

## Supplement III to the paper “Asymptotic cumulants of some information criteria” – Examples 2 and 3

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This article is to supplement Ogasawara (2016) with Examples 2 and 3 using expository computations.

### S6. Example 2: The normal distribution with the MLE of the mean and the known variance $\sigma^2$ under possible non-normality

#### S6.1 Preliminary results

$$f(x^* = x | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\}.$$

For  $j = 1, \dots, n$ ,

$$l_j = -\frac{1}{2} \log(2\pi\sigma^2) - \frac{(x_j - \mu)^2}{2\sigma^2}, \quad \bar{l} = n^{-1} \sum_{k=1}^n l_k,$$

$$l_{0j} = -\frac{1}{2} \log(2\pi\sigma^2) - \frac{(x_j - \mu_0)^2}{2\sigma^2}, \quad \bar{l}_0 = n^{-1} \sum_{k=1}^n l_{0k},$$

$$\bar{l}_0^* = E_g(l_{0j}) = E_f(l_{0j}) = E_g(\bar{l}_0) = E_f(\bar{l}) = -\frac{1}{2} \log(2\pi\sigma^2) - \frac{1}{2},$$

$$\theta = \mu, \theta_0 = \mu_0, \hat{\theta}_{ML} = \hat{\mu} = \bar{x}, n \text{var}_g(\hat{\theta}_{ML}) = n \text{var}_f(\hat{\theta}_{ML}) = \sigma^2,$$

$$\frac{\partial \bar{l}}{\partial \theta_0} = \frac{\bar{x} - \mu_0}{\sigma^2}, \quad \frac{\partial l_j}{\partial \theta_0} = \frac{x_j - \mu_0}{\sigma^2}, \quad \Lambda = \lambda = \frac{\partial^2 \bar{l}}{\partial \theta_0^2} = \frac{\partial^2 l_j}{\partial \theta_0^2} = -\frac{1}{\sigma^2} = -\bar{i}_0,$$

$$\Gamma = \gamma = E_g \left\{ \left( \frac{\partial l_j}{\partial \theta_0} \right)^2 \right\} = n E_g \left\{ \left( \frac{\partial \bar{l}}{\partial \theta_0} \right)^2 \right\} = n E_f \left\{ \left( \frac{\partial \bar{l}}{\partial \theta_0} \right)^2 \right\} = \frac{1}{\sigma^2} = \bar{i}_0,$$

(S6.1)

$$\text{tr}(-\Lambda^{-1}\Gamma) = -\lambda^{-1}\gamma = 1, \quad l_{0j} - \bar{l}_0^* = -\frac{1}{2\sigma^2} \{(x_j - \mu_0)^2 - \sigma^2\} \equiv -\frac{z_j^2 - 1}{2}$$

(note that  $\lambda = -\gamma$  even under non-normality),

$$\frac{\partial l_j}{\partial \theta_0} = \frac{z_j}{\sigma}, \quad \mathbf{J}_0^{(3)} = j_0^{(3)} = \frac{\partial^3 \bar{l}}{\partial \theta_0^3} = 0, \quad \mathbf{J}_0^{(4)} = j_0^{(4)} = \frac{\partial^4 \bar{l}}{\partial \theta_0^4} = 0.$$

Let  $\kappa_k \equiv \kappa_{gk}(z)$  ( $k=1, 2, \dots$ ) under possible non-normality with  $z = (x^* - \mu_0) / \sigma$ . Note that the notation  $\kappa_k$  is used only when the argument of  $\kappa_{gk}(\cdot)$  is  $z$  for simplicity.

$$v_0^{(A)} = 4(n-1)^{-1} \sum_{j=1}^n (l_{0j} - \bar{l}_0)^2 = 2(n^2 - n)^{-1} \sum_{j,k=1}^n (l_{0j} - l_{0k})^2 = O_p(1),$$

$$\hat{v}_{\text{ML}}^{(A)} = 4(n-1)^{-1} \sum_{j=1}^n (\hat{l}_{\text{ML}j} - \hat{l}_{\text{ML}})^2 = \hat{\alpha}_{\text{ML}2}^{(A)} = O_p(1),$$

$$\hat{l}_{\text{ML}j} = l_j \big|_{\theta = \hat{\theta}_{\text{ML}}}, \quad \hat{l}_{\text{ML}} = n^{-1} \sum_{j=1}^n \hat{l}_{\text{ML}j},$$

$$\kappa_{g2}(l_{0j}) = \text{var}_g(l_{0j}) = \frac{1}{4} \text{var}_g(z^2) = \frac{1}{4}(\kappa_4 + 2),$$

$$\begin{aligned} \kappa_{g3}(l_{0j}) &= -\frac{1}{8} \kappa_{g3}(z^2) = -\frac{1}{8} \text{E}_g \{(z^2 - 1)^3\} \\ &= -\frac{1}{8}(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8) \end{aligned}$$

(see (S6.8) given later), (S6.2)

$$\begin{aligned} \kappa_{g4}(l_{0j}) &= \frac{1}{16} \kappa_{g4}(z^2) \\ &= \frac{1}{16}(\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 240\kappa_3^2 + 48) \end{aligned}$$

(see also (S6.10) given later),

$$E_g \left( \frac{\partial l_j}{\partial \theta_0} l_{0j} \right) = E_g \left\{ \frac{z}{\sigma} \left( -\frac{z^2}{2} \right) \right\} = -\frac{\kappa_3}{2\sigma},$$

$$\begin{aligned} E_g \{ (l_{0j} - \bar{l}_0^*)^4 \} &= \frac{1}{16} E_g \{ (z^2 - 1)^4 \} = \frac{1}{16} E_g (z^8 - 4z^6 + 6z^4 - 4z^2 + 1) \\ &= \frac{1}{16} \{ \kappa_8 + 28\kappa_6 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 210\kappa_4 + 280\kappa_3^2 + 105 \\ &\quad - 4(\kappa_6 + 15\kappa_4 + 10\kappa_3^2 + 15) + 6(\kappa_4 + 3) - 4 + 1 \} \\ &= \frac{1}{16} (\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 156\kappa_4 + 240\kappa_3^2 + 60) \end{aligned}$$

(see also (S6.10)),

$$\begin{aligned} E_g \left\{ \frac{\partial l_j}{\partial \theta_0} (l_{0j} - \bar{l}_0^*)^2 \right\} &= E_g \left\{ \frac{z}{\sigma} \left( -\frac{z^2 - 1}{2} \right)^2 \right\} = \frac{1}{4\sigma} E_g (z^5 - 2z^3 + z) \\ &= \frac{1}{4\sigma} (\kappa_5 + 10\kappa_3 - 2\kappa_3) = \frac{1}{4\sigma} (\kappa_5 + 8\kappa_3), \end{aligned}$$

$$\begin{aligned} E_g \left\{ (l_{0j} - \bar{l}_0^*) \left( \frac{\partial l_j}{\partial \theta_0} \right)^2 \right\} &= E_g \left( -\frac{z^2 - 1}{2} \frac{z^2}{\sigma^2} \right) = -\frac{1}{2\sigma^2} E_g (z^4 - z^2) \\ &= -\frac{1}{2\sigma^2} (\kappa_4 + 2), \end{aligned}$$

$$E_g (v_0^{(A)}) = \alpha_{ML2}^{(A)} = \text{var}_g (-2l_{0j}) = \kappa_4 + 2,$$

$$E_g \left( \frac{\partial v^{(A)}}{\partial \theta_0} \right) = 8E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) = -\frac{4\kappa_3}{\sigma}, \quad E_g \left( \frac{\partial^2 v^{(A)}}{\partial \theta_0^2} \right) = 8\gamma = \frac{8}{\sigma^2}$$

(the result  $8\gamma$  holds under canonical parametrization),

$$E_g \left\{ \left( \frac{\partial v^{(A)}}{\partial \theta_0} \right)^2 \right\} = 64 \left\{ E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\}^2 + O(n^{-1}) = \frac{16\kappa_3^2}{\sigma^2} + O(n^{-1}),$$

$$\begin{aligned} n \operatorname{cov}_g(m_v, \bar{l}_0) &= n \operatorname{cov}_g(v_0^{(A)}, \bar{l}_0) = 4E_g \{(l_{0j} - \bar{l}_0^*)^3\} = 4\kappa_{g3}(l_{0j}) \\ &= -\frac{1}{2}(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8), \end{aligned}$$

$$n \operatorname{cov}_g \left( \frac{\partial \bar{l}}{\partial \theta_0}, m_v \right) = 4E_g \left\{ \frac{\partial l_j}{\partial \theta_0} (l_{0j} - \bar{l}_0^*)^2 \right\} = \frac{1}{\sigma}(\kappa_5 + 8\kappa_3),$$

$$\begin{aligned} n \operatorname{avar}_g(m_v) &= 16 \left[ E_g \{(l_{0j} - \bar{l}_0^*)^4\} - [E_g \{(l_{0j} - \bar{l}_0^*)^2\}]^2 \right] \\ &= 16 \left[ \frac{1}{16} E_g \{(z^2 - 1)^4\} - \left\{ \frac{1}{4} E_g \{(z^2 - 1)^2\} \right\}^2 \right] \\ &= E_g \{(z^2 - 1)^4\} - [E_g \{(z^2 - 1)^2\}]^2 \\ &= \kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 156\kappa_4 + 240\kappa_3^2 + 60 - (\kappa_4 + 2)^2 \\ &= \kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 240\kappa_3^2 + 56, \end{aligned}$$

$$n \operatorname{cov}_g \left( \frac{\partial v^{(A)}}{\partial \theta_0}, \frac{\partial \bar{l}_0}{\partial \theta_0} \right) = 8E_g \left\{ (l_{0j} - \bar{l}_0^*) \left( \frac{\partial l_j}{\partial \theta_0} \right)^2 \right\} = -\frac{4}{\sigma^2}(\kappa_4 + 2),$$

$$n \operatorname{cov}_g \left( \frac{\partial v^{(A)}}{\partial \theta_0}, \bar{l}_0 \right) = 8E_g \left\{ (l_{0j} - \bar{l}_0^*)^2 \frac{\partial l_j}{\partial \theta_0} \right\} = \frac{2}{\sigma}(\kappa_5 + 8\kappa_3),$$

$$\mathbf{m}_v^{(1)} = \left( m_v, \frac{\partial \bar{l}}{\partial \theta_0} \right)',$$

$$\mathbf{m}_v^{(2)} = \left[ m_v^2, m_v \frac{\partial \bar{l}}{\partial \theta_0}, 0, \left( \frac{\partial \bar{l}}{\partial \theta_0} \right)^2, \left\{ \frac{\partial v^{(A)}}{\partial \theta_0} - E_g \left( \frac{\partial v^{(A)}}{\partial \theta_0} \right) \right\} \otimes \frac{\partial \bar{l}}{\partial \theta_0} \right]',$$

$$\begin{aligned} \mathbf{v}^{(1)} &= \frac{1}{2}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \left\{ -1, \mathbb{E}_g \left( \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right) \lambda^{-1} \right\}' \\ &= \frac{1}{2}(\kappa_4 + 2)^{-3/2} \left\{ -1, -\frac{4\kappa_3}{\sigma} \left( -\frac{1}{\sigma^2} \right)^{-1} \right\}' = \frac{1}{2}(\kappa_4 + 2)^{-3/2} (-1, 4\sigma\kappa_3)' \\ \text{(under normality } \mathbf{v}^{(1)} &= (-2^{-5/2}, 0)'), \end{aligned}$$

$$\begin{aligned} \mathbf{v}^{(2)} &= \left[ \begin{array}{l} \frac{3}{8}(\alpha_{\text{ML}2}^{(A)})^{-5/2}, -\frac{3}{4}(\alpha_{\text{ML}2}^{(A)})^{-5/2} \mathbb{E}_g \left( \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right) \lambda^{-1}, \\ -\frac{1}{2}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \mathbb{E}_g \left( \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right) (\lambda^{-1})^2, \left[ \begin{array}{l} -\frac{1}{4}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \mathbb{E}_g \left( \frac{\partial^2 \mathbf{v}^{(A)}}{\partial \theta_0^2} \right) \\ + \frac{3}{8}(\alpha_{\text{ML}2}^{(A)})^{-5/2} \mathbb{E}_g \left\{ \left( \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right)^2 \right\} + O(n^{-1}) \end{array} \right] (\lambda^{-1})^2, \frac{1}{2}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \lambda^{-1} \end{array} \right]' \\ &= \left[ \begin{array}{l} \frac{3}{8}(\kappa_4 + 2)^{-5/2}, -3(\kappa_4 + 2)^{-5/2} \kappa_3 \sigma, 2(\kappa_4 + 2)^{-3/2} \kappa_3 \sigma^3, \\ \left\{ -2(\kappa_4 + 2)^{-3/2} + 6(\kappa_4 + 2)^{-5/2} \kappa_3^2 \right\} \sigma^2, -\frac{\sigma^2}{2}(\kappa_4 + 2)^{-3/2} \end{array} \right]' \end{aligned}$$

$$\text{(under normality } \mathbf{v}^{(2)} = (3 \times 2^{-11/2}, 0, 0, -2^{-1/2} \sigma^2, -2^{-5/2} \sigma^2)'),$$

$$\begin{aligned} n^2 \mathbb{E}_g \{ (\bar{l}_{\text{ML}}^{(1)})^2 \mathbf{m}_v^{(1)} \} &= 4 \left[ 4\kappa_{g4}(l_{0j}), \mathbb{E}_g \left\{ (l_{0j} - \bar{l}_0^*)^2 \frac{\partial l_j}{\partial \theta_0} \right\} \right] \\ &= \left\{ 16\kappa_{g4}(l_{0j}), \frac{1}{\sigma}(\kappa_5 + 8\kappa_3) \right\} \end{aligned}$$

$$\text{(under normality } n^2 \mathbb{E}_g \{ (\bar{l}_{\text{ML}}^{(1)})^2 \mathbf{m}_v^{(1)} \} = (48, 0)),$$

$$\begin{aligned}
nE_g(\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} &= -2 \left\{ n \text{cov}_g(\bar{l}_0, m_v), n \text{cov}_g\left(\bar{l}_0, \frac{\partial \bar{l}}{\partial \theta_0}\right) \right\} \mathbf{v}^{(1)} \\
&= -2 \left\{ 4\kappa_{g3}(l_{0j}), -\frac{\kappa_3}{2\sigma} \right\} \frac{1}{2} (\kappa_4 + 2)^{-3/2} (-1, 4\sigma\kappa_3)' \\
&= -(\kappa_4 + 2)^{-3/2} \{-4\kappa_{g3}(l_{0j}) - 2\kappa_3^2\} \\
&= -(\kappa_4 + 2)^{-3/2} \left\{ \frac{1}{2} (\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8) - 2\kappa_3^2 \right\} \\
&= -\frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)
\end{aligned}$$

(under normality  $nE_g(\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} = -2^{1/2}$ ),

$$-2E_g(\hat{\bar{l}}_{ML} - \hat{\bar{l}}_{ML}^*) = n^{-1} 2 \text{tr}(\Lambda^{-1} \Gamma) + n^{-2} (c_1 + c_2 + c_3) + O(n^{-3})$$

$$= n^{-1} b_1 + n^{-2} b_2 + O(n^{-3}) = -n^{-1} 2 \quad (b_2 = 0, O(n^{-3}) = 0),$$

$b_1 = -2$  even under non-normality, and  $b_2 = 0$  comes from  $c_2 = c_3 = 0$

under canonical parametrization and  $c_1 = 0$  due to  $j_0^{(3)} = j_0^{(4)} = 0$

( $O(n^{-3}) = 0$  stems from  $j_0^{(k)} = 0$ ,  $k = 5, 6, \dots$  defined similarly to  $j_0^{(k)}$ ,  $k = 3, 4$ ),

$$\bar{l}_{ML}^{(1)} = -2(\bar{l}_0 - \bar{l}_0^*) = \frac{1}{\sigma^2} \left\{ n^{-1} \sum_{j=1}^n (x_j - \mu_0)^2 - \sigma^2 \right\} \equiv \bar{z}^2 - 1,$$

$$\bar{l}_{ML}^{(2)} = \frac{\partial \bar{l}}{\partial \theta_0} \lambda^{-1} \frac{\partial \bar{l}}{\partial \theta_0} = \left( -\frac{1}{\sigma^2} \right)^{-1} \left( \frac{\bar{x} - \mu_0}{\sigma^2} \right)^2 = -\frac{(\bar{x} - \mu_0)^2}{\sigma^2} \equiv -\bar{z}^2 \quad (\text{S6.3})$$

$$(\bar{z} \equiv (\bar{x} - \mu_0) / \sigma), \quad nE_g(\bar{l}_{ML}^{(2)}) = \lambda^{-1} \gamma = -1, \quad \bar{l}_{ML}^{(3)} = \bar{l}_{ML}^{(4)} = 0,$$

where  $\bar{l}_{ML}^{(3)} = \bar{l}_{ML}^{(4)} = 0$  is due to  $j_0^{(3)} = j_0^{(4)} = 0$ .

## S6.2 $n^{-1} \text{AIC}_{ML}$

$$n^{-1} \text{AIC}_{ML} = -2\hat{\bar{l}}_{ML} + n^{-1} 2q = -2\hat{\bar{l}}_{ML} + n^{-1} 2.$$

**S6.2.1 Asymptotic cumulants of  $n^{-1}\text{AIC}_{\text{ML}}$  before studentization**

For estimation of  $-2\text{E}_g(\widehat{\bar{l}}_{\text{ML}}^*)$ ,

$$\begin{aligned} \kappa_{g1}\{n^{-1}\text{AIC}_{\text{ML}} + 2\text{E}_g(\widehat{\bar{l}}_{\text{ML}}^*)\} \\ = -2\text{E}_g(\widehat{\bar{l}}_{\text{ML}} - \widehat{\bar{l}}_{\text{ML}}^*) + n^{-1}2q = -n^{-1}2 + n^{-1}2 = 0 \end{aligned} \quad (\text{S6.4})$$

(exactly 0;  $\alpha_{\text{ML}1}^{(A)*} = \alpha_{\text{ML}\Delta 1}^{(A)*} = 0$ )

while for estimation of  $-2\bar{l}_0^*$ ,

$$\begin{aligned} \kappa_{g1}(n^{-1}\text{AIC}_{\text{ML}} + 2\bar{l}_0^*) &= n^{-1}(2q + \lambda^{-1}\gamma) + n^{-2}\{n^2\text{E}_g(\bar{l}_{\text{ML}}^{(3)} + \bar{l}_{\text{ML}}^{(4)})\} + \dots \\ &= n^{-1}(2-1) + n^{-2}\times 0 + n^{-3}\times 0 + \dots = n^{-1} \\ (\alpha_{\text{ML}1}^{(A)} &= 1, \alpha_{\text{ML}\Delta 1}^{(A)} = 0). \end{aligned} \quad (\text{S6.5})$$

Using  $\bar{l}_{\text{ML}}^{(3)} = 0$ ,

$$\begin{aligned} \kappa_{g2}(n^{-1}\text{AIC}_{\text{ML}}) &= n^{-1}[n\text{E}_g\{(\bar{l}_{\text{ML}}^{(1)})^2\}] + n^{-2}[2n^2\text{E}_g(\bar{l}_{\text{ML}}^{(1)}\bar{l}_{\text{ML}}^{(2)}) \\ &\quad + n^2\text{E}_g\{(\bar{l}_{\text{ML}}^{(2)})^2\} - \{n\text{E}_g(\bar{l}_{\text{ML}}^{(2)})\}^2] + O(n^{-3}) \\ &= n^{-1}\frac{1}{\sigma^4}n\text{E}_g\left[\left\{n^{-1}\sum_{j=1}^n(x_j - \mu_0)^2 - \sigma^2\right\}^2\right] \\ &\quad + n^{-2}\left[\frac{2n^2}{\sigma^4}\text{E}_g\left[\left\{n^{-1}\sum_{j=1}^n(x_j - \mu_0)^2 - \sigma^2\right\}(\bar{x} - \mu_0)^2\right] \right. \\ &\quad \left. + \frac{n^2}{\sigma^4}\text{E}_g\{(\bar{x} - \mu_0)^4\} - (-1)^2\right] + O(n^{-3}), \end{aligned} \quad (\text{S6.6})$$

where the first term on the right-hand side of (S6.6) is

$$\begin{aligned} n^{-1}[n\text{E}_g\{(\bar{l}_{\text{ML}}^{(1)})^2\}] &= n^{-1}[n\text{E}_g\{(z^2 - 1)^2\}] \\ &= n^{-1}[\text{E}_g(z^4) - \{\text{E}_g(z^2)\}^2] = n^{-1}(\kappa_4 + 3 - 1) = n^{-1}(\kappa_4 + 2), \end{aligned}$$

the first term in  $\left[\begin{smallmatrix} \cdot \\ (A) \end{smallmatrix}\right]_{(A)}$  of (S6.6) is

$$\begin{aligned}
2n^2 E_g(\bar{I}_{ML}^{(1)} \bar{I}_{ML}^{(2)}) &= -\frac{2n^2}{\sigma^4} E_g \left[ \left\{ n^{-1} \sum_{j=1}^n (x_j - \mu_0)^2 - \sigma^2 \right\} (\bar{x} - \mu_0)^2 \right] \\
&= -\frac{2n^2}{\sigma^4} \left[ E_g \left[ n^{-1} \sum_{j=1}^n (x_j - \mu_0)^2 n^{-2} \left\{ \sum_{j=1}^n (x_j - \mu_0) \right\}^2 \right] - n^{-1} \sigma^4 \right] \\
&= -\frac{2n^2}{\sigma^4} \left[ n^{-3} n E_g \{(x^* - \mu_0)^4\} + n^{-3} (n^2 - n) [E_g \{(x^* - \mu_0)^2\}]^2 - n^{-1} \sigma^4 \right] \\
&= -\frac{2}{\sigma^4} \{ \kappa_{g4}(x^*) + 3\sigma^4 - \sigma^4 \} = -2(\kappa_4 + 2),
\end{aligned}$$

the second term in  $[\cdot]_{(A)}$  of (S6.6) is

$$n^2 E_g \{ (\bar{I}_{ML}^{(1)})^2 \} = \frac{n^2}{\sigma^4} E_g \{ (\bar{x} - \mu_0)^4 \} = 3 + O(n^{-1})$$

(under normality  $n^2 E_g \{ (\bar{I}_{ML}^{(1)})^2 \} = 3$  without the remainder term).

Consequently,

$$\begin{aligned}
\kappa_{g2}(n^{-1} \text{AIC}_{ML}) &= n^{-1}(\kappa_4 + 2) + n^{-2} \{-2(\kappa_4 + 2) + 3 - 1\} + O(n^{-3}) \\
&= n^{-1}(\kappa_4 + 2) - n^{-2} 2(\kappa_4 + 1) + O(n^{-3}) = n^{-1} \alpha_{ML2}^{(A)} + n^{-2} \alpha_{ML\Delta 2}^{(A)} + O(n^{-3}) \quad (\text{S6.7})
\end{aligned}$$

(under normality  $\alpha_{ML2}^{(A)} = 2$  and  $\alpha_{ML\Delta 2}^{(A)} = -2$ ).

$$\begin{aligned}
\kappa_{g3}(n^{-1} \text{AIC}_{ML}) &= n^{-2} [ n^2 E_g \{ (\bar{I}_{ML}^{(1)})^3 \} + 3n^2 E_g \{ (\bar{I}_{ML}^{(1)})^2 \bar{I}_{ML}^{(2)} \} \\
&\quad - 3n E_g \{ \bar{I}_{ML}^{(2)} \} \alpha_{ML2}^{(A)} ] + O(n^{-3}) \quad (\text{S6.8})
\end{aligned}$$

where the first term in  $[\cdot]$  of (S6.8) is

$$\begin{aligned}
n^2 E_g \{ (\bar{I}_{ML}^{(1)})^3 \} &= n^2 E_g \{ (z^2 - 1)^3 \} = n^2 \kappa_{g3}(z^2) = \kappa_{g3}(z^2) \\
&= E_g \{ (z^2 - 1)^3 \} = E_g(z^6 - 3z^4 + 3z^2 - 1) \\
&= \kappa_6 + 15\kappa_4 + 10\kappa_3^2 + 15 - 3(\kappa_4 + 3) + 3 - 1 = \kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8 \\
& (= E_g(z^6) - 3E_g(z^4) + 2; \text{Stuart \& Ort, 1994, Equation (3.38)}),
\end{aligned}$$



the sum of the second and third terms in  $[\cdot]$  of (S6.8) is

$$\begin{aligned}
 & 3n^2 E_g \{(\bar{I}_{ML}^{(1)})^2 \bar{I}_{ML}^{(2)}\} - 3n E_g (\bar{I}_{ML}^{(2)}) \alpha_{ML2}^{(A)} \\
 &= -\frac{6}{\sigma^2} [n E_g \{\bar{I}_{ML}^{(1)}(\bar{x} - \mu_0)\}]^2 + O(n^{-1}) \\
 &= -\frac{6}{\sigma^6} \left[ E_g \left\{ \sum_{j=1}^n \{(x_j - \mu_0)^2 - \sigma^2\} n^{-1} \sum_{k=1}^n (x_k - \mu_0) \right\} \right]^2 + O(n^{-1}) \\
 &= -6\kappa_3^2 + O(n^{-1}).
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \kappa_{g3}(n^{-1} \text{AIC}_{ML}) &= n^{-2}(\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8) + O(n^{-3}) = n^{-2} \alpha_{ML3}^{(A)} + O(n^{-3}) \\
 &= (n^{-2} [E_g(z^6) - 3E_g(z^4) - 6\{E_g(z^3)\}^2 + 2]) + O(n^{-3});
 \end{aligned}$$

$$\text{under normality } \alpha_{ML3}^{(A)} = 8). \quad (\text{S6.9})$$

Use  $\bar{I}_{ML}^{(3)} = 0$ , then

$$\begin{aligned}
 \kappa_{g4}(n^{-1} \text{AIC}_{ML}) &= n^{-3} \left[ n^3 \kappa_{g4}(\bar{I}_{ML}^{(1)}) + 4n^3 E_g \{(\bar{I}_{ML}^{(1)})^3 \bar{I}_{ML}^{(2)}\} \right. \\
 &\quad + 6n^3 E_g \{(\bar{I}_{ML}^{(1)})^2 (\bar{I}_{ML}^{(2)})^2\} - 4n E_g (\bar{I}_{ML}^{(2)}) \alpha_{ML3}^{(A)} \\
 &\quad \left. - 6\alpha_{ML2}^{(A)} \alpha_{ML\Delta 2}^{(A)} - 6\alpha_{ML2}^{(A)} \{n E_g (\bar{I}_{ML}^{(2)})\}^2 \right] + O(n^{-4}), \quad (\text{S6.10})
 \end{aligned}$$

where the first term in  $[\cdot]_{(A)}^{(A)}$  of (S6.10) is

$$\begin{aligned}
 n^3 \kappa_{g4}(\bar{I}_{ML}^{(1)}) &= n^3 \kappa_{g4}(\bar{z}^2) = \kappa_{g4}(z^2) = E_g \{(z^2 - 1)^4\} - 3[E_g \{(z^2 - 1)^2\}]^2 \\
 &= E_g (z^8 - 4z^6 + 6z^4 - 4z^2 + 1) - 3\{E_g (z^4 - 2z^2 + 1)\}^2 \\
 &= \kappa_8 + 28\kappa_6 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 210\kappa_4 + 280\kappa_3^2 + 105 \\
 &\quad - 4(\kappa_6 + 15\kappa_4 + 10\kappa_3^3 + 15) + 6(\kappa_4 + 3) - 4 + 1 - 3(\kappa_4 + 2)^2 \\
 &= \kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + (210 - 4 \times 15 + 6 - 12)\kappa_4 + 240\kappa_3^2 \\
 &\quad + 105 - 4 \times 15 + 6 \times 3 - 4 + 1 - 12 \\
 &= \kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 240\kappa_3^2 + 48,
 \end{aligned}$$

the second term in  $\left[ \begin{array}{c} \cdot \\ (A) \end{array} \right]_{(A)}$  of (S6.10) is

$$\begin{aligned} 4n^3 E_g \{ (\bar{l}_{ML}^{(1)})^3 \bar{l}_{ML}^{(2)} \} &= 4 \left[ n^2 E_g \{ (\bar{l}_{ML}^{(1)})^3 \} n E_g \{ \bar{l}_{ML}^{(2)} \} + 3\alpha_{ML2}^{(A)} n^2 E_g \{ \bar{l}_{ML}^{(1)} \bar{l}_{ML}^{(2)} \} \right. \\ &\quad \left. - 6n E_g \{ \bar{l}_{ML}^{(1)} \bar{z} \} n^2 E_g \{ (\bar{l}_{ML}^{(1)})^2 \bar{z} \} \right] + O(n^{-1}) \\ &= 4 \left[ \begin{array}{c} n^2 E_g \{ (\bar{l}_{ML}^{(1)})^3 \} (-1) + 3(\kappa_4 + 2) \{ -(\kappa_4 + 2) \} \\ (B) \end{array} \right. \\ &\quad \left. - 6\kappa_3 \frac{n^{-1}}{\sigma^5} E_g \left[ \sum_{j=1}^n \{ (x_j - \mu_0)^2 - \sigma^2 \} \sum_{k=1}^n \{ (x_k - \mu_0)^2 - \sigma^2 \} \sum_{l'=1}^n (x_{l'} - \mu_0) \right] \right]_{(B)} \\ &\quad + O(n^{-1}) \end{aligned}$$

(see the first term in  $\left[ \begin{array}{c} \cdot \\ (A) \end{array} \right]_{(A)}$  of (S6.6))

$$\begin{aligned} &= 4 \left[ \begin{array}{c} -n^2 E_g \{ (\bar{l}_{ML}^{(1)})^3 \} - 3(\kappa_4 + 2)^2 - 6\kappa_3 \frac{n^{-1}}{\sigma^5} E_g \left[ \left\{ \sum_{j=1}^n (x_j - \mu_0)^2 \right\}^2 \right. \right. \\ (B) \quad \left. \left. \times \sum_{l'=1}^n (x_{l'} - \mu_0) - 2n\sigma^2 \sum_{j=1}^n (x_j - \mu_0)^2 \sum_{l'=1}^n (x_{l'} - \mu_0) \right] \right] + O(n^{-1}) \\ &= 4 \left[ -n^2 E_g \{ (\bar{l}_{ML}^{(1)})^3 \} - 3(\kappa_4 + 2)^2 \right. \\ &\quad \left. - 6\kappa_3 n^{-1} \{ n E_g \{ z^5 \} + 2(n^2 - n) E_g \{ z^3 \} E_g \{ z^2 \} - 2n^2 E_g \{ z^3 \} \} \right] + O(n^{-1}) \\ &= 4 \left[ -(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8) - 3(\kappa_4^2 + 4\kappa_4 + 4) - 6\kappa_3(\kappa_5 + 10\kappa_3 - 2\kappa_3) \right] \\ &\quad + O(n^{-1}) \\ &= -4(\kappa_6 + 24\kappa_4 + 58\kappa_3^2 + 3\kappa_4^2 + 6\kappa_3\kappa_5 + 20) + O(n^{-1}) \end{aligned}$$

(see (S6.8); Stuart & Ort, 1994, Equation (3.38); note that  $n^3 = n + 3(n^2 - n) + n(n-1)(n-2)$ ).

the third term in  $\left[ \begin{array}{c} \cdot \\ (A) \end{array} \right]_{(A)}$  of (S6.10) is

$$\begin{aligned}
6n^3 E_g \{ (\bar{I}_{ML}^{(1)})^2 (\bar{I}_{ML}^{(2)})^2 \} &= 6n^3 E_g \{ (\bar{z}^2 - 1)^2 \bar{z}^4 \} \\
&= 6 [ n \text{var}_g (\bar{z}^2) 3 \{ n \text{var}_g (\bar{z}) \}^2 + 12 \{ n \text{cov}_g (\bar{z}^2, \bar{z}) \}^2 n \text{var}_g (\bar{z}) ] + O(n^{-1}) \\
&= 6 \{ (\kappa_4 + 2) \times 3 + 12 \kappa_3^2 \} + O(n^{-1}) = 18(\kappa_4 + 4\kappa_3^2 + 2) + O(n^{-1}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\kappa_{g^4} (n^{-1} \text{AIC}_{ML}) &= n^{-3} [ \kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 240\kappa_3^2 + 48 \\
&\quad - 4(\kappa_6 + 24\kappa_4 + 58\kappa_3^2 + 3\kappa_4^2 + 6\kappa_3\kappa_5 + 20) + 18(\kappa_4 + 4\kappa_3^2 + 2) \\
&\quad - 4 \times (-1)(\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8) - 6 \times (-2)(\kappa_4 + 2)(\kappa_4 + 1) \\
&\quad - 6(\kappa_4 + 2)(-1)^2 ] + O(n^{-4}) \\
&= n^{-3} [ \kappa_8 + (24 - 4 + 4)\kappa_6 + (56 - 24)\kappa_5\kappa_3 + (32 - 12 + 12)\kappa_4^2 \\
&\quad + (144 - 96 + 18 + 48 + 36 - 6)\kappa_4 + (240 - 4 \times 58 + 18 \times 4 + 4 \times 4)\kappa_3^2 \\
&\quad + 48 - 80 + 36 + 32 + 24 - 12 ] + O(n^{-4}) \quad (\text{S6.11}) \\
&= n^{-3} (\kappa_8 + 24\kappa_6 + 32\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 96\kappa_3^2 + 48) + O(n^{-4}) \\
&= n^{-3} \alpha_{ML4}^{(A)} + O(n^{-4}) \\
&\text{(under normality } \alpha_{ML4}^{(A)} = 48 \text{).}
\end{aligned}$$

### S6.2.2 Asymptotic cumulants of $n^{-1} \text{AIC}_{ML}$ after studentization for estimation of $-2\bar{I}_0^*$

$$t_{ML}^{(A)} = \frac{n^{1/2} (n^{-1} \text{AIC}_{ML} + 2\bar{I}_0^*)}{(\hat{v}_{ML}^{(A)})^{1/2}}.$$

$$\begin{aligned}
\kappa_{g^1} (t_{ML}^{(A)}) &= n^{-1/2} \{ \alpha_{ML1}^{(A)} (\alpha_{ML2}^{(A)})^{-1/2} + \alpha_{(\Delta r)ML1}^{(A)} \} + O(n^{-3/2}) \\
&= n^{-1/2} \{ \alpha_{ML1}^{(A)} (\alpha_{ML2}^{(A)})^{-1/2} + n E_g (\bar{I}_{ML}^{(1)} \mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} \} + O(n^{-3/2}) \\
&= n^{-1/2} \left\{ (\kappa_4 + 2)^{-1/2} - \frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\} \\
&\quad + O(n^{-3/2}) \quad (\text{S6.12}) \\
&= n^{-1/2} \alpha_{(t)ML1}^{(A)} + O(n^{-3/2})
\end{aligned}$$

$$\begin{aligned} (\alpha_{(\Delta t)ML1}^{(A)} &= -(1/2)(\kappa_4 + 2)^{-3/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8); \text{ under normality} \\ \alpha_{(t)ML1}^{(A)} &= 2^{-1/2} - 2^{1/2} \quad \text{and} \quad \alpha_{(\Delta t)ML1}^{(A)} = -2^{1/2}, \end{aligned}$$

$$\begin{aligned} & \kappa_{g2} (t_{ML}^{(A)}) \\ &= 1 + n^{-1} \left[ \begin{array}{l} \alpha_{ML\Delta 2}^{(A)} (\alpha_{ML2}^{(A)})^{-1} + 2n^2 E_g \{ (\bar{l}_{ML}^{(1)})^2 \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-1/2} \\ \text{(A)} \end{array} \right. \\ & \quad \left. + 2n^2 E_g \left[ \begin{array}{l} \bar{l}_{ML}^{(1)} (\bar{l}_{ML}^{(2)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} + \bar{l}_{ML}^{(1)} \mathbf{v}^{(2)}, \mathbf{m}_v^{(2)} + n^{-1} 2q \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)}) \\ \text{(B)} \end{array} \right] \right. \\ & \quad \left. \times (\alpha_{ML2}^{(A)})^{-1/2} \right. \\ & \quad \left. + n^2 E_g \{ 2\bar{l}_{ML}^{(2)} (\alpha_{ML2}^{(A)})^{-1/2} \bar{l}_{ML}^{(1)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} + (\bar{l}_{ML}^{(1)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)})^2 \} \right. \\ & \quad \left. - \{ 2n E_g (\bar{l}_{ML}^{(2)}) (\alpha_{ML2}^{(A)})^{-1/2} \alpha_{(\Delta t)ML1}^{(A)} + (\alpha_{(\Delta t)ML1}^{(A)})^2 \} \right] + O(n^{-2}). \end{aligned} \tag{S6.13}$$

(i) the first term in  $\left[ \begin{array}{l} \cdot \\ \text{(A)} \end{array} \right]$  of (S6.13) is

$$\alpha_{ML\Delta 2}^{(A)} (\alpha_{ML2}^{(A)})^{-1} = -2(\kappa_4 + 1)(\kappa_4 + 2)^{-1} \quad (= -1 \text{ under normality}),$$

(ii) the second term in  $\left[ \begin{array}{l} \cdot \\ \text{(A)} \end{array} \right]$  of (S6.13) is

$$\begin{aligned} & 2n^2 E_g \{ (\bar{l}_{ML}^{(1)})^2 \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-1/2} \\ &= 2 \left\{ 16\kappa_{g4} (l_{0j}), \frac{1}{\sigma} (\kappa_5 + 8\kappa_3) \right\} \frac{1}{2} (\kappa_4 + 2)^{-3/2} (-1, 4\sigma\kappa_3) (\kappa_4 + 2)^{-1/2} \\ &= (\kappa_4 + 2)^{-2} \{ -16\kappa_{g4} (l_{0j}) + 4\kappa_3 (\kappa_5 + 8\kappa_3) \} \\ &= (\kappa_4 + 2)^{-2} \{ -(\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 240\kappa_3^2 + 48) \\ & \quad + 4\kappa_3 (\kappa_5 + 8\kappa_3) \} \\ &= -(\kappa_4 + 2)^{-2} (\kappa_8 + 24\kappa_6 + 52\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 208\kappa_3^2 + 48) \\ & (= -12 \text{ under normality}), \end{aligned}$$

(iii) the first part in  $2n^{-2}E_g \left[ \cdot \right]_{(B)} (\alpha_{ML2}^{(A)})^{-1/2}$  of the third term in  $\left[ \cdot \right]_{(A)}^{(A)}$  of

(S6.13) is

$$\begin{aligned}
& 2n^2 E_g \left\{ \bar{l}_{ML}^{(1)} \bar{l}_{ML}^{(2)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \right\} (\alpha_{ML2}^{(A)})^{-1/2} \\
&= 2 \left[ n E_g \left( \bar{l}_{ML}^{(2)} \right) n E_g \left( \bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)} \right) \mathbf{v}^{(1)} \right. \\
&\quad \left. + 2n E_g \left( \bar{l}_{ML}^{(1)} \frac{\partial \bar{l}}{\partial \theta_0} \right) \lambda^{-1} \left\{ n E_g \left( \frac{\partial \bar{l}}{\partial \theta_0} m_v \right), \gamma \right\} \mathbf{v}^{(1)} \right] (\alpha_{ML2}^{(A)})^{-1/2} + O(n^{-1}) \\
&= 2 \left[ (-1) \left\{ -\frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\} \right. \\
&\quad \left. + 2 \frac{\kappa_3}{\sigma} (-\sigma^2) \left\{ n E_g \left( \frac{\partial \bar{l}}{\partial \theta_0} m_v \right), \gamma \right\} \frac{1}{2} (\kappa_4 + 2)^{-3/2} (-1, 4\sigma\kappa_3)' \right] (\kappa_4 + 2)^{-1/2} \\
&\quad + O(n^{-1}) \\
&= (\kappa_4 + 2)^{-2} \left[ \kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8 - 2\kappa_3 \sigma \left\{ \frac{1}{\sigma} (\kappa_5 + 8\kappa_3), \frac{1}{\sigma^2} \right\} (-1, 4\sigma\kappa_3)' \right] \\
&\quad + O(n^{-1}) \\
&= (\kappa_4 + 2)^{-2} [ \kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8 - 2\kappa_3 \{ -(\kappa_5 + 8\kappa_3) + 4\kappa_3 \} ] + O(n^{-1}) \\
&= (\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 14\kappa_3^2 + 2\kappa_3\kappa_5 + 8) + O(n^{-1}) \\
& (= 2 \text{ under normality}),
\end{aligned}$$

(iv) the central part in  $2n^{-2}E_g \left[ \cdot \right]_{(B)} (\alpha_{ML2}^{(A)})^{-1/2}$  of the third term in  $\left[ \cdot \right]_{(A)}^{(A)}$  of

(S6.13) is

$$\begin{aligned}
& 2n^2 E_g \left\{ \left( \bar{l}_{ML}^{(1)} \right)^2 \mathbf{v}^{(2)}, \mathbf{m}_v^{(2)} \right\} (\alpha_{ML2}^{(A)})^{-1/2} \\
&= 2 \left[ \alpha_{ML2}^{(A)} \left[ n \text{avar}_g(m_v), n \text{cov}_g \left( m_v, \frac{\partial \bar{l}}{\partial \theta_0} \right), 0, \gamma, n \text{cov}_g \left( \frac{\partial v^{(A)}}{\partial \theta_0}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right] \mathbf{v}^{(2)} \right. \\
&\quad \left. \right]_{(C)}
\end{aligned}$$



$$\begin{aligned}
&= 2 \left[ \begin{aligned}
&\{ (\kappa_4 + 2)(\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 240\kappa_3^2 + 56) \\
&+ 2(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)^2 \} \frac{3}{8} (\kappa_4 + 2)^{-3} \\
&+ \left\{ (\kappa_4 + 2) \frac{1}{\sigma} (\kappa_5 + 8\kappa_3) + 2(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8) \frac{\kappa_3}{\sigma} \right\} \{ -3(\kappa_4 + 2)^{-3} \kappa_3 \sigma \} \\
&+ \left\{ (\kappa_4 + 2) \frac{1}{\sigma^2} + 2 \frac{\kappa_3^2}{\sigma^2} \right\} \{ -2(\kappa_4 + 2)^{-2} + 6(\kappa_4 + 2)^{-3} \kappa_3^2 \} \sigma^2 \\
&+ \left\{ -(\kappa_4 + 2) \frac{4}{\sigma^2} (\kappa_4 + 2) - \frac{8}{\sigma} (\kappa_5 + 8\kappa_3) \frac{\kappa_3}{\sigma} \right\} \left[ \left\{ -\frac{\sigma^2}{2} (\kappa_4 + 2)^{-2} \right\} \right] + O(n^{-1})
\end{aligned} \right] \\
&= 2 \left[ \begin{aligned}
&\left( \kappa_4 + 2 \right)^{-3} \left\{ \frac{3}{4} (\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)^2 \right. \\
&\quad \left. - 6(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)\kappa_3^2 + 12\kappa_3^4 \right\} \\
&+ (\kappa_4 + 2)^{-2} \left\{ \frac{3}{8} (\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 240\kappa_3^2 + 56) \right. \\
&\quad \left. - 3(\kappa_5 + 8\kappa_3)\kappa_3 + 6\kappa_3^2 - 4\kappa_3^2 + 4(\kappa_5 + 8\kappa_3)\kappa_3 \right\} \\
&\left. + (\kappa_4 + 2)^{-1} (-2) + 2 \right] + O(n^{-1})
\end{aligned} \right] \tag{C}
\end{aligned}$$

$$= 2 \left[ \begin{array}{l} (\kappa_4 + 2)^{-3} \frac{3}{4} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\ + (\kappa_4 + 2)^{-2} \left\{ \frac{3}{8} (\kappa_8 + 24\kappa_6 + 34\kappa_4^2 + 152\kappa_4 + 56) + 22\kappa_3\kappa_3 + 100\kappa_3^2 \right\} \\ - 2(\kappa_4 + 2)^{-1} + 2 \end{array} \right] + O(n^{-1})$$

( $= 2(2^{-3} \times 48 + 2^{-2} \times 3 \times 7 - 2 \times 2^{-1} + 2) = 2\{6 + (21/4) + 1\} = 49/2$  under normality),

(v) the third part in  $2n^{-2} E_g \left[ \begin{array}{l} \cdot \\ (B) \end{array} \right] (\alpha_{ML2}^{(A)})^{-1/2}$  of the third term in  $\left[ \begin{array}{l} \cdot \\ (A) \end{array} \right]$  of (S6.13) is

$$\begin{aligned} & 2n^2 E_g \{ n^{-1} \bar{l}_{ML}^{(1)} 2q \mathbf{m}_v^{(1)'} \mathbf{v}^{(1)} \} (\alpha_{ML2}^{(A)})^{-1/2} = 4n E_g \{ \bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)'} \mathbf{v}^{(1)} \} (\alpha_{ML2}^{(A)})^{-1/2} \\ & = 4 \left\{ -\frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\} (\kappa_4 + 2)^{-1/2} \\ & = -2(\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \\ & (= -2 \times 2^{-2} \times 8 = -4 \text{ under normality}), \end{aligned}$$

(vi) the first half of the fourth term in  $\left[ \begin{array}{l} \cdot \\ (A) \end{array} \right]$  of (S6.13)

$$2n^2 E_g \{ \bar{l}_{ML}^{(2)} (\alpha_{ML2}^{(A)})^{-1/2} \bar{l}_{ML}^{(1)} \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)} \} \text{ is equal to the value in (iii),}$$

(vii) the second half of the fourth term in  $\left[ \begin{array}{l} \cdot \\ (A) \end{array} \right]$  of (S6.13) is

$$\begin{aligned} & n^2 E_g \{ (\bar{l}_{ML}^{(1)} \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)})^2 \} \\ & = \alpha_{ML2}^{(A)} \mathbf{v}^{(1)'} n \text{acov}_g (\mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} + 2 \{ n E_g (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)'} \mathbf{v}^{(1)}) \}^2 + O(n^{-1}) \end{aligned}$$



$$\begin{aligned}
&= (\kappa_4 + 2) \frac{1}{4} (\kappa_4 + 2)^{-3} (-1, 4\sigma\kappa_3) \\
&\quad \times \begin{bmatrix} n \text{avar}_g(m_v) & n \text{cov}_g(m_v, \partial \bar{l} / \partial \theta_0) \\ n \text{cov}_g(m_v, \partial \bar{l} / \partial \theta_0) & \gamma \end{bmatrix} \begin{pmatrix} -1 \\ 4\sigma\kappa_3 \end{pmatrix} \\
&\quad + 2 \times \frac{1}{4} (\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 + O(n^{-1}) \\
&= \frac{1}{4} (\kappa_4 + 2)^{-2} (-1, 4\sigma\kappa_3) \\
&\quad \times \begin{bmatrix} \kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 34\kappa_4^2 & \frac{1}{\sigma}(\kappa_5 + 8\kappa_3) \\ +152\kappa_4 + 240\kappa_3^2 + 56 & \\ \frac{1}{\sigma}(\kappa_5 + 8\kappa_3) & \frac{1}{\sigma^2} \end{bmatrix} \begin{pmatrix} -1 \\ 4\sigma\kappa_3 \end{pmatrix} \\
&\quad + \frac{1}{2} (\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 + O(n^{-1}) \\
&= \frac{1}{4} (\kappa_4 + 2)^{-2} \{ \kappa_8 + 24\kappa_6 + (56 - 8)\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 \\
&\quad + (240 - 64 + 16)\kappa_3^2 + 56 \} \\
&\quad + \frac{1}{2} (\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 + O(n^{-1}) \\
&= \frac{1}{4} (\kappa_4 + 2)^{-2} (\kappa_8 + 24\kappa_6 + 48\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 192\kappa_3^2 + 56) \\
&\quad + \frac{1}{2} (\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 + O(n^{-1}) \\
&\quad \left( = \frac{1}{4} \times \frac{1}{4} \times 56 + \frac{1}{2} \times \frac{1}{8} \times 64 = \frac{7}{2} + 4 = \frac{15}{2} \text{ under normality} \right),
\end{aligned}$$

(viii) the fifth term in  $\begin{bmatrix} \cdot \\ (A) \end{bmatrix}$  of (S6.13) is

$$\begin{aligned}
& -\{2nE_g(\bar{I}_{ML}^{(2)})(\alpha_{ML2}^{(A)})^{-1/2}\alpha_{(\Delta t)ML1}^{(A)} + (\alpha_{(\Delta t)ML1}^{(A)})^2\} \\
& = -\left[2(-1)(\kappa_4 + 2)^{-1/2}\left\{-\frac{1}{2}(\kappa_4 + 2)^{-3/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)\right\}\right. \\
& \quad \left. + \frac{1}{4}(\kappa_4 + 2)^{-3}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2\right] \\
& = -\left\{(\kappa_4 + 2)^{-2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) + \frac{1}{4}(\kappa_4 + 2)^{-3}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2\right\} \\
& (= -(2^{-2} \times 8 + (1/4)2^{-3} \times 8^2) = -4 \text{ under normality}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\kappa_{g2}(t_{ML}^{(A)}) & = 1 + n^{-1} \left[ \begin{array}{l} -2(\kappa_4 + 2)^{-1}(\kappa_4 + 1) \\ -(\kappa_4 + 2)^{-2}(\kappa_8 + 24\kappa_6 + 52\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 208\kappa_3^2 + 48) \\ + 2(\kappa_4 + 2)^{-2}(\kappa_6 + 12\kappa_4 + 14\kappa_3^2 + 2\kappa_5\kappa_3 + 8) \end{array} \right. \\
& + 2 \left[ \begin{array}{l} (\kappa_4 + 2)^{-3} \frac{3}{4}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\ + (\kappa_4 + 2)^{-2} \left\{ \frac{3}{8}(\kappa_8 + 24\kappa_6 + 34\kappa_4^2 + 152\kappa_4 + 56) + 22\kappa_5\kappa_3 + 100\kappa_3^2 \right\} \\ \left. - 2(\kappa_4 + 2)^{-1} + 2 \right] \end{array} \right] \quad (B)
\end{aligned}$$

(S6.14)

$$\begin{aligned}
& -2(\kappa_4 + 2)^{-2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \\
& + \frac{1}{4}(\kappa_4 + 2)^{-2}(\kappa_8 + 24\kappa_6 + 48\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 192\kappa_3^2 + 56) \\
& + \frac{1}{2}(\kappa_4 + 2)^{-3}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\
& - \left\{ (\kappa_4 + 2)^{-2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) + \frac{1}{4}(\kappa_4 + 2)^{-3}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \right\} \Bigg]_{(A)} \\
& + O(n^{-2}) \\
& = 1 + n^{-1} \left[ \begin{aligned} & 4 - 2(\kappa_4 + 2)^{-1}(\kappa_4 + 1 + 2) \\ & + (\kappa_4 + 2)^{-2} \{ (-1 + (3/4) + (1/4))\kappa_8 + (-24 + 2 + 18 - 2 + 6 - 1)\kappa_6 \\ & + (-52 + 4 + 44 + 12)\kappa_5\kappa_3 + (-32 + (3/2)17 + (17/2))\kappa_4^2 \\ & + (-144 + 24 + 6 \times 19 - 24 + 38 - 12)\kappa_4 \\ & + (-208 + 28 + 200 - 12 + 48 - 6)\kappa_3^2 - 48 + 16 + 6 \times 7 - 16 + 14 - 8 \} \\ & + (\kappa_4 + 2)^{-3} \left( \frac{3}{2} + \frac{1}{2} - \frac{1}{4} \right) (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \end{aligned} \right]_{(A)} + O(n^{-2}) \\
& = 1 + n^{-1} \left\{ \begin{aligned} & 2 - 2(\kappa_4 + 2)^{-1} + (\kappa_4 + 2)^{-2}(-\kappa_6 + 8\kappa_5\kappa_3 + 2\kappa_4^2 - 4\kappa_4 + 50\kappa_3^2) \\ & + (\kappa_4 + 2)^{-3} \frac{7}{4} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \end{aligned} \right\} + O(n^{-2}) \\
& = 1 + n^{-1} \alpha_{(t)ML\Delta_2}^{(A)} + O(n^{-2}) \quad (\alpha_{(t)ML_2}^{(A)} = 1) \\
& (= 1 + n^{-1} \{2 - 1 + (7/4)8\} + O(n^{-2})) = 1 + n^{-1}15 + O(n^{-2}) \quad \text{under normality),}
\end{aligned}$$

$$\begin{aligned} \kappa_{g3}(t_{ML}^{(A)}) &= n^{-1/2} \{ \alpha_{ML3}^{(A)} (\alpha_{ML2}^{(A)})^{-3/2} + 6\alpha_{(\Delta t)ML1}^{(A)} \} + O(n^{-3/2}) \\ &= \left\{ (\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8)(\kappa_4 + 2)^{-3/2} \right. \\ &\quad \left. - 6 \times \frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\} + O(n^{-3/2}) \\ &= -n^{-1/2} 2(\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 7\kappa_3^2 + 8) + O(n^{-3/2}) \\ &= n^{-1/2} \alpha_{(t)ML3}^{(A)} + O(n^{-3/2}) \end{aligned} \tag{S6.15}$$

(under normality  $\alpha_{(t)ML3}^{(A)} = -2 \times 2^{3/2} = -2^{5/2}$ ).

$$\begin{aligned} \kappa_{g4}(t_{ML}^{(A)}) &= n^{-1} \left[ \alpha_{ML4}^{(A)} (\alpha_{ML2}^{(A)})^{-2} + 4n^3 E_g \{ (\bar{l}_{ML}^{(1)})^4 \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-3/2} \right. \\ &\quad + 12n^3 E_g \{ (\bar{l}_{ML}^{(1)})^3 \bar{l}_{ML}^{(2)} \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-3/2} \\ &\quad + 6n^3 E_g \{ (\bar{l}_{ML}^{(1)})^4 (\mathbf{v}^{(1)'} \mathbf{m}_v^{(1)})^2 \} (\alpha_{ML2}^{(A)})^{-1} \\ &\quad + 4n^3 E_g \{ (\bar{l}_{ML}^{(1)})^3 \bar{l}_{ML}^{(2)} \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)} + (\bar{l}_{ML}^{(1)})^4 \mathbf{v}^{(2)'} \mathbf{m}_v^{(2)} \} (\alpha_{ML2}^{(A)})^{-3/2} \\ &\quad + [4n E_g (\bar{l}_{ML}^{(2)}) \alpha_{ML3}^{(A)} + 6\alpha_{ML2}^{(A)} \alpha_{ML\Delta 2}^{(A)} + 6\alpha_{ML2}^{(A)} \{ n E_g (\bar{l}_{ML}^{(2)}) \}^2 ] (\alpha_{ML2}^{(A)})^{-2} \\ &\quad - 4\{ \alpha_{(t)ML1}^{(A)} - 2q(\alpha_{ML2}^{(A)})^{-1/2} \} \alpha_{(t)ML3}^{(A)} \\ &\quad - 6\{ \alpha_{(t)ML\Delta 2}^{(A)} - 4qn E_g (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)'}) \mathbf{v}^{(1)} \} (\alpha_{ML2}^{(A)})^{-1/2} \} \\ &\quad \left. - 6\{ \alpha_{(t)ML1}^{(A)} - 2q(\alpha_{ML2}^{(A)})^{-1/2} \}^2 \right] + O(n^{-2}), \end{aligned} \tag{S6.16}$$

(i) the first term in  $\left[ \begin{matrix} \cdot \\ (A) \end{matrix} \right]_{(A)}$  of (S6.16) is

$$\begin{aligned} \alpha_{ML4}^{(A)} (\alpha_{ML2}^{(A)})^{-2} &= (\kappa_8 + 24\kappa_6 + 32\kappa_3\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 96\kappa_3^2 + 48) \\ &\quad \times (\kappa_4 + 2)^{-2} \end{aligned}$$

(= 12 under normality),

(ii) the second term in  $\begin{bmatrix} \cdot \\ \text{(A)} \end{bmatrix}$  of (S6.16) is

$$\begin{aligned}
& 4n^3 E_g \{ (\bar{l}_{ML}^{(1)})^4 \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-3/2} \\
&= 4 \begin{bmatrix} \text{(B)} \\ \text{(B)} \end{bmatrix} \left[ 6 \times 4 \alpha_{ML2}^{(A)} \left[ 4 \kappa_{g4}(l_{0j}), E_g \left\{ \frac{\partial l_j}{\partial \theta_0} (l_{0j} - \bar{l}_0^*)^2 \right\} \right] \right] \\
& - 4 \times 8 \kappa_{g3}(l_{0j}) \left\{ n \text{cov}_g(\bar{l}_{ML}^{(1)}, m_v), -2 E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\} \begin{bmatrix} \text{(B)} \\ \text{(B)} \end{bmatrix} \mathbf{v}^{(1)} (\alpha_{ML2}^{(A)})^{-3/2} + O(n^{-1}) \\
&= \begin{bmatrix} \text{(B)} \\ \text{(B)} \end{bmatrix} \left[ 96(\kappa_4 + 2) \right. \\
& \quad \times \left\{ \frac{1}{4}(\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 240\kappa_3^2 + 48), \frac{1}{4\sigma}(\kappa_5 + 8\kappa_3) \right\} \\
& \quad \left. - 128 \left\{ -\frac{1}{8}(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8) \right\} \left( \kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8, \frac{\kappa_3}{\sigma} \right) \right] \\
& \quad \times \frac{1}{2}(\kappa_4 + 2)^{-3/2} (-1, 4\sigma\kappa_3)' (\kappa_4 + 2)^{-3/2} + O(n^{-1}) \\
&= 12(\kappa_4 + 2)^{-2} \{ -(\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 240\kappa_3^2 + 48) \\
& \quad + 4(\kappa_5 + 8\kappa_3)\kappa_3 \} \\
& \quad + 8(\kappa_4 + 2)^{-3} \{ -(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)^2 + 4(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)\kappa_3^2 \} \\
& \quad + O(n^{-1}) \\
&= -12(\kappa_4 + 2)^{-2} (\kappa_8 + 24\kappa_6 + 52\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 208\kappa_3^2 + 48) \\
& \quad - 8(\kappa_4 + 2)^{-3} \{ (\kappa_6 + 12\kappa_4 + 8\kappa_3^2 + 8)^2 - 4\kappa_3^4 \} \\
& (= -12(1/4) \times 48 - 8(1/8) \times 64 = -144 - 64 = -208 \text{ under normality}),
\end{aligned}$$

(iii) the third term in  $\begin{bmatrix} \cdot \\ \text{(A)} \end{bmatrix}$  of (S6.16) is

$$\begin{aligned}
& 12n^3 E_g \{ (\bar{l}_{ML}^{(1)})^3 \bar{l}_{ML}^{(2)} \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-3/2} \\
&= 12 \left[ 3\alpha_{ML2}^{(A)} n^2 E_g (\bar{l}_{ML}^{(1)} \bar{l}_{ML}^{(2)} \mathbf{m}_v^{(1)'} \mathbf{v}^{(1)}) (\alpha_{ML2}^{(A)})^{-3/2} \right. \\
&\quad \left. + 6 \left\{ n \text{cov}_g \left( \bar{l}_{ML}^{(1)}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right\}^2 \lambda^{-1} n E_g (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)'} \mathbf{v}^{(1)}) (\alpha_{ML2}^{(A)})^{-3/2} \right] + O(n^{-1}) \\
&= 36n^2 E_g (\bar{l}_{ML}^{(1)} \bar{l}_{ML}^{(2)} \mathbf{m}_v^{(1)'} \mathbf{v}^{(1)}) (\alpha_{ML2}^{(A)})^{-1/2} \\
&\quad + 12 \times 6 \times (-2)^2 \left\{ E_g \left( l_{0j}, \frac{\partial l_j}{\partial \theta_0} \right) \right\}^2 (-\sigma^2) n E_g (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)'} \mathbf{v}^{(1)}) (\alpha_{ML2}^{(A)})^{-3/2} \\
&\quad + O(n^{-1}) \\
&= 18(\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 14\kappa_3^2 + 2\kappa_3\kappa_3 + 8) \\
&\quad + 72 \frac{\kappa_3^2}{\sigma^2} (-\sigma^2) \left\{ -\frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\} (\kappa_4 + 2)^{-3/2} + O(n^{-1}) \\
&= 18(\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 14\kappa_3^2 + 2\kappa_3\kappa_3 + 8) \\
&\quad + 36(\kappa_4 + 2)^{-3} \kappa_3^2 (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) + O(n^{-1}) \\
& (= 18(1/4) \times 8 = 36 \text{ under normality; note that the first term } 18(\kappa_4 + 2)^{-2} (\cdot) \\
&\text{on the left-hand side of the above last equation is 18 times the result of (iii) for} \\
&\text{(S6.13) of } \alpha_{(l)ML\Delta 2}^{(A)})
\end{aligned}$$

(iv) the fourth term in  $\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_{(A) (A)}$  of (S6.16) is

$$\begin{aligned}
& 6n^3 E_g \{ (\bar{l}_{ML}^{(1)})^4 (\mathbf{v}^{(1)'} \mathbf{m}_v^{(1)})^2 \} (\alpha_{ML2}^{(A)})^{-1} \\
&= 6 \left[ 3(\alpha_{ML2}^{(A)})^2 \mathbf{v}^{(1)'} n \text{acov}_g (\mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} \right. \\
&\quad \left. + 12\alpha_{ML2}^{(A)} \{ n \text{cov}_g (\bar{l}_{ML}^{(1)}, \mathbf{m}_v^{(1)'} \mathbf{v}^{(1)}) \}^2 \right] (\alpha_{ML2}^{(A)})^{-1} + O(n^{-1}) \\
&= 18 \times \frac{1}{4} (\kappa_4 + 2)^{-2} (\kappa_8 + 24\kappa_6 + 48\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 192\kappa_3^2 + 56) \\
&\quad + 36 \times \frac{1}{2} (\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 + O(n^{-1})
\end{aligned}$$

$$\begin{aligned}
&= \frac{9}{2}(\kappa_4 + 2)^{-2}(\kappa_8 + 24\kappa_6 + 48\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 192\kappa_3^2 + 56) \\
&\quad + 18(\kappa_4 + 2)^{-3}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 + O(n^{-1}) \\
&\left( = \frac{9}{2} \times \frac{1}{4} \times 56 + 18 \times \frac{1}{8} \times 64 = 63 + 144 = 207 \quad \text{under normality; see the result of} \right. \\
&\left. \text{(vii) for (S6.13) of } \alpha_{(r)\text{ML}\Delta 2}^{(A)}, \right)
\end{aligned}$$

(v) the first half of the fifth term in  $\left[ \begin{smallmatrix} \cdot \\ (A) \end{smallmatrix} \right]_{(A)}$  of (S6.16) is

$$\begin{aligned}
&4n^3 E_g \{ (\bar{l}_{\text{ML}}^{(1)})^3 \bar{l}_{\text{ML}}^{(2)} \mathbf{v}^{(1)} \mathbf{m}_v^{(1)} \} (\alpha_{\text{ML}2}^{(A)})^{-3/2} \\
& (= 36/3 = 12 \text{ under normality; one third of the value in (iii)})
\end{aligned}$$

(vi) the second half of the fifth term in  $\left[ \begin{smallmatrix} \cdot \\ (A) \end{smallmatrix} \right]_{(A)}$  of (S6.16) is

$$\begin{aligned}
&4n^3 E_g \{ (\bar{l}_{\text{ML}}^{(1)})^4 \mathbf{v}^{(2)} \mathbf{m}_v^{(2)} \} (\alpha_{\text{ML}2}^{(A)})^{-3/2} \\
&= 4 \left[ \begin{array}{l} \text{(B)} \\ 3(\alpha_{\text{ML}2}^{(A)})^2 \left\{ n \text{avar}_g(m_v), n \text{cov}_g \left( m_v, \frac{\partial \bar{l}}{\partial \theta_0} \right), 0, \gamma, n \text{cov}_g \left( \frac{\partial v^{(A)}}{\partial \theta_0}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right\} \\ + 12\alpha_{\text{ML}2}^{(A)} (-2)^2 \left[ \begin{array}{l} \text{(C)} \\ \{ n \text{cov}_g(\bar{l}_0, m_v) \}^2, n \text{cov}_g(\bar{l}_0, m_v) E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right), 0, \right. \\ \left. \left[ E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right]^2, n \text{cov}_g \left( \bar{l}_0, \frac{\partial v^{(A)}}{\partial \theta_0} \right) E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right] \end{array} \right] \mathbf{v}^{(2)} (\alpha_{\text{ML}2}^{(A)})^{-3/2} \\
&\quad \text{(C) (B)}
\end{array} \right]
\end{aligned}$$

$$\begin{aligned}
&= 12 \left[ \underset{(B)}{\left[ (\kappa_4 + 2)^2 \text{navar}_g(m_v) + 16(\kappa_4 + 2) \{n \text{cov}_g(\bar{l}_0, m_v)\}^2 \right] \frac{3}{8} (\kappa_4 + 2)^{-4}} \right. \\
&\quad \left. + \left\{ (\kappa_4 + 2)^2 n \text{cov}_g \left( m_v, \frac{\partial \bar{l}}{\partial \theta_0} \right) + 16(\kappa_4 + 2) n \text{cov}_g(\bar{l}_0, m_v) E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\} \right. \\
&\quad \quad \left. \times \{ -3(\kappa_4 + 2)^{-4} \kappa_3 \sigma \} \right. \\
&\quad \left. + \left[ (\kappa_4 + 2)^2 \gamma + 16(\kappa_4 + 2) \left\{ E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\}^2 \right] \right. \\
&\quad \quad \left. \times \{ -2(\kappa_4 + 2)^{-3} + 6(\kappa_4 + 2)^{-4} \kappa_3^2 \} \sigma^2 \right. \\
&\quad \left. + \left\{ (\kappa_4 + 2)^2 n \text{cov}_g \left( \frac{\partial v^{(\Lambda)}}{\partial \theta_0}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right. \right. \\
&\quad \left. \left. + 16(\kappa_4 + 2) n \text{cov}_g \left( \bar{l}_0, \frac{\partial v^{(\Lambda)}}{\partial \theta_0} \right) E_g \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\} \left\{ -\frac{\sigma^2}{2} (\kappa_4 + 2)^{-3} \right\} \right]_{(B)} \\
&\quad + O(n^{-1})
\end{aligned}$$



$$\begin{aligned}
&= 12 \left[ \begin{aligned} & \underbrace{\left\{ (\kappa_4 + 2)^{-2} (\kappa_8 + 24\kappa_6 + 56\kappa_3\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 240\kappa_3^2 + 56) \right.}_{(B)} \\ & \quad \left. + 4(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)^2 \right\} \frac{3}{8} \\ & + \left[ (\kappa_4 + 2)^{-2} (\kappa_5 + 8\kappa_3) \frac{1}{\sigma} \right. \\ & \quad \left. + 16(\kappa_4 + 2)^{-3} \left\{ -\frac{1}{2} (\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8) \right\} \left( -\frac{\kappa_3}{2\sigma} \right) \right] (-3\kappa_3\sigma) \\ & + \left\{ (\kappa_4 + 2)^{-2} \frac{1}{\sigma^2} + 16(\kappa_4 + 2)^{-3} \frac{\kappa_3^2}{4\sigma^2} \right\} \{-2(\kappa_4 + 2) + 6\kappa_3^2\} \sigma^2 \\ & + \left[ -\frac{4}{\sigma^2} + 16(\kappa_4 + 2)^{-2} \frac{2}{\sigma} (\kappa_5 + 8\kappa_3) \left( -\frac{\kappa_3}{2\sigma} \right) \right] \left( -\frac{\sigma^2}{2} \right) \underbrace{\Big]}_{(B)} + O(n^{-1}) \\ & = 12 \left[ \underbrace{(\kappa_4 + 2)^{-3} \left\{ \frac{3}{2} (\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)^2 \right.}_{(B)} \right. \\ & \quad \left. - 12(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)\kappa_3^2 + 24\kappa_3^4 \right\} \\ & + (\kappa_4 + 2)^{-2} \left\{ \frac{3}{8} (\kappa_8 + 24\kappa_6 + 56\kappa_3\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 240\kappa_3^2 + 56) \right. \\ & \quad \left. - 3(\kappa_5 + 8\kappa_3)\kappa_3 + 6\kappa_3^2 - 8\kappa_3^2 + 8(\kappa_5 + 8\kappa_3)\kappa_3 \right\} \\ & \quad \left. - 2(\kappa_4 + 2)^{-1} + 2 \right] \underbrace{\Big]}_{(B)} + O(n^{-1})
\end{aligned}$$



$$\begin{aligned}
& -6 \left\{ (\kappa_4 + 2)^{-1/2} - \frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) - 2g(\kappa_4 + 2)^{-1/2} \right\}^2 \\
& = -8 \left\{ 1 + \frac{1}{2} (\kappa_4 + 2)^{-1} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\} \\
& \quad \times (\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 7\kappa_3^2 + 8) \\
& \quad - 6 \{ \alpha_{(r)ML\Delta 2}^{(A)} + 2(\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \} \\
& \quad - 6 \left\{ 1 + \frac{1}{2} (\kappa_4 + 2)^{-1} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\}^2 (\kappa_4 + 2)^{-1} \\
& = -6 \alpha_{(r)ML\Delta 2}^{(A)} - (8 + 12 + 6)(\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \\
& \quad - \{ 8(\kappa_4 + 2)^{-2} \kappa_3^2 + 4(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \kappa_3^2 \} \\
& \quad - 6(\kappa_4 + 2)^{-1} - \left( 4 + \frac{3}{2} \right) (\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\
& = -6 \{ 2 - 2(\kappa_4 + 2)^{-1} + (\kappa_4 + 2)^{-2} (-\kappa_6 + 8\kappa_5\kappa_3 + 2\kappa_4^2 - 4\kappa_4 + 50\kappa_3^2) \\
& \quad + (\kappa_4 + 2)^{-3} (7/4)(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \} \\
& \quad - 26(\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) - 8(\kappa_4 + 2)^{-2} \kappa_3^2 - 6(\kappa_4 + 2)^{-1} \\
& \quad - (11/2)(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\
& \quad - 4(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \kappa_3^2 \\
& = -12 + 6(\kappa_4 + 2)^{-1} + (\kappa_4 + 2)^{-2} \{ (6 - 26)\kappa_6 - 48\kappa_5\kappa_3 - 12\kappa_4^2 \\
& \quad + (24 - 26 \times 12)\kappa_4 + (-6 \times 50 - 26 \times 6 - 8)\kappa_3^2 - 26 \times 8 \} \\
& \quad - \left( \frac{21}{2} + \frac{11}{2} \right) (\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\
& \quad - 4(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \kappa_3^2 \\
& = -12 + 6(\kappa_4 + 2)^{-1} \\
& \quad + (\kappa_4 + 2)^{-2} (-20\kappa_6 - 48\kappa_5\kappa_3 - 12\kappa_4^2 - 288\kappa_4 - 464\kappa_3^2 - 208) \\
& \quad - 16(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\
& \quad - 4(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \kappa_3^2 \\
& (= -12 + 3 - 52 - 128 = -189 \text{ under normality}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
 & \kappa_{g^4}^{(A)}(t_{ML}^{(A)}) \\
 = & n^{-1} \left[ \begin{aligned}
 & (\kappa_4 + 2)^{-2} (\kappa_8 + 24\kappa_6 + 32\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 96\kappa_3^2 + 48) \\
 & - 12(\kappa_4 + 2)^{-2} (\kappa_8 + 24\kappa_6 + 52\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 208\kappa_3^2 + 48) \\
 & - 8(\kappa_4 + 2)^{-3} \{ (\kappa_6 + 12\kappa_4 + 8\kappa_3^2 + 8)^2 - 4\kappa_3^4 \} \\
 & \quad + 24(\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 14\kappa_3^2 + 2\kappa_5\kappa_3 + 8) \\
 & \quad + 48(\kappa_4 + 2)^{-3} \kappa_3^2 (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \\
 & + \frac{9}{2} (\kappa_4 + 2)^{-2} (\kappa_8 + 24\kappa_6 + 48\kappa_5\kappa_3 + 34\kappa_4^2 + 152\kappa_4 + 192\kappa_3^2 + 56) \\
 & + 18(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\
 & \quad + 18(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\
 & \quad + (\kappa_4 + 2)^{-2} \left\{ \frac{9}{2} (\kappa_8 + 24\kappa_6 + 34\kappa_4^2 + 152\kappa_4 + 56) + 312\kappa_5\kappa_3 + 1536\kappa_3^2 \right\} \\
 & \quad - 24(\kappa_4 + 2)^{-1} + 24 \\
 & - (\kappa_4 + 2)^{-2} (4\kappa_6 + 78\kappa_4 + 12\kappa_4^2 + 16\kappa_3^2 + 44) \\
 & - 12 + 6(\kappa_4 + 2)^{-1} \\
 & \quad + (\kappa_4 + 2)^{-2} (-20\kappa_6 - 48\kappa_5\kappa_3 - 12\kappa_4^2 - 288\kappa_4 - 464\kappa_3^2 - 208) \\
 & \quad - 16(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \\
 & \quad - 4(\kappa_4 + 2)^{-3} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)\kappa_3^2 \Big] + O(n^{-2})
 \end{aligned} \right] \tag{S6.17}
 \end{aligned}$$

$$\begin{aligned}
&= n^{-1} \left[ \begin{aligned} &12 - 18(\kappa_4 + 2)^{-1} \\ &+ (\kappa_4 + 2)^{-2} \{ (1 - 12 + 9)\kappa_8 + (24 - 12 \times 24 + 24 + 9 \times 24 - 4 - 20)\kappa_6 \\ &\quad + (32 - 12 \times 52 + 48 + 9 \times 24 + 312 - 48)\kappa_3\kappa_3 \\ &\quad + (32 - 12 \times 32 + 9 \times 34 - 12 - 12)\kappa_4^2 \\ &\quad + (144 - 12 \times 144 + 24 \times 12 + 9 \times 152 - 78 - 288)\kappa_4 \\ &\quad + (96 - 12 \times 208 + 24 \times 14 + 9 \times 96 + 1536 - 16 - 464)\kappa_3^2 \\ &\quad + 48 - 12 \times 48 + 24 \times 8 + 9 \times 56 - 44 - 208 \} \\ &+ (\kappa_4 + 2)^{-3} \{ -8(\kappa_6 + 12\kappa_4 + 8\kappa_3^2 + 8)^2 + 32\kappa_3^4 \\ &\quad + (48 - 4)(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)\kappa_3^2 \\ &\quad + (36 - 16)(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \} \end{aligned} \right] + O(n^{-2}) \\
&= n^{-1} \left[ \begin{aligned} &12 - 18(\kappa_4 + 2)^{-1} \\ &+ (\kappa_4 + 2)^{-2} (-2\kappa_8 - 48\kappa_6 - 64\kappa_3\kappa_3 - 70\kappa_4^2 - 294\kappa_4 - 144\kappa_3^2 - 84) \\ &+ (\kappa_4 + 2)^{-3} \{ 12(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 + 12(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)\kappa_3^2 \} \end{aligned} \right] \\
&\quad + O(n^{-2}) \\
&= n^{-1} \alpha_{(t)ML4}^{(A)} + O(n^{-2}) \\
& (= 12 - 9 - 21 + 96 = 78 \text{ under normality}).
\end{aligned}$$

**S6.2.3 A result for estimation of  $-2\bar{I}_0^*$**

$$\begin{aligned}
 & n \operatorname{acov}_g \left\{ n^{-1} \operatorname{AIC}_{\text{ML}}, \hat{\alpha}_{(t)\text{ML}1}^{(A)} + \frac{\hat{\alpha}_{(t)\text{ML}3}^{(A)}}{6} (z_{\hat{\alpha}}^2 - 1) \right\} \\
 &= n \operatorname{acov}_g \left\{ n^{-1} \operatorname{AIC}_{\text{ML}}, \hat{\alpha}_{\text{ML}1}^{(A)} (\hat{\alpha}_{\text{ML}2}^{(A)})^{-1/2} + \hat{\alpha}_{(\Delta t)\text{ML}1}^{(A)} z_{\hat{\alpha}}^2 \right. \\
 &\quad \left. + \frac{1}{6} \hat{\alpha}_{\text{ML}3}^{(A)} (\hat{\alpha}_{\text{ML}2}^{(A)})^{-3/2} (z_{\hat{\alpha}}^2 - 1) \right\}, \tag{S6.18}
 \end{aligned}$$

(i) where the first term on the right-hand side of (S6.18) is

$$\begin{aligned}
 & n \operatorname{acov}_g \{ n^{-1} \operatorname{AIC}_{\text{ML}}, \hat{\alpha}_{\text{ML}1}^{(A)} (\hat{\alpha}_{\text{ML}2}^{(A)})^{-1/2} \} \\
 &= n \operatorname{acov}_g \{ \bar{l}_{\text{ML}}^{(1)}, (\hat{\alpha}_{\text{ML}2}^{(A)})^{-1/2} \} = n \mathbf{E}_g \{ \bar{l}_{\text{ML}}^{(1)} \mathbf{m}_v^{(1)} \} \mathbf{v}^{(1)} \\
 &= -\frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8),
 \end{aligned}$$

(ii) the second term on the right-hand side of (S6.18) is

$$\begin{aligned}
 & n \operatorname{acov}_g \{ n^{-1} \operatorname{AIC}_{\text{ML}}, \hat{\alpha}_{(\Delta t)\text{ML}1}^{(A)} z_{\hat{\alpha}}^2 \} \\
 &= n \operatorname{acov}_g \{ \bar{l}_{\text{ML}}^{(1)}, n \mathbf{E}_g \{ \bar{l}_{\text{ML}}^{(1)} \mathbf{m}_v^{(1)} \} \mathbf{v}^{(1)} \} z_{\hat{\alpha}}^2 \\
 &= n \operatorname{acov}_g \left[ \bar{l}_{\text{ML}}^{(1)}, (\hat{\alpha}_{\text{ML}2}^{(A)})^{-3/2} \{ 4\widehat{\kappa}_{g3}(l_{0j}) + 2\hat{\kappa}_3^2 \} \right] z_{\hat{\alpha}}^2
 \end{aligned}$$

with

$$\begin{aligned}
 4\widehat{\kappa}_{g3}(l_{0j}) &\equiv 4n^{-1} \sum_{j=1}^n \left\{ \hat{l}_{\text{ML}j} - \left( -\frac{1}{2} - \frac{1}{2} \log(2\pi\sigma^2) \right) \right\}^3 \\
 \hat{l}_{\text{ML}j} &= -\{(x_j - \bar{x})^2 / (2\sigma^2)\} - (1/2) \log(2\pi\sigma^2) \\
 &= -\frac{1}{2\sigma^6} n^{-1} \sum_{j=1}^n \{(x_j - \bar{x})^2 - \sigma^2\}^3 \\
 &= -\frac{1}{2\sigma^6} \left[ n^{-1} \sum_{j=1}^n \{(x_j - \mu_0)^2 - \sigma^2\}^3 \right. \\
 &\quad \left. - 6n^{-1} \sum_{j=1}^n \{(x_j - \mu_0)^2 - \sigma^2\}^2 (x_j - \mu_0)(\bar{x} - \mu_0) \right] + O_p(n^{-1})
 \end{aligned}$$

$$= -\frac{1}{2} \left[ \bar{z}^6 - 3\bar{z}^4 + 3\bar{z}^2 - 1 - 6\{\bar{z}^5 - 2\bar{z}^3 + (\bar{z})_{O_p(n^{-1/2})}\}(\bar{z})_{O_p(n^{-1/2})} \right] + O_p(n^{-1})$$

$$\left( \bar{z}^k \equiv n^{-1} \sum_{j=1}^n \left( \frac{x_j - \mu_0}{\sigma} \right)^k \right)$$

and

$$2\hat{\kappa}_3^2 = 2 \left\{ n^{-1} \sum_{j=1}^n \left( \frac{x_j - \bar{x}}{\sigma} \right)^3 \right\}^2 = 2 \left\{ n^{-1} \sum_{j=1}^n \left( \frac{x_j - \mu_0}{\sigma} \right)^3 \right\}^2$$

$$- 12n^{-1} \sum_{j=1}^n \left( \frac{x_j - \mu_0}{\sigma} \right)^3 n^{-1} \sum_{k=1}^n \left( \frac{x_k - \mu_0}{\sigma} \right)^2 \frac{\bar{x} - \mu_0}{\sigma} + O_p(n^{-1})$$

$$= 2\bar{z}^3{}^2 - 12\bar{z}^3 \bar{z}^2 (\bar{z})_{O_p(n^{-1/2})} + O_p(n^{-1}),$$

that is

$$n \text{acov}_g \left[ \bar{l}_{ML}^{(1)}, (\hat{\alpha}_{ML2}^{(A)})^{-3/2} \{4\widehat{\kappa}_{g3}(\bar{l}_{0j}) + 2\hat{\kappa}_3^2\} \right] z_{\bar{\alpha}}^2$$

$$= n \text{acov}_g \left\{ -2\bar{l}_0, 3(\alpha_{ML2}^{(A)})^{-1} (\hat{\alpha}_{ML2}^{(A)})^{-1/2} \{4\kappa_{g3}(\bar{l}_{0j}) + 2\kappa_3^2\} z_{\bar{\alpha}}^2 \right.$$

$$\left. + n \text{acov}_g \left\{ -2\bar{l}_0, 4\widehat{\kappa}_{g3}(\bar{l}_{0j}) + 2\hat{\kappa}_3^2 \right\} (\kappa_4 + 2)^{-3/2} z_{\bar{\alpha}}^2 \right.$$

$$= 3(\alpha_{ML2}^{(A)})^{-1} \left\{ -\frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\}$$

$$\times \left\{ -\frac{1}{2} (\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8) + 2\kappa_3^2 \right\} z_{\bar{\alpha}}^2$$

$$+ n \text{acov}_g \left[ \bar{z}^2, -\frac{1}{2} \left\{ \bar{z}^6 - 3\bar{z}^4 + 3\bar{z}^2 - 1 - 6(\bar{z}^5 - 2\bar{z}^3 + \bar{z})\bar{z} \right\} \right.$$

$$\left. + 2\bar{z}^3{}^2 - 12\bar{z}^3 \bar{z}^2 \bar{z} \right] (\kappa_4 + 2)^{-3/2} z_{\bar{\alpha}}^2,$$

$$\begin{aligned}
&= \frac{3}{4}(\kappa_4 + 2)^{-5/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 z_{\bar{\alpha}}^2 \\
&\quad + \left[ -\frac{1}{2}\{(\sigma_8 - \sigma_6) - 3(\sigma_6 - \sigma_4) + 3(\sigma_4 - 1)\} + 3(\sigma_5 - 2\sigma_3)\sigma_3 \right. \\
&\quad \quad \left. + 4\sigma_3(\sigma_5 - \sigma_3) - 12\sigma_3^2 \right] (\kappa_4 + 2)^{-3/2} z_{\bar{\alpha}}^2 \\
&= \left[ \frac{3}{4}(\kappa_4 + 2)^{-5/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \right. \\
&\quad \left. + (\kappa_4 + 2)^{-3/2} \left\{ -\frac{1}{2}(\sigma_8 - 4\sigma_6 + 6\sigma_4 - 3) + 7\sigma_5\sigma_3 - 22\sigma_3^2 \right\} \right] z_{\bar{\alpha}}^2,
\end{aligned}$$

where

$$\begin{aligned}
\sigma_k &\equiv E_g(z^k), \quad \sigma_2 = \kappa_2 = 1, \quad \sigma_3 = \kappa_3, \quad \sigma_4 = \kappa_4 + 3, \\
\text{cov}_g(\bar{z}^2, \bar{z}^2) &= n^{-3} E_g \left\{ \sum_{j=1}^n (z_j^2 - 1) \sum_{k=1}^n z_k \sum_{l=1}^n z_l \right\} \\
&= n^{-3} (n\sigma_4 + n^2 - n - n^2) = n^{-2} (\sigma_4 - 1) = O(n^{-2}).
\end{aligned}$$

(iii) the third term on the right-hand side of (S6.18)

$$\begin{aligned}
&\text{Using } \hat{\alpha}_{\text{ML3}}^{(A)} \equiv \hat{\kappa}_6 + 12\hat{\kappa}_4 + 4\hat{\kappa}_3^2 + 8 = \hat{\sigma}_6 - 3\hat{\sigma}_4 - 6\hat{\sigma}_3^2 + 2 \quad \text{with} \\
&\hat{\sigma}_k \equiv n^{-1} \sum_{j=1}^n \{(x_j - \bar{x}) / \sigma\}^k, \text{ we have} \\
&n \text{acov}_g \left\{ n^{-1} \text{AIC}_{\text{ML}}, \frac{1}{6} \hat{\alpha}_{\text{ML3}}^{(A)} (\hat{\alpha}_{\text{ML2}}^{(A)})^{-3/2} (z_{\bar{\alpha}}^2 - 1) \right\} \\
&= \left[ n \text{acov}_g \left\{ \bar{z}^2, \bar{z}^6 - 3\bar{z}^4 - 6\bar{z}^3{}^2 - 6(\bar{z}^5 - 2\bar{z}^3 - 6\bar{z}^3 \bar{z}^2) \bar{z} \right\} (\alpha_{\text{ML2}}^{(A)})^{-3/2} \right. \\
&\quad \left. + n \text{acov}_g \left\{ \bar{I}_{\text{ML}}^{(1)}, 3(\alpha_{\text{ML2}}^{(A)})^{-1} (\hat{\alpha}_{\text{ML2}}^{(A)})^{-1/2} \right\} (\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8) \right] \frac{z_{\bar{\alpha}}^2 - 1}{6}
\end{aligned}$$



$$\begin{aligned}
&= \left[ (\kappa_4 + 2)^{-3/2} \{ (\sigma_8 - \sigma_6) - 3(\sigma_6 - \sigma_4) - 12\sigma_3(\sigma_5 - \sigma_3) \right. \\
&\quad \left. - 6(\sigma_5 - 2\sigma_3 - 6\sigma_3)\sigma_3 \} \right. \\
&\quad \left. + 3(\kappa_4 + 2)^{-1} \left\{ -\frac{1}{2}(\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right\} \right. \\
&\quad \left. \times (\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8) \right] \frac{z_{\hat{\alpha}}^2 - 1}{6} \\
&= \left[ (\kappa_4 + 2)^{-3/2} (\sigma_8 - 4\sigma_6 + 3\sigma_4 - 18\sigma_3\sigma_5 + 60\sigma_3^2) \right. \\
&\quad \left. - \frac{3}{2}(\kappa_4 + 2)^{-5/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)(\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8) \right] \frac{z_{\hat{\alpha}}^2 - 1}{6}.
\end{aligned}$$

Consequently,

$$n \text{acov}_g \left\{ n^{-1} \text{AIC}_{\text{ML}}, \hat{\alpha}_{(t)\text{ML}1}^{(A)} + \frac{\hat{\alpha}_{(t)\text{ML}3}^{(A)}}{6} (z_{\hat{\alpha}}^2 - 1) \right\} \quad (\text{S6.19})$$

is the sum of the values in (i), (ii) and (iii) given above

(under normality, (S6.19) is 0 since  $\hat{\alpha}_{(t)\text{ML}1}^{(A)} = \alpha_{(t)\text{ML}1}^{(A)} = 2^{-1/2} - 2^{1/2}$  and  $\hat{\alpha}_{(t)\text{ML}3}^{(A)} = \alpha_{(t)\text{ML}3}^{(A)} = -2^{5/2}$  are fixed values).

**S6.2.4 Asymptotic cumulants of  $n^{-1} \text{AIC}_{\text{ML}}$  after studentization for estimation of  $-2\text{E}_g(\hat{l}_{\text{ML}}^*)$**

$$t_{\text{ML}}^{(A)*} = \frac{n^{1/2} \{ n^{-1} \text{AIC}_{\text{ML}} + 2\text{E}_g(\hat{l}_{\text{ML}}^*) \}}{(\hat{v}_{\text{ML}}^{(A)})^{1/2}}.$$

$$\begin{aligned}
\kappa_{g1}(t_{ML}^{(A)*}) &= n^{-1/2} \{ \alpha_{(t)ML1}^{(A)} + \lambda^{-1} \gamma (\alpha_{ML2}^{(A)})^{-1/2} \} + O(n^{-3/2}) \\
&= n^{-1/2} \left\{ (\kappa_4 + 2)^{-1/2} - \frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \right. \\
&\quad \left. - (\kappa_4 + 2)^{-1/2} \right\} + O(n^{-3/2}) \\
&= -n^{-1/2} \frac{1}{2} (\kappa_4 + 2)^{-3/2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) + O(n^{-3/2}) \\
&= n^{-1/2} \alpha_{(t)ML1}^{(A)*} + O(n^{-3/2}) \quad (\alpha_{(t)ML1}^{(A)*} = \alpha_{(\Delta t)ML1}^{(A)}) \\
(\text{under normality } \alpha_{(t)ML1}^{(A)*} &= \alpha_{(\Delta t)ML1}^{(A)} = -2^{1/2}),
\end{aligned} \tag{S6.20}$$

$$\begin{aligned}
\kappa_{g2}(t_{ML}^{(A)*}) &= 1 + n^{-1} \{ \alpha_{(t)ML\Delta 2}^{(A)} + 2\lambda^{-1} \gamma (\alpha_{ML2}^{(A)})^{-1/2} n E_g (\bar{I}_{ML}^{(1)'} \mathbf{m}_v^{(1)'}) \mathbf{v}^{(1)} \} + O(n^{-2}) \\
&= 1 + n^{-1} \{ \alpha_{(t)ML\Delta 2}^{(A)} + (\kappa_4 + 2)^{-2} (\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \} + O(n^{-2}) \\
&= 1 + n^{-1} \alpha_{(t)ML\Delta 2}^{(A)*} + O(n^{-2}), \\
(\text{under normality } \alpha_{(t)ML\Delta 2}^{(A)*} &= \alpha_{(\Delta t)ML\Delta 2}^{(A)} + 2),
\end{aligned}$$

$$\begin{aligned}
\kappa_{g3}(t_{ML}^{(A)*}) &= n^{-1/2} \alpha_{(t)ML3}^{(A)} + O(n^{-3/2}) \quad (\alpha_{(t)ML3}^{(A)*} = \alpha_{(t)ML3}^{(A)}), \\
\kappa_{g4}(t_{ML}^{(A)*}) &= n^{-1} \alpha_{(t)ML4}^{(A)} + O(n^{-2}) \quad (\alpha_{(t)ML4}^{(A)*} = \alpha_{(t)ML4}^{(A)}).
\end{aligned}$$

### S6.2.5 A result for estimation of $-2E_g(\hat{t}_{ML}^*)$

$$\begin{aligned}
n \text{acov}_g \left\{ n^{-1} \text{AIC}_{ML}, \hat{\alpha}_{(t)ML1}^{(A)*} + \frac{\hat{\alpha}_{(t)ML3}^{(A)}}{6} (z_{\hat{\alpha}}^2 - 1) \right\} \\
= n \text{acov}_g \left\{ n^{-1} \text{AIC}_{ML}, \hat{\alpha}_{(\Delta t)ML1}^{(A)} z_{\hat{\alpha}}^2 + \frac{1}{6} \hat{\alpha}_{ML3}^{(A)} (\hat{\alpha}_{ML2}^{(A)})^{-3/2} (z_{\hat{\alpha}}^2 - 1) \right\}. \tag{S6.21}
\end{aligned}$$

This reduces to the case of Subsection S6.2.3, where the first term of (i) for (S6.18) is zero. So, (S6.21) is equal to the sum of the second and third terms given in (ii) and (iii) for (S6.18), respectively.

### S6.2.6 Higher-order bias correction of $n^{-1} \text{AIC}_{ML}$

Since the correction term of  $n^{-1}\text{AIC}_{\text{ML}}$  i.e.,  $2n^{-1}$  is exact even under non-normality, the higher-order correction is unnecessary.

**S6.3**  $n^{-1}\text{TIC}_{\text{ML}}^{(j)}(j=1, 2)$

Since  $\lambda = \hat{\lambda} = -1/\sigma^2$  and  $\gamma = \hat{\gamma} = 1/\sigma^2$ ,

$$\begin{aligned} n^{-1}\text{TIC}_{\text{ML}}^{(j)} &= -2\hat{l}_{\text{ML}} + n^{-1}2(-\hat{\lambda}^{-1}\hat{\gamma}) = -2\hat{l}_{\text{ML}} + n^{-1}2 \\ &= n^{-1}\text{AIC}_{\text{ML}}(j=1, 2). \end{aligned} \tag{S6.22}$$

That is, the results in Subsection S6.2 for  $n^{-1}\text{AIC}_{\text{ML}}$  also hold for  $n^{-1}\text{TIC}_{\text{ML}}^{(j)}(j=1, 2)$ .

### S7. Example 3: The Bernoulli distribution with the WSE of the logit under correct model specification

#### S7.1 Preliminary results

$$\Pr(x^* = x | \theta) = \pi^x (1 - \pi)^{1-x} \quad (x = 0, 1), \quad \pi = \frac{1}{1 + \exp(-\theta)},$$

$$l = \log \left\{ \prod_{j=1}^n \pi^{x_j} (1 - \pi)^{1-x_j} \right\} = \sum_{j=1}^n l_j, \quad \bar{l} = n^{-1} l,$$

$$l_j = x_j \log \frac{\pi}{1 - \pi} + \log(1 - \pi) = x_j \theta + \log(1 - \pi),$$

$$l_{0j} = x_j \theta_0 + \log(1 - \pi_0), \quad \pi_0 = \frac{1}{1 + \exp(-\theta_0)}, \quad \bar{l}_0 = n^{-1} \sum_{j=1}^n l_{0j},$$

$$l_W^{(e)} = \log \left[ \left\{ \prod_{j=1}^n \pi^{x_j} (1 - \pi)^{1-x_j} \right\} \{ \pi(1 - \pi) \}^{a/2} \right], \quad (S7.1)$$

$$\bar{l}_W^{(e)} = n^{-1} l_W^{(e)}$$

$$= (\bar{x} + n^{-1} 0.5a) \log \pi + (1 - \bar{x} + n^{-1} 0.5a) \log(1 - \pi),$$

$$\bar{l}_{0W}^{(e)} = (\bar{x} + n^{-1} 0.5a) \log \pi_0 + (1 - \bar{x} + n^{-1} 0.5a) \log(1 - \pi_0),$$

where the superscript (e) indicates that the quantity is for estimation of  $\theta_0$  by the weighted score method and  $a$  is the sum of the pseudocounts for two categories,

$$\bar{l}_0^* = E_f(l_{0j}) = E_f(\bar{l}_0) = \pi_0 \theta_0 + \log(1 - \pi_0),$$

$$\frac{\partial \bar{l}}{\partial \theta_0} = \left( \frac{\bar{x}}{\pi_0} - \frac{1 - \bar{x}}{1 - \pi_0} \right) \pi_0 (1 - \pi_0) = \bar{x} - \pi_0, \quad \hat{\theta}_{ML} = \log \frac{\bar{x}}{1 - \bar{x}},$$

$$\hat{\pi}_{ML} = \bar{x}, \quad \Lambda = \lambda = \frac{\partial^2 \bar{l}}{\partial \theta_0^2} = \frac{\partial^2 l_j}{\partial \theta_0^2} = -\pi_0 (1 - \pi_0) = -\bar{l}_0,$$

$$\mathbf{\Gamma} = \boldsymbol{\gamma} = nE_f \left\{ \left( \frac{\partial \bar{l}}{\partial \theta_0} \right)^2 \right\} = n \text{var}_f(\bar{x}) = \pi_0(1 - \pi_0) = \bar{i}_0,$$

$$\text{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) = \lambda^{-1}\boldsymbol{\gamma} = -1,$$

$$l_{0j} - \bar{l}_0^* = (x_j - \pi_0) \log \frac{\pi_0}{1 - \pi_0} = (x_j - \pi_0)\theta_0,$$

$$\begin{aligned} \frac{\partial \bar{l}_w^{(e)}}{\partial \theta_0} &= \left( \frac{\bar{x} + n^{-1}0.5a}{\pi_0} - \frac{1 - \bar{x} + n^{-1}0.5a}{1 - \pi_0} \right) \pi_0(1 - \pi_0) \\ &= \bar{x} - \pi_0 + n^{-1}0.5a(1 - 2\pi_0) \quad (= \bar{x} + n^{-1}0.5a - (1 + n^{-1}a)\pi_0) \\ &= \bar{x} - \pi_0 + n^{-1}q_0^*, \end{aligned}$$

$$\hat{\pi}_w = \frac{\bar{x} + n^{-1}0.5a}{1 + n^{-1}a}, \quad \hat{\theta}_w = \log \frac{\hat{\pi}_w}{1 - \hat{\pi}_w} = \log \frac{\bar{x} + n^{-1}0.5a}{1 - \bar{x} + n^{-1}0.5a},$$

$$\mathbf{q}_0^* = q_0^* = 0.5a(1 - 2\pi_0), \quad \frac{\partial q^*}{\partial \theta_0} = -a\bar{i}_0,$$

$$\mathbf{J}_0^{(3)} = j_0^{(3)} = \frac{\partial^3 \bar{l}}{\partial \theta_0^3} = -(1 - 2\pi_0)\bar{i}_0 = -n^2 \kappa_{f_3}(\bar{x}) = -n^2 \kappa_{f_3} \left( \frac{\partial \bar{l}}{\partial \theta_0} \right),$$

$$\mathbf{J}_0^{(4)} = j_0^{(4)} = \frac{\partial^4 \bar{l}}{\partial \theta_0^4} = -(1 - 6\pi_0 + 6\pi_0^2)\bar{i}_0 = -n^3 \kappa_{f_4}(\bar{x}) = -n^3 \kappa_{f_4} \left( \frac{\partial \bar{l}}{\partial \theta_0} \right),$$

$$\hat{l}_w^{(e)} = (\bar{x} + n^{-1}0.5a) \log \hat{\pi}_w + (1 - \bar{x} + n^{-1}0.5a) \log(1 - \hat{\pi}_w),$$

$$\hat{\pi}_w = \frac{1}{1 + \exp(-\hat{\theta}_w)},$$

$$\hat{l}_w = \bar{x} \log \hat{\pi}_w + (1 - \bar{x}) \log(1 - \hat{\pi}_w),$$

$$\begin{aligned} \mathbf{I}_0^{(w)} &= I_0^{(w)} = -\lambda^{-1} \frac{\partial q^*}{\partial \theta_0} \left( -\lambda^{-1} \frac{\partial \bar{l}}{\partial \theta_0} \right) - j_0^{(3)} \lambda^{-3} q_0^* \frac{\partial \bar{l}}{\partial \theta_0} \\ &= \lambda^{-2} (-a \bar{i}_0) (\bar{x} - \pi_0) + (1 - 2\pi_0) \bar{i}_0 (-\bar{i}_0)^{-3} 0.5a(1 - 2\pi_0) (\bar{x} - \pi_0) \\ &= -a \{ \bar{i}_0^{-1} + 0.5 \bar{i}_0^{-2} (1 - 2\pi_0)^2 \} (\bar{x} - \pi_0) \quad (\text{see (A1.1)}), \end{aligned}$$

$$\begin{aligned} -2E_f(\hat{l}_w - \hat{l}_w^*) &= 2n^{-1} \text{tr}(\Lambda^{-1} \Gamma) + n^{-2} (c_1 + c_2 + c_3) + O(n^{-3}) \\ &= n^{-1} b_1 + n^{-2} b_2 + O(n^{-3}) \end{aligned}$$

$$b_1 = 2\text{tr}(\Lambda^{-1} \Gamma) = -2q = -2,$$

$b_2 = c_1$  ( $c_2 = c_3 = 0$  under canonical parametrization; see Corollaries 1 and 2 with (A1.5))

$$\begin{aligned} &= -2(1 - 2\pi_0)^2 \bar{i}_0^{-1} + (1 - 6\pi_0 + 6\pi_0^2) \bar{i}_0^{-1} \\ &\quad - 2 \frac{\partial q^*}{\partial \theta_0} \lambda^{-2} \gamma + 2j_0^{(3)} \lambda^{-3} \gamma q_0^* + O(n^{-1}) \\ &= -2(1 - 2\pi_0)^2 \bar{i}_0^{-1} + (1 - 6\pi_0 + 6\pi_0^2) \bar{i}_0^{-1} \\ &\quad + 2a \bar{i}_0 \bar{i}_0^{-1} + 2(1 - 2\pi_0) \bar{i}_0 \bar{i}_0^{-2} 0.5a(1 - 2\pi_0) \\ &= -2(1 - 2\pi_0)^2 \bar{i}_0^{-1} + (1 - 6\pi_0 + 6\pi_0^2) \bar{i}_0^{-1} \\ &\quad + a \{ (1 - 2\pi_0)^2 \bar{i}_0^{-1} + 2 \} \\ &= -(1 - 2\pi_0)^2 \bar{i}_0^{-1} - 2 + a \{ (1 - 2\pi_0)^2 \bar{i}_0^{-1} + 2 \} \\ &= (a - 1) \{ (1 - 2\pi_0)^2 \bar{i}_0^{-1} + 2 \}, \end{aligned}$$

$$\bar{l}_{\text{ML}}^{(1)} = -2(\bar{l}_0 - \bar{l}_0^*) = -2(\bar{x} - \pi_0) \log \frac{\pi_0}{1 - \pi_0} = -2(\bar{x} - \pi_0) \theta_0,$$

$$\bar{l}_{\text{ML}}^{(2)} = \frac{\partial \bar{l}}{\partial \theta_0} \lambda^{-1} \frac{\partial \bar{l}}{\partial \theta_0} = -(\bar{x} - \pi_0)^2 \bar{i}_0^{-1}, \quad nE_f(\bar{l}_{\text{ML}}^{(2)}) = \lambda^{-1} \gamma = -1,$$

$$\begin{aligned}
\bar{l}_{\text{ML}}^{(3)} &= \frac{1}{3} j_0^{(3)} \lambda^{-3} \left( \frac{\partial \bar{l}}{\partial \theta_0} \right)^3 = \frac{1}{3} (1 - 2\pi_0) \bar{i}_0^{-3} (\bar{x} - \pi_0)^3 \\
&= \frac{1}{3} (1 - 2\pi_0) \bar{i}_0^{-2} (\bar{x} - \pi_0)^3, \\
n^2 E_f(\bar{l}_{\text{ML}}^{(3)}) &= \frac{1}{3} (1 - 2\pi_0) \bar{i}_0^{-2} n^2 E_f\{(\bar{x} - \pi_0)^3\} = \frac{1}{3} (1 - 2\pi_0)^2 \bar{i}_0^{-1} \\
&= \frac{1}{3} \{\text{sk}(x^*)\}^2,
\end{aligned}$$

where  $\text{sk}(\cdot)$  is the skewness of a variable.

$$\begin{aligned}
\bar{l}_{\text{ML}}^{(4)} &= \frac{1}{4} (j_0^{(3)})^2 \lambda^{-5} \left( \frac{\partial \bar{l}}{\partial \theta_0} \right)^4 - \frac{1}{12} j_0^{(4)} \lambda^{-4} \left( \frac{\partial \bar{l}}{\partial \theta_0} \right)^4 \\
n^2 E_f(\bar{l}_{\text{ML}}^{(4)}) &= -\frac{3}{4} \{\text{sk}(x^*)\}^2 + \frac{1}{4} \text{kt}(x^*) + O(n^{-1}) \\
&= -\frac{1}{4} \{3(1 - 2\pi_0)^2 - (1 - 6\pi_0 + 6\pi_0^2)\} \bar{i}_0^{-1} + O(n^{-1}) \\
&= -\frac{1}{4} (6\pi_0^2 - 6\pi_0 + 2) \bar{i}_0^{-1} + O(n^{-1}) = \frac{3}{2} - \frac{1}{2} \bar{i}_0^{-1} + O(n^{-1}),
\end{aligned}$$

where  $\text{kt}(\cdot)$  is the excess kurtosis of a variable,

$$\begin{aligned}
n^2 E_f(\bar{l}_{\text{ML}}^{(3)} + \bar{l}_{\text{ML}}^{(4)}) &= -\frac{5}{12} \{\text{sk}(x^*)\}^2 + \frac{1}{4} \text{kt}(x^*) + O(n^{-1}) \\
&= \left\{ -\frac{5}{12} (1 - 2\pi_0)^2 + \frac{1}{4} (1 - 6\pi_0 + 6\pi_0^2) \right\} \bar{i}_0^{-1} + O(n^{-1}) \\
&= \frac{1}{12} (-2 + 2\pi_0 - 2\pi_0^2) \bar{i}_0^{-1} + O(n^{-1}) = \frac{1}{6} (1 - \bar{i}_0^{-1}) + O(n^{-1}), \\
-n^{-2} \mathbf{q}_0^* \Lambda^{-1} \mathbf{q}_0^* &= -n^{-2} (q_0^*)^2 \lambda^{-1} = n^{-2} \frac{a^2}{4} (1 - 2\pi_0)^2 \bar{i}_0^{-1}.
\end{aligned}$$

$$\text{For } t_W^{(A)} = \frac{n^{1/2}(n^{-1}\text{AIC}_W + 2\bar{l}_0^*)}{(\hat{v}_W^{(A)})^{1/2}},$$

we have

$$\hat{v}_W^{(A)} = 4(n-1)^{-1} \sum_{j=1}^n (\hat{l}_{Wj} - \hat{\bar{l}}_W)^2 \equiv \hat{\alpha}_{W2}^{(A)} = O_p(1),$$

$$\hat{l}_{Wj} = l_j |_{\theta=\hat{\theta}_W}, \quad \hat{\bar{l}}_W = n^{-1} \sum_{j=1}^n \hat{l}_{Wj},$$

$$\begin{aligned} v_0^{(A)} &= 4(n-1)^{-1} \sum_{j=1}^n (l_{0j} - \bar{l}_0)^2 = 4\theta_0^2(n-1)^{-1} \sum_{j=1}^n (x_j - \bar{x})^2 \\ &\equiv 4\theta_0^2 u_x^2, \end{aligned}$$

$$E_f(v_0^{(A)}) = 4\theta_0^2 \bar{i}_0 = \alpha_{ML2}^{(A)} \quad (E_f(u_x^2) = \text{var}_f(x^*) = \pi_0(1 - \pi_0) = \bar{i}_0),$$

$$\text{var}_f(l_{0j}) = \theta_0^2 \text{var}_f(x_j) = \theta_0^2 \bar{i}_0,$$

$$\kappa_{f3}(l_{0j}) = \theta_0^3 \kappa_{f3}(x_j) = \theta_0^3 (1 - 2\pi_0) \bar{i}_0,$$

$$\kappa_{f4}(l_{0j}) = \theta_0^4 \kappa_{f4}(x_j) = \theta_0^4 (1 - 6\pi_0 + 6\pi_0^2) \bar{i}_0,$$

$$\begin{aligned} E_f\{(l_{0j} - \bar{l}_0^*)^4\} &= \theta_0^4 [\kappa_{f4}(x_j) + 3\{\text{var}_f(x_j)\}^2] \\ &= \theta_0^4 \{1 - 6\pi_0 + 6\pi_0^2 + 3\pi_0(1 - \pi_0)\} \bar{i}_0 \\ &= \theta_0^4 (1 - 3\pi_0 + 3\pi_0^2) \bar{i}_0, \end{aligned}$$

$$E_f \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) = \theta_0 \text{var}_f(x_j) = \theta_0 \bar{i}_0, \tag{S7.2}$$



$$E_f \left\{ \frac{\partial l_j}{\partial \theta_0} (l_{0j} - \bar{l}_0^*)^2 \right\} = \theta_0^2 \kappa_{f3}(x_j) = \theta_0^2 (1 - 2\pi_0) \bar{i}_0,$$

$$E_f \left\{ (l_{0j} - \bar{l}_0^*) \left( \frac{\partial l_j}{\partial \theta_0} \right)^2 \right\} = \theta_0 \kappa_{f3}(x_j) = \theta_0 (1 - 2\pi_0) \bar{i}_0,$$

$$E_f \left( \frac{\partial v_0^{(A)}}{\partial \theta_0} \right) = 8E_f \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) = 8\theta_0 \text{var}_f(x_j) = 8\theta_0 \bar{i}_0,$$

$$E_f \left( \frac{\partial^2 v_0^{(A)}}{\partial \theta_0^2} \right) = 8\gamma = 8\bar{i}_0$$

( $8\gamma$  is due to canonical parametrization),

$$E_f \left\{ \left( \frac{\partial v_0^{(A)}}{\partial \theta_0} \right)^2 \right\} = 64 \left\{ E_f \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\}^2 + O(n^{-1}) = 64\theta_0^2 \bar{i}_0^2 + O(n^{-1}),$$

$$\begin{aligned} n \text{cov}_f(m_v, \bar{l}_0) &= n \text{cov}_f(v_0^{(A)}, \bar{l}_0) = 4\kappa_{f3}(l_{0j}) \\ &= 4\theta_0^3 \kappa_{f3}(x_j) = 4\theta_0^3 (1 - 2\pi_0) \bar{i}_0, \end{aligned}$$

$$n \text{cov}_f \left( \frac{\partial \bar{l}_0}{\partial \theta_0}, m_v \right) = 4E_f \left\{ (l_{0j} - \bar{l}_0^*)^2 \frac{\partial l_j}{\partial \theta_0} \right\} = 4\theta_0^2 (1 - 2\pi_0) \bar{i}_0,$$

$$\begin{aligned} n \text{avar}_f(m_v) &= 16 \left[ E_f \{ (l_{0j} - \bar{l}_0^*)^4 \} - [E_f \{ (l_{0j} - \bar{l}_0^*)^2 \}]^2 \right] \\ &= 16 \{ \theta_0^4 (1 - 3\pi_0 + 3\pi_0^2) \bar{i}_0 - \theta_0^4 \bar{i}_0^2 \} \\ &= 16\theta_0^4 \{ 1 - 3\pi_0 + 3\pi_0^2 - \pi_0(1 - \pi_0) \} \bar{i}_0 = 16\theta_0^4 (1 - 2\pi_0)^2 \bar{i}_0, \end{aligned}$$

$$\begin{aligned} n \text{cov}_f \left( \frac{\partial v_0^{(A)}}{\partial \theta_0}, \frac{\partial \bar{l}}{\partial \theta_0} \right) &= 8E_f \left\{ (l_{0j} - \bar{l}_0^*) \left( \frac{\partial l_j}{\partial \theta_0} \right)^2 \right\} = 8\theta_0 \kappa_{f3}(x_j) \\ &= 8\theta_0 (1 - 2\pi_0) \bar{i}_0 \end{aligned}$$

$$n \text{cov}_f \left( \frac{\partial v_0^{(A)}}{\partial \theta_0}, \bar{l}_0 \right) = 8E_f \left\{ (l_{0j} - \bar{l}_0^*)^2 \frac{\partial l_j}{\partial \theta_0} \right\}$$

$$= 8\theta_0^2 \kappa_{f3}(x_j) = 8\theta_0^2 (1 - 2\pi_0) \bar{i}_0,$$

$$n^2 E_f \{ (\bar{l}_{ML}^{(1)})^2 \mathbf{m}_v^{(1)} \}' = \left[ 16\kappa_{f4}(l_{0j}), 4E_f \left\{ (l_{0j} - \bar{l}_0^*)^2 \frac{\partial l_j}{\partial \theta_0} \right\} \right]$$

$$= \{ 16\theta_0^4 (1 - 6\pi_0 + 6\pi_0^2) \bar{i}_0, 4\theta_0^2 (1 - 2\pi_0) \bar{i}_0 \}$$

(see the result of the second term for (S1.7)),

$$\mathbf{m}_v^{(1)} = \left( m_v, \frac{\partial \bar{l}}{\partial \theta_0} \right)' = \{ 4\theta_0^2 (u_x^2 - \bar{i}_0), \bar{x} - \pi_0 \}' ,$$

$$\mathbf{m}_v^{(2)} = \left[ m_v^2, m_v \frac{\partial \bar{l}}{\partial \theta_0}, 0, \left( \frac{\partial \bar{l}}{\partial \theta_0} \right)^2, \left\{ \frac{\partial v^{(A)}}{\partial \theta_0} - E_f \left( \frac{\partial v^{(A)}}{\partial \theta_0} \right) \right\} \otimes \frac{\partial \bar{l}}{\partial \theta_0} \right]'$$

$$= \{ 16\theta_0^4 (u_x^2 - \bar{i}_0)^2, 4\theta_0^2 (u_x^2 - \bar{i}_0) (\bar{x} - \pi_0), 0, (\bar{x} - \pi_0)^2, 8\theta_0 (u_x^2 - \bar{i}_0) (\bar{x} - \pi_0) \}' ,$$

$$\mathbf{v}^{(1)} = \frac{1}{2} (\alpha_{ML2}^{(A)})^{-3/2} \left\{ -1, E_f \left( \frac{\partial v^{(A)}}{\partial \theta_0} \right) \lambda^{-1} \right\}'$$

$$= \frac{1}{2} (4\theta_0^2 \bar{i}_0)^{-3/2} \{ -1, 8\theta_0 \bar{i}_0 (-\bar{i}_0)^{-1} \}' = -\frac{1}{16} (\theta_0^2 \bar{i}_0)^{-3/2} (1, 8\theta_0)' ,$$

$$\begin{aligned}
 \mathbf{v}^{(2)} &= \left[ \begin{array}{l} \frac{3}{8}(\alpha_{\text{ML}2}^{(A)})^{-5/2}, -\frac{3}{4}(\alpha_{\text{ML}2}^{(A)})^{-5/2} \text{E}_f \left( \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right) \lambda^{-1}, \\ -\frac{1}{2}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \text{E}_f \left( \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right) (\lambda^{-1})^2, \left[ \frac{1}{4}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \text{E}_f \left( \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right) \lambda^{-1} j_0^{(3)} \right. \\ -\frac{1}{4}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \text{E}_g \left( \frac{\partial^2 \mathbf{v}^{(A)}}{\partial \theta_0^2} \right) + \frac{3}{8}(\alpha_{\text{ML}2}^{(A)})^{-5/2} \text{E}_g \left\{ \left( \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right)^2 \right\} \\ \left. \left. + O(n^{-1}) \right] (\lambda^{-1})^2, \frac{1}{2}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \lambda^{-1} \right]^{(A)} \\
 &= \left[ \begin{array}{l} \frac{3}{8}(4\theta_0^2 \bar{i}_0)^{-5/2}, -\frac{3}{4}(4\theta_0^2 \bar{i}_0)^{-5/2} 8\theta_0 \bar{i}_0 (-\bar{i}_0)^{-1}, -\frac{1}{2}(4\theta_0^2 \bar{i}_0)^{-3/2} 8\theta_0 \bar{i}_0 \bar{i}_0^{-2}, \\ \left[ \frac{1}{4}(4\theta_0^2 \bar{i}_0)^{-3/2} 8\theta_0 \bar{i}_0 (-\bar{i}_0)^{-1} \{-(1-2\pi_0)\bar{i}_0\} - \frac{1}{4}(4\theta_0^2 \bar{i}_0)^{-3/2} 8\bar{i}_0 \right. \\ \left. + \frac{3}{8}(4\theta_0^2 \bar{i}_0)^{-5/2} 64\theta_0^2 \bar{i}_0^2 \right] \bar{i}_0^{-2}, \frac{1}{2}(4\theta_0^2 \bar{i}_0)^{-3/2} (-\bar{i}_0)^{-1} \right]^{(A)} \\
 &= \left[ \begin{array}{l} \frac{3}{256} \theta_0^{-5} \bar{i}_0^{-5/2}, \frac{3}{16} \theta_0^{-4} \bar{i}_0^{-5/2}, -\frac{1}{2} \theta_0^{-2} \bar{i}_0^{-5/2}, \\ \left\{ \frac{1}{4} \theta_0^{-2} \bar{i}_0^{-1/2} (1-2\pi_0) - \frac{1}{4} \theta_0^{-3} \bar{i}_0^{-1/2} + \frac{3}{4} \theta_0^{-3} \bar{i}_0^{-1/2} \right\} \bar{i}_0^{-2}, -\frac{1}{16} \theta_0^{-3} \bar{i}_0^{-5/2} \right] \\
 &= \left[ \frac{3}{256} \theta_0^{-5}, \frac{3}{16} \theta_0^{-4}, -\frac{1}{2} \theta_0^{-2}, \frac{1}{4} \theta_0^{-2} (1-2\pi_0) + \frac{1}{2} \theta_0^{-3}, -\frac{1}{16} \theta_0^{-3} \right] \bar{i}_0^{-5/2} \\
 &\equiv \mathbf{v}^{(2)*} \bar{i}_0^{-5/2},
 \end{array} \right.
 \end{aligned}$$

$$\begin{aligned}
nE_f(\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} &= -2 \left\{ n \text{cov}_f(\bar{l}_0, m_v), n \text{cov}_f\left(\bar{l}_0, \frac{\partial \bar{l}}{\partial \theta_0}\right) \right\} \mathbf{v}^{(1)} \\
&= -2 \left\{ 4\kappa_{f3}(l_{0j}), n \text{cov}_f\left(\bar{l}_0, \frac{\partial \bar{l}}{\partial \theta_0}\right) \right\} \mathbf{v}^{(1)} \\
&= -\{8\theta_0^3(1-2\pi_0)\bar{i}_0, 2\theta_0\bar{i}_0\} \mathbf{v}^{(1)} \\
&= \{8\theta_0^3(1-2\pi_0)\bar{i}_0, 2\theta_0\bar{i}_0\} (1, 8\theta_0)' \frac{1}{16} (\theta_0^2 \bar{i}_0)^{-3/2} \\
&= \frac{1}{2} (1-2\pi_0) \bar{i}_0^{-1/2} + \theta_0^{-1} \bar{i}_0^{-1/2} = \left\{ \frac{1}{2} (1-2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2},
\end{aligned}$$

$$(\hat{v}_W^{(A)})^{-1/2} = (\alpha_{ML2}^{(A)})^{-1/2} + n^{-1} \eta_W^{(v)} + \sum_{j=1}^2 \mathbf{v}^{(j)}' \mathbf{m}_v^{(j)} + O_p(n^{-3/2}),$$

$$\begin{aligned}
\eta_W^{(v)} &= \frac{1}{2} (\alpha_{ML2}^{(A)})^{-3/2} E_f \left( \frac{\partial v^{(A)}}{\partial \theta_0} \right) \lambda^{-1} q_0^* \quad (\text{see (S1.2)}) \\
&= \frac{1}{2} (4\theta_0^2 \bar{i}_0)^{-3/2} 8\theta_0 \bar{i}_0 (-\bar{i}_0)^{-1} 0.5a(1-2\pi_0) \\
&= -\frac{a}{4} \theta_0^{-2} \bar{i}_0^{-3/2} (1-2\pi_0).
\end{aligned}$$

## S7.2 $n^{-1} \text{AIC}_W$

$$n^{-1} \text{AIC}_W = -2\hat{l}_W + n^{-1} 2q = -2\hat{l}_W + n^{-1} 2,$$

where note that  $\hat{l}_W$  is used rather than  $\hat{l}_W^{(e)}$ .

### S7.2.1 Asymptotic cumulants of $n^{-1} \text{AIC}_{ML}$ before studentization

For estimation of  $-2E_f(\hat{l}_W^*)$ ,

$$\begin{aligned}
& \kappa_{f_1} \{n^{-1} \text{AIC}_w + 2E_f(\bar{l}_w^*)\} \\
&= n^{-1}(2\lambda^{-1}\gamma + 2q) + n^{-2}b_2 + O(n^{-3}) \\
&= n^{-1} \times 0 + n^{-2}b_2 + O(n^{-3}) = n^{-2}c_1 + O(n^{-3}) \\
&= n^{-2}(a-1)\{(1-2\pi_0)^2 \bar{i}_0^{-1} + 2\} + O(n^{-3})
\end{aligned} \tag{S7.3}$$

( $b_2 = c_1$  due to canonical parametrization;  $\alpha_{w_1}^{(A)*} = \alpha_{ML1}^{(A)*} = b_1 = 0$ ,

$$\alpha_{w\Delta 1}^{(A)*} = b_2 = c_1),$$

while for estimation of  $-2\bar{l}_0^*$ ,

$$\begin{aligned}
& \kappa_{f_1}(n^{-1} \text{AIC}_w + 2\bar{l}_0^*) \\
&= n^{-1}(\lambda^{-1}\gamma + 2q) + n^{-2}\{n^2 E_f(\bar{l}_{ML}^{(3)} + \bar{l}_{ML}^{(4)}) + (q_0^*)^2 \bar{i}_0^{-1}\} + O(n^{-3}) \\
&= n^{-1} + n^{-2} \left\{ \frac{1}{6}(1 - \bar{i}_0^{-1}) + \frac{a^2}{4}(1 - 2\pi_0)^2 \bar{i}_0^{-1} \right\} + O(n^{-3}) \\
&= n^{-1} \alpha_{ML1}^{(A)} + n^{-2} \alpha_{w\Delta 1}^{(A)} + O(n^{-3}) \quad (\alpha_{w_1}^{(A)} = \alpha_{ML1}^{(A)}).
\end{aligned} \tag{S7.4}$$

$$\begin{aligned}
& \kappa_{f_2}(n^{-1} \text{AIC}_w) = n^{-1}[nE_f\{(\bar{l}_{ML}^{(1)})^2\}] + n^{-2}[2n^2 E_f(\bar{l}_{ML}^{(1)} \bar{l}_{ML}^{(2)}) \\
& \quad + 2n^2 E_f(\bar{l}_{ML}^{(1)} \bar{l}_{ML}^{(3)}) + n^2 E_f\{(\bar{l}_{ML}^{(2)})^2\} - \{nE_f(\bar{l}_{ML}^{(2)})\}^2] + O(n^{-3}) \\
&= n^{-1} 4n \text{var}_f(\bar{x})\theta_0^2 + n^{-2} \left[ 4n^2 \kappa_{f_3}(\bar{x})\theta_0 \bar{i}_0^{-1} \right. \\
& \quad \left. - \frac{4}{3}(1 - 2\pi_0)\theta_0 \bar{i}_0^{-2} n^2 E_f\{(\bar{x} - \pi_0)^4\} + 2\lambda^{-2}\gamma^2 \right] + O(n^{-3})
\end{aligned} \tag{S7.5}$$

$$\begin{aligned}
&= n^{-1} 4\theta_0^2 \bar{i}_0 + n^{-2} \{4(1 - 2\pi_0)\theta_0 - 4(1 - 2\pi_0)\theta_0 + 2\} + O(n^{-3}) \\
&= n^{-1} 4\theta_0^2 \bar{i}_0 + n^{-2} 2 + O(n^{-3}) = n^{-1} \alpha_{ML2}^{(A)} + n^{-2} \alpha_{ML\Delta 2}^{(A)} + O(n^{-3})
\end{aligned}$$

$$(\alpha_{w_2}^{(A)} = \alpha_{ML2}^{(A)}, \alpha_{w\Delta 2}^{(A)} = \alpha_{ML\Delta 2}^{(A)}).$$

$$\begin{aligned}
\kappa_{f_3}(n^{-1}\text{AIC}_w) &= n^{-2} [ n^2 E_f \{ (\bar{l}_{\text{ML}}^{(1)})^3 \} + 3n^2 E_f \{ (\bar{l}_{\text{ML}}^{(1)})^2 \bar{l}_{\text{ML}}^{(2)} \} \\
&\quad - 3n E_f (\bar{l}_{\text{ML}}^{(2)}) \alpha_{\text{ML}2}^{(A)} ] + O(n^{-3}) \\
&= n^{-2} \left[ n^2 E_f \{ (\bar{l}_{\text{ML}}^{(1)})^3 \} + 6\lambda^{-1} \left\{ n \text{cov}_f \left( \bar{l}_{\text{ML}}^{(1)}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right\}^2 \right] + O(n^{-3}) \\
&= n^{-2} \left[ n^2 E_f \{ (\bar{l}_{\text{ML}}^{(1)})^3 \} + 24\lambda^{-1} \theta_0^2 \left\{ n \text{cov}_f \left( \bar{x}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right\}^2 \right] + O(n^{-3}) \\
&= n^{-2} \{ -8\theta_0^3 n^2 \kappa_{f_3}(\bar{x}) - 24\bar{i}_0^{-1} \theta_0^2 \bar{i}_0^2 \} + O(n^{-3}) \\
&= n^{-2} \{ -8\theta_0^3 (1 - 2\pi_0) - 24\theta_0^2 \} \bar{i}_0 + O(n^{-3}) \\
&= n^{-2} \alpha_{\text{ML}3}^{(A)} + O(n^{-3}) \quad (\alpha_{\text{W}3}^{(A)} = \alpha_{\text{ML}3}^{(A)}).
\end{aligned} \tag{S7.6}$$

$$\begin{aligned}
\kappa_{f_4}(n^{-1}\text{AIC}_w) &= n^{-3} [ n^3 \kappa_{f_4}(\bar{l}_{\text{ML}}^{(1)}) + 4n^3 E_f \{ (\bar{l}_{\text{ML}}^{(1)})^3 \bar{l}_{\text{ML}}^{(2)} \} \\
&\quad + 6n^3 E_f \{ (\bar{l}_{\text{ML}}^{(1)})^2 (\bar{l}_{\text{ML}}^{(2)})^2 \} + 4n^3 E_f \{ (\bar{l}_{\text{ML}}^{(1)})^3 \bar{l}_{\text{ML}}^{(3)} \} - 4n E_f (\bar{l}_{\text{ML}}^{(2)}) \alpha_{\text{ML}3}^{(A)} \\
&\quad - 6\alpha_{\text{ML}2}^{(A)} \alpha_{\text{ML}\Delta 2}^{(A)} - 6\alpha_{\text{ML}2}^{(A)} \{ n E_f (\bar{l}_{\text{ML}}^{(2)}) \}^2 ] + O(n^{-4}) \\
&= n^{-3} \left[ 16\theta_0^4 n^3 \kappa_{f_4}(\bar{x}) - 32\lambda^{-1} \theta_0^3 n^3 E_f \{ (\bar{x} - \pi_0)^5 \} \right. \\
&\quad + 24\lambda^{-2} \theta_0^2 n^3 E_f \{ (\bar{x} - \pi_0)^6 \} + 4 \times (-8) \frac{1}{3} \theta_0^3 (1 - 2\pi_0) \bar{i}_0^{-2} n^3 E_f \{ (\bar{x} - \pi_0)^6 \} \\
&\quad \left. - 4 \times (-1) \alpha_{\text{ML}3}^{(A)} - 6\alpha_{\text{ML}2}^{(A)} \alpha_{\text{ML}\Delta 2}^{(A)} - 6\alpha_{\text{ML}2}^{(A)} (-1)^2 \right] + O(n^{-4})
\end{aligned}$$

$$\begin{aligned}
&= n^{-3} \left[ 16\theta_0^4(1-6\pi_0+6\pi_0^2)\bar{i}_0 + 32\theta_0^3\bar{i}_0^{-1}10\bar{i}_0(1-2\pi_0)\bar{i}_0 \right. \\
&\quad + 24 \times 15\theta_0^2\bar{i}_0^{-2}\bar{i}_0^3 - \frac{32}{3} \times 15\theta_0^3(1-2\pi_0)\bar{i}_0^{-2}\bar{i}_0^3 \\
&\quad \left. - 4\{8\theta_0^3(1-2\pi_0) + 24\theta_0^2\}\bar{i}_0 - 6 \times 4\theta_0^2\bar{i}_0 \times 2 - 6 \times 4\theta_0^2\bar{i}_0 \right] \quad (S7.7) \\
&\quad + O(n^{-4}) \\
&= n^{-3} \{ 16\theta_0^4(1-6\pi_0+6\pi_0^2) + (320-160-32)\theta_0^3(1-2\pi_0) \\
&\quad + (360-96-48-24)\theta_0^2 \} \bar{i}_0 + O(n^{-4}) \\
&= n^{-3} \{ 16\theta_0^4(1-6\pi_0+6\pi_0^2) + 128\theta_0^3(1-2\pi_0) + 192\theta_0^2 \} \bar{i}_0 + O(n^{-4}) \\
&= n^{-3} \alpha_{ML4}^{(A)} + O(n^{-4}) \quad (\alpha_{W4}^{(A)} = \alpha_{ML4}^{(A)}).
\end{aligned}$$

### S7.2.2 Asymptotic cumulants of $n^{-1}AIC_W$ after studentization for estimation of $-2\bar{l}_0^*$

$$t_W^{(A)} = \frac{n^{1/2}(n^{-1}AIC_W + 2\bar{l}_0^*)}{(\hat{v}_W^{(A)})^{1/2}}.$$

$$\begin{aligned}
\kappa_{f1}(t_W^{(A)}) &= n^{-1/2} \{ \alpha_{ML1}^{(A)} (\alpha_{ML2}^{(A)})^{-1/2} + \alpha_{(\Delta t)ML1}^{(A)} \} + O(n^{-3/2}) \\
&= n^{-1/2} \{ \alpha_{ML1}^{(A)} (\alpha_{ML2}^{(A)})^{-1/2} + nE_f(\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)'}) \mathbf{v}^{(1)} \} + O(n^{-3/2}) \\
&= n^{-1/2} \left[ 1 \times (4\theta_0^2\bar{i}_0)^{-1/2} + \left\{ \frac{1}{2}(1-2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2} \right] + O(n^{-3/2}) \\
&= n^{-1/2} \left\{ \frac{3}{2}\theta_0^{-1} + \frac{1}{2}(1-2\pi_0) \right\} \bar{i}_0^{-1/2} + O(n^{-3/2}) = n^{-1/2} \alpha_{(t)ML1}^{(A)} + O(n^{-3/2}) \quad (S7.8)
\end{aligned}$$

$$(\alpha_{(t)W1}^{(A)} = \alpha_{(t)ML1}^{(A)});$$

$$\alpha_{(\Delta t)ML1}^{(A)} = nE_f(\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)'}) \mathbf{v}^{(1)} = \{ (1/2)(1-2\pi_0) + \theta_0^{-1} \} \bar{i}_0^{-1/2}.$$

$$\begin{aligned}
& \kappa_{f_2}(t_W^{(A)}) \\
&= 1 + n^{-1} \left[ \alpha_{ML\Delta 2}^{(A)} (\alpha_{ML2}^{(A)})^{-1} + 2n^2 E_f \{ (\bar{l}_{ML}^{(1)})^2 \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-1/2} \right. \\
& \quad \left. + 2n^2 E_f \left[ \bar{l}_{ML}^{(1)} \{ \bar{l}_{ML}^{(2)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} + \bar{l}_{ML}^{(1)} (n^{-1} \eta_W^{(v)} + \mathbf{v}^{(2)}, \mathbf{m}_v^{(2)}) + n^{-1} 2q \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \} \right. \right. \\
& \quad \quad \left. \left. \times (\alpha_{ML2}^{(A)})^{-1/2} \right. \right. \\
& \quad \left. + n^2 E_f \{ 2 \bar{l}_{ML}^{(2)} (\alpha_{ML2}^{(A)})^{-1/2} \bar{l}_{ML}^{(1)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} + (\bar{l}_{ML}^{(1)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)})^2 \} \right. \\
& \quad \left. - \{ 2n E_f (\bar{l}_{ML}^{(2)}) (\alpha_{ML2}^{(A)})^{-1/2} \alpha_{(Af)ML1}^{(A)} + (\alpha_{(Af)ML1}^{(A)})^2 \} \right] + O(n^{-2}), \tag{S7.9}
\end{aligned}$$

(i) the first term in  $\left[ \begin{smallmatrix} \cdot \\ (A) \end{smallmatrix} \right]_{(A)}$  of (S7.9) is

$$\alpha_{ML\Delta 2}^{(A)} (\alpha_{ML2}^{(A)})^{-1} = 2(4\theta_0^2 \bar{i}_0)^{-1} = \frac{1}{2} \theta_0^{-2} \bar{i}_0^{-1},$$

(ii) the second term in  $\left[ \begin{smallmatrix} \cdot \\ (A) \end{smallmatrix} \right]_{(A)}$  of (S7.9) is

$$\begin{aligned}
& 2n^2 E_f \{ (\bar{l}_{ML}^{(1)})^2 \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-1/2} \\
&= 2 \left[ 16\kappa_{f_4}(l_{0j}), 4E_f \left\{ (l_{0j} - \bar{l}_0^*)^2 \frac{\partial l_j}{\partial \theta_0} \right\} \right] \mathbf{v}^{(1)} (\alpha_{ML2}^{(A)})^{-1/2} \quad (\text{see (S1.7)}) \\
&= 2 \{ 16\theta_0^4 (1 - 6\pi_0 + 6\pi_0^2) \bar{i}_0, 4\theta_0^2 (1 - 2\pi_0) \bar{i}_0 \} \\
& \quad \times \left( -\frac{1}{16} \right) (\theta_0^2 \bar{i}_0)^{-3/2} (1, 8\theta_0)' (4\theta_0^2 \bar{i}_0)^{-1/2} \\
&= -\{ \theta_0^4 (1 - 6\pi_0 + 6\pi_0^2) \bar{i}_0 + 2\theta_0^3 (1 - 2\pi_0) \bar{i}_0 \} (\theta_0^2 \bar{i}_0)^{-2} \\
&= -\{ 1 - 6\pi_0 + 6\pi_0^2 + 2\theta_0^{-1} (1 - 2\pi_0) \} \bar{i}_0^{-1} \\
&= -\{ (1 - 2\pi_0)^2 - 2\bar{i}_0 + 2\theta_0^{-1} (1 - 2\pi_0) \} \bar{i}_0^{-1} \\
&= -\{ (1 - 2\pi_0)^2 + 2\theta_0^{-1} (1 - 2\pi_0) \} \bar{i}_0^{-1} + 2,
\end{aligned}$$



(iii) the first part of  $2n^2 \mathbf{E}_f [ \cdot ]_{(B) (B)} (\alpha_{ML2}^{(A)})^{-1/2}$  in (S7.9) is

$$\begin{aligned}
& 2n^2 \mathbf{E}_f \{ (\bar{l}_{ML}^{(1)} \bar{l}_{ML}^{(2)}) \mathbf{m}_v^{(1)} \} \mathbf{v}^{(1)} (\alpha_{ML2}^{(A)})^{-1/2} \\
&= -4 \left[ \lambda^{-1} \gamma \left\{ n \text{cov}_f (\bar{l}_0, m_v), \mathbf{E}_f \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\} \right. \\
&\quad \left. + 2 \mathbf{E}_f \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \lambda^{-1} \left\{ n \text{cov}_f \left( \frac{\partial \bar{l}}{\partial \theta_0}, m_v \right), \gamma \right\} \right] \mathbf{v}^{(1)} (\alpha_{ML2}^{(A)})^{-1/2} \\
&+ O(n^{-1}) \\
&= -4 \left[ - \{ 4\theta_0^3 (1 - 2\pi_0), \theta_0 \bar{l}_0 \} + 2\theta_0 \bar{l}_0 (-\bar{l}_0)^{-1} \{ 4\theta_0^2 (1 - 2\pi_0) \bar{l}_0, \bar{l}_0 \} \right] \\
&\quad \times \left( -\frac{1}{16} \right) (\theta_0^2 \bar{l}_0)^{-3/2} (1, 8\theta_0)' (4\theta_0^2 \bar{l}_0)^{-1/2} + O(n^{-1}) \\
&= -\frac{1}{8} \left[ \{ 4\theta_0^3 (1 - 2\pi_0), \theta_0 \} \bar{l}_0 + 2\theta_0 \{ 4\theta_0^2 (1 - 2\pi_0), 1 \} \bar{l}_0 \right] \\
&\quad \times (1, 8\theta_0)' (\theta_0^2 \bar{l}_0)^{-2} + O(n^{-1}) \\
&= -\frac{1}{8} \left[ 4\theta_0^3 (1 - 2\pi_0) + 8\theta_0^2 + 2\theta_0 \{ 4\theta_0^2 (1 - 2\pi_0) + 8\theta_0 \} \right] \theta_0^{-4} \bar{l}_0^{-1} + O(n^{-1}) \\
&= -\left\{ \frac{3}{2} \theta_0^{-1} (1 - 2\pi_0) + 3\theta_0^{-2} \right\} \bar{l}_0^{-1} + O(n^{-1}),
\end{aligned}$$

(iv) the central part of  $2n^2 \mathbf{E}_f [ \cdot ]_{(B) (B)} (\alpha_{ML2}^{(A)})^{-1/2}$  in (S7.9) is

$$\begin{aligned}
& 2n^2 \mathbf{E}_f \{ (\bar{l}_{ML}^{(1)})^2 (n^{-1} \eta_w^{(v)} + \mathbf{v}^{(2)}, \mathbf{m}_v^{(2)}) \} (\alpha_{ML2}^{(A)})^{-1/2} \\
&= 2\alpha_{ML2}^{(A)} \eta_w^{(v)} (\alpha_{ML2}^{(A)})^{-1/2} + 2\alpha_{ML2}^{(A)} \mathbf{E}_f (\mathbf{m}_v^{(2)}) \mathbf{v}^{(2)} (\alpha_{ML2}^{(A)})^{-1/2} \\
&+ 2 \times (-2)^2 \left[ 2 \{ n \text{cov}_f (\bar{l}_0, m_v) \}^2, 2n \text{cov}_f (\bar{l}_0, m_v) \mathbf{E}_f \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right), 0, \right. \\
&\quad \left. 2 \left\{ \mathbf{E}_f \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\}^2, 2n \text{cov}_f \left( \bar{l}_0, \frac{\partial \mathbf{v}^{(A)}}{\partial \theta_0} \right) \mathbf{E}_f \left( l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right] \mathbf{v}^{(2)} (\alpha_{ML2}^{(A)})^{-1/2} \\
&+ O(n^{-1})
\end{aligned}$$

$$\begin{aligned}
&= 2(\alpha_{\text{ML}2}^{(A)})^{1/2} \eta_{\text{W}}^{(v)} + 2(\alpha_{\text{ML}2}^{(A)})^{1/2} \\
&\quad \times \left[ n \text{avar}_f(m_v), n \text{cov}_f \left( m_v, \frac{\partial \bar{l}}{\partial \theta_0} \right), 0, \gamma, n \text{cov}_f \left( \frac{\partial v^{(A)}}{\partial \theta_0}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right] \mathbf{v}^{(2)} \\
&\quad + 8 \left[ 2\{4\theta_0^3(1-2\pi_0)\bar{i}_0\}^2, 8\theta_0^3(1-2\pi_0)\bar{i}_0\theta_0\bar{i}_0, 0, 2(\theta_0\bar{i}_0)^2, \right. \\
&\quad \quad \left. 16\theta_0^2(1-2\pi_0)\bar{i}_0\theta_0\bar{i}_0 \right] \mathbf{v}^{(2)} (\alpha_{\text{ML}2}^{(A)})^{-1/2} + O(n^{-1}), \\
&= 2(4\theta_0^2\bar{i}_0)^{1/2} \left\{ -\frac{a}{4}\theta_0^{-2}\bar{i}_0^{-3/2}(1-2\pi_0) \right\} \\
&\quad + 2 \left[ 16\theta_0^4(1-2\pi_0)^2\bar{i}_0, 4\theta_0^2(1-2\pi_0)\bar{i}_0, 0, \bar{i}_0, 8\theta_0(1-2\pi_0)\bar{i}_0 \right] \\
&\quad \quad \times \mathbf{v}^{(2)*} \bar{i}_0^{-5/2} (4\theta_0^2\bar{i}_0)^{1/2} \\
&\quad + 8 \left[ 32\theta_0^6(1-2\pi_0)^2\bar{i}_0^2, 8\theta_0^4(1-2\pi_0)\bar{i}_0^2, 0, 2\theta_0^2\bar{i}_0^2, \right. \\
&\quad \quad \left. 16\theta_0^3(1-2\pi_0)\bar{i}_0^2 \right] \mathbf{v}^{(2)*} \bar{i}_0^{-5/2} (4\theta_0^2\bar{i}_0)^{-1/2} + O(n^{-1}) \\
&= -\frac{a}{4}\theta_0^{-1}\bar{i}_0^{-1}(1-2\pi_0) + \left[ 64\theta_0^5(1-2\pi_0)^2\bar{i}_0^{-1}, 16\theta_0^3(1-2\pi_0)\bar{i}_0^{-1}, \right. \\
&\quad \quad \left. 0, 4\theta_0\bar{i}_0^{-1}, 32\theta_0^2(1-2\pi_0)\bar{i}_0^{-1} \right] \mathbf{v}^{(2)*} \quad (\text{use } \mathbf{v}^{(2)} = \mathbf{v}^{(2)*} \bar{i}_0^{-5/2}) \\
&\quad + \left[ 128\theta_0^5(1-2\pi_0)^2\bar{i}_0^{-1}, 32\theta_0^3(1-2\pi_0)\bar{i}_0^{-1}, \right. \\
&\quad \quad \left. 0, 8\theta_0\bar{i}_0^{-1}, 64\theta_0^2(1-2\pi_0)\bar{i}_0^{-1} \right] \mathbf{v}^{(2)*} + O(n^{-1}) \\
&= -\frac{a}{4}\theta_0^{-1}\bar{i}_0^{-1}(1-2\pi_0) + (64+128)\theta_0^5(1-2\pi_0)^2\bar{i}_0^{-1} \frac{3}{256}\theta_0^{-5} \\
&\quad + (16+32)\theta_0^3(1-2\pi_0)\bar{i}_0^{-1} \frac{3}{16}\theta_0^{-4} \\
&\quad + (4+8)\theta_0\bar{i}_0^{-1} \left\{ \frac{1}{4}\theta_0^{-2}(1-2\pi_0) + \frac{1}{2}\theta_0^{-3} \right\} \\
&\quad + (32+64)\theta_0^2(1-2\pi_0)\bar{i}_0^{-1} \left( -\frac{1}{16}\theta_0^{-3} \right) + O(n^{-1}) \\
&= -\frac{a}{4}\theta_0^{-1}\bar{i}_0^{-1}(1-2\pi_0) + \frac{9}{4}(1-2\pi_0)^2\bar{i}_0^{-1} + 9\theta_0^{-1}(1-2\pi_0)\bar{i}_0^{-1} \\
&\quad + 3\theta_0^{-1}(1-2\pi_0)\bar{i}_0^{-1} + 6\theta_0^{-2}\bar{i}_0^{-1} - 6\theta_0^{-1}(1-2\pi_0)\bar{i}_0^{-1} + O(n^{-1}) \\
&= \left\{ \frac{9}{4}(1-2\pi_0)^2 + \left( -\frac{a}{4} + 6 \right) \theta_0^{-1}(1-2\pi_0) + 6\theta_0^{-2} \right\} \bar{i}_0^{-1} + O(n^{-1}),
\end{aligned}$$

(v) the third part of  $2n^2 E_f \left[ \cdot \right]_{(B) (B)} (\alpha_{ML2}^{(A)})^{-1/2}$  in (S7.9) is

$$\begin{aligned} 2n^2 E_f (n^{-1} 2q \bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)})' \mathbf{v}^{(1)} (\alpha_{ML2}^{(A)})^{-1/2} &= 4n E_f (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)})' \mathbf{v}^{(1)} (\alpha_{ML2}^{(A)})^{-1/2} \\ &= 4 \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2} (4\theta_0^2 \bar{i}_0)^{-1/2} = \{ (1 - 2\pi_0) \theta_0^{-1} + 2\theta_0^{-2} \} \bar{i}_0^{-1}, \end{aligned}$$

(vi) the first half of the fourth term in  $\left[ \cdot \right]_{(A) (A)}$  of (S7.9) is

equal to the result of (iii) i.e.,

$$n^2 E_f \{ 2 \bar{l}_{ML}^{(2)} (\alpha_{ML2}^{(A)})^{-1/2} \bar{l}_{ML}^{(1)} \mathbf{v}^{(1)} \mathbf{m}_v^{(1)} \} = - \left\{ \frac{3}{2} \theta_0^{-1} (1 - 2\pi_0) + 3\theta_0^{-2} \right\} \bar{i}_0^{-1} + O(n^{-1}),$$

(vii) the second half of the fourth term in  $\left[ \cdot \right]_{(A) (A)}$  of (S7.9) is

$$\begin{aligned} &n^2 E_f \{ (\bar{l}_{ML}^{(1)} \mathbf{v}^{(1)})' \mathbf{m}_v^{(1)} \}^2 \\ &= \alpha_{ML2}^{(A)} \mathbf{v}^{(1)}' n \text{acov}_f (\mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} + 2 \{ n E_f (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)})' \mathbf{v}^{(1)} \}^2 + O(n^{-1}) \\ &= \alpha_{ML2}^{(A)} \mathbf{v}^{(1)}' \begin{bmatrix} n \text{avar}_f (m_v) & n \text{cov}_f (m_v, \partial \bar{l} / \partial \theta_0) \\ n \text{cov}_f (m_v, \partial \bar{l} / \partial \theta_0) & \gamma \end{bmatrix} \mathbf{v}^{(1)} \\ &\quad + 2 \{ n E_f (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)})' \mathbf{v}^{(1)} \}^2 + O(n^{-1}) \\ &= 4\theta_0^2 \bar{i}_0 \frac{1}{16^2} (\theta_0^2 \bar{i}_0)^{-3} (1, 8\theta_0) \begin{pmatrix} 16\theta_0^4 (1 - 2\pi_0)^2 \bar{i}_0 & 4\theta_0^2 (1 - 2\pi_0) \bar{i}_0 \\ 4\theta_0^2 (1 - 2\pi_0) \bar{i}_0 & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ 8\theta_0 \end{pmatrix} \\ &\quad + 2 \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\}^2 \bar{i}_0^{-1} + O(n^{-1}) \\ &= \frac{1}{64} \theta_0^{-4} \bar{i}_0^{-2} \{ 16\theta_0^4 (1 - 2\pi_0)^2 + 2 \times 32\theta_0^3 (1 - 2\pi_0) + 64\theta_0^2 \} \bar{i}_0 \\ &\quad + 2 \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\}^2 \bar{i}_0^{-1} + O(n^{-1}) \end{aligned}$$

$$\begin{aligned}
&= \left\{ \frac{1}{4}(1-2\pi_0)^2 + \theta_0^{-1}(1-2\pi_0) + \theta_0^{-2} \right\} \bar{i}_0^{-1} + 2 \left\{ \frac{1}{2}(1-2\pi_0) + \theta_0^{-1} \right\}^2 \bar{i}_0^{-1} \\
&\quad + O(n^{-1}) \\
&= \left\{ \frac{3}{4}(1-2\pi_0)^2 + 3\theta_0^{-1}(1-2\pi_0) + 3\theta_0^{-2} \right\} \bar{i}_0^{-1} + O(n^{-1}),
\end{aligned}$$

(viii) the fifth term in  $\begin{bmatrix} \cdot \\ (A) \end{bmatrix}_{(A)}$  of (S7.9) is

$$\begin{aligned}
&-\{2nE_f(\bar{I}_{ML}^{(2)})(\alpha_{ML2}^{(A)})^{-1/2} \alpha_{(\Delta t)ML1}^{(A)} + (\alpha_{(\Delta t)ML1}^{(A)})^2\} \\
&= - \left[ 2(-1)(4\theta_0^2 \bar{i}_0)^{-1/2} \left\{ \frac{1}{2}(1-2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2} + \left\{ \frac{1}{2}(1-2\pi_0) + \theta_0^{-1} \right\}^2 \bar{i}_0^{-1} \right] \\
&= \left[ \theta_0^{-1} \left\{ \frac{1}{2}(1-2\pi_0) + \theta_0^{-1} \right\} - \left\{ \frac{1}{2}(1-2\pi_0) + \theta_0^{-1} \right\}^2 \right] \bar{i}_0^{-1} \\
&= - \left\{ \frac{1}{4}(1-2\pi_0)^2 + \frac{1}{2}\theta_0^{-1}(1-2\pi_0) \right\} \bar{i}_0^{-1}.
\end{aligned}$$

Then, from (i) to (viii),

$$\begin{aligned}
\kappa_{f2}(t_W^{(A)}) &= 1 + n^{-1} \alpha_{(t)W\Delta 2}^{(A)} + O(n^{-2}), \\
\alpha_{(t)W\Delta 2}^{(A)} &= \frac{1}{2} \theta_0^{-2} \bar{i}_0^{-1} - \{(1-2\pi_0)^2 + 2\theta_0^{-1}(1-2\pi_0)\} \bar{i}_0^{-1} + 2 \\
&\quad - 2 \left\{ \frac{3}{2} \theta_0^{-1}(1-2\pi_0) + 3\theta_0^{-2} \right\} \bar{i}_0^{-1} \\
&\quad + \left\{ \frac{9}{4}(1-2\pi_0)^2 + \left( -\frac{a}{4} + 6 \right) \theta_0^{-1}(1-2\pi_0) + 6\theta_0^{-2} \right\} \bar{i}_0^{-1} \\
&\quad + \{(1-2\pi_0)\theta_0^{-1} + 2\theta_0^{-2}\} \bar{i}_0^{-1} \tag{S7.10} \\
&\quad + \left\{ \frac{3}{4}(1-2\pi_0)^2 + 3\theta_0^{-1}(1-2\pi_0) + 3\theta_0^{-2} \right\} \bar{i}_0^{-1} \\
&\quad - \left\{ \frac{1}{4}(1-2\pi_0)^2 + \frac{1}{2}\theta_0^{-1}(1-2\pi_0) \right\} \bar{i}_0^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \left[ \left( -1 + \frac{9}{4} + \frac{3}{4} - \frac{1}{4} \right) (1 - 2\pi_0)^2 \right. \\
&\quad \left. + \left\{ -2\theta_0^{-1} - 3\theta_0^{-1} + \left( -\frac{a}{4} + 6 \right) \theta_0^{-1} + \theta_0^{-1} + 3\theta_0^{-1} - \frac{1}{2}\theta_0^{-1} \right\} (1 - 2\pi_0) \right. \\
&\quad \left. + \frac{1}{2}\theta_0^{-2} - 6\theta_0^{-2} + 6\theta_0^{-2} + 2\theta_0^{-2} + 3\theta_0^{-2} \right] \bar{i}_0^{-1} + 2 \\
&= \left\{ \frac{7}{4}(1 - 2\pi_0)^2 + \left( -\frac{a}{4} + \frac{9}{2} \right) \theta_0^{-1} (1 - 2\pi_0) + \frac{11}{2}\theta_0^{-2} \right\} \bar{i}_0^{-1} + 2 \\
&(\alpha_{(t)W\Delta 2}^{(A)} \neq \alpha_{(t)ML\Delta 2}^{(A)} \text{ when } a \neq 0).
\end{aligned}$$

$$\begin{aligned}
\kappa_{f_3}(t_W^{(A)}) &= n^{-1/2} \{ \alpha_{ML3}^{(A)} (\alpha_{ML2}^{(A)})^{-3/2} + 6n^2 E_f \{ \bar{l}_{ML}^{(1)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \} \} + O(n^{-3/2}) \\
&= n^{-1/2} \{ \alpha_{ML3}^{(A)} (\alpha_{ML2}^{(A)})^{-3/2} + 6\alpha_{(\Delta)ML1}^{(A)} \} + O(n^{-3/2}) \\
&= n^{-1/2} \left[ -\{ 8\theta_0^3 (1 - 2\pi_0) + 24\theta_0^2 \} \bar{i}_0 (4\theta_0^2 \bar{i}_0)^{-3/2} + 6 \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2} \right] \\
&\quad + O(n^{-3/2}) \\
&= n^{-1/2} \{ (-1 + 3)(1 - 2\pi_0) + (-3 + 6)\theta_0^{-1} \} \bar{i}_0^{-1/2} + O(n^{-3/2}) \\
&= n^{-1/2} \{ 2(1 - 2\pi_0) + 3\theta_0^{-1} \} \bar{i}_0^{-1/2} + O(n^{-3/2}).
\end{aligned} \tag{S7.11}$$

$$\begin{aligned}
\kappa_{f_4}(t_W^{(A)}) &= n^{-1} \left[ \alpha_{ML4}^{(A)} (\alpha_{ML2}^{(A)})^{-2} + 4n^3 E_f \{ (\bar{l}_{ML}^{(1)})^4 \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-3/2} \right. \\
&\quad \left. + 12n^3 E_f \{ (\bar{l}_{ML}^{(1)})^3 \bar{l}_{ML}^{(2)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-3/2} \right. \\
&\quad \left. + 6n^3 E_f \{ (\bar{l}_{ML}^{(1)})^4 (\mathbf{v}^{(1)}, \mathbf{m}_v^{(1)})^2 \} (\alpha_{ML2}^{(A)})^{-1} \right. \\
&\quad \left. + 4n^3 E_f \{ (\bar{l}_{ML}^{(1)})^3 \bar{l}_{ML}^{(2)} \mathbf{v}^{(1)}, \mathbf{m}_v^{(1)} + (\bar{l}_{ML}^{(1)})^4 \mathbf{v}^{(2)}, \mathbf{m}_v^{(2)} \} (\alpha_{ML2}^{(A)})^{-3/2} \right]
\end{aligned} \tag{S7.12}$$

$$\begin{aligned}
& + \{4(\alpha_{\text{ML}1}^{(A)} - 2q)\alpha_{\text{ML}3}^{(A)} + 6\alpha_{\text{ML}2}^{(A)}\alpha_{\text{ML}\Delta 2}^{(A)} + 6\alpha_{\text{ML}2}^{(A)}(\alpha_{\text{ML}1}^{(A)} - 2q)^2\}(\alpha_{\text{ML}2}^{(A)})^{-2} \\
& - 4\{\alpha_{(t)\text{ML}1}^{(A)} - 2q(\alpha_{\text{ML}2}^{(A)})^{-1/2}\}\alpha_{(t)\text{ML}3}^{(A)} \\
& - 6\{\alpha_{(t)\text{ML}\Delta 2}^{(A)} - 4qnE_f(\bar{l}_{\text{ML}}^{(1)} \mathbf{m}_v^{(1)'})\mathbf{v}^{(1)}(\alpha_{\text{ML}2}^{(A)})^{-1/2}\} \\
& - 6\{\alpha_{(t)\text{ML}1}^{(A)} - 2q(\alpha_{\text{ML}2}^{(A)})^{-1/2}\}^2 \Big] + O(n^{-2}), \\
& \hspace{15em} \text{(A)}
\end{aligned}$$

(i) the first term in  $\left[ \begin{smallmatrix} \cdot \\ \text{(A)} \end{smallmatrix} \right]_{\text{(A)}}$  of (S7.12) is

$$\begin{aligned}
& \alpha_{\text{ML}4}^{(A)}(\alpha_{\text{ML}2}^{(A)})^{-2} \\
& = \{16\theta_0^4(1 - 6\pi_0 + 6\pi_0^2) + 128\theta_0^3(1 - 2\pi_0) + 192\theta_0^2\}\bar{i}_0(4\theta_0^2\bar{i}_0)^{-2} \\
& = \{1 - 6\pi_0 + 6\pi_0^2 + 8\theta_0^{-1}(1 - 2\pi_0) + 12\theta_0^{-2}\}\bar{i}_0^{-1} \\
& = \{(1 - 2\pi_0)^2 - 2\bar{i}_0 + 8\theta_0^{-1}(1 - 2\pi_0) + 12\theta_0^{-2}\}\bar{i}_0^{-1},
\end{aligned}$$

(ii) the second term in  $\left[ \begin{smallmatrix} \cdot \\ \text{(A)} \end{smallmatrix} \right]_{\text{(A)}}$  of (S7.12) is

$$\begin{aligned}
& 4n^3E_f\{(\bar{l}_{\text{ML}}^{(1)})^4 \mathbf{v}^{(1)'}\mathbf{m}_v^{(1)}\}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \\
& = 4[6\alpha_{\text{ML}2}^{(A)}n^2E_f\{(\bar{l}_{\text{ML}}^{(1)})^2 \mathbf{m}_v^{(1)'}\}\mathbf{v}^{(1)}(\alpha_{\text{ML}2}^{(A)})^{-3/2} \\
& \quad + 4(-2)^3\kappa_{f3}(l_{0j})nE_f(\bar{l}_{\text{ML}}^{(1)} \mathbf{m}_v^{(1)'})\mathbf{v}^{(1)}(\alpha_{\text{ML}2}^{(A)})^{-3/2}] + O(n^{-1}) \\
& = 24n^2E_f\{(\bar{l}_{\text{ML}}^{(1)})^2 \mathbf{m}_v^{(1)'}\}\mathbf{v}^{(1)}(\alpha_{\text{ML}2}^{(A)})^{-1/2} \\
& \quad - 128\theta_0^3(1 - 2\pi_0)\bar{i}_0\left\{\frac{1}{2}(1 - 2\pi_0) + \theta_0^{-1}\right\}\bar{i}_0^{-1/2}(4\theta_0^2\bar{i}_0)^{-3/2} + O(n^{-1}) \\
& = -12\{(1 - 2\pi_0)^2 + 2\theta_0^{-1}(1 - 2\pi_0)\}\bar{i}_0^{-1} + 24 \\
& \quad - 8(1 - 2\pi_0)(1 - 2\pi_0 + 2\theta_0^{-1})\bar{i}_0^{-1} + O(n^{-1}) \\
& = -\{20(1 - 2\pi_0)^2 + 40\theta_0^{-1}(1 - 2\pi_0)\}\bar{i}_0^{-1} + 24 + O(n^{-1}), \\
& \text{(note that the term } 24n^2E_f\{(\bar{l}_{\text{ML}}^{(1)})^2 \mathbf{m}_v^{(1)'}\}\mathbf{v}^{(1)}(\alpha_{\text{ML}2}^{(A)})^{-1/2} \text{ is 12 times the result} \\
& \text{in (ii) for (S7.9) of } \alpha_{(t)\text{ML}\Delta 2}^{(A)}\text{),}
\end{aligned}$$

(iii) the third term in  $\begin{bmatrix} \cdot \\ \text{(A)} \end{bmatrix}$  of (S7.12) is

$$\begin{aligned}
& 12n^3 E_f \left\{ (\bar{l}_{ML}^{(1)})^3 \bar{l}_{ML}^{(2)} \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)} \right\} (\alpha_{ML2}^{(A)})^{-3/2} \\
&= 12 \left[ \begin{aligned} & \left[ 3\alpha_{ML2}^{(A)} n E_f (\bar{l}_{ML}^{(2)}) + 6\lambda^{-1} \left\{ n \text{cov}_f \left( \bar{l}_{ML}^{(1)}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right\}^2 \right] E_f (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)'}) \mathbf{v}^{(1)} \\ & + 6\alpha_{ML2}^{(A)} \lambda^{-1} n \text{cov}_f \left( \bar{l}_{ML}^{(1)}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \left\{ n \text{cov}_f \left( \frac{\partial \bar{l}}{\partial \theta_0}, m_v \right), \gamma \right\} \mathbf{v}^{(1)} \right] (\alpha_{ML2}^{(A)})^{-3/2} \\ & \left. \begin{aligned} & + O(n^{-1}) \\ & = 12 \left[ \left\{ 3(4\theta_0^2 \bar{i}_0)(-1) + 6(-\bar{i}_0^{-1})4(\theta_0 \bar{i}_0)^2 \right\} \left\{ \frac{1}{2}(1-2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2} \right. \\ & \quad + 6(4\theta_0^2 \bar{i}_0)(-\bar{i}_0^{-1})(-2\theta_0 \bar{i}_0) \left\{ 4\theta_0^2(1-2\pi_0) \bar{i}_0, \bar{i}_0 \right\} \\ & \quad \times \left( -\frac{1}{16} \right) (\theta_0^2 \bar{i}_0)^{-3/2} (1, 8\theta_0)' \left. \right] (4\theta_0^2 \bar{i}_0)^{-3/2} + O(n^{-1}) \\ & = 12 \left[ -\{18(1-2\pi_0)\theta_0^2 + 36\theta_0\} \bar{i}_0^{-1/2} \right. \\ & \quad \left. + 48\theta_0^3 \bar{i}_0 \left( -\frac{1}{16} \right) (\theta_0^2 \bar{i}_0)^{-3/2} \{4\theta_0^2(1-2\pi_0) + 8\theta_0\} \bar{i}_0 \right] (4\theta_0^2 \bar{i}_0)^{-3/2} + O(n^{-1}) \\ & = \frac{3}{2} \left[ -\{18(1-2\pi_0)\theta_0^2 + 36\theta_0\} \theta_0^{-3} \bar{i}_0^{-1} \right. \\ & \quad \left. - 3\theta_0^{-3} \bar{i}_0^{-1} \{4\theta_0^2(1-2\pi_0) + 8\theta_0\} \right] + O(n^{-1}) \\ & = -\frac{3}{2} \{30\theta_0^{-1} \bar{i}_0^{-1}(1-2\pi_0) + 60\theta_0^{-2} \bar{i}_0^{-1}\} + O(n^{-1}) \\ & = -\{45\theta_0^{-1}(1-2\pi_0) + 90\theta_0^{-2}\} \bar{i}_0^{-1} + O(n^{-1}), \end{aligned} \right. \\
\end{aligned}$$

(iv) the fourth term in  $\begin{bmatrix} \cdot \\ \text{(A)} \end{bmatrix}$  of (S7.12) is

$$\begin{aligned}
& 6n^3 E_f \{ (\bar{I}_{ML}^{(1)})^4 (\mathbf{v}^{(1)'} \mathbf{m}_v^{(1)})^2 \} (\alpha_{ML2}^{(A)})^{-1} \\
&= 6 \left[ 3(\alpha_{ML2}^{(A)})^2 \mathbf{v}^{(1)'} n \text{acov}_f(\mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} \right. \\
&\quad \left. + 12\alpha_{ML2}^{(A)} \{ n \text{cov}_f(\bar{I}_{ML}^{(1)}, \mathbf{m}_v^{(1)'}) \mathbf{v}^{(1)} \}^2 \right] (\alpha_{ML2}^{(A)})^{-1} + O(n^{-1}) \\
&= 18\alpha_{ML2}^{(A)} \mathbf{v}^{(1)'} n \text{acov}_f(\mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} + 72 \{ n \text{cov}_f(\bar{I}_{ML}^{(1)}, \mathbf{m}_v^{(1)'}) \mathbf{v}^{(1)} \}^2 + O(n^{-1}) \\
&= 18 \left\{ \frac{1}{4} (1 - 2\pi_0)^2 + \theta_0^{-1} (1 - 2\pi_0) + \theta_0^{-2} \right\} \bar{i}_0^{-1} \\
&\quad + 36 \times 2 \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\}^2 \bar{i}_0^{-1} + O(n^{-1}) \\
&= \left\{ \left( \frac{9}{2} + 18 \right) (1 - 2\pi_0)^2 + (18 + 72)\theta_0^{-1} (1 - 2\pi_0) + (18 + 72)\theta_0^{-2} \right\} \bar{i}_0^{-1} \\
&\quad + O(n^{-1}) \\
&= \left\{ \frac{45}{2} (1 - 2\pi_0)^2 + 90\theta_0^{-1} (1 - 2\pi_0) + 90\theta_0^{-2} \right\} \bar{i}_0^{-1} + O(n^{-1})
\end{aligned}$$

(the two terms with 18 and 72 on the right-hand side of the second equation are 18 and 36 times the corresponding ones in (vii) for (S7.9) of  $\alpha_{(t)ML\Delta 2}^{(A)}$ , respectively),

(v) the first half of the fifth term in  $\begin{bmatrix} \cdot \\ (A) \end{bmatrix}_{(A)}$  of (S7.12) is

$$\begin{aligned}
& \text{one third of the result in (iii) for (S7.12), i.e.,} \\
& 4n^3 E_f \{ (\bar{I}_{ML}^{(1)})^3 \bar{I}_{ML}^{(2)} \mathbf{v}^{(1)'} \mathbf{m}_v^{(1)} \} (\alpha_{ML2}^{(A)})^{-3/2} \\
&= \frac{1}{3} [-\{45\theta_0^{-1} (1 - 2\pi_0) + 90\theta_0^{-2}\} \bar{i}_0^{-1}] + O(n^{-1}) \\
&= -\{15\theta_0^{-1} (1 - 2\pi_0) + 30\theta_0^{-2}\} \bar{i}_0^{-1} + O(n^{-1})
\end{aligned}$$

(vi) the second half of the fifth term in  $\begin{bmatrix} \cdot \\ (A) \end{bmatrix}_{(A)}$  of (S7.12) is



$$\begin{aligned}
& 4n^3 E_f \{ (\bar{I}_{ML}^{(1)})^4 \mathbf{v}^{(2)'} \mathbf{m}_v^{(2)} \} (\alpha_{ML2}^{(A)})^{-3/2} \\
&= 4 [ 3(\alpha_{ML2}^{(A)})^2 n E_f (\mathbf{m}_v^{(2)'} \mathbf{v}^{(2)}) + 6\alpha_{ML2}^{(A)} n^2 \text{acov}_f \{ (\bar{I}_{ML}^{(1)})^2, \mathbf{m}_v^{(2)'} \} \mathbf{v}^{(2)} ] \\
&\quad \times (\alpha_{ML2}^{(A)})^{-3/2} + O(n^{-1}) \\
&= 12(\alpha_{ML2}^{(A)})^{1/2} n E_f (\mathbf{m}_v^{(2)'} \mathbf{v}^{(2)}) + 24(\alpha_{ML2}^{(A)})^{-1/2} n^2 \text{acov}_f \{ (\bar{I}_{ML}^{(1)})^2, \mathbf{m}_v^{(2)'} \} \mathbf{v}^{(2)} \\
&\quad + O(n^{-1})
\end{aligned}$$

(the first two terms are 6 and 12 times the second and third terms on the right-hand side of the first equation of (iv) for (S7.9) of  $\alpha_{(t)ML\Delta 2}^{(A)}$ , respectively)

$$\begin{aligned}
&= (6 \times 64 + 12 \times 128)(1 - 2\pi_0)^2 \bar{i}_0^{-1} \frac{3}{256} \\
&\quad + (6 \times 16 + 12 \times 32)\theta_0^{-1}(1 - 2\pi_0)\bar{i}_0^{-1} \frac{3}{16} \\
&\quad + (6 \times 4 + 12 \times 8) \left\{ \frac{1}{4}\theta_0^{-1}(1 - 2\pi_0) + \frac{1}{2}\theta_0^{-2} \right\} \bar{i}_0^{-1} \\
&\quad - (6 \times 32 + 12 \times 64)\theta_0^{-1}(1 - 2\pi_0)\bar{i}_0^{-1} \frac{1}{16} + O(n^{-1}) \\
&= \frac{45}{2}(1 - 2\pi_0)^2 \bar{i}_0^{-1} + 90\theta_0^{-1}(1 - 2\pi_0)\bar{i}_0^{-1} + 30\theta_0^{-1}(1 - 2\pi_0)\bar{i}_0^{-1} \\
&\quad + 60\theta_0^{-2}\bar{i}_0^{-1} - 60\theta_0^{-1}(1 - 2\pi_0)\bar{i}_0^{-1} + O(n^{-1}) \\
&= \left\{ \frac{45}{2}(1 - 2\pi_0)^2 + 60\theta_0^{-1}(1 - 2\pi_0) + 60\theta_0^{-2} \right\} \bar{i}_0^{-1} + O(n^{-1}),
\end{aligned}$$

(vii) the sixth term in  $\left[ \begin{smallmatrix} \cdot \\ (A) \end{smallmatrix} \right]_{(A)}$  of (S7.12) is

$$\begin{aligned}
&\{ 4(\alpha_{ML1}^{(A)} - 2q)\alpha_{ML3}^{(A)} + 6\alpha_{ML2}^{(A)}\alpha_{ML\Delta 2}^{(A)} + 6\alpha_{ML2}^{(A)}(\alpha_{ML1}^{(A)} - 2q)^2 \} (\alpha_{ML2}^{(A)})^{-2} \\
&= [ 4 \times (1 - 2) \{ -8\theta_0^3(1 - 2\pi_0) - 24\theta_0^2 \} \bar{i}_0 + 6(4\theta_0^2 \bar{i}_0) 2 \\
&\quad + 6(4\theta_0^2 \bar{i}_0)(1 - 2)^2 ] (4\theta_0^2 \bar{i}_0)^{-2} \\
&= \left\{ 2\theta_0^{-1}(1 - 2\pi_0) + \left( 6 + 3 + \frac{3}{2} \right) \theta_0^{-2} \right\} \bar{i}_0^{-1} = \left\{ 2\theta_0^{-1}(1 - 2\pi_0) + \frac{21}{2}\theta_0^{-2} \right\} \bar{i}_0^{-1},
\end{aligned}$$

(viii) the sum of the seventh to ninth terms in  $\left[ \begin{matrix} \cdot \\ (A) \end{matrix} \right]_{(A)}$  of (S7.12) is

$$\begin{aligned}
 & -4\{\alpha_{(t)ML1}^{(A)} - 2q(\alpha_{ML2}^{(A)})^{-1/2}\}\alpha_{(t)ML3}^{(A)} \\
 & -6\{\alpha_{(t)ML\Delta 2}^{(A)} - 4qnE_f(\bar{l}_{ML}^{(1)}\mathbf{m}_v^{(1)})\mathbf{v}^{(1)}(\alpha_{ML2}^{(A)})^{-1/2}\} - 6\{\alpha_{(t)ML1}^{(A)} - 2q(\alpha_{ML2}^{(A)})^{-1/2}\}^2 \\
 & = -4\left[\left\{\frac{3}{2}\theta_0^{-1} + \frac{1}{2}(1-2\pi_0)\right\}\bar{i}_0^{-1/2} - 2(4\theta_0^2\bar{i}_0)^{-1/2}\right]\{2(1-2\pi_0) + 3\theta_0^{-1}\}\bar{i}_0^{-1/2} \\
 & - 6\left[\left\{\frac{7}{4}(1-2\pi_0)^2 + \left(-\frac{0}{4} + \frac{9}{2}\right)\theta_0^{-1}(1-2\pi_0) + \frac{11}{2}\theta_0^{-2}\right\}\bar{i}_0^{-1} + 2\right. \\
 & \quad \left. - 4\left\{\frac{1}{2}(1-2\pi_0) + \theta_0^{-1}\right\}\bar{i}_0^{-1/2}(4\theta_0^2\bar{i}_0)^{-1/2}\right] \tag{B} \\
 & - 6\left[\left\{\frac{3}{2}\theta_0^{-1} + \frac{1}{2}(1-2\pi_0)\right\}\bar{i}_0^{-1/2} - 2(4\theta_0^2\bar{i}_0)^{-1/2}\right]^2 \\
 & = -4\left\{\frac{1}{2}\theta_0^{-1} + \frac{1}{2}(1-2\pi_0)\right\}\{2(1-2\pi_0) + 3\theta_0^{-1}\}\bar{i}_0^{-1} \\
 & - 6\left[\left\{\frac{7}{4}(1-2\pi_0)^2 + \left(\frac{9}{2} - 1\right)\theta_0^{-1}(1-2\pi_0) + \left(\frac{11}{2} - 2\right)\theta_0^{-2}\right\}\bar{i}_0^{-1} + 2\right] \\
 & - 6\left\{\frac{1}{2}\theta_0^{-1} + \frac{1}{2}(1-2\pi_0)\right\}^2\bar{i}_0^{-1} \\
 & = -\{4(1-2\pi_0)^2 + 10\theta_0^{-1}(1-2\pi_0) + 6\theta_0^{-2}\}\bar{i}_0^{-1} \\
 & - \left\{\frac{21}{2}(1-2\pi_0)^2 + 21\theta_0^{-1}(1-2\pi_0) + 21\theta_0^{-2}\right\}\bar{i}_0^{-1} - 12 \\
 & - \left\{\frac{3}{2}(1-2\pi_0)^2 + 3\theta_0^{-1}(1-2\pi_0) + \frac{3}{2}\theta_0^{-2}\right\}\bar{i}_0^{-1}
 \end{aligned}$$

$$\begin{aligned}
&= - \left[ \left( 4 + \frac{21}{2} + \frac{3}{2} \right) (1 - 2\pi_0)^2 + (10 + 21 + 3)\theta_0^{-1}(1 - 2\pi_0) \right. \\
&\quad \left. + \left( 6 + 21 + \frac{3}{2} \right) \theta_0^{-2} \right] \bar{i}_0^{-1} - 12 \\
&= - \left\{ 16(1 - 2\pi_0)^2 + 34\theta_0^{-1}(1 - 2\pi_0) + \frac{57}{2}\theta_0^{-2} \right\} \bar{i}_0^{-1} - 12.
\end{aligned}$$

From (i) to (viii),

$$\begin{aligned}
\kappa_{f4}(t_w^{(A)}) &= n^{-1} \left[ \begin{aligned} &\{ (1 - 2\pi_0)^2 + 8\theta_0^{-1}(1 - 2\pi_0) + 12\theta_0^{-2} \} \bar{i}_0^{-1} - 2 \\ &- \{ 20(1 - 2\pi_0)^2 + 40\theta_0^{-1}(1 - 2\pi_0) \} \bar{i}_0^{-1} + 24 - \{ 45\theta_0^{-1}(1 - 2\pi_0) + 90\theta_0^{-2} \} \bar{i}_0^{-1} \\ &+ \left\{ \frac{45}{2}(1 - 2\pi_0)^2 + 90\theta_0^{-1}(1 - 2\pi_0) + 90\theta_0^{-2} \right\} \bar{i}_0^{-1} \\ &- \{ 15\theta_0^{-1}(1 - 2\pi_0) + 30\theta_0^{-2} \} \bar{i}_0^{-1} \\ &+ \left\{ \frac{45}{2}(1 - 2\pi_0)^2 + 60\theta_0^{-1}(1 - 2\pi_0) + 60\theta_0^{-2} \right\} \bar{i}_0^{-1} \\ &+ \left\{ 2\theta_0^{-1}(1 - 2\pi_0) + \frac{21}{2}\theta_0^{-2} \right\} \bar{i}_0^{-1} \\ &- \left[ \left\{ 16(1 - 2\pi_0)^2 + 34\theta_0^{-1}(1 - 2\pi_0) + \frac{57}{2}\theta_0^{-2} \right\} \bar{i}_0^{-1} - 12 \right] \end{aligned} \right] + O(n^{-2}) \\
&\hspace{15em} \text{(A)}
\end{aligned}$$

$$\begin{aligned}
&= n^{-1} \left[ \left[ \left( 1 - 20 + \frac{45}{2} + \frac{45}{2} - 16 \right) (1 - 2\pi_0)^2 \right. \right. \\
&\quad \left. \left. + (8 - 40 - 45 + 90 - 15 + 60 + 2 - 34) \theta_0^{-1} (1 - 2\pi_0) \right. \right. \\
&\quad \left. \left. + \left( 12 - 90 + 90 - 30 + 60 + \frac{21}{2} - \frac{57}{2} \right) \theta_0^{-2} \right] \bar{i}_0^{-1} \right. \\
&\quad \left. - 2 + 24 - 12 \right] + O(n^{-2}) \tag{S7.13} \\
&= n^{-1} [ \{ 10(1 - 2\pi_0)^2 + 26\theta_0^{-1}(1 - 2\pi_0) + 24\theta_0^{-2} \} \bar{i}_0^{-1} + 10 ] + O(n^{-2}) \\
&= n^{-1} \alpha_{(t)ML4}^{(A)} + O(n^{-2}).
\end{aligned}$$

### S7.2.3 A result for estimation of $-2\bar{l}_0^*$

$$\begin{aligned}
&n \text{acov}_f \left\{ n^{-1} \text{AIC}_W, \hat{\alpha}_{(t)W1}^{(A)} + \frac{\hat{\alpha}_{(t)W3}^{(A)}}{6} (z_{\bar{\alpha}}^2 - 1) \right\} \\
&= n \text{acov}_f \left\{ n^{-1} \text{AIC}_{ML}, \hat{\alpha}_{(t)ML1}^{(A)} + \frac{\hat{\alpha}_{(t)ML3}^{(A)}}{6} (z_{\bar{\alpha}}^2 - 1) \right\} \\
&= n \text{acov}_f \left\{ n^{-1} \text{AIC}_{ML}, \hat{\alpha}_{ML1}^{(A)} (\hat{\alpha}_{ML2}^{(A)})^{-1/2} + \hat{\alpha}_{(\Delta t)ML1}^{(A)} z_{\bar{\alpha}}^2 \right. \\
&\quad \left. + \frac{1}{6} \hat{\alpha}_{ML3}^{(A)} (\hat{\alpha}_{ML2}^{(A)})^{-3/2} (z_{\bar{\alpha}}^2 - 1) \right\}, \tag{S7.14}
\end{aligned}$$

(i) where the first term on the right-hand side of (S7.14) is

$$\begin{aligned}
&n \text{acov}_f \{ n^{-1} \text{AIC}_{ML}, \hat{\alpha}_{ML1}^{(A)} (\hat{\alpha}_{ML2}^{(A)})^{-1/2} \} \\
&= n \text{acov}_f \{ \bar{l}_{ML}^{(1)}, (\hat{\alpha}_{ML2}^{(A)})^{-1/2} \} = n E_f \{ (\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} \} \quad (\hat{\alpha}_{ML1}^{(A)} = \alpha_{ML1}^{(A)} = 1) \\
&= \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2},
\end{aligned}$$

(ii) the second term on the right-hand side of (S7.14) is

$$\begin{aligned}
 & n \operatorname{acov}_f \left\{ n^{-1} \operatorname{AIC}_{\text{ML}}, \hat{\alpha}_{(\text{A})\text{ML}1}^{(\text{A})} z_{\hat{\alpha}}^2 \right\} \\
 &= n \operatorname{acov}_f \left\{ \bar{l}_{\text{ML}}^{(1)}, n \widehat{\operatorname{E}}_f \left( \bar{l}_{\text{ML}}^{(1)} \mathbf{m}_v^{(1)'} \right) \widehat{\mathbf{v}}^{(1)} \right\} z_{\hat{\alpha}}^2 \\
 &= n \operatorname{acov}_f \left[ \bar{l}_{\text{ML}}^{(1)}, \left\{ \frac{1}{2} (1 - 2\hat{\pi}_{\text{ML}}) + \hat{\theta}_{\text{ML}}^{-1} \right\} \hat{i}_{\text{ML}}^{-1/2} \right] z_{\hat{\alpha}}^2 \\
 &= n \operatorname{acov}_f \left[ -2\theta_0 \bar{x}, \left\{ \frac{1}{2} (1 - 2\bar{x}) + \left( \log \frac{\bar{x}}{1 - \bar{x}} \right)^{-1} \right\} \{\bar{x}(1 - \bar{x})\}^{-1/2} \right] z_{\hat{\alpha}}^2 \\
 &= -2\theta_0 n \operatorname{var}_f(\bar{x}) \left[ \left\{ -1 - \theta_0^{-2} \left( \frac{1}{\pi_0} + \frac{1}{1 - \pi_0} \right) \right\} \bar{i}_0^{-1/2} \right. \\
 &\quad \left. + \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\} \left\{ -\frac{1}{2} \bar{i}_0^{-3/2} (1 - 2\pi_0) \right\} \right] z_{\hat{\alpha}}^2 \\
 &= 2\theta_0 \bar{i}_0 \left[ (1 + \theta_0^{-2} \bar{i}_0^{-1}) \bar{i}_0^{-1/2} + \left\{ \frac{1}{4} (1 - 2\pi_0)^2 \bar{i}_0^{-3/2} + \frac{1}{2} \theta_0^{-1} (1 - 2\pi_0) \bar{i}_0^{-3/2} \right\} \right] z_{\hat{\alpha}}^2 \\
 &= \left\{ \frac{1}{2} (1 - 2\pi_0)^2 \theta_0 \bar{i}_0^{-1/2} + (1 - 2\pi_0) \bar{i}_0^{-1/2} + 2(\theta_0 \bar{i}_0^{1/2} + \theta_0^{-1} \bar{i}_0^{-1/2}) \right\} z_{\hat{\alpha}}^2
 \end{aligned}$$

(iii) the third term on the right-hand side of (S7.14)

$$\begin{aligned}
 & n \operatorname{acov}_f \left\{ n^{-1} \operatorname{AIC}_{\text{ML}}, \frac{1}{6} \hat{\alpha}_{\text{ML}3}^{(\text{A})} (\hat{\alpha}_{\text{ML}2}^{(\text{A})})^{-3/2} (z_{\hat{\alpha}}^2 - 1) \right\} \\
 &= n \operatorname{acov}_f \left[ \bar{l}_{\text{ML}}^{(1)}, \left\{ -8\hat{\theta}_{\text{ML}}^3 (1 - 2\hat{\pi}_{\text{ML}}) - 24\hat{\theta}_{\text{ML}}^2 \right\} \hat{i}_{\text{ML}} (4\hat{\theta}_{\text{ML}}^2 \hat{i}_{\text{ML}})^{-3/2} \right] \frac{z_{\hat{\alpha}}^2 - 1}{6} \\
 &= 2\theta_0 n \operatorname{acov}_f \left[ \bar{x}, \left\{ 1 - 2\bar{x} + 3 \left( \log \frac{\bar{x}}{1 - \bar{x}} \right)^{-1} \right\} \{\bar{x}(1 - \bar{x})\}^{-1/2} \right] \frac{z_{\hat{\alpha}}^2 - 1}{6}
 \end{aligned}$$

$$\begin{aligned}
&= 2\theta_0 \bar{i}_0 \left[ \left\{ -2 - 3\theta_0^{-2} \left( \frac{1}{\pi_0} + \frac{1}{1-\pi_0} \right) \right\} \bar{i}_0^{-1/2} \right. \\
&\quad \left. + (1 - 2\pi_0 + 3\theta_0^{-1}) \left\{ -\frac{1}{2} \bar{i}_0^{-3/2} (1 - 2\pi_0) \right\} \right] \frac{z_{\bar{\alpha}}^2 - 1}{6} \\
&= -2\theta_0 \bar{i}_0 \left\{ (2 + 3\theta_0^{-2} \bar{i}_0^{-1}) \bar{i}_0^{-1/2} + \frac{1}{2} (1 - 2\pi_0 + 3\theta_0^{-1}) (1 - 2\pi_0) \bar{i}_0^{-3/2} \right\} \\
&\quad \times \frac{z_{\bar{\alpha}}^2 - 1}{6} \\
&= -\left\{ \theta_0 \bar{i}_0^{-1/2} (1 - 2\pi_0)^2 + 3\bar{i}_0^{-1/2} (1 - 2\pi_0) + 4\theta_0 \bar{i}_0^{-1/2} + 6\theta_0^{-1} \bar{i}_0^{-1/2} \right\} \frac{z_{\bar{\alpha}}^2 - 1}{6}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&n \text{acov}_f \left\{ n^{-1} \text{AIC}_w, \hat{\alpha}_{(t)W1}^{(A)} + \frac{\hat{\alpha}_{(t)W3}^{(A)}}{6} (z_{\bar{\alpha}}^2 - 1) \right\} \\
&= \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2} \\
&\quad + \left\{ \frac{1}{2} \theta_0 \bar{i}_0^{-1/2} (1 - 2\pi_0)^2 + \bar{i}_0^{-1/2} (1 - 2\pi_0) + 2\theta_0 \bar{i}_0^{-1/2} + 2\theta_0^{-1} \bar{i}_0^{-1/2} \right\} z_{\bar{\alpha}}^2 \\
&\quad - \left\{ \frac{1}{6} \theta_0 \bar{i}_0^{-1/2} (1 - 2\pi_0)^2 + \frac{1}{2} \bar{i}_0^{-1/2} (1 - 2\pi_0) + \frac{2}{3} \theta_0 \bar{i}_0^{-1/2} + \theta_0^{-1} \bar{i}_0^{-1/2} \right\} \\
&\quad \times (z_{\bar{\alpha}}^2 - 1)
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \left( \frac{1}{2} - \frac{1}{6} \right) z_{\tilde{\alpha}}^2 + \frac{1}{6} \right\} \theta_0 \bar{i}_0^{-1/2} (1 - 2\pi_0)^2 \\
&\quad + \left\{ \left( 1 - \frac{1}{2} \right) z_{\tilde{\alpha}}^2 + \frac{1}{2} + \frac{1}{2} \right\} \bar{i}_0^{-1/2} (1 - 2\pi_0) + \left\{ \left( 2 - \frac{2}{3} \right) z_{\tilde{\alpha}}^2 + \frac{2}{3} \right\} \theta_0 \bar{i}_0^{-1/2} \\
&\quad + \{(2-1)z_{\tilde{\alpha}}^2 + 1 + 1\} \theta_0^{-1} \bar{i}_0^{-1/2} \\
&= \left( \frac{1}{3} z_{\tilde{\alpha}}^2 + \frac{1}{6} \right) \theta_0 \bar{i}_0^{-1/2} (1 - 2\pi_0)^2 + \left( \frac{1}{2} z_{\tilde{\alpha}}^2 + 1 \right) \bar{i}_0^{-1/2} (1 - 2\pi_0) \\
&\quad + \left( \frac{4}{3} z_{\tilde{\alpha}}^2 + \frac{2}{3} \right) \theta_0 \bar{i}_0^{-1/2} + (z_{\tilde{\alpha}}^2 + 2) \theta_0^{-1} \bar{i}_0^{-1/2}.
\end{aligned} \tag{S7.15}$$

**S7.2.4 Asymptotic cumulants of  $n^{-1}\text{AIC}_{\text{ML}}$  after studentization for estimation of  $-2E_f(\hat{l}_W^*)$**

$$\begin{aligned}
t_W^{(A)*} &= \frac{n^{1/2} \{n^{-1}\text{AIC}_W + 2E_f(\hat{l}_W^*)\}}{(\hat{v}_W^{(A)})^{1/2}}. \\
\kappa_{f1}(t_W^{(A)*}) &= n^{-1/2} \{ \alpha_{(t)W1}^{(A)} + \lambda^{-1} \gamma(\alpha_{W2}^{(A)})^{-1/2} \} + O(n^{-3/2}) \\
&= n^{-1/2} \{ \alpha_{(t)ML1}^{(A)} + \lambda^{-1} \gamma(\alpha_{ML2}^{(A)})^{-1/2} \} + O(n^{-3/2}) \\
&= n^{-1/2} \alpha_{(\Delta t)ML1}^{(A)} + O(n^{-3/2}) \\
&= n^{-1/2} \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2} + O(n^{-3/2}) \\
(\alpha_{(t)W1}^{(A)*} &= \alpha_{(t)ML1}^{(A)*} = \alpha_{(\Delta t)ML1}^{(A)}),
\end{aligned} \tag{S7.16}$$

$$\begin{aligned}
\kappa_{f2}(t_W^{(A)*}) &= 1 + n^{-1} \alpha_{(t)W\Delta 2}^{(A)*} + O(n^{-2}) \\
&= 1 + n^{-1} \{ \alpha_{(t)W\Delta 2}^{(A)} + 2\lambda^{-1} \gamma(\alpha_{ML2}^{(A)})^{-1/2} n E_f(\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} \} + O(n^{-2}) \\
&= 1 + n^{-1} \{ \alpha_{(t)W\Delta 2}^{(A)} - 2(\alpha_{ML2}^{(A)})^{-1/2} n E_f(\bar{l}_{ML}^{(1)} \mathbf{m}_v^{(1)}) \mathbf{v}^{(1)} \} + O(n^{-2})
\end{aligned}$$

$$\begin{aligned}
&= 1 + n^{-1} \left[ \left\{ \frac{7}{4}(1-2\pi_0)^2 + \left( -\frac{a}{4} + \frac{9}{2} \right) \theta_0^{-1}(1-2\pi_0) + \frac{11}{2} \theta_0^{-2} \right\} \bar{i}_0^{-1} + 2 \right. \\
&\quad \left. - 2(4\theta_0^2 \bar{i}_0)^{-1/2} \left\{ \frac{1}{2}(1-2\pi_0) + \theta_0^{-1} \right\} \bar{i}_0^{-1/2} \right] + O(n^{-2}) \\
&= 1 + n^{-1} \left[ \left\{ \frac{7}{4}(1-2\pi_0)^2 + \left( -\frac{a}{4} + 4 \right) \theta_0^{-1}(1-2\pi_0) + \frac{9}{2} \theta_0^{-2} \right\} \bar{i}_0^{-1} + 2 \right] \\
&\quad + O(n^{-2}).
\end{aligned}$$

$$\kappa_{f3}(t_W^{(A)*}) = n^{-1/2} \alpha_{(t)ML3}^{(A)} + O(n^{-3/2}) \quad (\alpha_{(t)W3}^{(A)*} = \alpha_{(t)ML3}^{(A)*} = \alpha_{(t)ML3}^{(A)}),$$

$$\kappa_{f4}(t_W^{(A)*}) = n^{-1} \alpha_{(t)ML4}^{(A)} + O(n^{-2}) \quad (\alpha_{(t)W4}^{(A)*} = \alpha_{(t)ML4}^{(A)*} = \alpha_{(t)ML4}^{(A)}).$$

### S7.2.5 A result for estimation of $-2E_f(\hat{l}_{ML}^*)$

$$\begin{aligned}
&n \text{acov}_f \left\{ n^{-1} \text{AIC}_W, \hat{\alpha}_{(t)W1}^{(A)*} + \frac{\hat{\alpha}_{(t)W3}^{(A)}}{6} (z_{\bar{\alpha}}^2 - 1) \right\} \\
&= n \text{acov}_f \left\{ n^{-1} \text{AIC}_{ML}, \hat{\alpha}_{(\Delta t)ML1}^{(A)} z_{\bar{\alpha}}^2 + \frac{1}{6} \hat{\alpha}_{ML3}^{(A)} (\hat{\alpha}_{ML2}^{(A)})^{-3/2} (z_{\bar{\alpha}}^2 - 1) \right\}
\end{aligned}$$

is the sum of the second and third terms of (S7.14) for estimation of  $-2\bar{l}_0^*$  i.e.,

$$\begin{aligned}
&n \text{acov}_f \left\{ n^{-1} \text{AIC}_W, \hat{\alpha}_{(t)W1}^{(A)*} + \frac{\hat{\alpha}_{(t)W3}^{(A)}}{6} (z_{\bar{\alpha}}^2 - 1) \right\} \\
&= \left( \frac{1}{3} z_{\bar{\alpha}}^2 + \frac{1}{6} \right) \theta_0 \bar{i}_0^{-1/2} (1-2\pi_0)^2 + \left( \frac{1}{2} z_{\bar{\alpha}}^2 + \frac{1}{2} \right) \bar{i}_0^{-1/2} (1-2\pi_0) \\
&\quad + \left( \frac{4}{3} z_{\bar{\alpha}}^2 + \frac{2}{3} \right) \theta_0 \bar{i}_0^{-1/2} + (z_{\bar{\alpha}}^2 + 1) \theta_0^{-1} \bar{i}_0^{-1/2}.
\end{aligned} \tag{S7.17}$$

### S7.2.6 Higher-order bias correction of $n^{-1} \text{AIC}_W$

The higher-order bias correction of  $n^{-1} \text{AIC}_W$  under correct model



specification and canonical parametrization can be done by using  $b_2 = c_1$  as

$$\begin{aligned} n^{-1} \text{AIC}_W - n^{-2} \hat{b}_2 &= n^{-1} \text{AIC}_W - n^{-2} \hat{c}_1 \\ &= n^{-1} \text{AIC}_W - n^{-2} (a-1) \{ (1 - 2\hat{\pi}_{\text{ML}})^2 \hat{i}_{\text{ML}}^{-1} + 2 \} \\ &= n^{-1} \text{AIC}_W + n^{-2} (1-a) [(1 - 2\bar{x})^2 \{ \bar{x}(1 - \bar{x}) \}^{-1} + 2]. \end{aligned}$$

### S7.3 $n^{-1} \text{TIC}_{\text{ML}}^{(j)} (j=1, 2)$

Since  $n^{-1} \text{AIC}_{\text{ML}} = n^{-1} \text{TIC}_{\text{ML}}^{(j)} (j=1, 2)$  due to

$$n^{-1} \sum_{i=1}^n (x_i - \bar{x})^2 = \bar{x}(1 - \bar{x}) \quad \text{in this example, the results in Subsection S7.2}$$

for  $n^{-1} \text{AIC}_{\text{ML}}$  also hold for  $n^{-1} \text{TIC}_{\text{ML}}^{(j)} (j=1, 2)$ .

### References

- Ogasawara, H. (2016). Asymptotic cumulants of some information criteria. *Journal of the Japanese Society of Computational Statistics*, 29, 1-25.
- Stuart, A., & Ord, J. K. (1994). *Kendall's advanced theory of statistics: Distribution theory* (6th ed., Vol.1). London: Arnold.