HIGHER ORDER DERIVATIVES OF FUNCTIONS PARAMETRICALLY DEFINED

By

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THEOREM. Let n be a positive integer. If x and y are C^n-functions of t and $\frac{dx}{dt} \neq 0$ in a certain interval, then

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1}}{(\frac{dx}{dt})^{2n-1}} \sum_{k=1}^{n} \frac{\frac{d^k y}{dt^k}}{(k-1)!} \sum_{\substack{s_1+s_2+\cdots=n-1 \\ 1\cdot s_1+2\cdot s_2+\cdots=2n-k-1}} \frac{(-1)^{s_1}(2n-s_1-2)!\,(\frac{dx}{dt})^{s_1}(\frac{d^2x}{dt^2})^{s_2}\cdots}{(2!)^{s_2}s_2!(3!)^{s_3}s_3!\cdots}$$

is valid in the same interval.

We prepare several notations for the proof of Theorem. Set that

$$\mathfrak{P} = \{s \in {\mathbb{N}_0}^{\mathbb{N}};\, \#\{j \in \mathbb{N};\, s(j) > 0\} < \aleph_0\}, \quad \text{where } \mathbb{N}_0 = \{0\} \cup \mathbb{N}.$$

We denote for $k \in \mathbb{N}_0$ and $s \in \mathfrak{P}$,

$$M_k(s) = \textstyle\sum_{j=1}^{\infty} j^k s(j),$$

and for $l, n \in \mathbb{N}_0$,

$$\mathfrak{P}_l(n) = \{s \in \mathfrak{P};\, M_0(s) = l, M_1(s) = n\}.$$

An element s of $\mathfrak{P}_l(n)$ represents a partition of n into l parts:

$$n = \overbrace{1 + \cdots + 1}^{s(1)} + \overbrace{2 + \cdots + 2}^{s(2)} + \cdots .$$

We denote

$$x^{(j)} = \frac{d^j x}{dt^j},\ y^{(k)} = \frac{d^k y}{dt^k}$$

and for $s \in \mathfrak{P}$,

$$\nu(s) = \frac{(-1)^{s(1)}\{2\,M_0(s) - s(1)\}!}{(2!)^{s(2)}s(2)!(3!)^{s(3)}s(3)!\cdots}, \qquad \xi(s) = (x')^{s(1)}(x'')^{s(2)}\cdots .$$

Thus we are able to rewrite the theorem to the following

$$(1) \quad \frac{d^n y}{dx^n} = (-1)^{n-1}(x')^{-2n+1} \sum_{k=1}^{n} \frac{y^{(k)}}{(k-1)!} \sum_{s\in\mathfrak{P}_{n-1}(2n-k-1)} \nu(s)\,\xi(s).$$

PROOF OF (1). If $n = 1$, then we have

$$\sum_{s\in\mathfrak{P}_{n-1}(2n-2)} \nu(s)\,\xi(s) = \sum_{s\in\mathfrak{P}_0(0)} \nu(s)\,\xi(s) = 1,$$

since $s \in \mathfrak{P}_0(0)$ implies $s(j) = 0$, for all $j \in \mathbb{N}$. Accordingly, (1) becomes

$$dy/dx = y'/x'\,,$$

if $n = 1$. This is well known.

Suppose that (1) is folds for $n-1 (n \geqslant 2)$. That is

$$\frac{d^{n-1}y}{dx^{n-1}} = (-1)^{n-2}(x')^{-2n+3}\sum_{k=1}^{n-1}\frac{y^{(k)}}{(k-1)!}\sum_{s\in\mathfrak{P}_{n-2}(2n-k-3)}\nu(s)\,\xi(s). \tag{2}$$

From (2), we have

$$\begin{aligned}
\frac{d^n y}{dx^n} &= \frac{dt}{dx}\frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \qquad (3)\\
&= (x')^{-1}\Bigg[(-1)^{n-1}(2n-3)(x')^{-2n+2}x''\sum_{k=1}^{n-1}\frac{y^{(k)}}{(k-1)!}\sum_{s\in\mathfrak{P}_{n-2}(2n-k-3)}\nu(s)\,\xi(s)\\
&\qquad +(-1)^{n-2}(x')^{-2n+3}\sum_{k=1}^{n-1}\frac{y^{(k+1)}}{(k-1)!}\sum_{s\in\mathfrak{P}_{n-2}(2n-k-3)}\nu(s)\,\xi(s)\\
&\qquad +(-1)^{n-2}(x')^{-2n+3}\sum_{k=1}^{n-1}\frac{y^{(k)}}{(k-1)!}\sum_{s\in\mathfrak{P}_{n-2}(2n-k-3)}\nu(s)\,\{\xi(s)\}'\Bigg]\\
&= (-1)^{n-1}(x')^{-2n+1}\Bigg[\sum_{k=1}^{n-1}\frac{y^{(k)}}{(k-1)!}\sum_{s\in\mathfrak{P}_{n-2}(2n-k-3)}(2n-3)\nu(s)\,\xi(s)x''\\
&\qquad -\sum_{k=2}^{n}\frac{y^{(k)}}{(k-1)!}\sum_{s\in\mathfrak{P}_{n-2}(2n-k-2)}(k-1)\nu(s)\,\xi(s)x'\\
&\qquad -\sum_{k=1}^{n-1}\frac{y^{(k)}}{(k-1)!}\sum_{s\in\mathfrak{P}_{n-2}(2n-k-3)}\nu(s)\{\xi(s)\}'x'\Bigg].
\end{aligned}$$

We define s'' for $s \in \mathfrak{P}_{n-2}(2n-k-3)$ as the following

$$s''(m) = \begin{cases} s(2)+1 & \text{, if } m = 2, \\ s(m) & \text{, otherwise.} \end{cases}$$

Then we have $s'' \in \mathfrak{P}_{n-1}(2n-k-1)$, $\xi(s)x'' = \xi(s'')$ and

$$\nu(s) = \frac{2s''(2)}{\{2n-2-s''(1)\}\{2n-3-s''(1)\}}\nu(s'') .$$

So that

(4) $$\sum_{s\in\mathfrak{P}_{n-2}(2n-k-3)} (2n-3)\nu(s)\,\xi(s)x'' = \sum_{u\in\mathfrak{P}_{n-1}(2n-k-1)} \frac{(2n-3)\,2\,u(2)\,\nu(u)\,\xi(u)}{\{2n-2-u(1)\}\{2n-3-u(1)\}} .$$

We define s' for $s \in \mathfrak{P}_{n-2}(2n-k-2)$ as the following

$$s'(m) = \begin{cases} s(1)+1 & \text{, if } m = 1, \\ s(m) & \text{, otherwise.} \end{cases}$$

Then we have $s' \in \mathfrak{P}_{n-1}(2n-k-1)$, $\xi(s)x' = \xi(s')$ and

$$\nu(s) = -\frac{\nu(s')}{2n-2-s'(1)} .$$

If $k > 1$ and $u \in \mathfrak{P}_{n-1}(2n-k-1)$, then $u(1) > 0$. Hence

(5) $$\sum_{s\in\mathfrak{P}_{n-2}(2n-k-2)} (k-1)\nu(s)\,\xi(s)x' = -\sum_{u\in\mathfrak{P}_{n-1}(2n-k-1)} \frac{(k-1)\,\nu(u)\,\xi(u)}{2n-2-u(1)} ,$$

if $k > 1$.

Next we define s^{*j} for $s \in \mathfrak{P}_{n-2}(2n-k-3)$. If $j = 1$, we set

$$s^{*1} = s''.$$

If $s(j) > 0$ and $j > 1$, we set as the following

$$s^{*j}(m) = \begin{cases} s(1)+1 & \text{, if } m = 1, \\ s(j)-1 & \text{, if } m = j, \\ s(j+1)+1 & \text{, if } m = j+1, \\ s(m) & \text{, otherwise.} \end{cases}$$

We have $s^{*j} \in \mathfrak{P}_{n-1}(2n-k-1)$,

$$\nu(s) = \frac{2s^{*1}(2)}{\{2n-2-s^{*1}(1)\}\{2n-3-s^{*1}(1)\}}\nu(s^{*1})$$

and

$$\nu(s) = -\frac{(j+1)s^{*j}(j+1)}{\{2n-2-s^{*j}(1)\}\{s^{*j}(j)+1\}}\nu(s^{*j}),$$

if $s(j) > 0$ and $j > 1$. Further we get

(6)
$$\begin{aligned}
&\sum_{s \in \mathfrak{P}_{n-2}(2n-k-3)} \nu(s)\{\xi(s)\}' x' \\
&= \sum_{s \in \mathfrak{P}_{n-2}(2n-k-3)} \nu(s) \sum_{s(j)>0} s(j)\, \xi(s^{*j}) \\
&= \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \left[\frac{2\, u(1)\, u(2)\, \nu(u)\, \xi(u)}{\{2n-2-u(1)\}\{2n-3-u(1)\}} \right. \\
&\qquad \left. - \sum_{j>1, u(j+1)>0} \frac{(j+1)\, u(j+1)\, \nu(u)\, \xi(u)}{2n-2-u(1)} \right].
\end{aligned}$$

From (3), (4), (5), and (6), we have

$$\begin{aligned}
\frac{d^n y}{dx^n} &= (-1)^{n-1}(x')^{-2n+1} \\
&\cdot \left[\sum_{k=1}^{n-1} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \frac{(2n-3)\, 2\, u(2)\, \nu(u)\, \xi(u)}{\{2n-2-u(1)\}\{2n-3-u(1)\}} \right. \\
&+ \sum_{k=2}^{n} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \frac{(k-1)\, \nu(u)\, \xi(u)}{2n-2-u(1)} \\
&- \sum_{k=1}^{n-1} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \left\{ \frac{2u(1)u(2)}{\{2n-2-u(1)\}\{2n-3-u(1)\}} \right. \\
&\qquad \left.\left. - \sum_{j>1} \frac{(j+1)\, u(j+1)}{2n-2-u(1)} \right\} \nu(u)\, \xi(u) \right]
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{n-1}(x')^{-2n+1} \\
&\quad \cdot \left[\sum_{k=1}^{n-1} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \frac{2\,u(2)\,\nu(u)\,\xi(u)}{2n-2-u(1)} \right. \\
&\quad + \sum_{k=2}^{n} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \frac{(k-1)\,\nu(u)\,\xi(u)}{2n-2-u(1)} \\
&\quad \left. + \sum_{k=1}^{n-1} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \sum_{j>1} \frac{(j+1)\,u(j+1)}{2n-2-u(1)} \nu(u)\,\xi(u) \right] \\
&= (-1)^{n-1}(x')^{-2n+1} \\
&\quad \cdot \left[\sum_{k=1}^{n-1} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \sum_{j>0} \frac{(j+1)\,u(j+1)}{2n-2-u(1)} \nu(u)\,\xi(u) \right. \\
&\quad \left. + \sum_{k=2}^{n} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \frac{(k-1)\,\nu(u)\,\xi(u)}{2n-2-u(1)} \right] \\
&= (-1)^{n-1}(x')^{-2n+1} \\
&\quad \cdot \left[\sum_{k=1}^{n-1} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \frac{2n-k-1-u(1)}{2n-2-u(1)} \nu(u)\,\xi(u) \right. \\
&\quad \left. + \sum_{k=2}^{n} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \frac{(k-1)\,\nu(u)\,\xi(u)}{2n-2-u(1)} \right] \\
&= (-1)^{n-1}(x')^{-2n+1} \sum_{k=1}^{n} \frac{y^{(k)}}{(k-1)!} \sum_{u \in \mathfrak{P}_{n-1}(2n-k-1)} \nu(u)\,\xi(u).
\end{aligned}
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This completes the proof.

Remark. In the theorem, by set $y = t$, we get the formula for inverse functions(see [1]).

Reference

[1] Kaneiwa, R., The Formula for Higher Order Derivatives of Inverse Functions, The Review of Liberal Arts, No.131, 1-3 (2016), Otaru University of Commerce.