THE FORMULA FOR HIGHER ORDER DERIVATIVES OF INVERSE FUNCTIONS

By

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THEOREM. Let n be a positive integer. If y is a C^n-function of x and $\frac{dy}{dx} \neq 0$ in a certain interval, then

$$\frac{d^n x}{dy^n} = \frac{(-1)^{n-1}}{(\frac{dy}{dx})^{2n-1}} \sum_{\substack{s_1+s_2+\cdots=n-1 \\ 1\cdot s_1+2\cdot s_2+\cdots=2n-2}} \frac{(-1)^{s_1}(2n-s_1-2)!(\frac{dy}{dx})^{s_1}(\frac{d^2y}{dx^2})^{s_2}\cdots}{(2!)^{s_2}s_2!(3!)^{s_3}s_3!\cdots}$$

is valid in the same interval.

PROOF. Set that

(1) $$f(y,x) = h(x) - y$$

and let $x = g(y)$ be the inverse function of $y = h(x)$. Then $x = g(y)$ is the implicit function determined by $f(y,x) = 0$. By the formula in the note[1],

(2) $$g^{(n)}(y) = -\frac{1}{{f_x}^{2n-1}} \sum_{u \in \mathfrak{P}_{2n-1}(n,2n-2)} \nu(u) f(u;y,x),$$

where $\mathbb{N}_0 = \{0,1,2,\dots\}$, $N_2 = {\mathbb{N}_0}^2 \backslash \{(0,0)\}$,

$$\mathfrak{P}_l(n_1,n_2) = \{u \in {\mathbb{N}_0}^{N_2};\ \sum_{(i,j)\in N_2} u(i,j) = l,\ \sum_{(i,j)\in N_2} i\,u(i,j) = n_1,\ \sum_{(i,j)\in N_2} j\,u(i,j) = n_2\},$$

$$\nu(u) = \frac{(-1)^{u(0,1)}u(0,1)!\,(2n-u(0,1)-2)!\,n!}{\prod_{(i,j)\in N_2} (i!\,j!)^{u(i,j)}u(i,j)!}$$

and

$$f(u;y,x) = \prod_{(i,j)\in N_2} \{f^{(i,j)}(y,x)\}^{u(i,j)}.\ ^\dagger$$

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† $f^{(i,j)}(y,x) = \frac{\partial^{i+j}}{\partial y^i \partial x^j} f(y,x)$

Let K be a set such that

$$K = \{(i,j) \in N_2 \ ; \ i > 1 \text{ or } i = 1, \ j > 0\}.$$

From (1), we have for $(i,j) \in N_2$,

$$f^{(i,j)}(y,x) = \begin{cases} h^{(j)}(x) & \text{, if } i = 0, \ j > 0 \\ -1 & \text{, if } (i,j) = (1,0), \\ 0 & \text{, if } (i,j) \in K. \end{cases} \tag{3}$$

If $(i,j) \in K$ and $u(i,j) > 0$, then $f(u;y,x) = 0$.
Therefore,

$$g^{(n)}(y) = -\frac{1}{{f_x}^{2n-1}} \sum_{\substack{u \in \mathfrak{P}_{2n-1}(n,2n-2) \\ (i,j)\in K \Rightarrow u(i,j)=0}} \nu(u) f(u;y,x).$$

For the proof of the theorem, we needs following two lemmata.

Lemma 1. If $u \in \mathfrak{P}_{2n-1}(n, 2n-2)$ and $u(i,j) = 0$ for $(i,j) \in K$, then $u(1,0) = n$, $\sum_{j=1}^{\infty} u(0,j) = n-1$ and $\sum_{j=1}^{\infty} j\, u(0,j) = 2n-2$.

Lemma 2. If a sequence $(s_j)_{j \in \mathbb{N}}$ of non-negative integers satisfying $\sum s_j = n-1$ and $\sum j s_j = 2n-2$, then the map $u \in {\mathbb{N}_0}^{N_2}$ defined by

$$u(i,j) = \begin{cases} s_j & \text{, if } i = 0, \ j > 0 \\ n & \text{, if } (i,j) = (1,0), \\ 0 & \text{, if } (i,j) \in K \end{cases} \tag{4}$$

is an element of $\mathfrak{P}_{2n-1}(n, 2n-2)$.

We can easily get these lemmata. By Lemma 1 and Lemma 2,

$$g^{(n)}(y) = -\frac{1}{{f_x}^{2n-1}} \sum_{\substack{s_1+s_2+\cdots=n-1 \\ 1\cdot s_1+2\cdot s_2+\cdots=2n-2}} \nu(u) f(u;y,x).$$

with u defined by (4). Thus we have

$$\nu(u) = \frac{(-1)^{s_1} s_1!\,(2n-s_1-2)!\,n!}{(1!\,0!)^n\,n!\,(1!)^{s_1} s_1!\,(2!)^{s_2} s_2! \cdots} = \frac{(-1)^{s_1}\,(2n-s_1-2)!}{(2!)^{s_2} s_2!\,(3!)^{s_3} s_3! \cdots}$$

and

$$\begin{aligned} f(u;y,x) &= \{f^{(1,0)}(y,x)\}^n\,\{f^{(0,1)}(y,x)\}^{s_1}\,\{f^{(0,2)}(y,x)\}^{s_2}\cdots \\ &= (-1)^n\,\{h'(x)\}^{s_1}\,\{h''(x)\}^{s_2}\,\cdots \end{aligned}$$

This completes the proof.

Reference

[1] Kaneiwa, R., The Formula of Higher Order Derivatives of Implicit Functions, The Review of Liberal Arts, No.129, 1-19 (2015), Otaru University of Commerce.

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