Economic Review (Otaru University of Commerce),
Vol. 67, No. 4, 5-67, March, 2017.

Supplement II to the paper "Asymptotic cumulants of some information criteria" －Asymptotic cumulants of the studentized information criteria and Example 1

Haruhiko Ogasawara

This article is to supplement Ogasawara (2016) with asymptotic cumulants of the studentized information criteria and Example 1.

S1. Asymptotic cumulants of the studentized estimators of $-2\bar{l}_0^*$

S1.1 $n^{-1}\mathrm{AIC}_{\mathrm{W}}$

Asymptotic cumulants of $t_{\mathrm{W}}^{(\mathrm{A})} = n^{1/2}(n^{-1}\mathrm{AIC}_{\mathrm{W}} + 2\bar{l}_0^*) / (\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}$ under possible model misspecification are obtained in this section. Define

$$m_v \equiv v_0^{(\mathrm{A})} - \mathrm{E}_g(v_0^{(\mathrm{A})}) = v_0^{(\mathrm{A})} - \alpha_{\mathrm{ML2}}^{(\mathrm{A})} = O_p(n^{-1/2})\,. \qquad \text{(S1.1)}$$

Then, the reciprocal of the denominator of $t_{\mathrm{W}}^{(\mathrm{A})}$ is expanded as

$$\begin{aligned}
&(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{-1/2} \\
&= (v_0^{(\mathrm{A})})^{-1/2} - \frac{1}{2}(v_0^{(\mathrm{A})})^{-3/2}\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0) \\
&\quad - \left\{\frac{1}{4}(v_0^{(\mathrm{A})})^{-3/2}\frac{\partial^2 v^{(\mathrm{A})}}{(\partial \boldsymbol{\theta}_0{}')^{<2>}} - \frac{3}{8}(v_0^{(\mathrm{A})})^{-5/2}\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)^{<2>}\right\}(\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)^{<2>} \\
&\quad + O_p(n^{-3/2})
\end{aligned} \qquad \text{(S1.2)}$$

$$
\begin{aligned}
&= (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} - \frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} m_v + \frac{3}{8}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2} m_v^2 \\
&\quad - \left\{ \frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} - \frac{3}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2} m_v \right\} \\
&\qquad \times \left[\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right) + \left\{ \frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} - \mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right) \right\}_{O_p(n^{-1/2})} \right] \\
&\times \Bigg[_{(\mathrm{A})} - \left(\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_{O_p(n^{-1/2})} - n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \Bigg\{ \boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \\
&\qquad - \frac{1}{2}\boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \Bigg\}_{O_p(n^{-1})} \Bigg]_{(\mathrm{A})} \\
&- \Bigg[_{(\mathrm{B})} \frac{1}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial^2 v^{(\mathrm{A})}}{(\partial \boldsymbol{\theta}_0{}')^{<2>}} \right) - \frac{3}{8}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2}\mathrm{E}_g\left\{ \left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} \right)^{<2>} \right\} \Bigg]_{(\mathrm{B})} \\
&\times \left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} + O_p(n^{-3/2})
\end{aligned}
$$

$$= (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \left[\frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\left\{-m_v + \mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right\}\right]_{O_p(n^{-1/2})}$$

$$+\left\{n^{-1}\frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*\right\}_{O(n^{-1})}$$

$$+\Big[_{(\mathrm{A})}\ \frac{3}{8}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2}m_v^2 - \frac{3}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2}m_v\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}$$

$$-\frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} + \Big[_{(\mathrm{B})}\ \frac{1}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$

$$\times\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)}) - \frac{1}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial^2 v^{(\mathrm{A})}}{(\partial \boldsymbol{\theta}_0{}')^{<2>}}\right)$$

$$+\frac{3}{8}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2}\mathrm{E}_g\left\{\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)^{<2>}\right\}\ \Big]_{(\mathrm{B})}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)^{<2>}$$

$$+\frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\left\{\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} - \mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)\right\}\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\ \Big]_{(\mathrm{A})O_p(n^{-1})} + O_p(n^{-3/2})$$

$$\equiv (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + (n^{-1}\eta_{\mathrm{W}}^{(v)})_{O(n^{-1})} + \sum_{j=1}^{2}\mathbf{v}^{(j)}{}'\mathbf{m}_v^{(j)} + O_p(n^{-3/2}),$$

where

$$\eta_{\mathrm{W}}^{(v)} = \frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*,$$

$$\mathbf{m}_v^{(1)} = \left(m_v, \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'} \right)',$$

$$\mathbf{m}_v^{(2)} = \Bigg[m_v^2,\ m_v \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'}, \mathrm{vec}'(\mathbf{M}) \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'}, \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'} \right)^{<2>}, \qquad \text{(S1.3)}$$

$$\left\{ \frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} - \mathrm{E}_g \left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} \right) \right\} \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'} \Bigg]'$$

$$\mathbf{v}^{(1)} = \frac{1}{2} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \left\{ -1,\ \mathrm{E}_g \left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} \right) \boldsymbol{\Lambda}^{-1} \right\}',$$

$$\mathbf{v}^{(2)} = \Bigg[_{(\mathrm{A})} \ \frac{3}{8} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2},\ -\frac{3}{4} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2} \mathrm{E}_g \left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} \right) \boldsymbol{\Lambda}^{-1},$$

$$-\frac{1}{2} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \left[\left\{ \mathrm{E}_g \left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} \right) \boldsymbol{\Lambda}^{-1} \right\} \otimes \mathrm{vec}'(\boldsymbol{\Lambda}^{-1}) \right],$$

$$\Bigg[_{(\mathrm{B})} \ \frac{1}{4} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \mathrm{E}_g \left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} \right) \boldsymbol{\Lambda}^{-1} \mathrm{E}_g (\mathbf{J}_0^{(3)}) - \frac{1}{4} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \mathrm{E}_g \left(\frac{\partial^2 v^{(\mathrm{A})}}{(\partial \boldsymbol{\theta}_0{}')^{<2>}} \right)$$

$$+ \frac{3}{8} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2} \mathrm{E}_g \left\{ \left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'} \right)^{<2>} \right\} + O(n^{-1}) \ \Bigg]_{(\mathrm{B})} (\boldsymbol{\Lambda}^{-1})^{<2>},$$

$$\frac{1}{2} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \mathrm{vec}'(\boldsymbol{\Lambda}^{-1}) \ \Bigg]_{(\mathrm{A})}'.$$

Then, the expansion of $t_{\mathrm{W}}^{(\mathrm{A})}$ is summarized as

$$t_{\mathrm{W}}^{(\mathrm{A})} = n^{1/2}(n^{-1}\mathrm{AIC}_{\mathrm{W}} + 2\bar{l}_0^{*}) / (\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}$$

$$= n^{1/2}\left(\sum_{j=1}^{3}\bar{l}_{\mathrm{ML}}^{(j)} + n^{-1}2q\right)\left\{(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + (n^{-1}\eta_{\mathrm{W}}^{(v)})_{O(n^{-1})} + \sum_{j=1}^{2}\mathbf{v}^{(j)}{}'\mathbf{m}_{v}^{(j)}\right\} \quad \text{(S1.4)}$$

$$+ O_p(n^{-3/2})$$

$$\equiv n^{1/2}\left\{\sum_{j=1}^{3}(l_{\mathrm{WA}}^{(tj)})_{O_p(n^{-j/2})} + n^{-1}2q(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\right\} + O_p(n^{-3/2}),$$

$$l_{\mathrm{WA}}^{(t1)} = l_{\mathrm{MLA}}^{(t1)} = \bar{l}_{\mathrm{ML}}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2},$$

$$l_{\mathrm{WA}}^{(t2)} = l_{\mathrm{MLA}}^{(t2)} = \bar{l}_{\mathrm{ML}}^{(2)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_{v}^{(1)},$$

$$l_{\mathrm{WA}}^{(t3)} = \bar{l}_{\mathrm{ML}}^{(3)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + (\bar{l}_{\mathrm{ML}}^{(2)} + n^{-1}2q)\mathbf{v}^{(1)}{}'\mathbf{m}_{v}^{(1)} + \bar{l}_{\mathrm{ML}}^{(1)}(n^{-1}\eta_{\mathrm{W}}^{(v)} + \mathbf{v}^{(2)}{}'\mathbf{m}_{v}^{(2)}).$$

First, some preliminary results are given as

$\mathrm{E}_g(v_0^{(\mathrm{A})}) = \alpha_{\mathrm{ML2}}^{(\mathrm{A})}$ with

$$v_0^{(\mathrm{A})} = 4(n-1)^{-1}\sum_{j=1}^{n}(l_{0j} - \bar{l}_0)^2$$

$$= 4\{2(n^2-n)\}^{-1}\sum_{j,k=1}^{n}(l_{0j} - l_{0k})^2 = 2(n^2-n)^{-1}\sum_{j,k=1}^{n}(l_{0j} - l_{0k})^2, \quad \text{(S1.5)}$$

$$\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0}\right) = 4(n^2-n)^{-1}\mathrm{E}_g\left\{\sum_{j,k=1}^{n}(l_{0j} - l_{0k})\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0} - \frac{\partial l_k}{\partial \boldsymbol{\theta}_0}\right)\right\}$$

$$= 4(n^2-n)^{-1}2(n^2-n)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)$$

$$= 8\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right) = 8\,\mathrm{cov}_g\left(l_{0j}, \frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right),$$

$$\mathrm{E}_g\left(\frac{\partial^2 v^{(\mathrm{A})}}{(\partial \boldsymbol{\theta}_0)^{<2>}}\right)=4(n^2-n)^{-1}\mathrm{E}_g\left[\sum_{j,k=1}^{n}\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}-\frac{\partial l_k}{\partial \boldsymbol{\theta}_0}\right)^{<2>}\right.$$

$$\left.+(l_{0j}-l_{0k})\left(\frac{\partial^2 l_j}{(\partial \boldsymbol{\theta}_0)^{<2>}}-\frac{\partial^2 l_k}{(\partial \boldsymbol{\theta}_0)^{<2>}}\right)\right]$$

$$=4(n^2-n)^{-1}2(n^2-n)\left[\mathrm{E}_g\left\{\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)^{<2>}\right\}+\mathrm{E}_g\left\{l_{0j}\frac{\partial^2 l_j}{(\partial \boldsymbol{\theta}_0)^{<2>}}\right\}\right.$$

$$\left.-\mathrm{E}_g(l_{0j})\mathrm{E}_g\left\{\frac{\partial^2 l_j}{(\partial \boldsymbol{\theta}_0)^{<2>}}\right\}\right]$$

$$=8\mathrm{vec}\left[\boldsymbol{\Gamma}+\mathrm{E}_g\left\{l_{0j}\frac{\partial^2 l_j}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0{}'}\right\}-\bar{l}_0^{*}\boldsymbol{\Lambda}\right],$$

$$\mathrm{E}_g\left\{\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0}\right)^{<2>}\right\}=16(n^2-n)^{-2}\mathrm{E}_g\left[\left\{\sum_{j,k=1}^{n}(l_{0j}-l_{0k})\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}-\frac{\partial l_k}{\partial \boldsymbol{\theta}_0}\right)\right\}^{<2>}\right]$$

$$=16(n^2-n)^{-2}$$

$$\times\mathrm{E}_g\left\{\sum_{j,k,l^*,m^*=1}^{n}(l_{0j}-l_{0k})(l_{0l^*}-l_{0m^*})\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}-\frac{\partial l_k}{\partial \boldsymbol{\theta}_0}\right)\otimes\left(\frac{\partial l_{l^*}}{\partial \boldsymbol{\theta}_0}-\frac{\partial l_{m^*}}{\partial \boldsymbol{\theta}_0}\right)\right\}$$

$$=16(n^2-n)^{-2}(n^2-n)\mathrm{E}_g\left\{(l_{01}-l_{02})\left(\frac{\partial l_1}{\partial \boldsymbol{\theta}_0}-\frac{\partial l_2}{\partial \boldsymbol{\theta}_0}\right)\right.$$

$$\left.\otimes\sum_{l^*,m^*=1}^{n}(l_{0l^*}-l_{0m^*})\left(\frac{\partial l_{l^*}}{\partial \boldsymbol{\theta}_0}-\frac{\partial l_{m^*}}{\partial \boldsymbol{\theta}_0}\right)\right\}$$

$$= 16(n^2 - n)^{-1}\mathrm{E}_g \underset{(A)}{\Bigg[} \left\{ (l_{01} - l_{02})\left(\frac{\partial l_1}{\partial \boldsymbol{\theta}_0} - \frac{\partial l_2}{\partial \boldsymbol{\theta}_0}\right)\right.$$

$$\otimes \underset{(B)}{\Bigg[} 2(l_{01} - l_{02})\left(\frac{\partial l_1}{\partial \boldsymbol{\theta}_0} - \frac{\partial l_2}{\partial \boldsymbol{\theta}_0}\right) + 2(n-2)(l_{01} - l_{03})\left(\frac{\partial l_1}{\partial \boldsymbol{\theta}_0} - \frac{\partial l_3}{\partial \boldsymbol{\theta}_0}\right)$$

$$+ 2(n-2)(l_{02} - l_{03})\left(\frac{\partial l_2}{\partial \boldsymbol{\theta}_0} - \frac{\partial l_3}{\partial \boldsymbol{\theta}_0}\right)$$

$$+ \{n^2 - n - 2 - 4(n-2)\}(l_{03} - l_{04})\left(\frac{\partial l_3}{\partial \boldsymbol{\theta}_0} - \frac{\partial l_4}{\partial \boldsymbol{\theta}_0}\right) \underset{(B)}{\Bigg]} \underset{(A)}{\Bigg]}$$

$$= 16\left[\mathrm{E}_g \left\{ (l_{01} - l_{02})\left(\frac{\partial l_1}{\partial \boldsymbol{\theta}_0} - \frac{\partial l_2}{\partial \boldsymbol{\theta}_0}\right)\right\}\right]^{<2>} + O(n^{-1})$$

$$= 64\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)\right\}^{<2>} + O(n^{-1}) = 64\left\{\mathrm{cov}_g\left(l_{0j}, \frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)\right\}^{<2>} + O(n^{-1}),$$

$$n\,\mathrm{cov}_g(v_0^{(A)}, \bar{l}_0) = n\,\mathrm{cov}_g(m_v, \bar{l}_0)$$

$$= 2(n^2 - n)^{-1}\mathrm{E}_g\left\{\sum_{j,k,l^*=1}^{n}(l_{0j} - l_{0k})^2(l_{0l^*} - \bar{l}_0^*)\right\}$$

$$= 2(n^2 - n)^{-1}2(n^2 - n)\mathrm{E}_g\{(l_{0j} - \bar{l}_0^*)^3\} = 4\mathrm{E}_g\{(l_{0j} - \bar{l}_0^*)^3\},$$

$$n\,\mathrm{cov}_g\left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}, \bar{l}_0\right) = \mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0} l_{0j}\right),$$

$$
\begin{aligned}
& n\,\mathrm{cov}_g\left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}, m_v\right) \\
& = n\mathrm{E}_g\left[\left\{2(n^2-n)^{-1}\sum_{j,k=1}^{n}(l_{0j}-l_{0k})^2-\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\right\}n^{-1}\sum_{l^*=1}^{n}\frac{\partial l_{l^*}}{\partial \boldsymbol{\theta}_0}\right] \\
& = 2(n^2-n)^{-1}2(n^2-n)\mathrm{E}_g\left\{\sum_{j=1}^{n}(l_{0j}-\bar{l}_0^*)^2\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right\} \\
& = 4\mathrm{E}_g\left\{\sum_{j=1}^{n}(l_{0j}-\bar{l}_0^*)^2\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right\},
\end{aligned}
$$

$$
n\,\mathrm{avar}_g(m_v) = 16\left[\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^4\} - [\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^2\}]^2\right]
$$

(the asymptotic variance of an unbiased variance),

$$
\begin{aligned}
& \mathrm{vec}'\left\{n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0}, \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'}\right)\right\} \\
& = \mathrm{vec}'\Bigg[_{(\mathrm{A})}\; n\mathrm{E}_g\left[4(n^2-n)^{-1}\sum_{j,k=1}^{n}(l_{0j}-l_{0k})\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}-\frac{\partial l_k}{\partial \boldsymbol{\theta}_0}\right)-\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0}\right)\right] \\
& \qquad\qquad \times n^{-1}\sum_{l^*=1}^{n}\frac{\partial l_{l^*}}{\partial \boldsymbol{\theta}_0{}'}\Bigg]_{(\mathrm{A})} \\
& = \mathrm{vec}'\mathrm{E}_g\left\{4(l_{01}-l_{02})\left(\frac{\partial l_1}{\partial \boldsymbol{\theta}_0}-\frac{\partial l_2}{\partial \boldsymbol{\theta}_0}\right)\left(\frac{\partial l_1}{\partial \boldsymbol{\theta}_0}+\frac{\partial l_2}{\partial \boldsymbol{\theta}_0}\right)\right\} \\
& = 8\mathrm{vec}'\mathrm{E}_g\left\{(l_{0j}-\bar{l}_0^*)\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}\right\},
\end{aligned}
$$

$$4n\,\mathrm{cov}_g\{\bar{l}_0, \mathrm{vec}'(\mathbf{M})\} = 4\mathrm{E}_g\left\{(l_{0j} - \bar{l}_0^*)\frac{\partial^2 l_j}{(\partial\boldsymbol{\theta}_0')^{<2>}}\right\}$$

$$= 4\mathrm{E}_g\left\{l_{0j}\frac{\partial^2 l_j}{(\partial\boldsymbol{\theta}_0)^{<2>}}\right\} - \bar{l}_0^*\mathrm{vec}'(\boldsymbol{\Lambda}),$$

$$4n\,\mathrm{cov}_g\left(\bar{l}_0, \frac{\partial v^{(\mathrm{A})}}{\partial\boldsymbol{\theta}_0'}\right)$$

$$= 4n\mathrm{E}_g\left\{n^{-1}\sum_{a=1}^{n}(l_{0a} - \bar{l}_0^*)4(n^2-n)^{-1}\sum_{j,k=1}^{n}(l_{0j} - l_{0k})\left(\frac{\partial l_j}{\partial\boldsymbol{\theta}_0} - \frac{\partial l_k}{\partial\boldsymbol{\theta}_0}\right)\right\}$$

$$= 16\mathrm{E}_g\left[\{(l_{01} - \bar{l}_0^*) + (l_{02} - \bar{l}_0^*)\}(l_{01} - l_{02})\left(\frac{\partial l_1}{\partial\boldsymbol{\theta}_0} - \frac{\partial l_2}{\partial\boldsymbol{\theta}_0}\right)\right]$$

$$= 32\mathrm{E}_g\left\{(l_{0j} - \bar{l}_0^*)^2\frac{\partial l_j}{\partial\boldsymbol{\theta}_0}\right\}.$$

Then, we have

$$\kappa_{g1}(t_{\mathrm{W}}^{(\mathrm{A})}) = n^{1/2}\mathrm{E}_g(l_{\mathrm{MLA}}^{(t2)}) + n^{-1/2}2q(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + O(n^{-3/2})$$

$$= n^{-1/2}\left[\{n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)}) + 2q\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})\right] + O(n^{-3/2})$$

$$= n^{-1/2}\left[\alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \mathbf{v}^{(1)}{}'n\,\mathrm{cov}_g\left\{\left(v_0^{(\mathrm{A})}, \frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)', \bar{l}_{\mathrm{ML}}^{(1)}\right\}\right] + O(n^{-3/2})$$

$$= n^{-1/2}\left[\alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} - 2\mathbf{v}^{(1)}{}'n\,\mathrm{cov}_g\left\{\left(v_0^{(\mathrm{A})}, \frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)', \bar{l}_0\right\}\right] + O(n^{-3/2})$$

$$= n^{-1/2}\left[\alpha^{(\mathrm{A})}_{\mathrm{ML1}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2} - 2\mathbf{v}^{(1)}{}'\left\{4\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^3\},\ \mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}l_{0j}\right)\right\}'\right]$$

$$+O(n^{-3/2}) \tag{S1.6}$$

$$\equiv n^{-1/2}\{\alpha^{(\mathrm{A})}_{\mathrm{ML1}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2} + \alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}}\} + O(n^{-3/2})$$

$$\equiv n^{-1/2}\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}} + O(n^{-3/2})\quad (\alpha^{(\mathrm{A})}_{(t)\mathrm{W1}} = \alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}})$$

(recall that $\alpha^{(\mathrm{A})}_{\mathrm{ML1}} = n\mathrm{E}_g(\bar{l}^{(2)}_{\mathrm{ML}}) + 2q = \mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + 2q$; and note that $\mathrm{cov}_g(u^2, \bar{x}) = n^{-1}\kappa_{g3}(x^*)$, where u^2 is the usual unbiased variance, $\bar{x}$ is the sample mean, and x^* is the associated variable (see e.g., Ogasawara, 2009, Subsection 2.A.1)).

$$\kappa_{g2}(t^{(\mathrm{A})}_{\mathrm{W}}) = n\mathrm{E}_g[\{\bar{l}^{(1)}_{\mathrm{ML}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}^2]$$

$$+n^{-1}[\,2n^2\mathrm{E}_g(l^{(t1)}_{\mathrm{MLA}}l^{(t2)}_{\mathrm{MLA}}) + 2n^2\mathrm{E}_g(l^{(t1)}_{\mathrm{MLA}}l^{(t3)}_{\mathrm{MLA}}) + n^2\mathrm{E}_g\{(l^{(t2)}_{\mathrm{MLA}})^2\}$$

$$-\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}} - 2q(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}^2\,] + O(n^{-2})$$

$$= 1 + n^{-1}\Big[_{(\mathrm{A})}\ \alpha^{(\mathrm{A})}_{\mathrm{ML\Delta2}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1} + 2n^2\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^2\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}$$

$$+2n^2\mathrm{E}_g\Big[_{(\mathrm{B})}\ \bar{l}^{(1)}_{\mathrm{ML}}\{\,\bar{l}^{(2)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v + \bar{l}^{(1)}_{\mathrm{ML}}(n^{-1}\eta^{(v)}_{\mathrm{W}} + \mathbf{v}^{(2)}{}'\mathbf{m}^{(2)}_v) \tag{S1.7}$$

$$+n^{-1}2q\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v\}\ \Big]_{(\mathrm{B})}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}$$

$$+n^2\mathrm{E}_g\{2\bar{l}^{(2)}_{\mathrm{ML}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v + (\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v)^2\}$$

$$-\{2(\alpha^{(\mathrm{A})}_{\mathrm{ML1}} - 2q)(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} + (\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}})^2\}\ \Big]_{(\mathrm{A})} + O(n^{-2})$$

$$\equiv 1 + n^{-1}\alpha^{(\mathrm{A})}_{(t)\mathrm{W\Delta2}} + O(n^{-2})$$

$$(\alpha^{(\mathrm{A})}_{(t)\mathrm{W2}} = \alpha^{(\mathrm{A})}_{(t)\mathrm{ML2}} = 1,\ \alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}} - 2q(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2} = n\mathrm{E}_g(l^{(t2)}_{\mathrm{MLA}})),$$

where the first term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ is given from (4.5), the second term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ is

$$2n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=2\mathbf{v}^{(1)}{}'n^2\mathrm{E}_g\left\{\left(m_v,\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)'(\bar{l}_{\mathrm{ML}}^{(1)})^2\right\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=8\mathbf{v}^{(1)}{}'\mathrm{E}_g\left[\left\{2(n^2-n)^{-1}\sum_{j,k=1}^{n}(l_{0j}-l_{0k})^2-\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\right\}\sum_{l^*,m^*=1}^{n}(l_{0l^*}-\bar{l}_0^*)(l_{0m^*}-\bar{l}_0^*),\right.$$

$$\left.n^{-1}\sum_{j,k,l^*=1}^{n}\frac{\partial l_{0j}}{\partial\boldsymbol{\theta}_0{}'}(l_{0k}-\bar{l}_0^*)(l_{0l^*}-\bar{l}_0^*)\right]'(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=8\mathbf{v}^{(1)}{}'$$

$$\times\mathrm{E}_g\left[\{2(l_{01}-l_{02})^2-\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\}\{(l_{01}-\bar{l}_0^*)^2+(l_{02}-\bar{l}_0^*)^2+2(l_{01}-\bar{l}_0^*)(l_{02}-\bar{l}_0^*),\right.$$

$$\left.\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}(l_{0j}-\bar{l}_0^*)^2\right]'(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=8\mathbf{v}^{(1)}{}'\left[4\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^4\}-4\,\mathrm{var}_g(l_{01})\,\mathrm{var}_g(l_{02})-2\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\,\mathrm{var}_g(l_{01}),\right.$$

$$\left.\mathrm{E}_g\left\{\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}(l_{0j}-\bar{l}_0^*)^2\right\}\right]'(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=8\mathbf{v}^{(1)}{}'\left[4\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^4\}-12\{\mathrm{var}_g(l_{01})\}^2,\ \mathrm{E}_g\left\{\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}(l_{0j}-\bar{l}_0^*)^2\right\}\right]'$$
$$\times(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=8\mathbf{v}^{(1)}{}'\left[4\kappa_{g4}(l_{0j}),\ \mathrm{E}_g\left\{\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}(l_{0j}-\bar{l}_0^*)^2\right\}\right]'(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

(the scaled multivariate third cumulant of two means and an unbiased variance;

Ogasawara, 2009, Subsection 2.A.2.2),

the left term in $2n^{-2}\mathrm{E}_g \underset{\text{(B)}}{[} \cdot \underset{\text{(B)}}{]}$ (the third term in $\underset{\text{(A)}}{[} \cdot \underset{\text{(A)}}{]}$) is

$$2n^2\mathrm{E}_g\{\bar{l}_{\mathrm{ML}}^{(1)}\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=-4n^2\mathrm{E}_g\left\{(\bar{l}_0-\bar{l}_0^*)\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\mathbf{v}^{(1)}{}'\left(m_v,\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)'\right\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=-4\left[\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})\left\{n\,\mathrm{cov}_g(\bar{l}_0,m_v),\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}\right)\right\}\right.$$

$$\left.+2\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\left\{n\,\mathrm{cov}_g\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0},m_v\right),\boldsymbol{\Gamma}\right\}\right]\mathbf{v}^{(1)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}+O(n^{-1})$$

the central term in $2n^{-2}\mathrm{E}_g \underset{\text{(B)}}{[} \cdot \underset{\text{(B)}}{]}$ (the third term in $\underset{\text{(A)}}{[} \cdot \underset{\text{(A)}}{]}$) is

$$2n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2(n^{-1}\eta_{\mathrm{W}}^{(v)}+\mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)})\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=2\underset{\text{(C)}}{\Bigg[}\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\eta_{\mathrm{W}}^{(v)}+n^2\mathrm{E}_g\underset{\text{(D)}}{\Bigg[}(\bar{l}_{\mathrm{ML}}^{(1)})^2\left[m_v^2,\,m_v\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'},\mathrm{vec}'(\mathbf{M})\otimes\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'},\right.$$

$$\left.\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)^{<2>},\left\{\frac{\partial v^{(\mathrm{A})}}{\partial\boldsymbol{\theta}_0{}'}-\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\boldsymbol{\theta}_0{}'}\right)\right\}\otimes\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right]\underset{\text{(D)}}{\Bigg]}\mathbf{v}^{(2)}\underset{\text{(C)}}{\Bigg]}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=2\underset{\text{(C)}}{\Bigg[}\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\eta_{\mathrm{W}}^{(v)}+\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\underset{\text{(D)}}{\Bigg[}n\,\mathrm{avar}_g(m_v),n\,\mathrm{cov}_g\left(m_v,\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right),$$

$$\mathrm{E}_g\left\{\frac{\partial^2 l_j}{(\partial\boldsymbol{\theta}_0{}')^{<2>}}\otimes\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}\right\},\mathrm{vec}'(\boldsymbol{\Gamma}),\mathrm{vec}'n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\boldsymbol{\theta}_0},\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)\underset{\text{(D)}}{\Bigg]}\mathbf{v}^{(2)}$$

$$+2\underset{\text{(E)}}{\Bigg[}\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)}, m_v)\}^2, 4n\,\mathrm{cov}_g(\bar{l}_0, m_v)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\right),$$

$$4n\,\mathrm{cov}_g\{\bar{l}_0, \mathrm{vec}'(\mathbf{M})\}\otimes \mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\right), 4\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\right)\right\}^{<2>},$$

$$4n\,\mathrm{cov}_g\left(\bar{l}_0, \frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0'}\right)\otimes \mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\right)\underset{\text{(E)}}{\Bigg]}\mathbf{v}^{(2)}\underset{\text{(C)}}{\Bigg]}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + O(n^{-1}),$$

the right term in $2n^{-2}\mathrm{E}_g \underset{\text{(B)}}{[}\cdot\underset{\text{(B)}}{]}$ (the third term in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$) is

$$2n^2\mathrm{E}_g\{n^{-1}2q(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\} = 4q(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)},$$

the first half of the fourth term in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$ i.e.,

$n^2\mathrm{E}_g\{2\bar{l}_{\mathrm{ML}}^{(2)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}$ is equal to the left term in $2n^{-2}\mathrm{E}_g \underset{\text{(B)}}{[}\cdot\underset{\text{(B)}}{]}$

(the third term in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$),

the second half of the fourth term in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$ is

$$n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}$$

$$= \alpha_{\mathrm{ML2}}^{(\mathrm{A})}\mathbf{v}^{(1)}{}'n\,\mathrm{acov}_g(\mathbf{m}_v^{(1)})\mathbf{v}^{(1)} + 2\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)}, \mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\}^2 + O(n^{-1})$$

$$= \alpha_{\mathrm{ML2}}^{(\mathrm{A})}\mathbf{v}^{(1)}{}'\begin{bmatrix} n\,\mathrm{avar}_g(m_v) & n\,\mathrm{cov}_g(m_v, \partial\bar{l}/\partial\boldsymbol{\theta}_0') \\ n\,\mathrm{cov}_g(m_v, \partial\bar{l}/\partial\boldsymbol{\theta}_0) & \boldsymbol{\Gamma} \end{bmatrix}\mathbf{v}^{(1)}$$

$$+2\left[\left\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)}, m_v), -2\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\right)\right\}\mathbf{v}^{(1)}\right]^2 + O(n^{-1}).$$

$$\kappa_{g3}(t_{\mathrm{W}}^{(\mathrm{A})}) = n^{-1/2}[\ \alpha_{\mathrm{ML3}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} + 3n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1}$$
$$-3n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})\ ] + O(n^{-3/2})$$
$$= n^{-1/2}\{\alpha_{\mathrm{ML3}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} + 6n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})\} + O(n^{-3/2})$$
$$= n^{-1/2}\{\alpha_{\mathrm{ML3}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} + 6\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}\} + O(n^{-3/2}) \qquad \text{(S1.8)}$$
$$\equiv n^{-1/2}\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})} + O(n^{-3/2})\ \ (\alpha_{(t)\mathrm{W3}}^{(\mathrm{A})} = \alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})}).$$

$$\kappa_{g4}(t_{\mathrm{W}}^{(\mathrm{A})}) = n^{-1}\Big[\ \alpha_{\mathrm{ML4}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-2} + 4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$
$$+12n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$
$$+6n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4(\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1} \qquad \text{(S1.9)}$$
$$+4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)} + (\bar{l}_{\mathrm{ML}}^{(1)})^4\mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$
$$+\{4(\alpha_{\mathrm{ML1}}^{(\mathrm{A})} - 2q)\alpha_{\mathrm{ML3}}^{(\mathrm{A})} + 6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\alpha_{\mathrm{ML\Delta2}}^{(\mathrm{A})} + 6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}(\alpha_{\mathrm{ML1}}^{(\mathrm{A})} - 2q)^2\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-2}$$
$$-4\{\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})} - 2q(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})}$$
$$-6\{\alpha_{(t)\mathrm{ML\Delta2}}^{(\mathrm{A})} - 4qn\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}$$
$$-6\{\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})} - 2q(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}^2\ \Big]_{(\mathrm{A})} + O(n^{-2})$$
$$\equiv n^{-1}\alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})} + O(n^{-2})\ \ (\alpha_{(t)\mathrm{W4}}^{(\mathrm{A})} = \alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})})$$

(the specific contribution by the term $l_{\mathrm{WA}}^{(t3)}$ for the WSE over that by the MLE to the expansion of $t_{\mathrm{W}}^{(\mathrm{A})}$ (see (S1.4)) is canceled in the last expression, and the contribution by the term $n^{-1}2q$ is similarly canceled), where the first term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ is given from (4.5),

the second term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ is

$$4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4\,\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}$$
$$=4[\,6\alpha_{\mathrm{ML}2}^{(\mathrm{A})}n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\,\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}$$
$$+4n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\}n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})\,](\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}+O(n^{-1})$$
$$=4\Big[6\alpha_{\mathrm{ML}2}^{(\mathrm{A})}n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{m}_v^{(1)}{}'\}-32\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^3\}$$
$$\times\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},m_v),\,-2\,\mathrm{E}_g(l_{0j}\partial l_{0j}/\partial\boldsymbol{\theta}_0')\}\Big]\mathbf{v}^{(1)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}+O(n^{-1})$$

the third term in $\underset{(\mathrm{A})}{[}\,\cdot\,\underset{(\mathrm{A})}{]}$ is

$$12n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}$$
$$=12n^3\mathrm{E}_g\left\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\left(m_v,\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)\mathbf{v}^{(1)}\right\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}$$
$$=12\left[\left[\left\{3\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})+6n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)\boldsymbol{\Lambda}^{-1}n\,\mathrm{cov}_g\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0},\bar{l}_{\mathrm{ML}}^{(1)}\right)\right\}\right.\right.$$
$$\times\left\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},m_v),n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)\right\}$$
$$\left.+6\alpha_{\mathrm{ML}2}^{(\mathrm{A})}n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)\boldsymbol{\Lambda}^{-1}\left\{n\,\mathrm{cov}_g\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0},m_v\right),\boldsymbol{\Gamma}\right\}\right]\mathbf{v}^{(1)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}$$
$$+O(n^{-1}),$$

the fourth term in $\underset{(\mathrm{A})}{[}\,\cdot\,\underset{(\mathrm{A})}{]}$ is

$$6n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4(\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1}$$
$$=6[\;3(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^2\mathbf{v}^{(1)}{}'n\,\mathrm{cov}_g(\mathbf{m}_v^{(1)})\mathbf{v}^{(1)}$$
$$+12\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\}^2\;](\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1}+O(n^{-1})$$
$$=18\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\mathbf{v}^{(1)}{}'n\,\mathrm{cov}_g(\mathbf{m}_v^{(1)})\mathbf{v}^{(1)}+72\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\}^2+O(n^{-1})$$

and the fifth term in $\underset{(\mathrm{A})}{[}\,\cdot\,\underset{(\mathrm{A})}{]}$ is

$$4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)\prime}\mathbf{m}_v^{(1)}+(\bar{l}_{\mathrm{ML}}^{(1)})^4\mathbf{v}^{(2)\prime}\mathbf{m}_v^{(2)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$

$$=4\Big[_{(\mathrm{D})}\ n^3\mathrm{E}_g\left\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\left(m_v,\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)\right\}\mathbf{v}^{(1)}$$

$$+n^3\mathrm{E}_g\Big[_{(\mathrm{E})}\ (\bar{l}_{\mathrm{ML}}^{(1)})^4\Big[m_v^2,\ m_v\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'},\mathrm{vec}'(\mathbf{M})\otimes\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'},$$

$$\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)^{<2>},\left\{\frac{\partial v^{(\mathrm{A})}}{\partial\boldsymbol{\theta}_0{}'}-\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\boldsymbol{\theta}_0{}'}\right)\right\}\otimes\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\Big]\mathbf{v}^{(2)}\ \Big]_{(\mathrm{E})}\Big]_{(\mathrm{D})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$

$$=4\Big[_{(\mathrm{D})}\Big[_{(\mathrm{E})}\left\{3\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})+6n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}n\,\mathrm{cov}_g\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0},\bar{l}_{\mathrm{ML}}^{(1)}\right)\right\}$$

$$\times\left\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},m_v),n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)\right\}$$

$$+6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\left\{n\,\mathrm{cov}_g\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0},m_v\right),\boldsymbol{\Gamma}\right\}\ \Big]_{(\mathrm{E})}\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$

$$+3(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\Big[_{(\mathrm{F})}\ n\,\mathrm{avar}(m_v),n\,\mathrm{cov}_g\left(m_v,\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right),$$

$$\mathrm{E}_g\left\{\frac{\partial^2 l_j}{(\partial\boldsymbol{\theta}_0{}')^{<2>}}\otimes\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}\right\},\mathrm{vec}'(\boldsymbol{\Gamma}),\mathrm{vec}'n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\boldsymbol{\theta}_0},\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)\ \Big]_{(\mathrm{F})}\mathbf{v}^{(2)}$$

$$+12(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\Big[_{(\mathrm{G})}\ \{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},m_v)\}^2,4n\,\mathrm{cov}_g(\bar{l}_0,m_v)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}\right),$$

$$4n\,\mathrm{cov}_g\{\bar{l}_0,\mathrm{vec}'(\mathbf{M})\}\otimes\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}\right),4\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\boldsymbol{\theta}_0{}'}\right)\right\}^{<2>},$$

$$4n\,\mathrm{cov}_g\left(\overline{l}_0, \frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0'}\right) \otimes \mathrm{E}_g\left(l_{0j} \frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\right) \underset{(\mathrm{G})}{\Big]} \mathbf{v}^{(2)} \underset{(\mathrm{D})}{\Big]} + O(n^{-1}),$$

where the term $4 \underset{(\mathrm{E})}{[} \cdot \underset{(\mathrm{E})}{]} \mathbf{v}^{(1)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}$ is equal to one-third of the third term in $\underset{(\mathrm{A})}{[} \cdot \underset{(\mathrm{A})}{]}$ except the remainder term given earlier, and for the remaining part see the central term in $2n^{-2}\mathrm{E}_g \underset{(\mathrm{B})}{[} \cdot \underset{(\mathrm{B})}{]}$ (the third term in $\underset{(\mathrm{A})}{[} \cdot \underset{(\mathrm{A})}{]}$) of $\kappa_{g2}(t_{\mathrm{W}}^{(\mathrm{A})})$ of (S1.7).

S1.2 $n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)}\,(j=1,\ 2)$

Recall that

$$n^{-1}\mathrm{AIC}_{\mathrm{W}} = -2(\overline{l}_0^*)_{O(1)} + \sum_{k=1}^{4}(\overline{l}_{\mathrm{ML}}^{(k)})_{O_p(n^{-k/2})} - n^{-2}\mathbf{q}_0^*{}'\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*$$
$$+ n^{-1}2q + O_p(n^{-5/2}),$$

$$n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)} = -2(\overline{l}_0^*)_{O(1)} + \sum_{k=1}^{4}(\overline{l}_{\mathrm{ML}}^{(k)})_{O_p(n^{-k/2})} - n^{-2}\mathbf{q}_0^*{}'\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*$$
$$+ n^{-1}2\mathrm{tr}(-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + 2(n^{-1}\mathrm{tr}_{\Delta}^{(\mathrm{T}j)})_{O_p(n^{-3/2})} + 2(n^{-1}\mathrm{tr}_{\Delta\Delta}^{(\mathrm{T}j)})_{O_p(n^{-2})} + O_p(n^{-5/2}) \tag{S1.10}$$

$(j=1,\ 2)$

(see (3.1), (3.4), (3.5) and (4.4)) and

$$t_{\mathrm{W}}^{(\mathrm{T}j)} = \frac{n^{1/2}(n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)} + 2\overline{l}_0^*)}{(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}} \quad (\hat{v}_{\mathrm{W}}^{(\mathrm{T}\bullet)} = \hat{v}_{\mathrm{W}}^{(\mathrm{A})})$$

$$= n^{1/2}\left\{\sum_{k=1}^{3}\overline{l}_{\mathrm{ML}}^{(k)} + n^{-1}2\mathrm{tr}(-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + 2(n^{-1}\mathrm{tr}_{\Delta}^{(\mathrm{T}j)})_{O_p(n^{-3/2})}\right\}$$
$$\times\left\{(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} + n^{-1}\eta_{\mathrm{W}}^{(v)} + \sum_{k=1}^{2}\mathbf{v}^{(k)}{}'\mathbf{m}_v^{(k)}\right\} + O_p(n^{-3/2}) \tag{S1.11}$$

$$\equiv n^{1/2}\left[\sum_{k=1}^{2}(l_{\mathrm{WT}(j)}^{(tk)})_{O_p(n^{-k/2})} + \{n^{-1}2\mathrm{tr}(-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}\}_{O(n^{-1})}\right] + O_p(n^{-3/2}),$$

$$l_{\mathrm{WT}(j)}^{(t1)} = l_{\mathrm{MLT}(j)}^{(t1)} = l_{\mathrm{WA}}^{(t1)} = l_{\mathrm{MLA}}^{(t1)} = \bar{l}_{\mathrm{ML}}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2},$$
$$l_{\mathrm{WT}(j)}^{(t2)} = l_{\mathrm{MLT}(j)}^{(t2)} = l_{\mathrm{WA}}^{(t2)} = l_{\mathrm{MLA}}^{(t2)} = \bar{l}_{\mathrm{ML}}^{(2)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)\prime}\mathbf{m}_{v}^{(1)},$$
$$l_{\mathrm{WT}(j)}^{(t3)} = \bar{l}_{\mathrm{ML}}^{(3)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \{\bar{l}_{\mathrm{ML}}^{(2)} + n^{-1}2\mathrm{tr}(-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})\}\mathbf{v}^{(1)\prime}\mathbf{m}_{v}^{(1)}$$
$$+ n^{-1}2\mathrm{tr}_{\Delta}^{(\mathrm{T}j)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \bar{l}_{\mathrm{ML}}^{(1)}(n^{-1}\eta_{\mathrm{W}}^{(v)} + \mathbf{v}^{(2)\prime}\mathbf{m}_{v}^{(2)}) \quad (j=1,\ 2)$$

(compare (S1.11) with (S1.4) for $t_{\mathrm{W}}^{(\mathrm{A})}$, where the terms of orders $O_p(n^{-2})$ and $O(n^{-2})$ are not used in (S1.4) and (S1.11)).

Recall that the asymptotic cumulants of $n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)}(j=1,\ 2)$ given earlier which are different from those of $n^{-1}\mathrm{AIC}_{\mathrm{W}}$ (see (4.6) and (4.8)) are the first-order asymptotic cumulants of orders $O(n^{-1})$ and $O(n^{-2})$, and the higher-order added asymptotic variance of order $O(n^{-2})$.

From (S1.11),

$$\begin{aligned}\kappa_{g1}(t_{\mathrm{W}}^{(\mathrm{T}\bullet)}) &= n^{-1/2}\{\alpha_{\mathrm{ML1}}^{(\mathrm{T}\bullet)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}\} + O(n^{-3/2}) \\ &= n^{-1/2}\{\mathrm{tr}(-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}\} + O(n^{-3/2}) \\ &\equiv n^{-1/2}\alpha_{(t)\mathrm{ML1}}^{(\mathrm{T}\bullet)} + O(n^{-3/2})\end{aligned} \qquad \text{(S1.12)}$$

(the result is common to $t_{\mathrm{W}}^{(\mathrm{T}j)}(j=1,2)$), where $\mathrm{tr}(-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})(=\alpha_{\mathrm{ML1}}^{(\mathrm{T}\bullet)})$ was $\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})+2q\ (=\alpha_{\mathrm{ML1}}^{(\mathrm{A})})$ in the case of $t_{\mathrm{W}}^{(\mathrm{A})}$.

$$\begin{aligned}&\kappa_{g2}(t_{\mathrm{W}}^{(\mathrm{T}j)}) \\ &= 1 + n^{-1}\Big[_{(\mathrm{A})}\ \alpha_{\mathrm{W}\Delta 2}^{(\mathrm{T}j)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1} + 2n^{2}\mathrm{E}_{g}\{(\bar{l}_{\mathrm{ML}}^{(1)})^{2}\mathbf{v}^{(1)\prime}\mathbf{m}_{v}^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} \\ &\quad + 2n^{2}\mathrm{E}_{g}\Big[_{(\mathrm{B})}\ \bar{l}_{\mathrm{ML}}^{(1)}\{\ \bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)\prime}\mathbf{m}_{v}^{(1)} + \bar{l}_{\mathrm{ML}}^{(1)}(n^{-1}\eta_{\mathrm{W}}^{(v)} + \mathbf{v}^{(2)\prime}\mathbf{m}_{v}^{(2)}) \\ &\quad + n^{-1}2\mathrm{tr}(-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})n^{-1}\mathbf{v}^{(1)\prime}\mathbf{m}_{v}^{(1)}\}\ \Big]_{(\mathrm{B})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\end{aligned} \qquad \text{(S1.13)}$$

$$+n^2\mathrm{E}_g\{2\bar{l}_{\mathrm{ML}}^{(2)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}+(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}$$

$$\left.-[2\{\alpha_{\mathrm{ML}1}^{(\mathrm{T}\bullet)}-2\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma})\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})}+(\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})})^2]\right]_{(\mathrm{A})}+O(n^{-2})$$

$$\equiv 1+n^{-1}\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}j)}+O(n^{-2})\ \ (j=1,\ 2)$$
$$(\alpha_{(t)\mathrm{W}2}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{ML}2}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{W}2}^{(\mathrm{A})}=\alpha_{(t)\mathrm{ML}2}^{(\mathrm{A})}=1),$$

where $2\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma})$ was $2q$ in the case of $t_{\mathrm{W}}^{(\mathrm{A})}$ though $\alpha_{\mathrm{ML}1}^{(\mathrm{T}\bullet)}-2\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma})=\alpha_{\mathrm{ML}1}^{(\mathrm{A})}-2q=\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})=n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})$ is unchanged.

$$\kappa_{g3}(t_{\mathrm{W}}^{(\mathrm{T}\bullet)})=n^{-1/2}\alpha_{(t)\mathrm{ML}3}^{(\mathrm{A})}+O(n^{-3/2})$$
$$(\alpha_{(t)\mathrm{W}3}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{ML}3}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{W}3}^{(\mathrm{A})}=\alpha_{(t)\mathrm{ML}3}^{(\mathrm{A})}) \qquad \text{(S1.14)}$$

(the result is common to $t_{\mathrm{W}}^{(\mathrm{T}j)}(j=1,2)$).

$$\kappa_{g4}(t_{\mathrm{W}}^{(\mathrm{T}\bullet)})=n^{-1}\alpha_{(t)\mathrm{ML}4}^{(\mathrm{A})}+O(n^{-2})\ \ (\alpha_{(t)\mathrm{W}4}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{ML}4}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{W}4}^{(\mathrm{A})}=\alpha_{(t)\mathrm{ML}4}^{(\mathrm{A})}) \qquad \text{(S1.15)}$$

(the result is common to $t_{\mathrm{W}}^{(\mathrm{T}j)}(j=1,2)$ and is given as in (S1.9); see the parenthetical note after (S1.9)).

S2. Interval estimation of $-2\bar{l}_0^*$ with higher-order asymptotic accuracy

S2.1 $t_{\mathrm{W}}^{(\mathrm{A})}$

Ogasawara (2012, Equation (2.5)) gave the general result of the lower endpoint of the one-sided confidence interval (CI) with the third-order asymptotic accuracy. When $t_{\mathrm{W}}^{(\mathrm{A})}$ is used for estimation of $-2\bar{l}_0^*$, the result using the Cornish-Fisher expansion gives the following endpoint:

$$L(\tilde{\alpha};n^{-3/2}) \equiv n^{-1}\mathrm{AIC}_{\mathrm{W}} - n^{-1/2}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2} z_{\tilde{\alpha}}$$
$$-n^{-1}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\{\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})} + (\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2 - 1)\}$$
$$-n^{-3/2}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\left[\frac{1}{2}\left\{ \hat{\alpha}_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})} - 2(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} \right.\right.$$
$$\left.\times n\widehat{\mathrm{acov}}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}}, \hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})} + (\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2 - 1)\} \right\} z_{\tilde{\alpha}}$$
$$\left. + (\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})})^2\left(-\frac{z_{\tilde{\alpha}}^3}{18} + \frac{5}{36}z_{\tilde{\alpha}}\right) + \hat{\alpha}_{(t)\mathrm{ML4}}^{(\mathrm{A})}\left(\frac{z_{\tilde{\alpha}}^3}{24} - \frac{z_{\tilde{\alpha}}}{8}\right)\right], \qquad \text{(S2.1)}$$
$$\Pr\{-2\bar{l}_0^* \geq L(\tilde{\alpha};n^{-3/2})\} = \tilde{\alpha} + O(n^{-3/2}),$$
$$\int_{-\infty}^{z_{\tilde{\alpha}}} (1/\sqrt{2\pi})\exp(-z^2/2)\mathrm{d}z = \tilde{\alpha} \ (0 < \tilde{\alpha} < 1),$$

where the estimators with carets except $\widehat{\mathrm{acov}}_g(\cdot)$ are consistent ones of the population counterparts that do not depend on n; the first two terms on the right-hand side of the first equation of (2.1) give the endpoint of the usual Wald CI with the first-order accuracy; similarly, the first three terms give the endpoint with the second-order accuracy; and $\widehat{\mathrm{acov}}_g(\cdot)$ is the consistent estimator of $\mathrm{acov}_g(\cdot)$, which is the asymptotic covariance of order $O(n^{-1})$ for two variates and will be given next.

Note that

$$n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}}, \hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})} + (\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2 - 1)\}$$
$$= n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}}, \hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \hat{\alpha}_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} z_{\tilde{\alpha}}^2$$
$$+ (1/6)\hat{\alpha}_{\mathrm{ML3}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}(z_{\tilde{\alpha}}^2 - 1)\}, \qquad \text{(S2.2)}$$

since $\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})} = \hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \hat{\alpha}_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}$ and $\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})} = \hat{\alpha}_{\mathrm{ML3}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} + 6\hat{\alpha}_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}$, where $n^{-1}\mathrm{AIC}_{\mathrm{W}} = -2(\bar{l}_0^*)_{O(1)} - 2(\bar{l}_0 - \bar{l}_0^*)_{O_p(n^{-1/2})} + O_p(n^{-1})$ will be used. For the estimators in (S2.2), the estimator of $\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})$ is required, which is denoted by $\mathrm{tr}(\hat{\mathbf{\Lambda}}_{\mathrm{W}}^{-1}\hat{\mathbf{\Gamma}}_{\mathrm{W}})$ and $\mathrm{tr}(-\hat{\mathbf{I}}_{\mathrm{W}}^{(-\Lambda)-1}\hat{\mathbf{I}}_{\mathrm{W}}^{(\Gamma)})$, and will be defined in Subsections S2.1.1 and S2.1.2, respectively.

S2.1.1 The result using $\mathrm{tr}(\hat{\mathbf{\Lambda}}_{\mathrm{W}}^{-1}\hat{\mathbf{\Gamma}}_{\mathrm{W}})$

Preliminary results including repeated ones are

$$(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} = (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + (\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})_{O_p(n^{-1/2})} + O_p(n^{-1}),$$

$$(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} = (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} + 3(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1}(\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})_{O_p(n^{-1/2})} + O_p(n^{-1}),$$

$$\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})} = \alpha_{\mathrm{ML2}}^{(\mathrm{A})} - 2(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{3/2}(\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})_{O_p(n^{-1/2})} + O_p(n^{-1}), \tag{S2.3}$$

$$(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2} = (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2} - \alpha_{\mathrm{ML2}}^{(\mathrm{A})}(\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})_{O_p(n^{-1/2})} + O_p(n^{-1}),$$

$$n\,\mathrm{acov}_g\{\bar{l}_0,\ \mathrm{tr}(\hat{\mathbf{\Lambda}}_{\mathrm{W}}^{-1}\hat{\mathbf{\Gamma}}_{\mathrm{W}})\} = n\,\mathrm{cov}_g\{\bar{l}_0,\ -(\mathrm{tr}_{\Delta}^{(\mathrm{T1})})_{O_p(n^{-1/2})}\}$$

$$= -n\,\mathrm{cov}_g[\bar{l}_0,\ \mathrm{tr}\{(-\hat{\mathbf{\Lambda}}_{\mathbf{M}}^{-1(\Delta)})\mathbf{\Gamma} - \mathbf{\Lambda}^{-1}\mathbf{\Gamma}_{\mathbf{M}}^{(\Delta)}\}_{O_p(n^{-1/2})}]$$

$$= -n\,\mathrm{cov}_g\left[_{(\mathrm{A})}\ \bar{l}_0,\ \mathrm{tr}\left[_{(\mathrm{B})}\left[\mathbf{\Lambda}^{-1}\mathbf{M}\mathbf{\Lambda}^{-1} - \mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left\{\mathbf{\Lambda}^{-1}\otimes\left(\mathbf{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\mathbf{\theta}_0}\right)\right\}\right]\mathbf{\Gamma}\right.\right.$$

$$\left.\left. - \mathbf{\Lambda}^{-1}\left\{\mathbf{M}_{\mathbf{G}} - \sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})\left(\mathbf{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\mathbf{\theta}_0}\right)_j\right\}\ \right]_{(\mathrm{B})}\right]_{(\mathrm{A})},$$

$$= -\mathrm{vec}'(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}\mathbf{\Lambda}^{-1})\mathrm{vec}\{n\mathrm{cov}_g(\mathbf{M},\bar{l}_0)\}$$

$$+\,\mathrm{tr}\left[_{(\mathrm{A})}\ \mathrm{E}_g(\mathbf{J}_0^{(3)})\left[(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}\mathbf{\Lambda}^{-1})\otimes\left\{\mathbf{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial\mathbf{\theta}_0}l_{0j}\right)\right\}\right]\right.$$

$$\left. + \mathbf{\Lambda}^{-1}n\mathrm{E}_g(\mathbf{M}_{\mathbf{G}}\bar{l}_0) - \mathbf{\Lambda}^{-1}\sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})\left\{\mathbf{\Lambda}^{-1}n\mathrm{E}_g\left(\frac{\partial\bar{l}}{\partial\mathbf{\theta}_0}\bar{l}_0\right)\right\}_j\ \right]_{(\mathrm{A})},$$

$$n\,\mathrm{acov}_g(\bar{l}_0,\ \hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1})$$

$$= n\,\mathrm{cov}_g\left[\bar{l}_0,\ -\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}+\boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left\{\boldsymbol{\Lambda}^{-1}\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}\right]$$

$$= -\boldsymbol{\Lambda}^{-1}n\,\mathrm{cov}_g(\bar{l}_0,\ \mathbf{M})\boldsymbol{\Lambda}^{-1}+\boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left[\boldsymbol{\Lambda}^{-1}\otimes\left\{\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)\right\}\right],$$

$$n\,\mathrm{var}_g(\bar{l}_0)=\frac{1}{4}n\,\mathrm{var}_g(\hat{\bar{l}}_{\mathrm{ML}}^{(1)})=\frac{1}{4}\alpha_{\mathrm{ML2}}^{(\mathrm{A})},$$

$$\widehat{\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^3\}}=n^{-1}\sum_{j=1}^{n}\hat{l}_{\mathrm{W}j}^3-3n^{-1}\sum_{j=1}^{n}\hat{l}_{\mathrm{W}j}^2\hat{\bar{l}}_{\mathrm{W}}+2\hat{\bar{l}}_{\mathrm{W}}^3$$

$$=n^{-1}\sum_{j=1}^{n}l_{0j}^3+n^{-1}3\sum_{j=1}^{n}l_{0j}^2\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)$$

$$-3n^{-1}\sum_{j=1}^{n}\left\{l_{0j}^2+2l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)\right\}\left[\bar{l}_0+\left\{\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)\right\}_{O_p(n^{-1})}\right]$$

$$+2\bar{l}_0^3+\left\{6\bar{l}_0^2\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)\right\}_{O_p(n^{-1})}+O_p(n^{-1})$$

$$=n^{-1}\sum_{j=1}^{n}l_{0j}^3-n^{-1}3\sum_{j=1}^{n}l_{0j}^2\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}-3n^{-1}\sum_{j=1}^{n}l_{0j}^2\bar{l}_0$$

$$+6n^{-1}\sum_{j=1}^{n}l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\bar{l}_0+6\bar{l}_0^{*2}(\bar{l}_0-\bar{l}_0^*)+O_p(n^{-1}),$$

$$n\,\mathrm{acov}_g[\bar{l}_0,\widehat{\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^3\}}]=\mathrm{cov}_g(l_{0j},l_{0j}^3)-3\mathrm{E}_g\left(l_{0j}^2\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\right)\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)$$

$$-3\,\mathrm{cov}_g(l_{0j}^2,l_{0j})\bar{l}_0^*-3\mathrm{E}_g(l_{0j}^2)\mathrm{var}_g(l_{0j})$$

$$+6\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0'}\right)\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)\bar{l}_0^*+6\bar{l}_0^{*2}\,\mathrm{var}_g(l_{0j}),$$

$$\widehat{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)} = n^{-1}\sum_{j=1}^{n}\hat{l}_{\mathrm{W}j}\frac{\partial l_j}{\partial \hat{\boldsymbol{\theta}}_{\mathrm{W}}}$$

$$= n^{-1}\sum_{j=1}^{n}\left\{l_{0j} + \frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)\right\}\left\{\frac{\partial l_j}{\partial \boldsymbol{\theta}_0} + \frac{\partial^2 l_j}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0{}'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)\right\} + O_p(n^{-1})$$

$$= \left(n^{-1}\sum_{j=1}^{n} l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)_{O_p(1)} + \left\{(\bar{l}_0^{*}\boldsymbol{\Lambda} + \boldsymbol{\Gamma})\left(-\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}_{O_p(n^{-1/2})} + O_p(n^{-1})$$

$$= \left[n^{-1}\sum_{j=1}^{n}\{\bar{l}_0^{*} + (l_{0j} - \bar{l}_0^{*})\}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right]_{O_p(1)} - \left\{(\bar{l}_0^{*}\mathbf{I}_{(q)} + \boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right\}_{O_p(n^{-1/2})}$$

$$= \left\{n^{-1}\sum_{j=1}^{n}(l_{0j} - \bar{l}_0^{*})\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right\}_{O_p(1)} - \left(\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)_{O_p(n^{-1/2})} + O_p(n^{-1}),$$

$$\widehat{\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0}\right)} = 8\widehat{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)} \quad \text{(see (S1.5))},$$

$$n\,\mathrm{cov}_g\left\{\bar{l}_0, \widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)}\right\} = \mathrm{E}_g\left\{(l_{0j} - \bar{l}_0^{*})^2\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right\} - \boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right).$$

Then, the three asymptotic covariances giving the last result of (S2.2) are shown one by one as follows. The first asymptotic covariance for (S2.2) reduces to

$$\begin{aligned}
&n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}\\
&= -2n\,\mathrm{acov}_g[\bar{l}_0,\ \{\mathrm{tr}(\hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1}\hat{\boldsymbol{\Gamma}}_{\mathrm{ML}}) + 2q\}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}]\\
&= -2n\,\mathrm{acov}_g[\bar{l}_0,\ \{\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + 2q - (\mathrm{tr}_{\Delta}^{(\mathrm{T1})})_{O_p(n^{-1/2})}\}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}]\\
&= -2\{\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + 2q\}n\,\mathrm{acov}_g\{\bar{l}_0,\ (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}\\
&\qquad - 2n\,\mathrm{cov}_g(\bar{l}_0,\ -\mathrm{tr}_{\Delta}^{(\mathrm{T1})})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}.
\end{aligned} \tag{S2.4}$$

The second asymptotic covariance for (S2.2) is

$$
\begin{aligned}
& n\,\mathrm{acov}_g(n^{-1}\mathrm{AIC}_{\mathrm{W}},\, \hat{\alpha}^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} z^2_{\tilde{\alpha}}) \\
& = -2n\,\mathrm{acov}_g\left[\bar{l}_0, -2\hat{\mathbf{v}}^{(1)\prime}\left\{4\widehat{\mathrm{E}_g\{(l_{0j}-\bar{l}_0^{*})^3\}}, \widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)}\right\}'\right] z^2_{\tilde{\alpha}} \\
& = 2n\,\mathrm{acov}_g\left[\bar{l}_0, (\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\left\{-1, \widehat{\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right)}\hat{\boldsymbol{\Lambda}}^{-1}_{\mathrm{ML}}\right\}\right. \\
& \qquad\qquad \left.\times\left\{4\widehat{\mathrm{E}_g\{(l_{0j}-\bar{l}_0^{*})^3\}}, \widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)}\right\}'\right] z^2_{\tilde{\alpha}} \qquad \text{(S2.5)} \\
& = 2\left[\underset{(\mathrm{A})}{} n\,\mathrm{acov}_g\{\bar{l}_0, (\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\}\left\{-1, \mathrm{E}_g\left(\frac{\partial v^{(A)}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\right\}\right. \\
& \qquad\qquad \times\left\{4\mathrm{E}_g\{(l_{0j}-\bar{l}_0^{*})^3\}, \mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)\right\}' \\
& \quad + (\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\left\{\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right) n\,\mathrm{acov}_g(\bar{l}_0, \hat{\boldsymbol{\Lambda}}^{-1}_{\mathrm{ML}})\right. \\
& \qquad\qquad \left. + n\,\mathrm{acov}_g\left(\bar{l}_0, 8\widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)}\right)\boldsymbol{\Lambda}^{-1}\right\}\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)
\end{aligned}
$$

$$+(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\left\{-1, \mathrm{E}_g\left(\frac{\partial v^{(A)}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\right\}$$

$$\times\left[4n\,\mathrm{acov}_g[\bar{l}_0, \widehat{\mathrm{E}_g\{(l_{0j}-\bar{l}_0^*)^3\}}], n\,\mathrm{acov}_g\left\{\bar{l}_0, \widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}l_{0j}\right)}\right\}\right]'\Bigg]_{(\mathrm{A})} z_{\tilde{\alpha}}^2.$$

The third asymptotic covariance for (S2.2) is

$$n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC_W},\ (1/6)\hat{\alpha}_{\mathrm{ML3}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}(z_{\tilde{\alpha}}^2-1)\}$$

$$=-2n\mathrm{acov}_g\Bigg[_{(\mathrm{A})}\ \bar{l}_0, \Bigg[\ -8\widehat{\mathrm{E}_g\{(l_{0j})^3\}} \tag{S2.6}$$

$$+3\left\{\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})}\mathrm{tr}(\hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1}\hat{\boldsymbol{\Gamma}}_{\mathrm{ML}})+2\times 4\widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}l_{0j}\right)}\hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1}\widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)}\right\}$$

$$-3\mathrm{tr}(\hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1}\hat{\boldsymbol{\Gamma}}_{\mathrm{ML}})\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})}\ \Bigg](\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\ \Bigg]_{(\mathrm{A})}\frac{z_{\tilde{\alpha}}^2-1}{6}$$

$$=-2n\mathrm{acov}_g\Bigg[_{(\mathrm{A})}\ \bar{l}_0, \Bigg[\ -8\widehat{\mathrm{E}_g\{(l_{0j})^3\}}$$

$$+24\widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}l_{0j}\right)}\hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1}\widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)}\ \Bigg](\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\ \Bigg]_{(\mathrm{A})}\frac{z_{\tilde{\alpha}}^2-1}{6}$$

$$
\begin{aligned}
&= -2\Bigg[\bigg[-8n\,\mathrm{acov}_g[\bar{l}_0,\ \widehat{\mathrm{E}_g\{(l_{0j})^3\}}] \\
&\quad + 48n\,\mathrm{acov}_g\left\{\bar{l}_0,\ \widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)}\right\}\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0} l_{0j}\right) \\
&\quad + 24\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right) n\,\mathrm{acov}_g(\bar{l}_0, \hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1})\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0} l_{0j}\right)\bigg](\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \\
&+ \left[-8\mathrm{E}_g\{(l_{0j})^3\} + 24\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0} l_{0j}\right)\right] \\
&\qquad \times n\ \mathrm{acov}\{\bar{l}_0, (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\}\Bigg]_{(\mathrm{A})}\frac{z_{\tilde{\alpha}}^2 - 1}{6}.
\end{aligned}
$$

S2.1.2 The result using $\mathrm{tr}(-\hat{\mathbf{I}}_{\mathrm{W}}^{(-\boldsymbol{\Lambda})-1}\hat{\mathbf{I}}_{\mathrm{W}}^{(\boldsymbol{\Gamma})})$

Preliminary results are

$$
\begin{aligned}
&n\,\mathrm{acov}_g\{\bar{l}_0, \mathrm{tr}(-\hat{\mathbf{I}}_{\mathrm{W}}^{(-\boldsymbol{\Lambda})-1}\hat{\mathbf{I}}_{\mathrm{W}}^{(\boldsymbol{\Gamma})})\} = n\,\mathrm{acov}_g\{\bar{l}_0, -(\mathrm{tr}_{\Delta}^{(\mathrm{T2})})_{O_p(n^{-1/2})}\} \\
&= -n\,\mathrm{acov}_g[\bar{l}_0, \mathrm{tr}\{(-\boldsymbol{\Lambda}_{\mathbf{I}}^{-1(\Delta)})\boldsymbol{\Gamma} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}_{\mathbf{I}}^{(\Delta)}\}_{O_p(n^{-1/2})}] \\
&= -n\,\mathrm{acov}_g\Bigg[\bar{l}_0, \mathrm{tr}\bigg[-\boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left\{\boldsymbol{\Lambda}^{-1}\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}\boldsymbol{\Gamma} \\
&\qquad - \boldsymbol{\Lambda}^{-1}\left\{-\sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)_j\right\}\bigg]\Bigg]_{(\mathrm{A})}
\end{aligned}
\qquad \text{(S2.7)}
$$

$$=\mathrm{tr}\left[\mathrm{E}_g(\mathbf{J}_0^{(3)})\left[(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\otimes\left\{\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial\boldsymbol{\theta}_0}l_{0j}\right)\right\}\right]\right.$$

$$\left.-\boldsymbol{\Lambda}^{-1}\sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})\left\{\boldsymbol{\Lambda}^{-1}n\mathrm{E}_g\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\bar{l}_0\right)\right\}_j\right]_{(\mathrm{A})}$$

and

$$n\,\mathrm{acov}_g(\bar{l}_0,-\hat{\mathbf{I}}_{\mathrm{ML}}^{(-\mathrm{A})-1})=n\,\mathrm{acov}_g\left[\bar{l}_0,\ \boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left\{\boldsymbol{\Lambda}^{-1}\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}\right]$$

$$=\boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left[\boldsymbol{\Lambda}^{-1}\otimes\left\{\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial\boldsymbol{\theta}_0}l_{0j}\right)\right\}\right].$$

Then, the first asymptotic covariance for (S2.2) is

$$\begin{aligned}&n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}\\&=-2\{\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})+2q\}n\,\mathrm{acov}_g\{\bar{l}_0,\ (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}\\&\qquad-2n\,\mathrm{cov}_g(\bar{l}_0,\ -\mathrm{tr}_{\Delta}^{(\mathrm{T2})})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2},\end{aligned}\qquad(\mathrm{S2.8})$$

where the last term is different from that of (S2.4).

The second asymptotic covariance for (S2.2) is

$$
\begin{aligned}
& n\,\mathrm{acov}_g(n^{-1}\mathrm{AIC_W},\ \hat{\alpha}^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} z^2_{\tilde{\alpha}}) \\
&= 2\Bigg[n\,\mathrm{acov}_g\{\bar{l}_0, (\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\}\left\{-1, \mathrm{E}_g\left(\frac{\partial v^{(A)}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\right\} \\
&\qquad \times \left\{4\mathrm{E}_g\{(l_{0j}-\bar{l}_0^{*})^3\}, \mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)\right\}' \\
&\quad + (\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\left\{\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \boldsymbol{\theta}_0{}'}\right) n\,\mathrm{acov}_g(\bar{l}_0, -\hat{\mathbf{I}}^{(-\boldsymbol{\Lambda})-1}_{\mathrm{ML}}) \right. \\
&\qquad \left. + n\,\mathrm{acov}_g\left(\bar{l}_0, \widehat{8\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)}\right)\boldsymbol{\Lambda}^{-1}\right\}\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right) \\
&\quad + (\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\left\{-1, \mathrm{E}_g\left(\frac{\partial v^{(A)}}{\partial \boldsymbol{\theta}_0{}'}\right)\boldsymbol{\Lambda}^{-1}\right\} \\
&\quad \times \left[4n\,\mathrm{acov}_g[\bar{l}_0, \widehat{\mathrm{E}_g\{(l_{0j}-\bar{l}_0^{*})^3\}}], n\,\mathrm{acov}_g\left\{\bar{l}_0, \widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)}\right\}\right]' \Bigg]_{(\mathrm{A})} z^2_{\tilde{\alpha}},
\end{aligned}
\tag{S2.9}
$$

where $n\,\mathrm{acov}_g(\bar{l}_0, -\hat{\mathbf{I}}^{(-\boldsymbol{\Lambda})-1}_{\mathrm{ML}})$ was $n\,\mathrm{acov}_g(\bar{l}_0, \hat{\boldsymbol{\Lambda}}^{-1}_{\mathrm{ML}})$ in (S2.5).

The third asymptotic covariance for (S2.2) is

$$n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ (1/6)\hat{\alpha}_{\mathrm{ML3}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}(z_{\tilde{\alpha}}^2-1)\}$$

$$= -2\Bigg[\Bigg[-8n\,\mathrm{acov}_g[\bar{l}_0,\ \widehat{\mathrm{E}_g\{(l_{0j})^3\}}]$$

$$+48n\,\mathrm{acov}_g\left\{\bar{l}_0,\ \widehat{\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}l_{0j}\right)}\right\}\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)$$

$$+24\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}l_{0j}\right)n\,\mathrm{acov}_g(\bar{l}_0,-\hat{\mathbf{I}}_{\mathrm{ML}}^{(-\Lambda)-1})\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)\Bigg](\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$

$$+\left[-8\mathrm{E}_g\{(l_{0j})^3\}+24\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'}l_{0j}\right)\boldsymbol{\Lambda}^{-1}\mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}l_{0j}\right)\right]$$

$$\times n\ \mathrm{acov}\{\bar{l}_0,\ (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\}\Bigg]_{(\mathrm{A})}\frac{z_{\tilde{\alpha}}^2-1}{6}. \tag{S2.10}$$

where $n\,\mathrm{acov}_g(\bar{l}_0,-\hat{\mathbf{I}}_{\mathrm{ML}}^{(-\Lambda)-1})$ was $n\,\mathrm{acov}_g(\bar{l}_0,\hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1})$ in (S2.6).

S2.2 $t_{\mathrm{W}}^{(\mathrm{T}j)}(j=1,2)$

The endpoint using $t_{\mathrm{W}}^{(\mathrm{T}j)}(j=1,2)$ corresponding to (S2.1) for $t_{\mathrm{W}}^{(\mathrm{A})}$ is

$$L(\tilde{\alpha};n^{-3/2}) \equiv n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)} - n^{-1/2}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}z_{\tilde{\alpha}}$$
$$-n^{-1}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\{\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{T}\bullet)}+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2-1)\}$$
$$-n^{-3/2}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\left[\frac{1}{2}\left\{\ \hat{\alpha}_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}j)} - 2(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\right.\right. \tag{S2.11}$$
$$\left.\times n\,\widehat{\mathrm{acov}}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{T}\bullet)}+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2-1)\}\ \right\}z_{\tilde{\alpha}}$$

$$+(\hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{ML3}})^2\left(-\frac{z^3_{\tilde{\alpha}}}{18}+\frac{5}{36}z_{\tilde{\alpha}}\right)+\hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{ML4}}\left(\frac{z^3_{\tilde{\alpha}}}{24}-\frac{z_{\tilde{\alpha}}}{8}\right)\Bigg]\ (j=1,2),$$

where $\hat{\alpha}^{(\mathrm{T}\bullet)}_{(t)\mathrm{ML1}}$ and $\hat{\alpha}^{(\mathrm{T}j)}_{(t)\mathrm{W\Delta2}}$ were $\hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{ML1}}$ and $\hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{W\Delta2}}$ in (S2.1) for $t^{(\mathrm{A})}_{\mathrm{W}}$, respectively; and $\hat{\alpha}^{(\mathrm{T}\bullet)}_{(t)\mathrm{ML1}}$ is an estimator of $\alpha^{(\mathrm{T}\bullet)}_{(t)\mathrm{ML1}}$ common to $n^{-1}\mathrm{TIC}^{(j)}_{\mathrm{W}}\,(j=1,2)$.

First, we have

$$\begin{aligned}&n\,\mathrm{acov}_g\{n^{-1}\mathrm{TIC}^{(j)}_{\mathrm{W}},\hat{\alpha}^{(\mathrm{T}\bullet)}_{(t)\mathrm{ML1}}+(\hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{ML3}}/6)(z^2_{\tilde{\alpha}}-1)\}\\&=n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\hat{\alpha}^{(\mathrm{T}\bullet)}_{\mathrm{ML1}}(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}+\hat{\alpha}^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}}z^2_{\tilde{\alpha}}\\&\qquad+(1/6)\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML3}}(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}(z^2_{\tilde{\alpha}}-1)\}\ (j=1,\ 2),\end{aligned}\tag{S2.12}$$

where

$\alpha^{(\mathrm{T}\bullet)}_{(t)\mathrm{ML1}}=\alpha^{(\mathrm{T}\bullet)}_{\mathrm{ML1}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}+\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}}=\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma})(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}+\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}}$

was $\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}}=\alpha^{(\mathrm{A})}_{\mathrm{ML1}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}+\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}}$

$=\{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})+2q\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}+\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}}$ in (S2.2) with the reversed sign of $\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})$ and an additional term $2q$. Note that in (S2.12) the first argument, $n^{-1}\mathrm{AIC}_{\mathrm{W}}$, is used in place of $n^{-1}\mathrm{TIC}^{(j)}_{\mathrm{W}}\,(j=1,\ 2)$ and consequently, that only the first covariance in (S2.12) is different from that in (S2.2). So, in the following, we show only different results.

S2.2.1 The result using $\mathrm{tr}(-\hat{\mathbf{\Lambda}}^{-1}_{\mathrm{W}}\hat{\mathbf{\Gamma}}_{\mathrm{W}})$

The first asymptotic covariance in (S2.12) is

$$\begin{aligned}&n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}^{(\mathrm{T}\bullet)}_{\mathrm{ML1}}(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}\\&=-2n\,\mathrm{acov}_g\{\bar{l}_0,\ \mathrm{tr}(-\hat{\mathbf{\Lambda}}^{-1}_{\mathrm{ML}}\hat{\mathbf{\Gamma}}_{\mathrm{ML}})(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}\\&=-2n\,\mathrm{acov}_g[\bar{l}_0,\ \{-\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})+(\mathrm{tr}^{(\mathrm{T1})}_{\Delta})_{O_p(n^{-1/2})}\}(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}]\\&=2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})n\,\mathrm{acov}_g\{\bar{l}_0,\ (\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}\\&\qquad-2n\,\mathrm{cov}_g(\bar{l}_0,\ \mathrm{tr}^{(\mathrm{T1})}_{\Delta})(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}.\end{aligned}\tag{S2.13}$$

S2.2.2 The result using $\mathrm{tr}(\hat{\mathbf{I}}_{\mathrm{W}}^{(-\Lambda)-1}\hat{\mathbf{I}}_{\mathrm{W}}^{(\Gamma)})$

The first asymptotic covariance in (S2.12) is

$$\begin{aligned}
& n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{T}\bullet)}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \\
&= -2n\,\mathrm{acov}_g\{\bar{l}_0,\ \mathrm{tr}(\hat{\mathbf{I}}_{\mathrm{W}}^{(-\Lambda)-1}\hat{\mathbf{I}}_{\mathrm{W}}^{(\Gamma)})(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \\
&= -2n\,\mathrm{acov}_g[\bar{l}_0,\ \{-\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) + (\mathrm{tr}_{\Delta}^{(\mathrm{T2})})_{O_p(n^{-1/2})}\}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}] \\
&= 2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})n\,\mathrm{acov}_g\{\bar{l}_0,\ (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \qquad \text{(S2.14)} \\
&\qquad - 2n\,\mathrm{cov}_g(\bar{l}_0,\ \mathrm{tr}_{\Delta}^{(\mathrm{T2})})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}.
\end{aligned}$$

S3. Asymptotic cumulants of the studentized estimators of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{W}}^{*})$

S3.1 $n^{-1}\mathrm{AIC}_{\mathrm{W}}$

Recall that

$$t_{\mathrm{W}}^{(\mathrm{A})} = \frac{n^{1/2}(n^{-1}\mathrm{AIC}_{\mathrm{W}} + 2\bar{l}_0^{*})}{(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}} \quad \text{and} \quad t_{\mathrm{W}}^{(\mathrm{A})*} = \frac{n^{1/2}\{n^{-1}\mathrm{AIC}_{\mathrm{W}} + 2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{W}}^{*})\}}{(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}} \qquad \text{(S3.1)}$$

where $2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{W}}^{*}) = 2\bar{l}_0^{*} + n^{-1}\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) + O(n^{-2})$. In this subsection, the asymptotic cumulants of $t_{\mathrm{W}}^{(\mathrm{A})*}$ corresponding to those of $t_{\mathrm{W}}^{(\mathrm{A})}$ are given. Only the following two asymptotic cumulants are different from the latter. First,

$$\begin{aligned}
\alpha_{(t)\mathrm{W1}}^{(\mathrm{A})*} &\equiv \alpha_{\mathrm{ML1}}^{(\mathrm{A})*}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} \\
&\equiv \{2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) + 2q\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} \qquad \text{(S3.2)}
\end{aligned}$$

for $t_{\mathrm{W}}^{(\mathrm{A})*}$ while earlier we had

$$\begin{aligned}
\alpha_{(t)\mathrm{W1}}^{(\mathrm{A})} &= \alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} \\
&= \{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) + 2q\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} \qquad \text{(S3.3)}
\end{aligned}$$

for $t_{\mathrm{W}}^{(\mathrm{A})}$. The above results give

$$\alpha_{(t)\mathrm{W1}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{W1}}^{(\mathrm{A})} + \mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}. \qquad \text{(S3.4)}$$

Note that in (S3.2) under correct model specification,
$\alpha_{\mathrm{ML1}}^{(\mathrm{A})*} = 2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) + 2q = 0$.

Second,

$$\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})*}=\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1}+2n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$
$$+2n^2\mathrm{E}_g\underset{(\mathrm{B})}{\Bigg[}\ \bar{l}_{\mathrm{ML}}^{(1)}\Big\{\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}+\bar{l}_{\mathrm{ML}}^{(1)}(n^{-1}\eta_{\mathrm{W}}^{(v)}+\mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)}) \qquad \text{(S3.5)}$$
$$+n^{-1}(2q+\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}))\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\Big\}\underset{(\mathrm{B})}{\Bigg]}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$
$$+n^2\mathrm{E}_g\{2\bar{l}_{\mathrm{ML}}^{(2)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}+(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}$$
$$-\{2(\alpha_{\mathrm{ML}1}^{(\mathrm{A})}-2q)(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})}+(\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})})^2\}$$

for $t_{\mathrm{W}}^{(\mathrm{A})*}$, where the factor $(2q+\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}))$ in $\underset{(\mathrm{B})}{[}\cdot\underset{(\mathrm{B})}{]}$ was $2q$ for $\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})}$ of $t_{\mathrm{W}}^{(\mathrm{A})}$ (see (S1.7); the factor $(\alpha_{\mathrm{ML}1}^{(\mathrm{A})}-2q)(=n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)}))$ in the last term $-\{\cdot\}$ is unchanged), which gives

$$\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})*}=\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})}+2\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}. \qquad \text{(S3.6)}$$

As mentioned earlier, the other asymptotic cumulants for $t_{\mathrm{W}}^{(\mathrm{A})*}$ are the same as those for $t_{\mathrm{W}}^{(\mathrm{A})}$:

$$\alpha_{(t)\mathrm{W}2}^{(\mathrm{A})*}=\alpha_{(t)\mathrm{ML}2}^{(\mathrm{A})*}=\alpha_{(t)\mathrm{W}2}^{(\mathrm{A})}=\alpha_{(t)\mathrm{ML}2}^{(\mathrm{A})}=1,$$
$$\alpha_{(t)\mathrm{W}j}^{(\mathrm{A})*}=\alpha_{(t)\mathrm{ML}\,j}^{(\mathrm{A})*}=\alpha_{(t)\mathrm{W}j}^{(\mathrm{A})}=\alpha_{(t)\mathrm{ML}\,j}^{(\mathrm{A})}\ (j=3,\ 4). \qquad \text{(S3.7)}$$

S3.2 $n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)}(j=1,2)$

Recall that

$$t_{\mathrm{W}}^{(\mathrm{T}j)}=\frac{n^{1/2}(n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)}+2\bar{l}_0^*)}{(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}} \qquad \text{(S3.8)}$$

and $t_{\mathrm{W}}^{(\mathrm{T}j)*}=\dfrac{n^{1/2}\{n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{W}}^*)\}}{(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}}\ (j=1,\ 2)$,

where as before $2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{W}}^*)=2\bar{l}_0^*+n^{-1}\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})+O(n^{-2})$. In this subsection, the asymptotic cumulants of $t_{\mathrm{W}}^{(\mathrm{T}j)*}$ corresponding to those of $t_{\mathrm{W}}^{(\mathrm{T}j)}$ are given. Note again that only the following two asymptotic cumulants are different.

First,

$$\alpha_{(t)\mathrm{W}1}^{(\mathrm{T}\bullet)*} \equiv \alpha_{\mathrm{ML}1}^{(\mathrm{T}\bullet)*}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})} = \alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})} \ \ (\alpha_{\mathrm{ML}1}^{(\mathrm{T}\bullet)*} = 0) \qquad \text{(S3.9)}$$

for $t_{\mathrm{W}}^{(\mathrm{T}\bullet)*}$ (the result is common to $t_{\mathrm{W}}^{(\mathrm{T}\,j)*}, j=1,\ 2$) while we had

$$\alpha_{(t)\mathrm{W}1}^{(\mathrm{T}\bullet)} = \alpha_{\mathrm{ML}1}^{(\mathrm{T}\bullet)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})} = \mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma})(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})} \quad \text{(S3.10)}$$

for $t_{\mathrm{W}}^{(\mathrm{T}\bullet)}$ giving $\alpha_{(t)\mathrm{W}1}^{(\mathrm{T}\bullet)*} = \alpha_{(t)\mathrm{W}1}^{(\mathrm{T}\bullet)} + \mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$.

Second

$$\begin{aligned}\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}\,j)*} &= \alpha_{\mathrm{W}\Delta 2}^{(\mathrm{T}\,j)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1} + 2n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} \\ &\quad + 2n^2\mathrm{E}_g\underset{(\mathrm{B})}{\Big[}\ \bar{l}_{\mathrm{ML}}^{(1)}\Big\{\ \bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)} + \bar{l}_{\mathrm{ML}}^{(1)}(n^{-1}\eta_{\mathrm{W}}^{(v)} + \mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)}) \\ &\qquad + n^{-1}\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma})\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\Big\}\ \underset{(\mathrm{B})}{\Big]}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} \\ &\quad + n^2\mathrm{E}_g\{2\bar{l}_{\mathrm{ML}}^{(2)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)} + (\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\} \\ &\quad - \{2(\alpha_{\mathrm{ML}1}^{(\mathrm{T}\bullet)} - 2\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma}))(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})} + (\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})})^2\}\end{aligned} \qquad \text{(S3.11)}$$

for $t_{\mathrm{W}}^{(\mathrm{T}\,j)*}(j=1,\ 2)$, where the factor $\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma})$ in $\underset{(\mathrm{B})}{[}\cdot\underset{(\mathrm{B})}{]}$ was $2\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma})$ for $\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}\,j)}$ of $t_{\mathrm{W}}^{(\mathrm{T}\,j)}$ (see (S1.13); the factor $(\alpha_{\mathrm{ML}1}^{(\mathrm{T}\bullet)} - 2\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma}))(= n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)}))$ in the last term $-\{\}$ is unchanged), which gives

$$\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}\,j)*} = \alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}\,j)} + 2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}. \qquad \text{(S3.12)}$$

As mentioned earlier, the other asymptotic cumulants for $t_{\mathrm{W}}^{(\mathrm{A})*}$ are the same as those for $t_{\mathrm{W}}^{(\mathrm{A})}$:

$$\begin{aligned}&\alpha_{(t)\mathrm{W}2}^{(\mathrm{T}\bullet)*} = \alpha_{(t)\mathrm{ML}2}^{(\mathrm{T}\bullet)*} = \alpha_{(t)\mathrm{W}2}^{(\mathrm{T}\bullet)} = \alpha_{(t)\mathrm{ML}2}^{(\mathrm{T}\bullet)} = 1, \\ &\alpha_{(t)\mathrm{W}k}^{(\mathrm{T}\bullet)*} = \alpha_{(t)\mathrm{ML}k}^{(\mathrm{T}\bullet)*} = \alpha_{(t)\mathrm{W}k}^{(\mathrm{T}\bullet)} = \alpha_{(t)\mathrm{ML}k}^{(\mathrm{T}\bullet)} = \alpha_{(t)\mathrm{W}k}^{(\mathrm{A})} = \alpha_{(t)\mathrm{ML}k}^{(\mathrm{A})}\ (k=3,\ 4)\end{aligned} \qquad \text{(S3.13)}$$

(the results are common to $t_{\mathrm{W}}^{(\mathrm{T}\,j)*}, j=1,\ 2$).

S4. Interval estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{W}}^{*})$ with higher-order asymptotic

accuracy

S4.1 $t_{\mathrm{W}}^{(\mathrm{A})*}$

In the endpoint $L(\tilde{\alpha};n^{-3/2})$ in (S2.1) by $t_{\mathrm{W}}^{(\mathrm{A})}$, replacing $\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})}$ and $\hat{\alpha}_{(t)\mathrm{W\Delta 2}}^{(\mathrm{A})}$ with $\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})*}$ and $\hat{\alpha}_{(t)\mathrm{W\Delta 2}}^{(\mathrm{A})*}$, respectively, we have the corresponding endpoint for $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{W}}^{*})$:

$$\begin{aligned}
L(\tilde{\alpha};n^{-3/2}) &= n^{-1}\mathrm{AIC}_{\mathrm{W}} - n^{-1/2}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2} z_{\tilde{\alpha}} \\
&- n^{-1}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\{\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})*} + (\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2 - 1)\} \\
&- n^{-3/2}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\left[\frac{1}{2}\left\{\hat{\alpha}_{(t)\mathrm{W\Delta 2}}^{(\mathrm{A})*} - 2(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\right.\right. \\
&\left.\times n\widehat{\mathrm{acov}}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}}, \hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})*} + (\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2 - 1)\}\right\} z_{\tilde{\alpha}} \qquad \text{(S4.1)} \\
&\left. + (\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})})^2\left(-\frac{z_{\tilde{\alpha}}^3}{18} + \frac{5}{36} z_{\tilde{\alpha}}\right) + \hat{\alpha}_{(t)\mathrm{ML4}}^{(\mathrm{A})}\left(\frac{z_{\tilde{\alpha}}^3}{24} - \frac{z_{\tilde{\alpha}}}{8}\right)\right].
\end{aligned}$$

This change gives the following changes in $n\widehat{\mathrm{acov}}_g\{\cdot\}$ of (S4.1).

S4.1.1 The result using $\mathrm{tr}(-\hat{\boldsymbol{\Lambda}}_{\mathrm{W}}^{-1}\hat{\boldsymbol{\Gamma}}_{\mathrm{W}})$

The result by $t_{\mathrm{W}}^{(\mathrm{A})*}$ corresponding to the first asymptotic covariance of (S2.2) by $t_{\mathrm{W}}^{(\mathrm{A})}$ is

$$\begin{aligned}
& n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}}, \hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{A})*}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \\
&= -2n\,\mathrm{acov}_g[\bar{l}_0, \{2\mathrm{tr}(\hat{\boldsymbol{\Lambda}}_{\mathrm{ML}}^{-1}\hat{\boldsymbol{\Gamma}}_{\mathrm{ML}}) + 2q\}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}] \\
&= -2n\,\mathrm{acov}_g[\bar{l}_0, \{2\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + 2q - 2(\mathrm{tr}_{\Delta}^{(\mathrm{T1})})_{O_p(n^{-1/2})}\}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}] \\
&= -4\{\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + q\}n\,\mathrm{acov}_g\{\bar{l}_0, (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \qquad \text{(S4.2)} \\
&\qquad - 4n\,\mathrm{cov}_g(\bar{l}_0, -\mathrm{tr}_{\Delta}^{(\mathrm{T1})})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2},
\end{aligned}$$

where the last two terms are different from those by $t_{\mathrm{W}}^{(\mathrm{A})}$ for estimation of

$-2\overline{l}_0^*$ (see (S2.4)).

S4.1.2 The result using $\mathrm{tr}(-\hat{\mathbf{I}}_{\mathrm{W}}^{(-\Lambda)-1}\hat{\mathbf{I}}_{\mathrm{W}}^{(\Gamma)})$

The result by $t_{\mathrm{W}}^{(\mathrm{A})*}$ corresponding to the first asymptotic covariance of (S2.2) by $t_{\mathrm{W}}^{(\mathrm{A})}$ is

$$
\begin{aligned}
&n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{A})*}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}\\
&=-2n\,\mathrm{acov}_g[\overline{l}_0,\ \{2\mathrm{tr}(-\hat{\mathbf{I}}_{\mathrm{W}}^{(-\Lambda)-1}\hat{\mathbf{I}}_{\mathrm{W}}^{(\Gamma)})+2q\}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}]\\
&=-4\{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})+q\}n\,\mathrm{acov}_g\{\overline{l}_0,\ (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \qquad \text{(S4.3)}\\
&\qquad -4n\,\mathrm{cov}_g(\overline{l}_0,\ -\mathrm{tr}_{\Delta}^{(\mathrm{T2})})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2},
\end{aligned}
$$

where as before the last two terms are different from those by $t_{\mathrm{W}}^{(\mathrm{A})}$ for estimation of $-2\overline{l}_0^*$ (see (S2.8)).

From (S2.4), (S2.8), (S4.2) (corresponding to (S2.4)) and (S4.3) (corresponding to (S2.8)), we have

$$
\begin{aligned}
&n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})*}+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2-1)\}\\
&=n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})}+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^2-1)\}\\
&\quad -2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})n\,\mathrm{acov}_g\{\overline{l}_0,\ (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \qquad \text{(S4.4)}\\
&\qquad -2n\,\mathrm{cov}_g(\overline{l}_0,\ -\mathrm{tr}_{\Delta}^{(\mathrm{T}j)})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\quad (j=1,\ 2),
\end{aligned}
$$

where j = 1 and 2 correspond to the results in Subsections S4.1.1 and S4.1.2, respectively.

S4.2 $t_{\mathrm{W}}^{(\mathrm{T}j)*}(j=1,\ 2)$

In the endpoint $L(\tilde{\alpha};n^{-3/2})$ in (S2.11) by $t_{\mathrm{W}}^{(\mathrm{T}j)}(j=1,\ 2)$, replacing $\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{T}\bullet)}$ and $\hat{\alpha}_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}j)}$ with $\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{T}\bullet)*}$ and $\hat{\alpha}_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}j)*}$, respectively, we have the corresponding endpoint for $-2\mathrm{E}_g(\hat{\overline{l}}_{\mathrm{W}}^*)$:

$$L(\tilde{\alpha};n^{-3/2})=n^{-1}\mathrm{TIC}_{\mathrm{W}}^{(j)}-n^{-1/2}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}z_{\tilde{\alpha}}$$
$$-n^{-1}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\{\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{T}\bullet)*}+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^{2}-1)\}$$
$$-n^{-3/2}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\left[\frac{1}{2}\left\{\hat{\alpha}_{(t)\mathrm{W}\Delta 2}^{(\mathrm{T}j)*}-2(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\right.\right.$$
$$\left.\times n\widehat{\mathrm{acov}}_{g}\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{T}\bullet)*}+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^{2}-1)\}\right\}z_{\tilde{\alpha}} \qquad \text{(S4.5)}$$
$$\left.+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})})^{2}\left(-\frac{z_{\tilde{\alpha}}^{3}}{18}+\frac{5}{36}z_{\tilde{\alpha}}\right)+\hat{\alpha}_{(t)\mathrm{ML4}}^{(\mathrm{A})}\left(\frac{z_{\tilde{\alpha}}^{3}}{24}-\frac{z_{\tilde{\alpha}}}{8}\right)\right]\ (j=1,2).$$

This change gives the following changes in $n\widehat{\mathrm{acov}}_{g}\{\cdot\}$ of (S2.11). Since $\hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{T}\bullet)*}=\alpha_{\mathrm{ML1}}^{(\mathrm{T}\bullet)*}=0$,

$$n\,\mathrm{acov}_{g}\{\bar{l}_{0},\ \hat{\alpha}_{\mathrm{ML1}}^{(\mathrm{T}\bullet)*}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}=0 \qquad \text{(S4.6)}$$

corresponding to the first asymptotic covariances in (S2.13) and (S2.14) for the estimators of $-2\bar{l}_{0}^{*}$ by $t_{\mathrm{W}}^{(\mathrm{T}j)}(j=1,\ 2)$, which gives

$$n\,\mathrm{acov}_{g}\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{T}\bullet)*}+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^{2}-1)\}$$
$$=n\,\mathrm{acov}_{g}\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{(t)\mathrm{ML1}}^{(\bullet\mathrm{T})}+(\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}/6)(z_{\tilde{\alpha}}^{2}-1)\}$$
$$-2\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})n\,\mathrm{acov}_{g}\{\bar{l}_{0},\ (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \qquad \text{(S4.7)}$$
$$-2n\,\mathrm{cov}_{g}(\bar{l}_{0},\ -\mathrm{tr}_{\Delta}^{(\mathrm{T}j)})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\quad (j=1,\ 2),$$

where j = 1 and 2 correspond to the results in Subsections S4.1.1 and S4.1.2, respectively (compare (S2.13) and (S2.14) with (S4.6)). Note that the last two terms in (S4.7) are equal to the corresponding terms in (S4.4) by $t_{\mathrm{W}}^{(\mathrm{A})}$ and $t_{\mathrm{W}}^{(\mathrm{A})*}$.

S5. Example 1: The exponential distribution with the MLE of its parameter when the gamma distribution, whose shape parameter is unequal to one, holds

S5.1 Preliminary results

$$f(x^* = x \mid \lambda_0) = \lambda_0 \exp(-\lambda_0 x) \ \ (x > 0),$$
$$g(x^* = x \mid \lambda_1, \alpha) = x^{\alpha-1}\lambda_1^{\alpha} \exp(-\lambda_1 x) / \Gamma(\alpha) \ \ (x > 0,\ \alpha \neq 1),$$
$$\theta_0 = \lambda_0 = \lambda_1 / \alpha,\ \ \hat{\theta}_{\mathrm{ML}} = 1 / \bar{x},$$
$$(\mathrm{E}_f(\bar{x}) = 1/\lambda_0,\ n\,\mathrm{var}_f(\bar{x}) = 1/\lambda_0^2),$$

$$\zeta_0 = (\alpha, \lambda_1)',$$
$$\mathrm{E}_g(\bar{x}) = \alpha/\lambda_1 = 1/\lambda_0,\ n\,\mathrm{var}_g(\bar{x}) = \alpha/\lambda_1^2, \tag{S5.1}$$
$$(n\,\mathrm{avar}_f(\hat{\theta}_{\mathrm{ML}}) = (\partial\hat{\theta}_{\mathrm{ML}}/\partial\bar{x}\,|_{\bar{x}=1/\lambda_0})^2 n\,\mathrm{var}_f(\bar{x}) = \lambda_0^4\lambda_0^{-2} = \lambda_0^2),$$
$$n\,\mathrm{avar}_g(\hat{\theta}_{\mathrm{ML}}) = (\partial\hat{\theta}_{\mathrm{ML}}/\partial\bar{x}\,|_{\bar{x}=1/\lambda_0})^2 n\,\mathrm{var}_g(\bar{x})$$
$$= \lambda_0^4(\alpha/\lambda_1^2) = (\lambda_1/\alpha)^4(\alpha/\lambda_1^2) = \lambda_1^2/\alpha^3.$$

For $j = 1,\ldots,n$,

$$l_{0j} = \log\{\lambda_0 \exp(-\lambda_0 x_j)\} = -\lambda_0 x_j + \log\lambda_0,\ \ \bar{l}_0 = -\lambda_0\bar{x} + \log\lambda_0,$$
$$\frac{\partial\bar{l}}{\partial\theta_0} = \frac{1}{\theta_0} - \bar{x} = -\left(\bar{x} - \frac{\alpha}{\lambda_1}\right),\ \boldsymbol{\Lambda} = \lambda = \frac{\partial^2\bar{l}}{\partial\theta_0^2} = -\frac{1}{\lambda_0^2} = -\bar{i}_0 = -\frac{\alpha^2}{\lambda_1^2},$$
$$\boldsymbol{\Gamma} = \gamma = n\mathrm{E}_g\left\{\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^2\right\} = n\mathrm{E}_g\left\{\left(\frac{1}{\theta_0} - \bar{x}\right)^2\right\} = \frac{\alpha}{\lambda_1^2},$$
$$(n\,\mathrm{var}_f(\bar{l}_0) = \lambda_0^2\,\mathrm{var}_f(x_j) = \lambda_0^2(1/\lambda_0^2) = 1),$$
$$n\,\mathrm{var}_g(\bar{l}_0) = \lambda_0^2\,\mathrm{var}_g(x_j) = \lambda_0^2(\alpha/\lambda_1^2) = (\lambda_1/\alpha)^2(\alpha/\lambda_1^2) = 1/\alpha,$$
$$\bar{l}_0^* = \mathrm{E}_g(\bar{l}_0) = \mathrm{E}_g(l_{0j}) = -\lambda_0(\alpha/\lambda_1) + \log\lambda_0 = \log(\lambda_1/\alpha) - 1,$$

$$\mathrm{E}_g\left\{\left(\frac{\partial l_j}{\partial \theta_0}\right)^3\right\} = -\kappa_{g3}(x^*) = -2\alpha / \lambda_1^3$$

(note the formula $\kappa_{gk}(x^*) = (k-1)!\alpha / \lambda_1^k \ (k = 1, 2, \ldots)$),

$$\mathrm{tr}(-\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) = -\lambda^{-1}\gamma = (\alpha^2 / \lambda_1^2)^{-1}(\alpha / \lambda_1^2) = 1/\alpha,$$

$$\mathbf{J}_0^{(3)} = j_0^{(3)} = \frac{\partial^3 \bar{l}}{\partial \lambda_0^3} = \frac{2}{\lambda_0^3} = \frac{2}{(\lambda_1 / \alpha)^3} = \frac{2\alpha^3}{\lambda_1^3},$$

$$\mathbf{J}_0^{(4)} = j_0^{(4)} = \frac{\partial^4 \bar{l}}{\partial \lambda_0^4} = -\frac{6}{\lambda_0^4} = -\frac{6}{(\lambda_1 / \alpha)^4} = -\frac{6\alpha^4}{\lambda_1^4},$$

$$l_{0j} - \bar{l}_0^* = -\lambda_0\{x_j - \mathrm{E}_g(x^*)\} = -\frac{\lambda_1}{\alpha}\left(x_j - \frac{\alpha}{\lambda_1}\right),$$

$$\kappa_{g3}(l_{0j}) = \left(-\frac{\lambda_1}{\alpha}\right)^3 \kappa_{g3}(x^*) = -\frac{\lambda_1^3}{\alpha^3}\frac{2\alpha}{\lambda_1^3} = -\frac{2}{\alpha^2},$$

$$\mathrm{E}_g\left(\frac{\partial l_{0j}}{\partial \theta_0} l_{0j}\right) = \frac{\lambda_1}{\alpha}\mathrm{var}_g(x^*) = \frac{\lambda_1}{\alpha}\frac{\alpha}{\lambda_1^2} = \frac{1}{\lambda_1},$$

$$\mathrm{E}_g\{(l_{0j} - \bar{l}_0^*)^4\} = \left(-\frac{\lambda_1}{\alpha}\right)^4 [\kappa_{g4}(x^*) + 3\{\mathrm{var}_g(x^*)\}^2]$$

$$= \frac{\lambda_1^4}{\alpha^4}\left\{\frac{6\alpha}{\lambda_1^4} + 3\left(\frac{\alpha}{\lambda_1^2}\right)^2\right\} = \frac{6}{\alpha^3} + \frac{3}{\alpha^2},$$

$$\mathrm{E}_g\left\{\frac{\partial l_j}{\partial \theta_0}(l_{0j} - \bar{l}_0^*)^2\right\} = -\left(-\frac{\lambda_1}{\alpha}\right)^2 \kappa_{g3}(x^*) = -\frac{\lambda_1^2}{\alpha^2}\frac{2\alpha}{\lambda_1^3} = -\frac{2}{\alpha\lambda_1},$$

$$\mathrm{E}_g\left\{(l_{0j} - \bar{l}_0^*)\left(\frac{\partial l_j}{\partial \theta_0}\right)^2\right\} = -\frac{\lambda_1}{\alpha}\kappa_{g3}(x^*) = -\frac{\lambda_1}{\alpha}\frac{2\alpha}{\lambda_1^3} = -\frac{2}{\lambda_1^2},$$

$$\kappa_{g4}(l_{0j}) = \left(-\frac{\lambda_1}{\alpha}\right)^4 \kappa_{g4}(x^*) = \frac{\lambda_1^4}{\alpha^4}\frac{6\alpha}{\lambda_1^4} = \frac{6}{\alpha^3},$$

$$v_0^{(\mathrm{A})} = 4(n-1)^{-1}\sum_{j=1}^{n}(l_{0j}-\overline{l}_0)^2 = 2(n^2-n)^{-1}\sum_{j,k=1}^{n}(l_{0j}-l_{0k})^2,$$

$$\mathrm{E}_g(v_0^{(\mathrm{A})}) = \alpha_{\mathrm{ML2}}^{(\mathrm{A})} = 4n\,\mathrm{var}_g(\overline{l}_0) = 4\,\mathrm{var}_g(l_{0j}) = 4/\alpha,$$

$$\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right) = 8\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right) = \frac{8}{\lambda_1},\ \mathrm{E}_g\left(\frac{\partial^2 v^{(\mathrm{A})}}{\partial\theta_0^2}\right) = 8\gamma = 8\frac{\alpha}{\lambda_1^2},$$

$$\mathrm{E}_g\left\{\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)^2\right\} = 64\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2 + O(n^{-1}) = \frac{64}{\lambda_1^2} + O(n^{-1}),$$

$$\mathbf{v}^{(1)} = \frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\left\{-1,\ \mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\lambda^{-1}\right\}' = \frac{1}{2}\left(\frac{4}{\alpha}\right)^{-3/2}\left\{-1,\ \frac{8}{\lambda_1}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-1}\right\}'$$

$$= \frac{\alpha^{3/2}}{16}\left(-1, -\frac{8\lambda_1}{\alpha^2}\right)' = -\left(\frac{\alpha^{3/2}}{16}, \frac{1}{2}\alpha^{-1/2}\lambda_1\right)',$$

$$\mathbf{v}^{(2)} = \Bigg[_{(\mathrm{A})}\ \frac{3}{8}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2},\ -\frac{3}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2}\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\lambda^{-1},$$

$$-\frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)(\lambda^{-1})^2,$$

$$\Bigg[_{(\mathrm{B})}\ \frac{1}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\lambda^{-1}\mathrm{E}_g(j_0^{(3)}) - \frac{1}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial^2 v^{(\mathrm{A})}}{\partial\theta_0^2}\right)$$

$$+\frac{3}{8}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2}\mathrm{E}_g\left\{\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)^2\right\} + O(n^{-1})\ \Bigg]_{(\mathrm{B})}(\lambda^{-1})^2, \frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\lambda^{-1}\ \Bigg]_{(\mathrm{A})}'$$

$$=\Big[\ \frac{3}{8}\left(\frac{4}{\alpha}\right)^{-5/2},\ -\frac{3}{4}\left(\frac{4}{\alpha}\right)^{-5/2}\frac{8}{\lambda_1}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-1},\ -\frac{1}{2}\left(\frac{4}{\alpha}\right)^{-3/2}\frac{8}{\lambda_1}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-2},$$

$$\left\{\frac{1}{4}\left(\frac{4}{\alpha}\right)^{-3/2}\frac{8}{\lambda_1}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-1}\frac{2\alpha^3}{\lambda_1^3}-\frac{1}{4}\left(\frac{4}{\alpha}\right)^{-3/2}8\frac{\alpha}{\lambda_1^2}+\frac{3}{8}\left(\frac{4}{\alpha}\right)^{-5/2}\frac{64}{\lambda_1^2}\right\}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-2},$$

$$\frac{1}{2}\left(\frac{4}{\alpha}\right)^{-3/2}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-1}\ \Big]'_{(\mathrm{A})}$$

$$=\left\{\frac{3\alpha^{5/2}}{256},\ \frac{3}{16}\alpha^{1/2}\lambda_1,-\frac{\alpha^{-5/2}}{2}\lambda_1^3,\left(-\frac{\alpha^{5/2}}{2\lambda_1^2}-\frac{\alpha^{5/2}}{4\lambda_1^2}+\frac{3}{4}\frac{\alpha^{5/2}}{\lambda_1^2}\right)\frac{\lambda_1^4}{\alpha^4},-\frac{\alpha^{-1/2}}{16}\lambda_1^2\right\}'$$

$$=\left(\frac{3\alpha^{5/2}}{256},\ \frac{3}{16}\alpha^{1/2}\lambda_1,-\frac{\alpha^{-5/2}}{2}\lambda_1^3,0,-\frac{\alpha^{-1/2}}{16}\lambda_1^2\right)'.$$

$$-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}-\hat{\bar{l}}^*_{\mathrm{ML}})=n^{-1}2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})+n^{-2}(c_1+c_2+c_3)+O(n^{-3})$$
$$=n^{-1}b_1+n^{-2}b_2+O(n^{-3}),$$

where since λ_0 is the canonical population parameter, $c_2=c_3=0$ and consequently $b_2=c_1$,

$$b_1=2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})=-\frac{2}{\alpha},$$

$$b_2=c_1=-2\left\{\sum_{a,b,c=1}^{q}(\mathbf{\Lambda}^{(2-2)})_{(c:a,b)}n^2\mathrm{E}_g\left(\frac{\partial\bar{l}}{\partial\theta_{0a}}\frac{\partial\bar{l}}{\partial\theta_{0b}}\frac{\partial\bar{l}}{\partial\theta_{0c}}\right)\right.$$
$$\left.+\sum_{a,b,c,d=1}^{q}(\mathbf{\Lambda}^{(3-4)})_{(d:a,b,c)}(\gamma_{ab}\gamma_{cd}+\gamma_{ac}\gamma_{bd}+\gamma_{ad}\gamma_{bc})\right\} \qquad \text{(S5.2)}$$

$$=-2\left[(1-3)\left(-\frac{1}{2}j_0^{(3)}\lambda^{-3}\right)\mathrm{E}_g\left\{\left(\frac{\partial l_j}{\partial\lambda_0}\right)^3\right\}+\left(\frac{1}{6}j_0^{(4)}\lambda^{-4}\right)3\gamma^2\right]$$

$$= -2\left[-2\left\{ -\frac{1}{2}\frac{2\alpha^3}{\lambda_1^3}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-3}\left(-\frac{2\alpha}{\lambda_1^3}\right)\right\} + \frac{1}{6}\left(-\frac{6\alpha^4}{\lambda_1^4}\right)\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-4} 3\left(\frac{\alpha}{\lambda_1^2}\right)^2 \right]$$

$$= -2\left(\frac{4}{\alpha^2} - \frac{3}{\alpha^2}\right) = -\frac{2}{\alpha^2},$$

$$\bar{l}_{\mathrm{ML}}^{(1)} = -2(\bar{l}_0 - \bar{l}_0^*) = -2[-\lambda_0\{\bar{x} - \mathrm{E}_g(\bar{x})\}] = \frac{2\lambda_1}{\alpha}\left(\bar{x} - \frac{\alpha}{\lambda_1}\right),$$

$$\bar{l}_{\mathrm{ML}}^{(2)} = \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} = -\frac{\lambda_1^2}{\alpha^2}\left(\bar{x} - \frac{\alpha}{\lambda_1}\right)^2,$$

$$n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)}) = -\frac{\lambda_1^2}{\alpha^2} n\mathrm{E}_g\left\{\left(\bar{x} - \frac{\alpha}{\lambda_1}\right)^2\right\} = -\frac{\lambda_1^2}{\alpha^2}\mathrm{var}_g(x_j)$$

$$= -\frac{\lambda_1^2}{\alpha^2}\frac{\alpha}{\lambda_1^2} = -\frac{1}{\alpha} \quad (= \mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})),$$

$$\bar{l}_{\mathrm{ML}}^{(3)} = \frac{1}{3}\mathrm{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)^{<3>} = \frac{1}{3} j_0^{(3)}\lambda^{-3}\left(\frac{\partial \bar{l}}{\partial \theta_0}\right)^3$$

$$= \frac{1}{3}\frac{2\alpha^3}{\lambda_1^3}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-3}\left(\frac{\alpha}{\lambda_1} - \bar{x}\right)^3 = \frac{2}{3}\frac{\lambda_1^3}{\alpha^3}\left(\bar{x} - \frac{\alpha}{\lambda_1}\right)^3,$$

$$n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(3)}) = \frac{2}{3}\frac{\lambda_1^3}{\alpha^3}\kappa_{g3}(x^*) = \frac{2}{3}\frac{\lambda_1^3}{\alpha^3}\frac{2\alpha}{\lambda_1^3} = \frac{4}{3\alpha^2},$$

$$\bar{l}_{\mathrm{ML}}^{(4)} = \mathrm{vec}'(\boldsymbol{\Lambda})(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})^{<2>} - \frac{1}{12}\mathrm{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(4)})\}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)^{<4>}$$

$$= \lambda\left\{-\frac{1}{2}\lambda^{-1} j_0^{(3)}\left(\lambda^{-1}\frac{\partial \bar{l}}{\partial \theta_0}\right)^2\right\}^2 - \frac{1}{12} j_0^{(4)}\left(\lambda^{-1}\frac{\partial \bar{l}}{\partial \theta_0}\right)^4$$

$$= \left\{\frac{1}{4}\lambda^{-5}(j_0^{(3)})^2 - \frac{1}{12}\lambda^{-4} j_0^{(4)}\right\}\left(\frac{\partial \bar{l}}{\partial \theta_0}\right)^4,$$

$$n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(4)})=\left\{\frac{1}{4}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-5}\left(\frac{2\alpha^3}{\lambda_1^3}\right)^2-\frac{1}{12}\left(-\frac{\alpha^2}{\lambda_1^2}\right)^{-4}\left(-\frac{6\alpha^4}{\lambda_1^4}\right)\right\}3\left(\frac{\alpha}{\lambda_1^2}\right)^2$$
$$+O(n^{-1})$$
$$=\left(-1+\frac{1}{2}\right)3\alpha^{-2}+O(n^{-1})=-\frac{3}{2\alpha^2}+O(n^{-1}),$$
$$n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(3)}+\bar{l}_{\mathrm{ML}}^{(4)})=\left(\frac{4}{3}-\frac{3}{2}\right)\frac{1}{\alpha^2}+O(n^{-1})=-\frac{1}{6\alpha^2}+O(n^{-1}).$$

S5.2 $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$

$$n^{-1}\mathrm{AIC}_{\mathrm{ML}}=-2\hat{\bar{l}}_{\mathrm{ML}}+n^{-1}2q=-2\hat{\bar{l}}_{\mathrm{ML}}+n^{-1}2\,.$$

S5.2.1 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ **before studentization**

For estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*)$,

$$\kappa_{g1}\{n^{-1}\mathrm{AIC}_{\mathrm{ML}}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*)\}$$
$$=\kappa_{g1}\{n^{-1}\mathrm{AIC}_{\mathrm{ML}}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}})\}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*-\hat{\bar{l}}_{\mathrm{ML}})$$
$$=\{-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}})+n^{-1}2q+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}})\}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*-\hat{\bar{l}}_{\mathrm{ML}})$$
$$=n^{-1}2q-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}-\hat{\bar{l}}_{\mathrm{ML}}^*)$$
$$=n^{-1}2q+n^{-1}b_1+n^{-2}b_2+O(n^{-3})\ (b_1=2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})=2\lambda^{-1}\gamma=-2\alpha^{-1})\quad\text{(S5.3)}$$
$$=n^{-1}(2q+b_1)+n^{-2}c_1+O(n^{-3})$$
$$=n^{-1}(2-2\alpha^{-1})-n^{-2}2\alpha^{-2}+O(n^{-3})$$
$$=n^{-1}\alpha_{\mathrm{ML1}}^{(\mathrm{A})*}+n^{-2}\alpha_{\mathrm{ML}\Delta1}^{(\mathrm{A})*}+O(n^{-3}),$$

while for estimation of $-2\bar{l}_0^*$

$$\kappa_{g1}(n^{-1}\mathrm{AIC}_{\mathrm{ML}}+2\bar{l}_0^*)$$
$$=n^{-1}\{2q+\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma})\}+n^{-2}\{n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(3)}+\bar{l}_{\mathrm{ML}}^{(4)})\}+O(n^{-3})$$
$$=n^{-1}(2-\alpha^{-1})-n^{-2}(1/6)\alpha^{-2}+O(n^{-3})\quad\text{(S5.4)}$$
$$=n^{-1}\alpha_{\mathrm{ML1}}^{(\mathrm{A})}+n^{-2}\alpha_{\mathrm{ML}\Delta1}^{(\mathrm{A})}+O(n^{-3}),$$

$$\kappa_{g2}(n^{-1}\mathrm{AIC}_{\mathrm{ML}})$$
$$= n^{-1}[n\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(1)})^2\}] + n^{-2}[2n^2\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(1)}\overline{l}_{\mathrm{ML}}^{(2)}) + 2n^2\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(1)}\overline{l}_{\mathrm{ML}}^{(3)})$$
$$+ n^2\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(2)})^2\} - \{n\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(2)})\}^2] + O(n^{-3})$$
$$= n^{-1}\left(\frac{2\lambda_1}{\alpha}\right)^2 \mathrm{var}_g(x^*) + n^{-2}\left[2\frac{2\lambda_1}{\alpha}\left(-\frac{\lambda_1^2}{\alpha^2}\right)\kappa_{g3}(x^*)\right. \tag{S5.5}$$
$$\left. + 2\frac{2\lambda_1}{\alpha}\left(\frac{2}{3}\frac{\lambda_1^3}{\alpha^3}\right)3\{\mathrm{var}_g(x^*)\}^2 + \left(-\frac{\lambda_1^2}{\alpha^2}\right)^2 2\{\mathrm{var}_g(x^*)\}^2\right] + O(n^{-3})$$
$$= n^{-1}\frac{4}{\alpha} + n^{-2}\left\{-4\frac{\lambda_1^3}{\alpha^3}\left(2\frac{\alpha}{\lambda_1^3}\right) + \frac{8}{3}\frac{\lambda_1^4}{\alpha^4}3\left(\frac{\alpha}{\lambda_1^2}\right)^2 + 2\frac{\lambda_1^4}{\alpha^4}\left(\frac{\alpha}{\lambda_1^2}\right)^2\right\} + O(n^{-3})$$
$$= n^{-1}\frac{4}{\alpha} + n^{-2}\frac{2}{\alpha^2} + O(n^{-3}) = n^{-1}\alpha_{\mathrm{ML}2}^{(\mathrm{A})} + n^{-2}\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})} + O(n^{-3}),$$

$$\kappa_{g3}(n^{-1}\mathrm{AIC}_{\mathrm{ML}})$$
$$= n^{-2}[\, n^2\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(1)})^3\} + 3n^2\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(1)})^2\overline{l}_{\mathrm{ML}}^{(2)})\} - 3n\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(2)})\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\,]$$
$$+ O(n^{-3})$$
$$= n^{-2}\left[\left(\frac{2\lambda_1}{\alpha}\right)^3\kappa_{g3}(x^*) + 6\left(\frac{2\lambda_1}{\alpha}\right)^2\left(-\frac{\lambda_1^2}{\alpha^2}\right)\{\mathrm{var}_g(x^*)\}^2\right] + O(n^{-3})$$
$$= n^{-2}\left\{\frac{8\lambda_1^3}{\alpha^3}\frac{2\alpha}{\lambda_1^3} - \frac{24\lambda_1^4}{\alpha^4}\left(\frac{\alpha}{\lambda_1^2}\right)^2\right\} + O(n^{-3}) = -n^{-2}\frac{8}{\alpha^2} + O(n^{-3}) \tag{S5.6}$$
$$= n^{-2}\alpha_{\mathrm{ML}3}^{(\mathrm{A})} + O(n^{-3}).$$

$$\kappa_{g4}(n^{-1}\mathrm{AIC}_{\mathrm{ML}}) = n^{-3}[\, n^3\kappa_{g4}(\overline{l}_{\mathrm{ML}}^{(1)}) + 4n^3\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(1)})^3\overline{l}_{\mathrm{ML}}^{(2)}\}$$
$$+ 6n^3\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(1)})^2(\overline{l}_{\mathrm{ML}}^{(2)})^2\} + 4n^3\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(1)})^3\overline{l}_{\mathrm{ML}}^{(3)}\} - 4n\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(2)})\alpha_{\mathrm{ML}3}^{(\mathrm{A})}$$
$$- 6\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})} - 6\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\{n\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(2)})\}^2\,] + O(n^{-4}),$$

where

$$n^3\kappa_{g4}(\overline{l}^{(1)}_{\mathrm{ML}}) = \left(\frac{2\lambda_1}{\alpha}\right)^4 \kappa_{g4}(x^*) = \frac{16\lambda_1^4}{\alpha^4}\frac{6\alpha}{\lambda_1^4} = \frac{96}{\alpha^3},$$

$$4n^3\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\overline{l}^{(2)}_{\mathrm{ML}}\} = 4\left(\frac{2\lambda_1}{\alpha}\right)^3\left(-\frac{\lambda_1^2}{\alpha^2}\right)10\,\mathrm{var}_g(x^*)\kappa_{g3}(x^*) + O(n^{-1})$$

$$= -320\frac{\lambda_1^5}{\alpha^5}\frac{\alpha}{\lambda_1^2}\frac{2\alpha}{\lambda_1^3} + O(n^{-1}) = -\frac{640}{\alpha^3} + O(n^{-1}),$$

$$6n^3\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^2(\overline{l}^{(2)}_{\mathrm{ML}})^2\} = 6\left(\frac{2\lambda_1}{\alpha}\right)^2\left(-\frac{\lambda_1^2}{\alpha^2}\right)^2 15\{\mathrm{var}_g(x^*)\}^3 + O(n^{-1})$$

$$= 24\times 15\frac{\lambda_1^6}{\alpha^6}\left(\frac{\alpha}{\lambda_1^2}\right)^3 + O(n^{-1}) = \frac{360}{\alpha^3} + O(n^{-1}),$$

$$4n^3\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\overline{l}^{(3)}_{\mathrm{ML}}\} = 4\left(\frac{2\lambda_1}{\alpha}\right)^3\frac{2}{3}\frac{\lambda_1^3}{\alpha^3}15\{\mathrm{var}_g(x^*)\}^3 + O(n^{-1})$$

$$= 320\frac{\lambda_1^6}{\alpha^6}\left(\frac{\alpha}{\lambda_1^2}\right)^3 + O(n^{-1}) = \frac{320}{\alpha^3} + O(n^{-1}),$$

$$-4n\mathrm{E}_g(\overline{l}^{(2)}_{\mathrm{ML}})\alpha^{(\mathrm{A})}_{\mathrm{ML}3} = -4\left(-\frac{1}{\alpha}\right)\left(-\frac{8}{\alpha^2}\right) = -\frac{32}{\alpha^3},$$

$$-6\alpha^{(\mathrm{A})}_{\mathrm{ML}2}\alpha^{(\mathrm{A})}_{\mathrm{ML}\Delta 2} = -6\frac{4}{\alpha}\frac{2}{\alpha^2} = -\frac{48}{\alpha^3},$$

$$-6\alpha^{(\mathrm{A})}_{\mathrm{ML}2}\{n\mathrm{E}_g(\overline{l}^{(2)}_{\mathrm{ML}})\}^2 = -6\frac{4}{\alpha}\left(-\frac{1}{\alpha}\right)^2 = -\frac{24}{\alpha^3},$$

consequently,

$$\begin{aligned}&\kappa_{g4}(n^{-1}\mathrm{AIC}_{\mathrm{ML}})\\ &= \frac{n^{-3}}{\alpha^3}(96 - 640 + 360 + 320 - 32 - 48 - 24) + O(n^{-4})\\ &= n^{-3}\frac{32}{\alpha^3} + O(n^{-4}) = n^{-3}\alpha^{(\mathrm{A})}_{\mathrm{ML}4} + O(n^{-4}).\end{aligned} \qquad \text{(S5.7)}$$

S5.2.2 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ after studentization for

estimation of $-2\bar{l}_0^*$

$$t_{\mathrm{ML}}^{(\mathrm{A})} = \frac{n^{1/2}(n^{-1}\mathrm{AIC}_{\mathrm{ML}} + 2\bar{l}_0^*)}{(\hat{v}_{\mathrm{ML}}^{(\mathrm{A})})^{1/2}}.$$

$$\begin{aligned}
\kappa_{g1}(t_{\mathrm{ML}}^{(\mathrm{A})}) &= n^{-1/2}\{\alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}\} + O(n^{-3/2}) \\
&= n^{-1/2}\left[\alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} - 2\mathbf{v}^{(1)}{}'\left\{4\mathrm{E}_g\{(l_{0j} - \bar{l}_0^*)^3\},\ \mathrm{E}_g\left(\frac{\partial l_j}{\partial \boldsymbol{\theta}_0{}'} l_{0j}\right)\right\}'\right] \qquad \text{(S5.8)} \\
&\quad + O(n^{-3/2}) \\
&= n^{-1/2}\left[\left(2q - \frac{1}{\alpha}\right)\left(\frac{4}{\alpha}\right)^{-1/2} + 2\left(\frac{\alpha^{3/2}}{16}, \frac{1}{2}\alpha^{-1/2}\lambda_1\right)\left\{4\left(-\frac{2}{\alpha^2}\right), \frac{1}{\lambda_1}\right\}'\right] \\
&\quad + O(n^{-3/2}) \\
&= n^{-1/2}\left(2 - \frac{1}{\alpha}\right)\frac{\alpha^{1/2}}{2} + O(n^{-3/2}) = n^{-1/2}\left(\alpha^{1/2} - \frac{1}{2}\alpha^{-1/2}\right) + O(n^{-3/2}) \\
&\equiv n^{-1/2}\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})} + O(n^{-3/2}) \quad (\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} = n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}) = 0),
\end{aligned}$$

$$\begin{aligned}
&\kappa_{g2}(t_{\mathrm{ML}}^{(\mathrm{A})}) \\
&= 1 + n^{-1}\underset{(\mathrm{A})}{\Big[}\ \alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1} + 2n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} \\
&\quad + 2n^2\mathrm{E}_g\underset{(\mathrm{B})}{\Big[}\ \bar{l}_{\mathrm{ML}}^{(1)}(\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)} + \bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)} + n^{-1}2q\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})\ \underset{(\mathrm{B})}{\Big]} \\
&\qquad \times (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} \qquad \text{(S5.9)} \\
&\quad + n^2\mathrm{E}_g\{2\bar{l}_{\mathrm{ML}}^{(2)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)} + (\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\} \\
&\quad - \{2n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} + (\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})})^2\}\ \underset{(\mathrm{A})}{\Big]} + O(n^{-2}).
\end{aligned}$$

(i) the first term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S5.9)

$$\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1}=\frac{2}{\alpha^{2}}\left(\frac{4}{\alpha}\right)^{-1}=\frac{1}{2\alpha},$$

(ii) the second term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S5.9)

$$2n^{2}\mathrm{E}_{g}\{(\bar{l}_{\mathrm{ML}}^{(1)})^{2}\mathbf{v}^{(1)}{}'\mathbf{m}_{v}^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=8\mathbf{v}^{(1)}{}'\left[4\kappa_{g4}(l_{0j}),\ \mathrm{E}_{g}\left\{\frac{\partial l_{j}}{\partial\theta_{0}}(l_{0j}-\bar{l}_{0}^{*})^{2}\right\}\right]'(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=8\left\{-\left(\frac{\alpha^{3/2}}{16},\frac{1}{2}\alpha^{-1/2}\lambda_{1}\right)\right\}\left(4\frac{6}{\alpha^{3}},-\frac{2}{\alpha\lambda_{1}}\right)'\left(\frac{4}{\alpha}\right)^{-1/2}$$

$$=4\left(-\frac{3}{2}\alpha^{-3/2}+\alpha^{-3/2}\right)\alpha^{1/2}=-2\alpha^{-1},$$

(iii) the first part in $2n^{-2}\mathrm{E}_{g}\underset{(\mathrm{B})}{[}\cdot\underset{(\mathrm{B})}{]}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$ of the third term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S5.9)

$$2n^{2}\mathrm{E}_{g}(\bar{l}_{\mathrm{ML}}^{(1)}\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_{v}^{(1)})(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}$$

$$=-4\left[\lambda^{-1}\gamma\left\{n\,\mathrm{cov}_{g}(\bar{l}_{0},m_{v}),\mathrm{E}_{g}\left(l_{0j}\frac{\partial l_{j}}{\partial\theta_{0}}\right)\right\}\right.$$

$$\left.+2\mathrm{E}_{g}\left(l_{0j}\frac{\partial l_{j}}{\partial\theta_{0}}\right)\lambda^{-1}\left\{n\,\mathrm{cov}_{g}\left(\frac{\partial\bar{l}}{\partial\theta_{0}},m_{v}\right),\gamma\right\}\right]\mathbf{v}^{(1)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}+O(n^{-1})$$

$$=-4\left[0+2\frac{1}{\lambda_{1}}\left(-\frac{\alpha^{2}}{\lambda_{1}^{2}}\right)^{-1}\left\{4\left(-\frac{2}{\alpha\lambda_{1}}\right),\frac{\alpha}{\lambda_{1}^{2}}\right\}\right]\left\{-\left(\frac{\alpha^{3/2}}{16},\frac{1}{2}\alpha^{-1/2}\lambda_{1}\right)\right\}'\left(\frac{4}{\alpha}\right)^{-1/2}$$

$$+O(n^{-1})$$

$$=2\alpha^{1/2}\left(\frac{16}{\alpha^{3}},-\frac{2}{\alpha\lambda_{1}}\right)\left(\frac{\alpha^{3/2}}{16},\frac{1}{2}\alpha^{-1/2}\lambda_{1}\right)'+O(n^{-1})=0+O(n^{-1})=O(n^{-1}),$$

(iv) the central part in $2n^{-2}\mathrm{E}_g \underset{(\mathrm{B})}{[}\cdot\underset{(\mathrm{B})}{]}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$ of the third term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S5.9)

$$2n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$$

$$=2\underset{(\mathrm{C})}{\Bigg[}\ \alpha_{\mathrm{ML2}}^{(\mathrm{A})}\left[n\,\mathrm{avar}_g(m_v), n\,\mathrm{cov}_g\left(m_v, \frac{\partial\bar{l}}{\partial\theta_0}\right), 0, \gamma, n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}, \frac{\partial\bar{l}}{\partial\theta_0}\right)\right]\mathbf{v}^{(2)}$$

$$+2\underset{(\mathrm{D})}{\Bigg[}\ \{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)}, m_v)\}^2, 4n\,\mathrm{cov}_g(\bar{l}_0, m_v)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right), 0,$$

$$4\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2, 4n\,\mathrm{cov}_g\left(\bar{l}_0, \frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\ \underset{(\mathrm{D})}{\Bigg]}\mathbf{v}^{(2)}\ \underset{(\mathrm{C})}{\Bigg]}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$$

$$+O(n^{-1})$$

$$=2\underset{(\mathrm{C})}{\Bigg[}\ (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\left[16\left\{\left(\frac{6}{\alpha^3}+\frac{3}{\alpha^2}\right)-\frac{1}{\alpha^2}\right\}, 4\left(-\frac{2}{\alpha\lambda_1}\right), 0, \frac{\alpha}{\lambda_1^2}, 8\left(-\frac{2}{\lambda_1^2}\right)\right]$$

$$+2(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\left\{64\left(-\frac{2}{\alpha^2}\right)^2, 16\left(-\frac{2}{\alpha^2}\right)\frac{1}{\lambda_1}, 0, 4\frac{1}{\lambda_1^2}, 32\left(-\frac{2}{\alpha\lambda_1}\right)\frac{1}{\lambda_1}\right\}\ \underset{(\mathrm{C})}{\Bigg]}\mathbf{v}^{(2)}$$

$$+O(n^{-1})$$

$$=2\underset{(\mathrm{C})}{\Bigg[}\ \frac{2}{\alpha^{1/2}}\left(\frac{96}{\alpha^3}+\frac{32}{\alpha^2}, -\frac{8}{\alpha\lambda_1}, 0, \frac{\alpha}{\lambda_1^2}, -\frac{16}{\lambda_1^2}\right)$$

$$+\alpha^{1/2}\left(\frac{256}{\alpha^4}, -\frac{32}{\alpha^2\lambda_1}, 0, \frac{4}{\lambda_1^2}, -\frac{64}{\alpha\lambda_1^2}\right)\ \underset{(\mathrm{C})}{\Bigg]}$$

$$\times\left(\frac{3\alpha^{5/2}}{256}, \frac{3}{16}\alpha^{1/2}\lambda_1, -\frac{\alpha^{-5/2}}{2}\lambda_1^3, 0, -\frac{\alpha^{-1/2}}{16}\lambda_1^2\right)'+O(n^{-1})$$

$$=\left(\frac{4\times 96\times 3}{256}\alpha^{-1}+\frac{4\times 32\times 3}{256}-\frac{4\times 8\times 3}{16}\alpha^{-1}+\frac{4\times 16}{16}\alpha^{-1}\right)$$
$$+\left(\frac{2\times 256\times 3}{256}\alpha^{-1}-\frac{2\times 32\times 3}{16}\alpha^{-1}+\frac{2\times 64}{16}\alpha^{-1}\right)+O(n^{-1})$$
$$=\frac{9}{2}\alpha^{-1}+\frac{3}{2}-6\alpha^{-1}+4\alpha^{-1}+6\alpha^{-1}-12\alpha^{-1}+8\alpha^{-1}+O(n^{-1})$$
$$=\frac{9}{2}\alpha^{-1}+\frac{3}{2}+O(n^{-1}),$$

(v) the last part in $2n^{-2}\mathrm{E}_g \underset{\text{(B)}}{[} \cdot \underset{\text{(B)}}{]} (\alpha_{\text{ML2}}^{(\text{A})})^{-1/2}$ of the third term in $\underset{\text{(A)}}{[} \cdot \underset{\text{(A)}}{]}$ of (S5.9)

$$2n^2\mathrm{E}_g\{n^{-1}2q(\alpha_{\text{ML2}}^{(\text{A})})^{-1/2}\bar{l}_{\text{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}=4\left(\frac{4}{\alpha}\right)^{-1/2}\times 0=0,$$

(vi) the first half of the fourth term in $\underset{\text{(A)}}{[} \cdot \underset{\text{(A)}}{]}$ of (S5.9)
(this is equal to the result of (iii))

$$n^2\mathrm{E}_g\{2\bar{l}_{\text{ML}}^{(2)}(\alpha_{\text{ML2}}^{(\text{A})})^{-1/2}\bar{l}_{\text{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}=0+O(n^{-1})=O(n^{-1}),$$

(vii) the second half of the fourth term in $\underset{\text{(A)}}{[} \cdot \underset{\text{(A)}}{]}$ of (S5.9)

$$n^2\mathrm{E}_g\{(\bar{l}_{\text{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}$$
$$=\alpha_{\text{ML2}}^{(\text{A})}\mathbf{v}^{(1)}{}'\begin{bmatrix} n\,\mathrm{avar}_g(m_v) & n\,\mathrm{cov}_g(m_v,\partial\bar{l}/\partial\theta_0) \\ n\,\mathrm{cov}_g(m_v,\partial\bar{l}/\partial\theta_0) & \gamma \end{bmatrix}\mathbf{v}^{(1)}$$
$$+2\{n\mathrm{E}_g(\bar{l}_{\text{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\}^2+O(n^{-1})$$

$$=\frac{4}{\alpha}\left\{-\left(\frac{\alpha^{3/2}}{16},\frac{1}{2}\alpha^{-1/2}\lambda_1\right)\right\}\begin{bmatrix}16\left\{\left(\frac{6}{\alpha^3}+\frac{3}{\alpha^2}\right)-\frac{1}{\alpha^2}\right\} & 4\left(-\frac{2}{\alpha\lambda_1}\right)\\ 4\left(-\frac{2}{\alpha\lambda_1}\right) & \frac{\alpha}{\lambda_1^2}\end{bmatrix}$$

$$\times\left\{-\begin{pmatrix}\frac{\alpha^{3/2}}{16}\\ \frac{1}{2}\alpha^{-1/2}\lambda_1\end{pmatrix}\right\}+2\times 0+O(n^{-1})$$

$$=\frac{4}{\alpha}\left(\frac{6}{16}+\frac{\alpha}{8}-2\times\frac{1}{4}+\frac{1}{4}\right)+O(n^{-1})=\frac{1}{2\alpha}+\frac{1}{2}+O(n^{-1}),$$

(viii) the fifth term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S5.9)

$$-\{2n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}+(\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})})^2\}$$

$$=-\left\{2\left(-\frac{1}{\alpha}\right)\left(\frac{4}{\alpha}\right)^{-1/2}\times 0+0^2\right\}=0,$$

then

$$\kappa_{g2}(t_{\mathrm{ML}}^{(\mathrm{A})})=1+n^{-1}\left\{\frac{1}{2\alpha}-2\alpha^{-1}+\left(\frac{9}{2}\alpha^{-1}+\frac{3}{2}\right)+\left(\frac{1}{2\alpha}+\frac{1}{2}\right)\right\}+O(n^{-2})$$

$$=1+n^{-1}\left(\frac{7}{2}\alpha^{-1}+2\right)+O(n^{-2})=1+n^{-1}\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}+O(n^{-2}) \qquad \text{(S5.10)}$$

$(\alpha_{(t)\mathrm{ML2}}^{(\mathrm{A})}=1)$.

$$\kappa_{g3}(t_{\mathrm{ML}}^{(\mathrm{A})})=n^{-1/2}\{\alpha_{\mathrm{ML3}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}+6\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}\}+O(n^{-3/2})$$

$$=n^{-1/2}\left\{-\frac{8}{\alpha^2}\left(\frac{4}{\alpha}\right)^{-3/2}+0\right\}+O(n^{-3/2})=-n^{-1/2}\alpha^{-1/2}+O(n^{-3/2}) \qquad \text{(S5.11)}$$

$$=n^{-1/2}\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})}+O(n^{-3/2}).$$

$$
\begin{aligned}
&\kappa_{g4}(t_{\mathrm{ML}}^{(\mathrm{A})}) \\
&= n^{-1}\Big[\ \alpha_{\mathrm{ML4}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-2} + 4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4 \mathbf{v}^{(1)\prime}\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \\
&\quad + 12n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3 \bar{l}_{\mathrm{ML}}^{(2)} \mathbf{v}^{(1)\prime}\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \\
&\quad + 6n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4 (\mathbf{v}^{(1)\prime}\mathbf{m}_v^{(1)})^2\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1} \qquad \text{(S5.12)} \\
&\quad + 4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3 \bar{l}_{\mathrm{ML}}^{(2)} \mathbf{v}^{(1)\prime}\mathbf{m}_v^{(1)} + (\bar{l}_{\mathrm{ML}}^{(1)})^4 \mathbf{v}^{(2)\prime}\mathbf{m}_v^{(2)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \\
&\quad + \{4n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})\alpha_{\mathrm{ML3}}^{(\mathrm{A})} + 6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\alpha_{\mathrm{ML\Delta 2}}^{(\mathrm{A})} + 6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\{n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})\}^2\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-2} \\
&\quad - 4\{\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})} - 2q(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})} \\
&\quad - 6\{\alpha_{(t)\mathrm{ML\Delta 2}}^{(\mathrm{A})} - 4qn\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)\prime})\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} \\
&\quad - 6\{\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})} - 2q(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}^2\ \Big]_{(\mathrm{A})} + O(n^{-2}),
\end{aligned}
$$

(i) the first term in $[\,\cdot\,]_{(\mathrm{A})}$ of (S5.12)

$$
\alpha_{\mathrm{ML4}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-2} = \frac{32}{\alpha^3}\left(\frac{4}{\alpha}\right)^{-2} = \frac{2}{\alpha},
$$

(ii) the second term in $[\,\cdot\,]_{(\mathrm{A})}$ of (S5.12)

$$
\begin{aligned}
&4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4 \mathbf{v}^{(1)\prime}\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \\
&= 4\Big[\ 24\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\left[4\kappa_{g4}(l_{0j}),\ \mathrm{E}_g\left\{\frac{\partial l_j}{\partial \theta_0}(l_{0j} - \bar{l}_0^*)^2\right\}\right] \\
&\qquad - 32\kappa_{g3}(l_{0j})n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)\prime})\ \Big]_{(\mathrm{B})}\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} + O(n^{-1})
\end{aligned}
$$

$$=\left\{96\times\frac{4}{\alpha}\left(4\times\frac{6}{\alpha^{3}},-\frac{2}{\alpha\lambda_1}\right)-\mathbf{0}'\right\}\left\{-\left(\frac{\alpha^{3/2}}{16},\frac{1}{2}\alpha^{-1/2}\lambda_1\right)\right\}'\left(\frac{4}{\alpha}\right)^{-3/2}+O(n^{-1})$$

$$=\left(-96\times 96\frac{\alpha^{-5/2}}{16}+96\times 8\times\frac{1}{2}\alpha^{-5/2}\right)\frac{\alpha^{3/2}}{8}+O(n^{-1})$$

$$=(-6\times 12+48)\alpha^{-1}+O(n^{-1})=-24\alpha^{-1}+O(n^{-1}),$$

(iii) the third term in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$ of (S5.12)

$$12n^{3}\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^{3}\bar{l}^{(2)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_{v}\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}$$

$$=12\underset{\text{(B)}}{\Bigg[}\left[3\alpha^{(\mathrm{A})}_{\mathrm{ML2}}\lambda^{-1}\gamma+6\left\{n\,\mathrm{cov}_g\left(\bar{l}^{(1)}_{\mathrm{ML}},\frac{\partial\bar{l}}{\partial\theta_0}\right)\right\}^{2}\lambda^{-1}\right]$$

$$\times\left\{n\,\mathrm{cov}_g(\bar{l}^{(1)}_{\mathrm{ML}},m_v),n\,\mathrm{cov}_g\left(\bar{l}^{(1)}_{\mathrm{ML}},\frac{\partial\bar{l}}{\partial\theta_0}\right)\right\}$$

$$+6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}n\,\mathrm{cov}_g\left(\bar{l}^{(1)}_{\mathrm{ML}},\frac{\partial\bar{l}}{\partial\theta_0}\right)\lambda^{-1}\left\{n\,\mathrm{cov}_g\left(\frac{\partial\bar{l}}{\partial\theta_0},m_v\right),\gamma\right\}\underset{\text{(B)}}{\Bigg]}\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}$$

$$+O(n^{-1})$$

$$=12\left[\mathbf{0}'+6\times\frac{4}{\alpha}\left(-\frac{2}{\lambda_1}\right)\left(-\frac{\alpha^{2}}{\lambda_1^{2}}\right)^{-1}\left\{4\left(-\frac{2}{\alpha\lambda_1}\right),\frac{\alpha}{\lambda_1^{2}}\right\}\right]$$

$$\times\left\{-\left(\frac{\alpha^{3/2}}{16},\frac{1}{2}\alpha^{-1/2}\lambda_1\right)\right\}'\left(\frac{4}{\alpha}\right)^{-3/2}+O(n^{-1})$$

$$=12\frac{48\lambda_1}{\alpha^3}\left(\frac{-8}{\alpha\lambda_1},\frac{\alpha}{\lambda_1^2}\right)\left\{-\left(\frac{\alpha^{3/2}}{16},\frac{1}{2}\alpha^{-1/2}\lambda_1\right)\right\}'\left(\frac{4}{\alpha}\right)^{-3/2}+O(n^{-1})$$

$$=12(24-24)\alpha^{-5/2}\frac{\alpha^{3/2}}{8}+O(n^{-1})=0+O(n^{-1})$$

(iv) the fourth term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S5.12)

$$6n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4(\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1}$$

$$=6[\,3\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\mathbf{v}^{(1)}{}'n\,\mathrm{acov}_g(\mathbf{m}_v^{(1)})\mathbf{v}^{(1)}+12\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\}^2\,]+O(n^{-1})$$

$$=6\left\{3\left(\frac{1}{2\alpha}+\frac{1}{2}\right)+12\times 0\right\}+O(n^{-1})=\frac{9}{\alpha}+9+O(n^{-1})$$

(see (vii) for (S5.9)),

(v) the fifth term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S5.12)

recalling that $n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}=0$, and using (iii) and (iv) for (S5.9),

$$4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}+(\bar{l}_{\mathrm{ML}}^{(1)})^4\mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}$$

$$=4\underset{(\mathrm{B})}{\Bigg[}\underset{(\mathrm{C})}{\Bigg[}\left[3\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\lambda^{-1}\gamma+6\left\{n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\theta_0}\right)\right\}^2\lambda^{-1}\right]$$

$$\times\left\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},m_v),n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\theta_0}\right)\right\}$$

$$+6\alpha_{\mathrm{ML}2}^{(\mathrm{A})}n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\theta_0}\right)\lambda^{-1}\left\{n\,\mathrm{cov}_g\left(\frac{\partial\bar{l}}{\partial\theta_0},m_v\right),\gamma\right\}\underset{(\mathrm{C})}{\Bigg]}\mathbf{v}^{(1)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}$$

$$+\left\{3\times\frac{2}{\alpha^{1/2}}\left(\frac{96}{\alpha^3}+\frac{32}{\alpha^2},-\frac{8}{\alpha\lambda_1},0,\frac{\alpha}{\lambda_1^2},-\frac{16}{\lambda_1^2}\right)\right.$$

$$\left.+12\left(\frac{4}{\alpha}\right)^{-1/2}\left(\frac{256}{\alpha^4},-\frac{32}{\alpha^2\lambda_1},0,\frac{4}{\lambda_1^2},-\frac{64}{\alpha\lambda_1^2}\right)\right\}$$

$$\times\left(\frac{3\alpha^{5/2}}{256},\ \frac{3}{16}\alpha^{1/2}\lambda_1,-\frac{\alpha^{-5/2}}{2}\lambda_1^3,0,-\frac{\alpha^{-1/2}}{16}\lambda_1^2\right)'\ \Bigg]_{(\mathrm{B})}+O(n^{-1})$$

$$=4\Bigg[_{(\mathrm{B})}\ 0+\left(\frac{6\times96\times3}{256}\alpha^{-1}+\frac{6\times32\times3}{256}-\frac{6\times8\times3}{16}\alpha^{-1}+\frac{6\times16}{16}\alpha^{-1}\right.$$

$$\left.+\frac{6\times256\times3}{256}\alpha^{-1}-\frac{6\times32\times3}{16}\alpha^{-1}+\frac{6\times64}{16}\alpha^{-1}\right)\ \Bigg]_{(\mathrm{B})}+O(n^{-1})$$

$$=4\left\{\left(\frac{27}{4}-9+6+18-36+24\right)\alpha^{-1}+\frac{9}{4}\right\}+O(n^{-1})=39\alpha^{-1}+9+O(n^{-1}),$$

where $4\underset{(\mathrm{C})}{[}\ \cdot\ \underset{(\mathrm{C})}{]}\mathbf{v}^{(1)}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}$ is one third of (iii) for (S5.12), then

$$\kappa_{g4}(t_{\mathrm{ML}}^{(\mathrm{A})})=n^{-1}\Bigg[_{(\mathrm{A})}\ 2\alpha^{-1}-24\alpha^{-1}+(9\alpha^{-1}+9)+(39\alpha^{-1}+9)$$

$$+(4\times8+6\times4\times2+6\times4)\alpha^{-3}\left(\frac{4}{\alpha}\right)^{-2}$$

$$-4\left\{\alpha^{1/2}-\frac{1}{2}\alpha^{-1/2}-2\left(\frac{4}{\alpha}\right)^{-1/2}\right\}(-\alpha^{-1/2}) \qquad \text{(S5.13)}$$

$$-6\left(\frac{7}{2}\alpha^{-1}+2-4\times0\right)-6\left\{\alpha^{1/2}-\frac{1}{2}\alpha^{-1/2}-2\left(\frac{4}{\alpha}\right)^{-1/2}\right\}^2\ \Bigg]_{(\mathrm{A})}+O(n^{-2})$$

$$= n^{-1}\left\{\left(-13+39+\frac{13}{2}-2-21-\frac{3}{2}\right)\alpha^{-1}+9+9-12\right\}+O(n^{-2})$$

$$= n^{-1}(8\alpha^{-1}+6)+O(n^{-2}) = n^{-1}\alpha^{(\mathrm{A})}_{(t)\mathrm{ML4}}+O(n^{-2}),$$

where for the factor $(4\times 8+6\times 4\times 2+6\times 4)\alpha^{-3}=(32+48+24)\alpha^{-3}$ see the three results just before (S5.7).

S5.2.3 A result for estimation of $-2\overline{l}_0^*$

$$n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{ML1}}+\frac{\hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{ML3}}}{6}(z_{\tilde{\alpha}}^2-1)\right\}$$

$$= n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}^{1/2}-\frac{\hat{\alpha}^{-1/2}}{2}-\frac{\hat{\alpha}^{-1/2}}{6}(z_{\tilde{\alpha}}^2-1)\right\}$$

$$= n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}^{1/2}-\left(\frac{1}{3}+\frac{z_{\tilde{\alpha}}^2}{6}\right)\hat{\alpha}^{-1/2}\right\} \qquad \text{(S5.14)}$$

$$= n\,\mathrm{acov}_g(n^{-1}\mathrm{AIC}_{\mathrm{ML}},\hat{\alpha})\frac{1}{2}\left\{\alpha^{-1/2}+\left(\frac{1}{3}+\frac{z_{\tilde{\alpha}}^2}{6}\right)\alpha^{-3/2}\right\}=0$$

since $n\,\mathrm{acov}_g(n^{-1}\mathrm{AIC}_{\mathrm{ML}},\hat{\alpha})=0$, which is derived in the following.

$$n\,\mathrm{acov}_g(\hat{\boldsymbol{\zeta}}) = n\,\mathrm{acov}_g\{(\hat{\alpha},\ \hat{\lambda}_1)'\} = -\boldsymbol{\Lambda}_{\zeta_0}^{-1}$$

$$= \begin{pmatrix} \alpha/\lambda_1^2 & 1/\lambda_1 \\ 1/\lambda_1 & \psi'(\alpha) \end{pmatrix} \Big/ \left[\frac{1}{\lambda_1^2}\{\alpha\psi'(\alpha)-1\}\right] \qquad \text{(S5.15)}$$

$$= \begin{pmatrix} \alpha & \lambda_1 \\ \lambda_1 & \psi'(\alpha)\lambda_1^2 \end{pmatrix} \Big/ \{\alpha\psi'(\alpha)-1\},$$

where $\psi'(\alpha)\equiv\partial\psi(\alpha)/\partial\alpha$ and $\psi(\alpha)\equiv\partial\log\Gamma(\alpha)/\partial\alpha=\Gamma'(\alpha)/\Gamma(\alpha)$ is the digamma function.

The above result was given for clarity and for later use with $\psi''(\alpha)\equiv\partial^2\psi(\alpha)/\partial\alpha^2$ though not directly used here. Define

$$l_{\zeta_0 j} \equiv \log\{x_j^{\alpha-1}\lambda_1^{\alpha}\exp(-\lambda_1 x_j)/\Gamma(\alpha)\}$$
$$=(\alpha-1)\log x_j+\alpha\log\lambda_1-\lambda_1 x_j-\log\Gamma(\alpha),$$

then

$$\frac{\partial l_{\zeta j}}{\partial\alpha}=\log x_j+\log\lambda_1-\psi(\alpha),\quad \frac{\partial l_{\zeta j}}{\partial\lambda_1}=\frac{\alpha}{\lambda_1}-x_j,$$

while recall that $l_{0j}=\log\lambda_0-\lambda_0 x_j$ and $\frac{\partial l_j}{\partial\lambda_0}=\frac{1}{\lambda_0}-x_j=\frac{\alpha}{\lambda_1}-x_j\left(=\frac{\partial l_{\zeta j}}{\partial\lambda_1}\right)$.

Consequently,

$$n\,\mathrm{acov}_g(n^{-1}\mathrm{AIC}_{\mathrm{WL}},\hat{\alpha})=\mathrm{E}_g\left\{-\boldsymbol{\Lambda}_{\zeta_0}^{-1}\frac{\partial l_{\zeta j}}{\partial\boldsymbol{\zeta}_0}(-2)l_{0j}\right\}_1$$

$$=\mathrm{E}_g\left\{-\boldsymbol{\Lambda}_{\zeta_0}^{-1}\begin{pmatrix}\log x_j-\mathrm{E}_g(\log x_j)\\ \frac{\alpha}{\lambda_1}-x_j\end{pmatrix}2\lambda_0 x_j\right\}_1=\mathrm{E}_g\left\{-\boldsymbol{\Lambda}_{\zeta_0}^{-1}(-2\lambda_0)(\boldsymbol{\Gamma}_{\zeta_0})_{\cdot 2}\right\}_1$$

$$=-2\lambda_0\{-\boldsymbol{\Lambda}_{\zeta_0}^{-1}(-\boldsymbol{\Lambda}_{\zeta_0})_{\cdot 2}\}_1=-2\lambda_0\begin{pmatrix}0\\1\end{pmatrix}_1=0$$

giving (S5.14), where $(\cdot)_1$ is the first element of a vector; and $(\cdot)_{\cdot 2}$ is the second column of a matrix.

S5.2.4 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ after studentization for estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*)$

$$t_{\mathrm{ML}}^{(\mathrm{A})*}=\frac{n^{1/2}\{n^{-1}\mathrm{AIC}_{\mathrm{ML}}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*)\}}{(\hat{v}_{\mathrm{ML}}^{(\mathrm{A})})^{1/2}}.$$

$$
\begin{aligned}
\kappa_{g1}(t_{\mathrm{ML}}^{(\mathrm{A})*}) &= n^{-1/2}\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})*} + O(n^{-3/2}) \\
&= n^{-1/2}\{\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})} + \lambda^{-1}\gamma(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} + O(n^{-3/2}) \\
&= n^{-1/2}\left\{\alpha^{1/2} - \frac{1}{2}\alpha^{-1/2} + \left(-\frac{1}{\alpha}\right)\left(\frac{4}{\alpha}\right)^{-1/2}\right\} + O(n^{-3/2}) \\
&= n^{-1/2}(\alpha^{1/2} - \alpha^{-1/2}) + O(n^{-3/2}),
\end{aligned}
\tag{S5.17}
$$

$$
\begin{aligned}
\kappa_{g2}(t_{\mathrm{ML}}^{(\mathrm{A})*}) &= 1 + n^{-1}\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})*} + O(n^{-2}) \\
&= 1 + n^{-1}\{\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})} + 2\lambda^{-1}\gamma(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\} + O(n^{-2}) \\
&= 1 + n^{-1}(\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})} + 0) + O(n^{-2}) = 1 + n^{-1}\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})} + O(n^{-2})
\end{aligned}
$$

(in this example $\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})}$),

$$
\kappa_{g3}(t_{\mathrm{ML}}^{(\mathrm{A})*}) = n^{-1/2}\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})} + O(n^{-3/2}) \quad (\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})}),
$$

$$
\kappa_{g4}(t_{\mathrm{ML}}^{(\mathrm{A})*}) = n^{-1}\alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})} + O(n^{-2}) \quad (\alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})}).
$$

S5.2.5 A result for estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*)$

$$
\begin{aligned}
&n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})*} + \frac{\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}}{6}(z_{\tilde{\alpha}}^2 - 1)\right\} \\
&= n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}^{1/2} - \hat{\alpha}^{-1/2} - \frac{\hat{\alpha}^{-1/2}}{6}(z_{\tilde{\alpha}}^2 - 1)\right\} = 0
\end{aligned}
\tag{S5.18}
$$

as in (S5.14) of Subsection S5.2.3 for estimation of $-2\bar{l}_0^*$.

S5.2.6 Higher-order bias correction of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ under correct model specification

When a statistical model for $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ is incorrect, the bias correction of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ reduces to those of $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\ (j = 1,\ 2)$, which will be dealt with later. So, in this subsection correct model specification is assumed. The bias terms under model misspecification in (S5.2) can also be used with $\alpha = 1$

under correct model specification. That is,

$$b_1 = 2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) = -2q = -2 \text{ or } b_1 = -2/\alpha \text{ when } \alpha = 1 \qquad \text{(S5.19)}$$

and using $c_2 = c_3 = 0$ in this example,

$$\begin{aligned} b_2 = c_1 &= -2\left\{ \sum_{a,b,c=1}^{q} (\mathbf{\Lambda}^{(2-2)})_{(c:a,b)} n^2 \mathrm{E}_g\left(\frac{\partial \bar{l}}{\partial \theta_{0a}} \frac{\partial \bar{l}}{\partial \theta_{0b}} \frac{\partial \bar{l}}{\partial \theta_{0c}} \right) \right. \\ &\quad \left. + \sum_{a,b,c,d=1}^{q} (\mathbf{\Lambda}^{(3-4)})_{(d:a,b,c)} (\gamma_{ab}\gamma_{cd} + \gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc}) \right\} \\ &= -2\kappa_{f3}\left(\bar{i}_0^{-1} \frac{\partial l_j}{\partial \lambda_0} \right) \kappa_{f3}(-x_j) + \kappa_{f4}\{\bar{i}_0^{-1/2}(-x_j)\} \\ &= -2\bar{i}_0^{-3}\{\kappa_{f3}(-x_j)\}^2 + \bar{i}_0^{-2}\kappa_{f4}(-x_j) \\ &= -2\left(\frac{1}{\lambda_0^2}\right)^{-3}\left(-\frac{2}{\lambda_0^3}\right)^2 + \left(\frac{1}{\lambda_0^2}\right)^{-2}\left(\frac{6}{\lambda_0^4}\right) = -8 + 6 = -2, \end{aligned} \qquad \text{(S5.20)}$$

which is also obtained from $b_2 = -2\alpha^{-2} = -2$ when $\alpha = 1$ (see (S5.2)).

Define $n^{-1}\mathrm{AIC}_{\mathrm{ML}\to O(n^{-2})}$ as the bias-corrected $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ up to order $O(n^{-2})$. Then,

$$n^{-1}\mathrm{AIC}_{\mathrm{ML}\to O(n^{-2})} = -2\hat{\bar{l}}_{\mathrm{ML}} + n^{-1}2 + n^{-2}2 = n^{-1}\mathrm{AIC}_{\mathrm{ML}} + n^{-2}2\,. \qquad \text{(S5.21)}$$

S5.3 $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j = 1, 2)$

Since the gamma distribution is used as a true distribution in this example, define

$$\begin{aligned} n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(1)} = n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(2)} &= -2\hat{\bar{l}}_{\mathrm{ML}} - n^{-1}2\hat{\lambda}_{\mathrm{ML}}^{-1}\hat{\gamma}_{\mathrm{ML}} \\ &= -2\hat{\bar{l}}_{\mathrm{ML}} + n^{-1}(2/\hat{\alpha}). \end{aligned} \qquad \text{(S5.22)}$$

The notation $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\cdot)} (= n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)},\ j = 1, 2)$ will also be used.

S5.3.1 Asymptotic cumulants of $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j = 1, 2)$ before studentization

For estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^{*})$,

$$\kappa_{g1}\{n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\cdot)}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^{*})\}=n^{-2}\alpha_{\mathrm{ML}\Delta 1}^{(\mathrm{T}\cdot)*}+O(n^{-3})\ \ (\alpha_{\mathrm{ML}1}^{(\mathrm{T}\cdot)*}=0)$$
$$=n^{-2}\{\alpha_{\mathrm{ML}\Delta 1}^{(\mathrm{A})*}+2n\mathrm{E}_g(\mathrm{tr}_{\Delta\Delta}^{(\mathrm{T}\cdot)})\}+O(n^{-3})=n^{-2}(\alpha_{\mathrm{ML}\Delta 1}^{(\mathrm{A})*}+d^{(\mathrm{T}\cdot)})+O(n^{-3}) \tag{S5.23}$$
$$=n^{-2}\left[-\frac{2}{\alpha^2}+\frac{\alpha\psi''(\alpha)+\psi'(\alpha)}{\alpha\{\alpha\psi'(\alpha)-1\}^2}\right],$$

where $\mathrm{tr}_{\Delta\Delta}^{(\mathrm{T}\cdot)}=\mathrm{tr}_{\Delta\Delta}^{(\mathrm{T}j)}$ and $d^{(\mathrm{T}\cdot)}=d^{(\mathrm{T}j)}(j=1,\ 2)$; and the expression of $d^{(\mathrm{T}\cdot)}$ will be derived soon, while for estimation of $-2\bar{l}_0^{*}$,

$$\kappa_{g1}(n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\cdot)}+2\bar{l}_0^{*})=n^{-1}\alpha_{\mathrm{ML}1}^{(\mathrm{T}\cdot)}+n^{-2}\alpha_{\mathrm{ML}\Delta 1}^{(\mathrm{T}\cdot)}+O(n^{-3})$$
$$=n^{-1}(-2\lambda^{-1}\gamma+\lambda^{-1}\gamma)+n^{-2}\{\alpha_{\mathrm{ML}\Delta 1}^{(\mathrm{A})}+2n\mathrm{E}_g(\mathrm{tr}_{\Delta\Delta}^{(\mathrm{T}\cdot)})\}+O(n^{-3})$$
$$=n^{-1}(-\lambda^{-1}\gamma)+n^{-2}\{n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(3)}+\bar{l}_{\mathrm{ML}}^{(4)})+d^{(\mathrm{T}\cdot)}\}+O(n^{-3}) \tag{S5.24}$$
$$=n^{-1}\frac{1}{\alpha}+n^{-2}\left[-\frac{1}{6\alpha^2}+\frac{\alpha\psi''(\alpha)+\psi'(\alpha)}{\alpha\{\alpha\psi'(\alpha)-1\}^2}\right]+O(n^{-3})$$

($\alpha_{\mathrm{ML}1}^{(\mathrm{T}\cdot)}=\alpha^{-1}\neq\alpha_{\mathrm{ML}1}^{(\mathrm{A})}=2-\alpha^{-1}$ when $\alpha\neq 1$).

For $d^{(\mathrm{T}\cdot)}$, expand n times the correction term $2(-\hat{\lambda}^{-1}\hat{\gamma})=2/\hat{\alpha}$ in $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\cdot)}$ as

$$\frac{2}{\hat{\alpha}}=\frac{2}{\alpha}-\frac{2}{\alpha^2}(\hat{\alpha}-\alpha)+\frac{2}{\alpha^3}(\hat{\alpha}-\alpha)^2+O_p(n^{-3/2}).$$

Then,

$$\mathrm{E}_g\left(\frac{2}{\hat{\alpha}}\right)=\frac{2}{\alpha}+n^{-1}\left\{-\frac{2}{\alpha^2}n\,\mathrm{abias}_g(\hat{\alpha})+\frac{2}{\alpha^3}n\,\mathrm{avar}_g(\hat{\alpha})\right\}+O(n^2)$$
$$=\frac{2}{\alpha}+n^{-1}d^{(\mathrm{T}\cdot)}+O(n^2),$$

where $n\,\mathrm{avar}_g(\hat{\alpha})=\alpha/\{\alpha\psi'(\alpha)-1\}$ (see (S5.15)) and n times the asymptotic bias of order $O(n^{-1})$ for $\hat{\alpha}$ denoted by $n\,\mathrm{abias}_g(\hat{\alpha})$ is given below.

Noting that

$$\mathbf{I}_{\zeta_0}\equiv-\mathbf{\Lambda}_{\zeta_0}=\begin{pmatrix}\psi'(\alpha) & -1/\lambda_1\\ -1/\lambda_1 & \alpha/\lambda_1^2\end{pmatrix},$$

$$\mathbf{I}_{\zeta_0}^{-1} = -\mathbf{\Lambda}_{\zeta_0}^{-1} = \begin{pmatrix} \alpha & \lambda_1 \\ \lambda_1 & \psi'(\alpha)\lambda_1^2 \end{pmatrix} / \{\alpha\psi'(\alpha)-1\} \quad \text{(see (S5.15))},$$

$$-\frac{\partial^3 l_{\zeta j}}{\partial\zeta_0\partial\zeta_0{}'\partial\alpha} = \begin{pmatrix} \psi''(\alpha) & 0 \\ 0 & 1/\lambda_1^2 \end{pmatrix}, \quad -\frac{\partial^3 l_{\zeta j}}{\partial\zeta_0\partial\zeta_0{}'\partial\lambda_1} = \begin{pmatrix} 0 & 1/\lambda_1^2 \\ 1/\lambda_1^2 & -2\alpha/\lambda_1^3 \end{pmatrix}$$

and $\mathbf{I}_{\zeta_0} = \mathbf{\Gamma}_{\zeta_0} \equiv \mathrm{E}_g\left(\frac{\partial l_{\zeta j}}{\partial\zeta_0}\frac{\partial l_{\zeta j}}{\partial\zeta_0{}'}\right) = -\mathbf{\Lambda}_{\zeta_0}$, we have

$$n\,\mathrm{abias}_g(\hat{\zeta}_{\mathrm{ML}}) = -\frac{1}{2}\mathbf{\Lambda}_{\zeta_0}^{-1}\mathrm{E}_g(\mathbf{J}_{\zeta_0}^{(3)})\mathrm{vec}(\mathbf{\Lambda}_{\zeta_0}^{-1}\mathbf{\Gamma}_{\zeta_0}\mathbf{\Lambda}_{\zeta_0}^{-1})$$

$$= -\frac{1}{2}\mathbf{\Lambda}_{\zeta_0}^{-1}\mathrm{E}_g(\mathbf{J}_{\zeta_0}^{(3)})\mathrm{vec}(-\mathbf{\Lambda}_{\zeta_0}^{-1})$$

$$= -\frac{1}{2}\frac{1}{\{\alpha\psi'(\alpha)-1\}^2}\begin{pmatrix} \alpha & \lambda_1 \\ \lambda_1 & \psi'(\alpha)\lambda_1^2 \end{pmatrix}\begin{pmatrix} \psi''(\alpha) & 0 & 0 & 1/\lambda_1^2 \\ 0 & 1/\lambda_1^2 & 1/\lambda_1^2 & -2\alpha/\lambda_1^3 \end{pmatrix}$$

$$\times\{\alpha, \lambda_1, \lambda_1, \psi'(\alpha)\lambda_1^2\}'$$

$$= -\frac{1}{2}\frac{1}{\{\alpha\psi'(\alpha)-1\}^2}\begin{pmatrix} \alpha & \lambda_1 \\ \lambda_1 & \psi'(\alpha)\lambda_1^2 \end{pmatrix}\begin{pmatrix} \alpha\psi''(\alpha)+\psi'(\alpha) \\ (2/\lambda_1)-\{2\alpha\,\psi'(\alpha)/\lambda_1\} \end{pmatrix}$$

$$= -\frac{1}{2\{\alpha\psi'(\alpha)-1\}^2}\begin{pmatrix} \alpha^2\psi''(\alpha)-\alpha\psi'(\alpha)+2 \\ \lambda_1\alpha\psi''(\alpha)+3\lambda_1\psi'(\alpha)-2\lambda_1\alpha\{\psi'(\alpha)\}^2 \end{pmatrix}$$

i.e., $n\,\mathrm{abias}_g(\hat{\alpha}) = -\frac{\alpha^2\psi''(\alpha)-\alpha\psi'(\alpha)+2}{2\{\alpha\psi'(\alpha)-1\}^2}$

and $n\,\mathrm{abias}_g(\hat{\lambda}_1) = -\frac{\lambda_1[\alpha\psi''(\alpha)+3\psi'(\alpha)-2\alpha\{\psi'(\alpha)\}^2]}{2\{\alpha\psi'(\alpha)-1\}^2}$.

From the above results,

$$
\begin{aligned}
d^{(\mathrm{T}\bullet)} &= -\frac{2}{\alpha^2} n\,\mathrm{abias}_g(\hat{\alpha}) + \frac{2}{\alpha^3} n\,\mathrm{avar}_g(\hat{\alpha}) \\
&= \frac{\alpha^2\psi''(\alpha) - \alpha\psi'(\alpha) + 2}{\alpha^2\{\alpha\psi'(\alpha)-1\}^2} + \frac{2}{\alpha^2\{\alpha\psi'(\alpha)-1\}} \\
&= \frac{\alpha^2\psi''(\alpha) + \alpha\psi'(\alpha)}{\alpha^2\{\alpha\psi'(\alpha)-1\}^2} = \frac{\alpha\psi''(\alpha) + \psi'(\alpha)}{\alpha\{\alpha\psi'(\alpha)-1\}^2}
\end{aligned} \tag{S5.26}
$$

follows.

$$
\begin{aligned}
\kappa_{g2}(n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\bullet)}) &= n^{-1}\alpha_{\mathrm{ML}2}^{(\mathrm{A})} + n^{-2}\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{T}\bullet)} + O(n^{-3}) \\
&= n^{-1}\alpha_{\mathrm{ML}2}^{(\mathrm{A})} + n^{-2}\{\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})} + 4n\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^{(1)}\mathrm{tr}_{\Delta}^{(\mathrm{T}\bullet)})\} + O(n^{-3}),
\end{aligned}
$$

where $4n\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^{(1)}\mathrm{tr}_{\Delta}^{(\mathrm{T}\bullet)}) = -8n\,\mathrm{acov}\left(\bar{l}_0, \dfrac{1}{\hat{\alpha}}\right) = 0$, consequently

$$
\kappa_{g2}(n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\bullet)}) = n^{-1}\alpha_{\mathrm{ML}2}^{(\mathrm{A})} + n^{-2}\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})} + O(n^{-3}) \tag{S5.27}
$$

(in this example $\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{T}\bullet)} = \alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})}$).

$$
\begin{aligned}
\kappa_{g3}(n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\bullet)}) &= n^{-2}\alpha_{\mathrm{ML}3}^{(\mathrm{A})} + O(n^{-3}), \\
\kappa_{g4}(n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\bullet)}) &= n^{-3}\alpha_{\mathrm{ML}4}^{(\mathrm{A})} + O(n^{-4}),
\end{aligned} \tag{S5.28}
$$

where the results of (S5.28) hold generally.

S5.3.2 Asymptotic cumulants of $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j=1,\,2)$ after studentization for estimation of $-2\bar{l}_0^{*}$

$$
t_{\mathrm{ML}}^{(\mathrm{T}\bullet)} = \frac{n^{1/2}(n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\bullet)} + 2\bar{l}_0^{*})}{(\hat{v}_{\mathrm{ML}}^{(\mathrm{A})})^{1/2}}.
$$

$$
\begin{aligned}
\kappa_{g1}(t_{\mathrm{ML}}^{(\mathrm{T}\bullet)}) &= n^{-1/2}\alpha_{(t)\mathrm{ML}1}^{(\mathrm{T}\bullet)} + O(n^{-3/2}) \\
&= n^{-1/2}\{-\lambda^{-1}\gamma(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})}\} + O(n^{-3/2}) \\
&= n^{-1/2}\left\{\frac{1}{\alpha}\left(\frac{4}{\alpha}\right)^{-1/2} + 0\right\} + O(n^{-3/2}) \quad (\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})} = 0) \\
&= n^{-1/2}\frac{1}{2\alpha^{1/2}} + O(n^{-3/2}),
\end{aligned} \tag{S.29}
$$

where the result is generally common to $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j=1,\,2)$.

$$\begin{aligned}\kappa_{g2}(t_{\mathrm{ML}}^{(\mathrm{T}\bullet)}) &= 1+n^{-1}\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{T}\bullet)}+O(n^{-2})\\ &= 1+n^{-1}\{\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}+4\{(-\lambda^{-1}\gamma)-q\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\\ &\quad +O(n^{-2})\\ &= 1+n^{-1}\{\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}+4\{(-\lambda^{-1}\gamma)-q\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2}\times 0\}+O(n^{-2})\\ &= 1+n^{-1}\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}+O(n^{-2}),\end{aligned} \tag{S5.30}$$

where $\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}$ holds in this example.

$$\begin{aligned}\kappa_{g3}(t_{\mathrm{ML}}^{(\mathrm{T}\bullet)}) &= n^{-1/2}\alpha_{(t)\mathrm{ML}3}^{(\mathrm{A})}+O(n^{-3/2})\ \ (\alpha_{(t)\mathrm{ML}3}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{ML}3}^{(\mathrm{A})}),\\ \kappa_{g4}(t_{\mathrm{ML}}^{(\mathrm{T}\bullet)}) &= n^{-1}\alpha_{(t)\mathrm{ML}4}^{(\mathrm{A})}+O(n^{-2})\ \ (\alpha_{(t)\mathrm{ML}4}^{(\mathrm{T}\bullet)}=\alpha_{(t)\mathrm{ML}4}^{(\mathrm{A})}),\end{aligned} \tag{S5.31}$$

where the results of (S5.31) hold generally.

S5.3.3 A result for estimation of $-2\bar{l}_0^*$

$$\begin{aligned}& n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}_{(t)\mathrm{ML}1}^{(\mathrm{T}\bullet)}+\frac{\hat{\alpha}_{(t)\mathrm{ML}3}^{(\mathrm{A})}}{6}(z_{\hat{\alpha}}^2-1)\right\}\\ &= n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \frac{\hat{\alpha}^{-1/2}}{2}-\frac{\hat{\alpha}^{-1/2}}{6}(z_{\hat{\alpha}}^2-1)\right\}=0\\ &(n\,\mathrm{acov}_g(n^{-1}\mathrm{AIC}_{\mathrm{ML}},\,\hat{\alpha})=0),\end{aligned} \tag{S5.32}$$

which is a result generally common to $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j=1,\,2)$.

S5.3.4 Asymptotic cumulants of $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j=1,\,2)$ after studentization for estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*)$

$$t_{\mathrm{ML}}^{(\mathrm{T}\bullet)*}=\frac{n^{1/2}\{n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\bullet)}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^*)\}}{(\hat{v}_{\mathrm{ML}}^{(\mathrm{A})})^{1/2}}.$$

$$\kappa_{g1}(t_{\mathrm{ML}}^{(\mathrm{T}\bullet)*}) = n^{-1/2}\alpha_{(t)\mathrm{ML}1}^{(\mathrm{T}\bullet)*} + O(n^{-3/2})$$
$$= n^{-1/2}\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})} + O(n^{-3/2}) \quad (\alpha_{\mathrm{ML}1}^{(\mathrm{T}\bullet)*} = 0) \qquad \text{(S5.33)}$$
$$= 0 + O(n^{-3/2}) = O(n^{-3/2}) \ (\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})} = 0),$$

which is a result generally common to $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}(j = 1,\ 2)$.

$$\kappa_{g2}(t_{\mathrm{ML}}^{(\mathrm{T}\bullet)*}) = 1 + n^{-1}\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{T}\bullet)*} + O(n^{-2})$$
$$= 1 + n^{-1}\{\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{T}\bullet)} + 2(\lambda^{-1}\gamma)(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)} + O(n^{-2}) \qquad \text{(S5.34)}$$
$$= 1 + n^{-1}(\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{T}\bullet)} + 0) + O(n^{-2}) = 1 + n^{-1}\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})} + O(n^{-2}),$$

where $\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{T}\bullet)*} = \alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}$ holds in this example.

$$\kappa_{g3}(t_{\mathrm{ML}}^{(\mathrm{T}\bullet)*}) = n^{-1/2}\alpha_{(t)\mathrm{ML}3}^{(\mathrm{A})} + O(n^{-3/2}) \ \ (\alpha_{(t)\mathrm{ML}3}^{(\mathrm{T}\bullet)*} = \alpha_{(t)\mathrm{ML}3}^{(\mathrm{A})}),$$
$$\kappa_{g4}(t_{\mathrm{ML}}^{(\mathrm{T}\bullet)*}) = n^{-1}\alpha_{(t)\mathrm{ML}4}^{(\mathrm{A})} + O(n^{-2}) \ \ (\alpha_{(t)\mathrm{ML}4}^{(\mathrm{T}\bullet)*} = \alpha_{(t)\mathrm{ML}4}^{(\mathrm{A})}), \qquad \text{(S5.35)}$$

where the results of (S5.35) hold generally.

S5.3.5 A result for estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^{*})$

$$n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}_{(t)\mathrm{ML}1}^{(\mathrm{T}\bullet)*} + \frac{\hat{\alpha}_{(t)\mathrm{ML}3}^{(\mathrm{A})}}{6}(z_{\tilde{\alpha}}^2 - 1)\right\}$$
$$= n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ 0 - \frac{\hat{\alpha}^{-1/2}}{6}(z_{\tilde{\alpha}}^2 - 1)\right\} = 0 \qquad \text{(S5.36)}$$

as in (S5.32) of Subsection S5.3.3, which is a result generally common to $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}(j = 1,\ 2)$.

S5.3.6 Higher-order bias correction of $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}(j = 1,\ 2)$ under model misspecification

Recall that $c_1 = -2\alpha^{-2}$ and $d^{(\mathrm{T}\bullet)} = \dfrac{\alpha\psi''(\alpha) + \psi'(\alpha)}{\alpha\{\alpha\psi'(\alpha) - 1\}^2}$ (see (S5.23)) and define $\hat{c}_1$ and $\hat{d}^{(\mathrm{T}\bullet)}$ as their consistent estimators, respectively. Let $n^{-1}\mathrm{TIC}_{\mathrm{ML}\to O(n^{-2})}^{(\bullet)}$ as the bias-corrected $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(\bullet)}$ up to order $O(n^{-2})$. Then, using $c_2 = c_3 = 0$ in this case,

$$n^{-1}\mathrm{TIC}^{(\cdot)}_{\mathrm{ML}\to O(n^{-2})} = n^{-1}\mathrm{TIC}^{(\cdot)}_{\mathrm{ML}} - n^{-2}(\hat{c}_1 + \hat{c}_2 + \hat{c}_3 + \hat{d}^{(\mathrm{T}\cdot)})$$

$$= n^{-1}\mathrm{TIC}^{(\cdot)}_{\mathrm{ML}} - n^{-2}(\hat{c}_1 + \hat{d}^{(\mathrm{T}\cdot)})$$

$$= n^{-1}\mathrm{TIC}^{(\cdot)}_{\mathrm{ML}} + n^{-2}\left[2\hat{\alpha}^{-2} - \frac{\hat{\alpha}\psi''(\hat{\alpha}) + \psi'(\hat{\alpha})}{\hat{\alpha}\{\hat{\alpha}\psi'(\hat{\alpha}) - 1\}^2}\right].$$

References

Ogasawara, H. (2009). Supplement to Ogasawara's papers on "the ADF pivotal statistic", "mean and covariance structure analysis", and "maximal reliability". *Economic Review* (*Otaru University of Commerce*), *60* (1), 21-44. http://www.res.otaru-uc.ac.jp/~hogasa/, Permalink: http://hdl.handle.net/10252/3454.

Ogasawara, H. (2012). Cornish-Fisher expansions using sample cumulants and monotonic transformations. *Journal of Multivariate Analysis, 103*, 1-18.

Ogasawara, H. (2016). Asymptotic cumulants of some information criteria. *Journal of the Japanese Society of Computational Statistics, 29*. (in press)