1}} s_{k+1}^{\lambda_{k+1}} \cdots s_m^{\lambda_m} \\ &\quad \cdot (s_0 - xs_m)^{\mu_0} (s_0 - s_1)^{\mu_1} \cdots (s_{k-1} - 1)^{\mu_k} (1 - s_{k+1})^{\mu_{k+1}} \cdots (s_{m-1} - s_m)^{\mu_m} \\ &= s_{k+1}^{\lambda_{k+1}} \cdots s_m^{\lambda_m} s_0^{\lambda_0} \cdots s_{k-1}^{\lambda_{k-1}} \cdot (s_{k-1} - 1)^{\mu_k} (1 - s_{k+1})^{\mu_{k+1}} \\ &\quad \cdot (s_{k+1} - s_{k+2})^{\mu_{k+2}} \cdots (s_{m-1} - s_m)^{\mu_m} (s_0 - xs_m)^{\mu_0} (s_0 - s_1)^{\mu_1} \cdots (s_{k-2} - s_{k-1})^{\mu_{k-1}}, \end{aligned}$$

in terms of the coordinates $s_j = T_j/T_k$ $(0 \le j \le m,\ j \ne k)$. Thus,

$$L_{k+1}, \dots, L_m, L_0, \dots, L_{k-1} \quad \text{and} \quad H_k, \dots, H_m, H_0, \dots, H_{k-1}$$

in $\mathbb{P}^m - L_k$ behave similarly to

$$L_1, \dots, L_m \quad \text{and} \quad H_0, \dots, H_m$$

in $\mathbb{P}^m - L_0$. Then the lemma follows from Lemma 3.3. □

Remark 3.5. For $1 \leq j \leq m$, H_j is defined by $T_{j-1} - T_j = 0$. On the other hand, H_0 is defined by $T_0 - xT_m (= T_0 - xT_{0-1}) = 0$. As mentioned below, this difference gives the sign ε_{ij} in (7).

In particular, we have $\#\Phi_k = \#\Psi_k = 2m$. We put

$$\Phi_k^{(m)} := \{\{D_1, \ldots, D_m\} \subset \Phi_k \mid D_i \neq D_j \ (i \neq j),\ D_1 \cap \cdots \cap D_m \neq \emptyset\}$$

($\Psi_k^{(m)}$ is also defined in a similar way). Then Lemma 3.4 implies

$$\Phi_k^{(m)} = \{\{H_p\}_{p \in I} \cup \{L_{q,q+1,\ldots,k-1}\}_{q \notin I} \mid I \subset \{k+1, k+2, \ldots, m, 0, \ldots, k-1\}\},$$
$$\Psi_k^{(m)} = \{\{H_p\}_{p \in I} \cup \{L_{k,k+1,\ldots,q-1}\}_{q \notin I} \mid I \subset \{k+1, k+2, \ldots, m, 0, \ldots, k-1\}\}.$$

Finally, we evaluate the intersection numbers of the φ_i and ψ_j by using the results in [**8**].

Proof of Theorem 3.1. First, we obtain (4), since it is clear that

$$\Phi_i^{(m)} \cap \Phi_j^{(m)} = \Psi_i^{(m)} \cap \Psi_j^{(m)} = \emptyset \quad (i \neq j).$$

Second, we have

$$\begin{aligned} I_c(\varphi_k, \varphi_k) &= (2\pi\sqrt{-1})^m \sum_{I \subset \{k+1,k+2,\ldots,k-1\}} \prod_{i \in I} \frac{1}{\mu_i} \cdot \prod_{j \notin I} \frac{1}{\lambda_{j,j+1,\ldots,k-1}} \\ &= (2\pi\sqrt{-1})^m \sum_{I \subset \{k+1,k+2,\ldots,k-1\}} \prod_{i \in I} \frac{1}{b_i - a_i} \cdot \prod_{j \notin I} \frac{1}{a_j - b_k}. \end{aligned} \tag{8}$$

By induction on m, we can show that

$$\sum_{I \subset \{k+1,k+2,\ldots,k-1\}} \prod_{i \in I} \frac{1}{b_i - a_i} \prod_{j \notin I} \frac{1}{a_j - b_k} = \prod_{\substack{0 \leq l \leq m \\ l \neq k}} \frac{b_l - b_k}{(a_l - b_k)(b_l - a_l)},$$

which reduces (8) to (5). The equality (6) can be shown in a similar way. Finally, we prove (7). Because of

$$\Phi_i^{(m)} \cap \Psi_i^{(m)} = \{\{H_0, \ldots, H_m\} - \{H_i\}\},$$
$$\Phi_i^{(m)} \cap \Psi_j^{(m)} = \{\{H_0, \ldots, H_m, L_{j,j+1,\ldots,i-1}\} - \{H_i, H_j\}\} \quad (i \neq j),$$

we have

$$I_c(\varphi_i, \psi_i) = \varepsilon'_{ii} \cdot (2\pi\sqrt{-1})^m \prod_{l \neq i} \frac{1}{\mu_l} = \varepsilon'_{ii} \cdot (2\pi\sqrt{-1})^m \prod_{l \neq i} \frac{1}{b_l - a_l},$$

$$\begin{aligned} I_c(\varphi_i, \psi_j) &= \varepsilon'_{ij} \cdot (2\pi\sqrt{-1})^m \cdot \frac{1}{\lambda_{j,j+1,\ldots,i-1}} \prod_{l \neq i,j} \frac{1}{\mu_l} \\ &= \varepsilon'_{ij} \cdot (2\pi\sqrt{-1})^m \cdot \frac{1}{a_j - b_i} \prod_{l \neq i,j} \frac{1}{b_l - a_l} \quad (i \neq j), \end{aligned}$$

where $\varepsilon'_{ij} = \pm 1$. Let us show that

$$\varepsilon'_{ij} = \begin{cases} 1 & i = 0 \text{ or } j = 0 \text{ or } i = j, \\ -1 & \text{otherwise.} \end{cases} \tag{9}$$

When we evaluate the intersection number $I_c(\varphi_i, \psi_j)$, it is sufficient to consider blowing up along only

$$L_{i+1} \cap L_{i+2} \cap \cdots \cap L_{i-1},\ L_{i+2} \cap \cdots \cap L_{i-1}, \ldots,\ L_{i-2} \cap L_{i-1}$$

in the coordinate system of $\mathbb{P}^m - L_i$, since the pole divisor of φ_i is a normal crossing divisor following this blowing-up process. Put $\Phi_i^{(m)} \cap \Psi_j^{(m)} = \{\{G_1, \ldots, G_m\}\}$, and let g_l be the defining linear form of G_l. By taking appropriate coordinates $t'_1, \ldots, t'_m$, we can explicitly express $\varphi_i \cdot \prod_l g_l$ and $\psi_j \cdot \prod_l g_l$ around the intersection point $G_1 \cap \cdots \cap G_m$.

(i) For $i = j = 0$, we have

$$g_l = 1 - t'_l \quad (1 \le l \le m),$$

$$\varphi_0 \cdot \prod_l g_l = \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots t'_m}, \qquad \psi_0 \cdot \prod_l g_l = dt'_1 \wedge \cdots \wedge dt'_m,$$

and the intersection point $G_1 \cap \cdots \cap G_m$ is expressed as

$$t'_l = 1 \quad (1 \le l \le m).$$

(ii) For $i = 0$ and $j \neq 0$, we have

$$g_j = t'_j, \quad g_l = 1 - t'_l \quad (l \neq j),$$

$$\varphi_0 \cdot \prod_l g_l = \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots \widehat{t'_j} \cdots t'_m (1 - t'_j)}, \qquad \psi_j \cdot \prod_l g_l = \frac{dt'_1 \wedge \cdots \wedge dt'_m}{1 - x t'_1 \cdots t'_m},$$

and the intersection point $G_1 \cap \cdots \cap G_m$ is expressed as

$$t'_j = 0, \quad t'_l = 1 \quad (l \neq j).$$

(iii) For $i \neq 0$ and $j = i$, we have

$$g_{m+1-i} = t'_{m+1-i} - x, \quad g_l = 1 - t'_l \quad (l \neq m + 1 - i),$$

$$\varphi_i \cdot \prod_l g_l = x \cdot \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots t'_m}, \qquad \psi_i \cdot \prod_l g_l = dt'_1 \wedge \cdots \wedge dt'_m,$$

and the intersection point $G_1 \cap \cdots \cap G_m$ is expressed as

$$t'_{m+1-i} = x, \quad t'_l = 1 \quad (l \neq m + 1 - i).$$

(iv) For $i \neq 0$ and $j = 0$, we have

$$g_{m+1-i} = t'_{m+1-i}, \quad g_l = 1 - t'_l \quad (l \neq m - i + 1),$$

$$\varphi_i \cdot \prod_l g_l = x \cdot \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots \widehat{t'_{m+1-i}} \cdots t'_m (t'_{m+1-i} - x)},$$

$$\psi_0 \cdot \prod_l g_l = \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots t'_m - 1}$$

and the intersection point $G_1 \cap \cdots \cap G_m$ is expressed as

$$t'_{m+1-i} = 0, \quad t'_l = 1 \quad (l \neq m + 1 - i).$$

(v) For $i \neq 0$ and $j \neq 0, i$, we have

$$g_{j-i} = t'_{j-i}, \quad g_{m+1-i} = t'_{m+1-i} - x, \quad g_l = 1 - t'_l \quad (l \neq j-i, m+1-i),$$

$$\varphi_i \cdot \prod_l g_l = x \cdot \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots \widehat{t'_{j-i}} \cdots t'_m (1 - t'_{j-i})}, \quad \psi_j \cdot \prod_l g_l = \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots t'_m - 1}$$

(note that if $j < i$, then we regard $j - i$ as $m + 1 + j - i$), and the intersection point $G_1 \cap \cdots \cap G_m$ is expressed as

$$t'_{j-i} = 0, \quad t'_{m+1-i} = x, \quad t'_l = 1 \quad (l \neq j-i, m+1-i).$$

Thus, (9) is proved, and we complete the proof of (7). □

4. Twisted homology groups and intersection numbers

In this section, we construct $m + 1$ twisted cycles in M, corresponding to the solutions (1) to ${}_{m+1}E_m$.

For $0 \leq k \leq m$, we set

$$M_k := \mathbb{C}^m - \left(\bigcup_{j=1}^{m} \left((z_j = 0) \cup (1 - z_j = 0) \right) \cup \left(z_k - x \prod_{j \neq k} z_j = 0 \right) \right),$$

where $z_1, \ldots, z_m$ are coordinates of $\mathbb{C}^m$, and z_0 is regarded as 1. Let u_k be a multivalued function and let ϕ_k be an m-form on M_k, defined as follows: for $1 \leq k \leq m$,

$$u_k := \prod_{j \neq k} z_j^{a_j - b_k} (1 - z_j)^{b_j - a_j} \cdot z_k^{a_k} (1 - z_k)^{-a_0} \left(z_k - x \prod_{j \neq k} z_j \right)^{b_k - a_k},$$

$$\phi_k := \frac{dz_1 \wedge \cdots \wedge dz_m}{z_k \cdot \prod_{j \neq k} (1 - z_j) \cdot (z_k - x \prod_{j \neq k} z_j)},$$

and for $k = 0$,

$$u_0 := \prod_{i=1}^{m} z_i^{a_i} (1 - z_i)^{b_i - a_i} \cdot \left(1 - x \prod_{i=1}^{m} z_i \right)^{-a_0} \qquad \phi_0 := \frac{dz_1 \wedge \cdots \wedge dz_m}{\prod_{i=1}^{m} (z_i (1 - z_i))}.$$

We construct a twisted cycle $\tilde{\Delta}_k$ loaded by u_k in M_k. Let x and ε be positive real numbers satisfying

$$\varepsilon < \frac{1}{2}, \quad x < \frac{\varepsilon}{(1 + \varepsilon)^{m-1}}$$

(for example, if

$$\varepsilon = \frac{1}{3}, \quad 0 < x < \frac{1}{3} \cdot \left(\frac{3}{4} \right)^{m-1}$$

this condition holds). Thus, the direct product

$$\sigma_k := \{ (z_1, \ldots, z_m) \in \mathbb{R}^m \mid \varepsilon \leq z_j \leq 1 - \varepsilon \ (1 \leq j \leq m) \}$$

of m intervals is contained in the bounded domain

$$\left\{ (z_1, \ldots, z_m) \in \mathbb{R}^m \,\middle|\, 0 < z_j < 1 \ (1 \leq j \leq m), \ z_k > x \cdot \prod_{j \neq k} z_j \right\}.$$

The orientation of σ_k is induced from the natural embedding $\mathbb{R}^m \subset \mathbb{C}^m$.

By using the ε-neighborhoods of $C_1 := (z_1 = 0), \dots, C_m := (z_m = 0)$, $C_{m+1} := (1 - z_1 = 0), \dots, C_{2m} := (1 - z_m = 0)$, we construct a twisted cycle $\tilde{\Delta}_k$ from $\sigma_k \otimes u_k$ in a way similar to that used in **[5]** and **[6]**. Here the branch of u_k on σ_k is defined by the principal value.

Let $U \subset \mathbb{R}^m$ be the direct product $(0, 1)^m$ of m intervals, which is bounded by $C_1, \dots, C_{2m}$, and contains σ_k. For $J \subset \{1, \dots, 2m\}$, we set $C_J := \bigcap_{j \in J} C_j$, $U_J := U \cap C_J$, and $T_J := \varepsilon$-neighborhood of U_J. Then we have

$$\sigma_k = U - \bigcup_J T_J.$$

Using these neighborhoods T_J, we can construct a twisted cycle $\tilde{\Delta}_k$ in the same manner as in Section 3.2.4 of **[1]** (C and U correspond to the notation in **[1]**, H and Δ, respectively). When $k \neq 0$, for fixed positive real numbers $z_j \in (0, 1)$ $(j \neq k)$, the point $(z_1, \dots, z_m)$ satisfying $z_k - x \prod_{j \neq k} z_j = 0$ belongs to T_k, since the solution $x \prod_{j \neq k} z_j$ of the equation $z_k - x \prod_{j \neq k} z_j = 0$ in z_k belongs to $\mathbb{R}$ and satisfies

$$x \cdot \prod_{j \neq k} z_j < x < \varepsilon.$$

Thus, if $(z_1, \dots, z_m)$ moves along a circle with radius ε surrounding C_k, the branch of

$$z_k^{a_k} \Big(z_k - x \prod_{j \neq k} z_j \Big)^{b_k - a_k}$$

is changed. The difference between the branches of u_k at the ending and starting points of this circle is

$$\exp(2\pi\sqrt{-1}a_k) \cdot \exp(2\pi\sqrt{-1}(b_k - a_k)) = \exp(2\pi\sqrt{-1}b_k).$$

Those of the circles surrounding C_{m+k}, C_i, C_{m+i} $(i \neq k)$ are

$$\exp(2\pi\sqrt{-1}(-a_0)), \quad \exp(2\pi\sqrt{-1}(a_i - b_k)), \quad \exp(2\pi\sqrt{-1}(b_i - a_i)),$$

respectively.

We give the expression of $\tilde{\Delta}_k$ briefly. For $j = 1, \dots, 2m$, let c_j be the $(m-1)$-face of σ_k given by $\sigma_k \cap T_j$, and let S_j be a positively oriented circle with radius ε in the orthogonal complement of C_j, starting from the projection of c_j to this space and surrounding C_j. Then $\tilde{\Delta}_k$ is written as

$$\sigma_k \otimes u_k + \sum_{\emptyset \neq J \subset \{1,\dots,2m\}} \prod_{j \in J} \frac{1}{d_j} \cdot \Big(\Big(\bigcap_{j \in J} c_j \Big) \times \prod_{j \in J} S_j \Big) \otimes u_k,$$

where

$$d_k := \beta_k - 1, \quad d_{m+k} := \alpha_0^{-1} - 1, \quad d_i := \alpha_i \beta_k^{-1} - 1, \quad d_{m+i} := \beta_i \alpha_i^{-1} - 1 \quad (i \neq k),$$

and $\alpha_j := e^{2\pi\sqrt{-1}a_j}$, $\beta_j := e^{2\pi\sqrt{-1}b_j}$. Note that we define an appropriate orientation for each $(\bigcap_{j \in J} c_j) \times \prod_{j \in J} S_j$; see **[1]** for details.

Remark 4.1. If $k=0$, the difference between the branches of u_0 at the ending and starting points of the circles surrounding C_i, C_{m+i} $(1 \le i \le m)$ are

$$\exp(2\pi\sqrt{-1}a_i), \quad \exp(2\pi\sqrt{-1}(b_i - a_i)),$$

respectively. Since $(0,1)^m \cap \{z \mid 1 - x\prod_i z_i = 0\} = \emptyset$, the twisted cycle $\tilde{\Delta}_0$ is the usual regularization of $(0,1)^m \otimes u_0$.

PROPOSITION 4.2. *We have*

$$\int_{\tilde{\Delta}_0} u_0\phi_0 = \prod_{i=1}^{m} \frac{\Gamma(a_i)\Gamma(b_i - a_i)}{\Gamma(b_i)} \cdot {}_{m+1}F_m\left(\begin{matrix} a_0, \dots, a_m \\ b_1, \dots, b_m \end{matrix}; x\right),$$

$$\int_{\tilde{\Delta}_r} u_r\phi_r = \frac{\Gamma(b_r - 1)\Gamma(1 - a_0)}{\Gamma(b_r - a_0)} \cdot \prod_{\substack{1 \le j \le m \\ j \ne r}} \frac{\Gamma(a_j - b_r + 1)\Gamma(b_j - a_j)}{\Gamma(b_j - b_r + 1)}$$

$$\cdot {}_{m+1}F_m\left(\begin{matrix} a_0 - b_r + 1, \dots, a_m - b_r + 1 \\ b_1 - b_r + 1, \dots, 2 - b_r, \dots, b_m - b_r + 1 \end{matrix}; x\right) \quad (1 \le r \le m).$$

Proof. In a way similar to that used for Proposition 4.3 of **[5]** or Proposition 4.3 of **[6]**, we can show this proposition by expanding the left-hand sides with respect to x. Note that we use the equalities

$$\int_{\tilde{\Delta}_0} \prod_{i=1}^{m} z_i^{a_i+n-1}(1 - z_i)^{b_i - a_i - 1}\, dz = \prod_{i=1}^{m} \frac{\Gamma(a_i + n)\Gamma(b_i - a_i)}{\Gamma(b_i + n)},$$

$$\int_{\tilde{\Delta}_r} \prod_{j \ne r} z_j^{a_j - b_r + n}(1 - z_j)^{b_j - a_j - 1} \cdot z_r^{b_r - 2 - n}(1 - z_r)^{-a_0}\, dz$$

$$= \prod_{j \ne r} \frac{\Gamma(a_j - b_r + n + 1)\Gamma(b_j - a_j)}{\Gamma(b_j - b_r + n + 1)} \frac{\Gamma(b_r - 1 - n)\Gamma(1 - a_0)}{\Gamma(b_r - a_0 - n)},$$

for a natural number n and $1 \le r \le m$. The second equality follows from the fact that the twisted cycle $\tilde{\Delta}_r$ of the integral can be identified with the usual regularization of the domain $(0,1)^m$, loaded by the multivalued function

$$\prod_{j \ne r} z_j^{a_j - b_r + n}(1 - z_j)^{b_j - a_j - 1} \cdot z_r^{b_r - 2 - n}(1 - z_r)^{-a_0}$$

on $\mathbb{C}^m - \bigcup_{j=1}^m ((z_j = 0) \cup (1 - z_j = 0))$. □

We define a bijection $\iota_k : M_k \to M$ by

$$\iota_0(z_1, \dots, z_m) := (t_1, \dots, t_m); \quad t_s = \prod_{i=1}^{s} z_i,$$

$$\iota_r(z_1, \dots, z_m) := (t_1, \dots, t_m); \quad t_s = \prod_{j=1}^{s} z_j \quad (s < r), \quad t_s = \frac{z_r}{x \cdot \prod_{j=s+1}^{m} z_j} \quad (s \ge r),$$

where $1 \le r \le m$. We also define branches of the multivalued function u on real chambers in M. Let $D_r \subset \mathbb{R}^m$ $(1 \le r \le m)$ be the chamber defined by

$$t_j > 0 \quad (1 \le j \le m), \quad 1 - xt_m > 0, \quad t_{j-1} - t_j > 0 \quad (j \ne r), \quad t_{r-1} - t_r < 0,$$

where we regard t_0 as 1. On D_r, the arguments of the factors of u are given as follows.

t_j	$1-xt_m$	$t_{j-1}-t_j$ $(j\neq r)$	$t_{r-1}-t_r$
0	0	0	$-\pi$

Recall that on $D=\{(t_1,\dots,t_m)\in\mathbb{R}^m \mid 0<t_m<t_{m-1}<\cdots<t_1<1\}$, all of the arguments of the factors of u are 0.

THEOREM 4.3. *We define a twisted cycle Δ_k in M by*

$$\Delta_k := (\iota_k)_*(\tilde{\Delta}_k).$$

Then we have

$$\int_{\Delta_0} u\varphi_0 = \prod_{i=1}^{m} \frac{\Gamma(a_i)\Gamma(b_i-a_i)}{\Gamma(b_i)}\cdot f_0,$$

$$\int_{\Delta_r} u\varphi_0 = e^{-\pi\sqrt{-1}(b_r-a_r-1)}\frac{\Gamma(b_r-1)\Gamma(1-a_0)}{\Gamma(b_r-a_0)}\cdot \prod_{\substack{1\le j\le m\\ j\neq r}} \frac{\Gamma(a_j-b_r+1)\Gamma(b_j-a_j)}{\Gamma(b_j-b_r+1)}\cdot f_r,$$

where $1\le r\le m$.

Proof. By pulling back $u\varphi_0$ under ι_0, we can show the first claim. We prove the second one. On Δ_r, we have

$$u=e^{-\pi\sqrt{-1}(b_r-a_r)}(t_r-t_{r-1})^{b_r-a_r}(1-xt_m)^{-a_0}\cdot\prod_{j=1}^{m} t_j^{a_j-b_{j+1}}\cdot\prod_{\substack{1\le j\le m\\ j\neq r}}(t_{j-1}-t_j)^{b_j-a_j},$$

where the argument of each factor is zero on $\iota_r(\sigma_r)\subset D_r$. We consider the pullback of $u\varphi_0$ under ι_r:

$$u(\iota_r(z)) = e^{-\pi\sqrt{-1}(b_r-a_r)}\cdot x^{-b_r}\cdot u_r(z),$$
$$\iota_r^*\varphi_0 = -x\cdot\phi_r.$$

By Proposition 4.2, we thus have

$$\int_{\Delta_r} u\varphi_0 = -e^{-\pi\sqrt{-1}(b_r-a_r)}x^{1-b_r}\int_{\tilde{\Delta}_r} u_r\phi_r = e^{-\pi\sqrt{-1}(b_r-a_r-1)}\cdot(\Gamma\text{-factors})\cdot f_r. \qquad \square$$

Remark 4.4. For $1\le r\le m$, the twisted cycle Δ_r is different from the regularization of $D_r\otimes u$ as elements in $H_m(\mathcal{C}_\bullet(M,u))$.

Remark 4.5. Let $\iota'_r: M_r\to M_0$ $(1\le r\le m)$ be the map defined as

$$\iota'_r(z_1,\dots,z_m) := (w_1,\dots,w_m);\quad w_r=\frac{z_r}{x\prod_{j\neq r} z_j},\quad w_s=z_s\quad (s\neq r).$$

Then it is easy to see that $\iota_r=\iota_0\circ\iota'_r$.

The replacement $u\mapsto u^{-1}=1/u$ and the same construction as used for Δ_k give the twisted cycle $\Delta_k^\vee$, which represents an element in $H_m(\mathcal{C}_\bullet(M,u^{-1}))$. We evaluate the intersection numbers of the twisted cycles $\{\Delta_k\}_{k=0}^m$ and $\{\Delta_k^\vee\}_{k=0}^m$.

THEOREM 4.6.
(i) *For $k \neq l$, we have $I_h(\Delta_k, \Delta_l^\vee) = 0$.*
(ii) *The self-intersection numbers of the Δ_k are as follows:*

$$I_h(\Delta_0, \Delta_0^\vee) = \prod_{i=1}^{m} \frac{\alpha_i(1-\beta_i)}{(1-\alpha_i)(\alpha_i-\beta_i)},$$

$$I_h(\Delta_r, \Delta_r^\vee) = \prod_{\substack{1\le j\le m\\ j\neq r}} \frac{\alpha_j(\beta_r-\beta_j)}{(\beta_r-\alpha_j)(\alpha_j-\beta_j)} \frac{\alpha_0-\beta_r}{(1-\beta_r)(\alpha_0-1)} \quad (1 \le r \le m).$$

Proof. This theorem can also be shown in a way similar to the method used for Theorem 4.6 of [**5**] or Theorem 4.6 of [**6**]. Keys to the proof of (ii) are $I_h(\Delta_k, \Delta_k^\vee) = I_h(\tilde{\Delta}_k, \tilde{\Delta}_k^\vee)$ and the fact that the self-intersection number of a twisted cycle constructed from the direct product of intervals is obtained as the product of those of the loaded intervals. □

COROLLARY 4.7. *Under the condition (3), the twisted cycles $\Delta_0, \ldots, \Delta_m$ form a basis of $H_m(\mathcal{C}_\bullet(M, u))$.*

Proof. The determinant of the intersection matrix $H := (I_h(\Delta_i, \Delta_j))_{i,j=0,\ldots,m}$ does not vanish. □

Remark 4.8. In Section 3 of [**9**], there are the twisted cycles $D_1^{(0)}, \ldots, D_m^{(0)}, D_{m+1}^{(0)}$, which correspond to the solutions $f_1, \ldots, f_m, f_0$, respectively. By the variable change

$$p : (t_1, \ldots, t_m) \longmapsto \left(\frac{1}{t_1}, \ldots, \frac{1}{t_m}\right),$$

our integral representation (2) coincides with that in [**9**]. It is easy to see that

$$\Delta_0 = (-1)^m p_*(D_{m+1}^{(0)}), \quad \Delta_r = (-1)^m \frac{\beta_r-\alpha_0}{\alpha_0(\beta_r-1)} p_*(D_r^{(0)}) \quad (1 \le r \le m)$$

as elements in $H_m(\mathcal{C}_\bullet(M, u))$.

5. Twisted period relations

The compatibility of the intersection forms and the pairings obtained by integrations (see [**4**]) implies the twisted period relation:

$$C = \Pi_\omega \, {}^t H^{-1} \, {}^t \Pi_{-\omega},$$

where $\Pi_{\pm\omega}$ are defined as

$$\Pi_\omega := \left(\int_{\Delta_j} u\varphi_i\right)_{i,j}, \quad \Pi_{-\omega} := \left(\int_{\Delta_j^\vee} u^{-1}\varphi_i\right)_{i,j},$$

and C and H are the intersection matrices (see the proofs of Corollaries 3.2 and 4.7). Comparing the (i, j)-entries of both sides, we obtain the following theorem.

THEOREM 5.1. *We have*

$$I_c(\varphi_i, \varphi_j) = \sum_{k=0}^{m} \frac{1}{I_h(\Delta_k, \Delta_k^\vee)} \cdot \int_{\Delta_k} u\varphi_i \cdot \int_{\Delta_k^\vee} u^{-1}\varphi_j. \quad (10)$$

By using our results, we can reduce the twisted period relations (10) to quadratic relations between the ${}_{m+1}F_m$. We write down one of them as a corollary.

COROLLARY 5.2. *The equality (10) for $i = j = 0$ is reduced to*

$$\prod_{l=1}^{m} \frac{b_l - b_0}{a_l - b_0} = \prod_{l=1}^{m} \frac{b_l}{a_l} \cdot {}_{m+1}F_m\left(\begin{matrix} \boldsymbol{a} \\ \boldsymbol{b} \end{matrix}; x\right) \cdot {}_{m+1}F_m\left(\begin{matrix} -\boldsymbol{a} \\ -\boldsymbol{b} \end{matrix}; x\right)$$
$$+ \sum_{r=1}^{m} x^2 \cdot \frac{a_0(a_0 - b_r)(b_r - a_r)}{b_r(b_r^2 - 1)} \cdot \prod_{\substack{1 \le l \le m \\ l \ne r}} \frac{a_l - b_r}{b_l - b_r}$$
$$\cdot {}_{m+1}F_m\left(\begin{matrix} \boldsymbol{a}^{r,+} \\ \boldsymbol{b}^{r,+} \end{matrix}; x\right) \cdot {}_{m+1}F_m\left(\begin{matrix} \boldsymbol{a}^{r,-} \\ \boldsymbol{b}^{r,-} \end{matrix}; x\right),$$

where

$$\boldsymbol{a} := (a_0, \ldots, a_m), \quad \boldsymbol{b} := (b_1, \ldots, b_m),$$
$$\boldsymbol{a}^{r,\pm} := (1, \ldots, 1) \pm (a_0 - b_r, \ldots, a_m - b_r),$$
$$\boldsymbol{b}^{r,\pm} := (1, \ldots, 1) \pm (b_1 - b_r, \ldots, \pm 1 - b_r, \ldots, b_m - b_r).$$

Proof. By Theorem 4.3, we can express the integrals in (10) as products of Γ-factors and ${}_{m+1}F_m$. By Theorems 3.1, 4.6, and the formula $\Gamma(w)\Gamma(1 - w) = \pi/\sin(\pi w)$, we obtain the corollary. □

Remark 5.3. If we assume the condition (3) and

$$a_i - a_j \notin \mathbb{Z} \quad (0 \le i < j \le m),$$

then $\psi_0, \ldots, \psi_m$ also form a basis of $H^m(M, \nabla_\omega)$, because of Theorem 3.1. Considering $I_c(\varphi_i, \psi_j)$, $I_c(\psi_i, \varphi_j)$, or $I_c(\psi_i, \psi_j)$, we obtain other twisted period relations.

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