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Continuous Double-Sided Auctions in Foreign Exchange Markets

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ABSTRACT

This work deals with the intra-day determination of foreign exchange rates. We have two objectives. The first is to suggest a microstructure model of the foreign exchange markets. The second is to explain certain empirical issues, using this model.

Auctions in foreign exchange markets are continuous and double-sided. In a continuous auction, there is no specific length of time during which quantities of demand and supply are defined. Therefore, we model random arrivals of buyers and sellers as Poisson processes and define per-unit-time expected number of arrivals (arrival intensity) of buyers or sellers. In a double-sided auction, buyers (sellers) compete with other buyers (sellers). This competition complicates a trader's decision process. We circumvent this difficulty by adopting the concept of arrival intensities.

Our model combines an individual agent's optimization problem with an auction setting, which models interactions among heterogeneous agents. By solving the agent's optimization problem, we show the first local extremum (FLE) of the expected time path of the exchange rate, not any other local extrema, determines an agent's current action.

Since agents' actions depend on their expected FLE values, the distribution of the expected FLE values among the agents indicates the distribution of actions. Keynes' metaphor, which compares the problem of predicting asset prices to guessing the winner of a beauty contest, can be applied to estimating the distribution of the expected FLE values.

In the second part of the model, by taking agents' heterogeneous expectations into consideration, we derive the formula for the expected time path of the exchange rate with given values for arrival intensities of retail transactions. In the course of finding the formula, the effects of two demand and supply components, namely heterogeneous expectations and retail transactions, are identified.

This distinction of effects is then applied to explain, for example, the positive relationship between volatility and trading volume, which has been empirically detected in equity markets but not in foreign exchange markets. The model also suggests that the degree of heterogeneity of expectations affects the bid-ask spread.

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PART 3.

TERMS, SYMBOLS AND ASSUMPTIONS

TERMS

AGGREGATE RETAIL DEMAND AND SUPPLY : sums of all the agent's $R_1(t)$ and $R_2(t)$.

ARRIVAL: If an agent quotes his price or if he notifies the broker of his intention to trade at the price being quoted by someone else, we call such an event or the agent himself *an arrival.*

ARRIVAL INTENSITY: The expected number of the arrival per unit time is called *the arrival intensity.*

BANKRUPTCY AVOIDANCE: This is a criterion such that an agent always maintains the probability of bankruptcy below a given level. In other words, the agent does not speculate, if the probability of catastrophic loss exceeds the given level.

BEARISH, BULLISH: An agent is called *bearish* (*bullish*) at epoch t_0 if $\xi(t)$ has an interval $[t_0, T_0]$ such that $\frac{d\xi(t)}{dt} < 0$ ($\frac{d\xi(t)}{dt} > 0$) for $t \in [t_0, T_0]$.

EPOCH: A point on a time axis is called *an epoch.*

CONTINUOUS AUCTION: The continuous auction implies (1) buyers and sellers may quote their respective prices at any epoch, (2) whenever a buyer and seller pair agree upon the price, a transaction takes place, and (3) upon the completion of the transaction, the buyer and seller pair leave the market and the bidding is continued among the remaining traders and new entrants. For the continuous auction, there is not a specific length of time during which quantities of demand and supply are defined.

DAYLIGHT LIMIT: The maximum magnitudes of the open position which are allowed during the business day. The daylight limit is exogenously given to the agent by his bank.

FLE: The first local extremum of the agent's expected time path of the exchange rate.

MARKET MAKERS: Agents who quote both buying and selling prices at the same time.

MARKET RATES: Bid rate and offered rate together are called the *market rates.*

OVERNIGHT LIMIT: The restriction on the open position at the end of the business day is called *overnight limit.* This is stricter than the daylight limit.

POSITION: We call the agent's level of inventory his *position* and zero inventory level *a square position.* A nonzero inventory is called *an open position.*

RESILIENCY OF MARKET: A market has resiliency if temporary price changes due to temporary order imbalances quickly attract new orders to the market. (Schwartz, 1988)

SPECIALIST: A member of a stock exchange who maintains a fair and orderly market in one or more securities; buying or selling for the specialist's own account to counteract temporary imbalances in supply and demand. (John Downes and Jordan Elliot Goodman, *Dictionary of Finance and Investment Terms*, Barron's, 1985)

STATE OF THE CRYSTAL GLASS: If the agent is in the state of the crystal glass, he is confident enough of his expectation to speculate based on it.

STATE OF THE FROSTED GLASS: If the agent is in the state of the frosted glass, he does not want to assume an open position. This is because if the agent has the open position, there is a substantial possibility that an adverse shift of the exchange rate will incur loss to the agent.

STATIONARY HETEROGENEITY: A situation where distribution functions $H_t(x)$ and $G_t(x)$ coincide.

TRANSITION 1: An agent's level of confidence in his own expectation moves from the state of the frosted glass to the state of the crystal glass.

TRANSITION 2: An agent loses confidence in his expectation. This is a transition from the state of the crystal glass to the state of the frosted glass. The agent wants to square his position and he becomes a buyer or seller, depending on his position at that moment.

SYMBOLS

$[t_1, t_2]$: A closed interval between epoch t_1 and t_2.

$A(t)$: The minimum selling prices which are being quoted in the market at epoch t.

$AH(t)$: The aggregate heterogeneity transactions $AH(t)$ mean excess transaction quantity of buyers over sellers who hit the market rates due to heterogeneous expectations.

$AR(t)$: The aggregate retail transactions. $AR(t) \equiv ARD(t) - ARS(t)$ and $E[AR(t)] = (\lambda_{a1} - \lambda_{b1})t$.

$ARD(t)$: The aggregate retail demand (agents' selling to customers). This is a cumulative value over an interval $[0, t]$.

$ARS(t)$: The aggregate retail supply (agents' buying from the customers). This is a cumulative value over $[0, t]$.

$B(t)$: The maximum buying prices which are being quoted in the market at epoch t

$ED(t)$: The excess demand at epoch t; $ED(t) \equiv AR(t) + AH(t)$.

$\mathcal{F}_{t_0}$: Information which the agent has at epoch t_0. The expectation is conditional on the information obtained by epoch t_0, $\mathcal{F}_{t_0}$.

N: A total number of agents in this economy which is given exogenously.

$N_c(t)$: Number of agents who stay in the crystal glass at epoch t. This is a random variable and $N = N_c + N_f$.

$N_f(t)$: Number of agents who stay in the frosted glass at epoch t.

$R_1(t)$: Cumulative quantity purchased from customers during $[0, t]$.

$R_2(t)$: Cumulative quantity sold to customers during $[0, t]$.

$S(t)$: Price at which the last transaction was made before, and at epoch t.

$S_1(t)$: Transaction price at which an agent sells at epoch t.

$S_2(t)$: Transaction price at which an agent buys at epoch t.

$S_i^*(t)$: A desired value for $S_i(t)$ for $i = 1, 2$.

$Z_1(t)$: Cumulative quantity purchased from the market during $[0, t]$.

$Z_2(t)$: Cumulative quantity sold to the market during $[0, t]$. If $dZ_1(t)$ and $dZ_2(t)$ are not zero, they are quantities which agent bought and sold in the market at epoch t.

$Z_i^*(t)$: Desired values of $Z_i(t)$ for $i = 1, 2$.

$Z(t)$: The agent's position at epoch t. $Z(t) \equiv Z_1(t) - Z_2(t) + z_0 + R_1(t) - R_2(t)$ where z_0 is an initial value of $Z(t)$ at epoch 0. $Z(t)$ is a random variable and it takes values from a finite subset of integers, for example, $\{-10, -9, ..., 0, 1, ..., 9, 10\}$.

$Z^*(t)$: Desired value of $Z(t)$. The agent can control $Z(t)$ by increasing $Z_1(t)$ or $Z_2(t)$ but an instantaneous adjustment of $Z_1(t)$ or $Z_2(t)$ is not always possible. The desired value of $Z(t)$ becomes the agent's decision variable, depending on what action the agent takes. The constraints are imposed on $Z^*(t)$.

$X(t)$: A value of **FLE** for an agent who has Transition 1 at epoch t. It is assumed that when the agent determines the value for FLE, it is a random drawing according to $G_t(x)$.

$Y(t)$: Middle point of the market rates at epoch t. This is a random variable.

$G_t(x)$: Distribution function from which the new arrival's $X(t)$ is drawn.

$H_t(x)$: $H_t(x)$ is a sample distribution function of $X(t)$ of agents who exist in the market at epoch t.

α_i: Arrival intensity of retail transactions. The expected number of arrivals of retail customers per unit time.

β_i: Compound arrival intensity of retail transactions. $\beta_i = \alpha_i E[C] = \int_0^1 R_i(t)\,dt \quad for \quad i = 1, 2.$ The expected quantity of the retail transactions per unit time.

Γ: A set of feasible actions.

Γ_h: A subset of actions which is feasible when an agent sets his position at a desired level, hitting the market rate right away.

Γ_w: A subset of actions which is feasible when an agent sets his quotation at a desired level, waiting for his quotation to be hit.

λ_a: An arrival intensity of buyers who hit the offered rate. $\lambda_a \equiv \lambda_{a1} + \lambda_{a2}$

λ_{a1}: An arrival intensity of buyers who hit the offered rate due to the retail selling.

λ_{a2}: An arrival intensity of buyers who hit the offered rate due to the heterogeneous expectations.

λ_b: An arrival intensity of sellers who hit the bid rate. $\lambda_b \equiv \lambda_{b1} + \lambda_{b2}$.

λ_{b1}: An arrival intensity of sellers who hit the bid rate due to retail buying.

λ_{b2}: An arrival intensity of sellers who hit the bid rate due to heterogeneous expectations.

Ω_p: A finite subset of positive integers whose element is a value which the exchange rate may take. The exchange rate is the price of US dollars in terms of the local currency. For the sake of simplicity, the actual exchange rates with decimal points are redefined to positive integers. For example, $\Omega_1 = \{1, 2, ..., 200\}$.

Ω_q: A finite subset of non-negative integers, including 0, whose elements are values which cumulative quantities of retail and wholesale transactions of the agent may take.

$\Pi[0, t_0]$: An agent's profit over $[0, t_0]$.

$\xi(t)$: Expected value of $Y(t)$ conditional on information available at a given epoch, say, t_0. $\xi(t) \equiv E[Y(t) \mid \mathcal{F}_{t_0}]$

$\xi_a(t)$: Expected offered rate at epoch t.

$\xi_b(t)$: Expected bid rate at epoch t.

$\theta_1, \ \theta_2$: Parameters for the exponential distributions. $f_1(t) = \theta_1 \exp(-\theta_1 t)$ and $f_2(t) = \theta_2 \exp(-\theta_2 t)$ are density functions for the length of time which individual agents stay in the frosted-glass and the crystal-glass state respectively.

ASSUMPTIONS

A$(4-1)$: At epoch t, a retail selling price is $A(t) + c$ and a retail buying price is $B(t) - c$, where c is an exogenously given constant.

A(4 − 2): Arrivals of retail buyers and sellers constitute respective *compound Poisson processes.*

A(4 − 3): The daylight limit is L.

A(4 − 4): The overnight limit is 0.

A(4 − 5): Individual local markets around the world have their specific business hours. The start and the end of the business hours of a local market overlap with neighboring local markets. Some of the agents have branches in the neighboring local markets. At the end of the business day, some of the agents who have the overseas branches remains as market maker. If their positions are open when the transactions in our local market are completed, these market makers have transactions with their branches. The prices applied for these inter-branch transactions are the same as the market rates at the last epoch. If the market maker ends with a short (long) position, he buys from (sells to) the overseas branch at the last offered (bid) rate of our local market.

A(4 − 6): When the agent calculates his expected profit, the set of values which $S_1^*(t)$ and $S_2^*(t)$ can take at given epoch t in the future consists of $\xi_a(t)$ and $\xi_b(t)$ only.

A(4 − 7): $\xi(t)$ is not influenced by Γ.

A(4 − 8): The agent is risk neutral.

A(4 − 9): The agent expects that the exchange rate will go up more than the bid ask spread. The agent's expectation is as follows: At epoch t_0, there is an interval $[t_0, T_0]$ such that $\frac{d\xi(t)}{dt} > 0$ for $t_0 < t < T_0$ and $\xi_b(T) > \xi_a(t_0)$.

A(4 − 10): When the offered rate is hit, the bid and offered rates jump upward by 0, v or $2v$ with equal probabilities while maintaining the bid and ask spread at v.

A(4 − 11): If the agent has a bullish expectation such that for $0 < t < T$, $\frac{d\xi(t)}{dt} > 0$ and $|\xi(T) - \xi(0)| > 2u$, then the agent chooses $\{\gamma_1\}$.

A(4 − 12): The agent does not expect the exchange rate to move more than the bid ask spread for a while: At epoch t_0, there exists an interval $[t_0, T_0]$ such that $\frac{d\xi(t)}{dt} = 0$ or $\frac{d\xi(t)}{dt} \neq 0$ but $|\xi(T) - \xi(t_0)| < 2u$.

A(4 − 10): Each arrival trades either one or two units.

A(6 − 1): The transition between the two states is a renewal process whose renewal epochs follow exponential distributions.

A(6 − 2): The quantity of each arrival is unity.

A(6 − 3): Each agent's daylight limit is equal to one transaction unit.

A(6 − 4): $H_t(x)$ and $G_t(x)$ are not equal.

A(6 − 4)′: $G_t(x)$ and $H_t(x)$ are the same distributions.

A(6 − 5): $G_t(x)$ and $H_t(x)$ are uniform distributions; $G_t(x) = \frac{x}{k}, H_t(x) = \frac{x-m_1}{k}$

A(6 − 6): All the agents have the same bid-ask spread.

A(7 − 1): All the agents make the same estimate about the variance of $G_t(x)$ and $H_t(x)$ and $E[N_c]$.

A(7 − 2): The aggregate retail demand and supply have the same arrival rates; $\lambda_{a1} = \lambda_{b1}$.

LIST OF FIGURES

PART 1.

1. INTRODUCTION

This work deals with the intra-day determination of foreign exchange rates. We have two objectives. The first is to establish a beachhead in the analysis of the *microstructure* of the foreign exchange market. The microstructure consists of studies about details of transaction processes. This subject has been researched for equity markets. However, only a few models[1] have been suggested for the microstructure of the foreign exchange markets. The models for the equity markets are not directly applicable to the foreign exchange market, because the equity market is essentially a retail market while the foreign exchange market is a wholesale market. The existing models for the foreign exchange market do not consider interactions between the wholesalers.

Empirically macroeconomic models are not better at approximating the foreign exchange rate than a random walk hypothesis. Our motivation to study the microstructure of the foreign exchange market stems from the fact no one has yet investigated whether individual traders' optimizations in the market are consistent with macroeconomic models. Our model shows that optimizing traders try to exploit any fluctuation of the transaction price no matter what

[1]See Allen (1977) and Garman (1976). Garman (1976) is applicable although not specialized in the foreign exchange market.

factor is causing the fluctuation, including misunderstandings of other agents. It implies that there are not enough stabilizing forces in the market to bring the exchange rate to a level determined by macroeconomic factors.

The exchange rate is determined by continuous and double-sided auction. Such an auction constitutes a process of price formation in continuous time, which is also a process of dissemination of information in a speculative market. The continuous auction is a subject which is inherently incompatible with comparative statics equilibrium analysis. Studying the auction process of the foreign exchange market necessarily prompts a theoretical challenge. We introduce a queueing-theoretic approach. Namely, instead of demand and supply per period, we define expected numbers of arrivals of buyers and sellers per unit time. With this novel approach as a main feature of our beachhead model, we specify an agent's optimization problem and develop a model of the auction among heterogeneous agents.

The second objective of this paper is the application of our model to empirical issues. Using our model, we explain empirical observations, including the relationship between the size of bid-ask spread and price volatility. As a further application, we expand our model by allowing some agents to manipulate the market although there are many agents in the market. Then we suggest that we can construct a model such that Stackelberg behaviors of some agents can cause bandwagon effects[2] among the agents. So far the bandwagon effects in the foreign exchange market have been taken as psychological phenomena by academicians such as Baillie and McMahon (1989).

[2]This is described in Section 3.8.

The following are the empirical observations which we want to explain with our model: (1) Exchange rate volatility is an important factor in explaining variations in bid-ask spread (Glassman, 1987). (2) The relationship between the bid-ask spread and trading volume is sometimes positive and often statistically insignificant (Glassman, 1987). (3) A clear relationship between volatility and trading volume does not exist, judging from (1) and (2). On the contrary, in the equity market, a positive correlation between trading volume and the absolute values of price change exists (Karpoff, 1987). (4) Bandwagon effects appear sometimes. (5) A sequence of transaction prices shows a trend.

Figure 1 summarizes the structure of the foreign exchange market of our model. We call foreign exchange traders of banks *agents.* The market is a wholesale market and its constituents are one broker and many agents connected by telephone. Spot US dollars are traded against another currency and the bidding takes place with a broker during business hours. The market is located in one country and has specific business hours. There exist overseas markets whose business hours may or may not overlap with our local market. Interactions with overseas markets are left implicit in this work as we focus on our local market.

All agents continuously monitor the bidding, but not all of them are quoting their buying or selling prices. If an agent quotes his price to the broker, or if he notifies the broker about his intention to trade at the price being quoted by someone else, we call such an event, or the agent himself, *an arrival.* The agents arrive at the market as buyers or sellers and leave the market when their transactions are realized or when they cancel their quotations. The broker announces to the market the maximum buying price and the minimum selling price among the valid quotations.

Our model consists of two parts, firstly, an individual agent's optimization problem with a given expectation about transaction prices and, secondly, auction which is interactions among heterogeneous agents. The agent's optimization problem is expected profit maximization with a given expectation. In our setup, the agent trades in the market and meanwhile he trades with randomly arriving retail customers. In addition, the auction is continuous and double-sided. The double-sided auction complicates the agent's optimization problem by offering various choices to the agent. There are too many alternative actions and the probabilities for the consequences which each choice of action may bring are intractable. Therefore, we solve the optimization problem by limiting the set of feasible choices for the agent to a restricted set. As a conclusion of the individual agent's optimization problem, we show how the first local extremum (FLE) of the expected time path of the exchange rate determines the agent's action now.

In the second part of the model, optimizing agents interact in the auction process. They are assumed to be heterogeneous with respect to the expectation of an intra-day time path of the exchange rate and also with respect to retail transactions. Since agents' actions depend on their expected FLE values, the distribution of the expected FLE values is the distribution of the actions. The quantity which an agent wants to trade is also treated as a random variable. The heterogeneous agents constitute *a statistical ensemble* (Garman (1976)). Taking the heterogeneity of the agents into account, we derive the expected time path of the exchange rate and the market maker's optimal quotations which have been taken as given in the individual agent's optimization problem. In the course of the derivation of the expectations, effects of two sources of arrivals, heterogeneous expectations and retail transactions are identified. We use our model to prove propositions about the empirical issues. and present our analysis as proofs for propositions.

2. FEATURES OF THE MODEL

2.1. THEORETICAL FEATURES

The continuous double-sided auction in the foreign exchange market cannot be modeled with comparative statics. In addition, as in other financial markets, when traders form expectations about the exchange rate, they take into account what expectations other traders have. Estimates of others' expectations determine the traders' actions and, hence, the transaction prices. These interactions of expectations, which Keynes compared to guessing a winner of the beauty contest, have not been incorporated into microstructure models. To establish a beachhead in the analysis of the microstructure of foreign exchange markets, we need analytical method which have not been applied in this context. Our model contains the following features: arrival intensities of the buyers and sellers; heterogeneity of information; a process of revising expectations; Poisson arrival processes of the retail customers; agents acting as super Keynesian; a market maker behavior that depends on expectation.

The auction in the foreign exchange market is continuous and double-sided. The continuous auction implies (1) buyers and sellers may quote their respective prices at any epoch; (2) whenever a buyer and seller pair agrees upon a price, a transaction takes place, and; (3) upon completion of the transaction, the buyer and seller pair leaves the market, and the bidding is continued among the remaining traders and new entrants. For the continuous auction, there is no specific length of time during which quantities of demand and supply are defined. Instead of demand and supply, we model random arrivals of buyers and sellers as Poisson processes and we define per-unit-time arrival rates of buyers and sellers. The expected number of arrivals per unit time is called *the arrival intensity.* This describes the first feature of our model.

The second feature deals with how to measuring the heterogeneity of agents. Since our model is in continuous time, agents form expectations for continuous time paths, instead of exchange rates of the end of discrete periods. Agents are heterogeneous in regard to the expected time path. Along the expected time path, not every point is equally important for the agent's decision making. As is argued in the discussion about the agent's optimization problem, only *the first local extremum* (**FLE**) of the expected time path matters for an agent's present transaction decisions. We assign a distribution function to the values of FLE expected by the agents. Defining a distribution function for the expected FLE values is the second feature of our model. This distribution function reflects the percentage of the agents sharing a given expected FLE value and, hence, the percentage of the agents who will take the same actions.

The third feature of our model is that the agent revises expectations over time. We assume that the agent's level of confidence shifts between two states, which we call *state of crystal glass* and *state of frosted glass*, respectively. If the agent is in a crystal-glass state, he is confident enough of his expectation to speculate on it. If the agent is in a frosted glass state, he does not want to assume an open position. When the transition from the crystal glass to the frosted glass occurs, the agent loses confidence in his expectation and wants to square his position. He becomes a buyer or seller, depending on his position at that moment. When the transition is from frosted to crystal, the agent becomes buyer, seller or market maker, depending on his new expectation.

We add a further specification to these transitions of expectations. We assume that for each state the lengths of time the agent stays in this state can be described by an exponential distribution. In other words, we model state transitions as a renewal process of switching between two states, where renewal times follow the exponential distribution assigned to the respective state. These expectation transitions explain why agents arrive at the market asynchronously. In practice, not everyone is quoting his prices nor having open position at any given time. With this formal model of revising expectations, we can differentiate volatile periods from quiet periods by changing parameter values of the renewal process.

Agents trade with retail customers who are merchants and investors. The fourth feature of our model is that the random arrivals of these retail customers are assumed to constitute Poisson processes which are exogenously given to the agent. Whenever the retail customer arrives, the agent trades with the customer and the agent's position changes. Then the agent may want to adjust his position by trading in the market. The random arrival of retail customers provides another reason for agents to arrive asynchronously. By specifying retail demand and supply as Poisson processes, we can explicitly derive their expected quantities for a given interval of time. Also Poisson processes have the convenient property that the sum of Poisson processes is also a Poisson process whose arrival intensity is the sum of the intensities of the individual processes. Therefore, when we aggregate the individual agents' retail demand

and supply, the aggregated quantities also constitute Poisson process. The expected values for the aggregate retail demand and supply can be obtained and used in the analysis.

The fifth feature of our model is super Keynesian. A sequence of the transaction prices is determined by the individual agents' perception about what the transaction prices will be. Formation of the expectation about future spot rate involves estimating the other agents' expectations. Individual agents' transaction decisions are based on such estimates. Keynes's analogy of guessing a winner for a beauty contest applies here. Everybody is looking around and trying to guess what others are thinking about the future path of the exchange rate. In this sense, the agents in our model can be called *super Keynesian*.[3] As long as one can take advantage of the other agents' expectations, it does not matter whether their expectations match your forecast. Our super Keynesian agents estimate the distribution function of expected FLE values, so they can take advantage of the heterogeneity of the expectations.

The sixth feature of our model is that agents' decision to act as market makers depends on their expected FLE values. If an agent's FLE equals the current transaction prices, the agent becomes the market maker. In other words, if the agent expects that the transaction price will remain around the current level, his optimal action is quoting both buying and selling prices at the same time, in order to take advantage of the different expectations among the agents. The bid-ask spread will become his profit as the transactions continue. The sixth feature is not an assumption but a logical consequence of our model.

[3]This term is coined by Donald Schilling.

2.2. TRADING MECHANISMS TO BE MODELED

Our model's assumptions are based on a realistic trading mechanism. In the following, we present the assumptions and our rationale for them. In our model, the local market has a broker and all auctions take place with the broker. The agents, who are banks, may assume open positions depending on their expectations on the time path of the exchange rate. The agent's cost of the open position, *i.e.,* the cost of holding nonzero inventory, is zero, except for an overnight open position. Transactions with overseas banks and forward transactions are implicit in our model. The overseas banks enter our model as retail customers of our agents. The forward market is eliminated from the analysis. Interest rates in US dollars and our local currency are exogenously given.

In the actual foreign exchange market, there are two methods by which banks bid and trade. The first method is auctions through brokers. The second is direct dealings between banks. In the Tokyo market, 50% of the transactions are through the brokers (Takahashi, 1989) and in New York, 30% are through brokers. In both methods, delivery of spot currencies is done two business days after the contract. Therefore, the open position during the business day does not incur cost to the agent.

In the case of auctions through brokers, banks are connected to brokers by telephone. When they want to trade, banks notify the broker of their quotations and some additional information. Among these quotations, the broker announces to the market the maximum buying price and the minimum selling price. Names of buyers and sellers are withheld. When a buyer and seller pair agrees upon the price, the transaction takes place.

The other method to bidding and trading is direct dealings between banks. A bank inquires of another bank its quotations (both bid and asked rates) and if the inquiring bank wants to trade at one of the quotations, a transaction takes place. Direct dealings are done only between banks which have contracted to do so bilaterally. Transaction take place in units of 10 million US dollars while in the broker's market most of the transactions are 3 to 5 million US dollars. When a bank inquires about another bank's quotations, as a matter of principle, the inquired bank does not refuse to quote. What can the inquired bank do, if it does not want to buy (or sell) any more because of its position or expectation? It quotes a less competitive buying (selling) price and a more competitive selling (buying) price, relative to the other banks' quotations. However, this strategy does not always work. Even if the quotation is less competitive, some inquiring banks still may sell (buy) US dollars to the inquired bank in order to force it to have the longer (shorter) position than the inquired bank wants. The reason of the inquiring bank's action is that as an antagonist of the inquired bank the inquiring bank is wishing that the inquired bank would start selling (buying) a large quantity so the transaction price will fall (rise). If the inquiring bank's strategy works, then it would buy (sell) back US dollars profitably. As this example shows, direct dealings are dependent on the other bank's strategies and they make the trader's inventory control more difficult. Besides, the inquiring bank has to shop around in order to find the most competitive quotation. However, this method saves banks brokerage fees. Banks who conduct direct dealings are larger ones. They also participate in a bid process with brokers. Almost all of the transactions between banks in different local markets take place by direct dealings.

Since the best quotations of the brokers are known to all the participating banks, a less competitive price quoted for direct dealings is not likely to be realized as a transaction price. On the contrary, the best quotations for direct dealings are not known to all of the banks. In this paper we avoid this complication. We assume that our local foreign exchange market consists of one broker and N banks ($N = 100$, for example). No direct dealings are allowed between banks in our local market. Direct dealings with overseas banks are treated as retail transactions of the agents who are inquired of their quotations.

Banks have restrictions on inventory levels of foreign currencies during and at the end of a business day. These restrictions are called *daylight* and *overnight* limits. The overnight limit is stricter than the daylight limit. We model these restrictions. They have significant consequences for the arrival rate of the buyers and sellers at the market. For example, when you observe one bank selling a large quantity of US dollars, the bank may be either expending its inventory or selling short. If it is short selling, the bank will have to buy back US dollars sooner or later in order to satisfy the overnight limit, even if the exchange rate changes unfavorably. Other traders can take advantage of this, if they know that short selling has taken place.

Outright forward transactions do not exist virtually. Almost all of the forward transactions are done as a part of swap transactions (see Appendix A). The swap contract is similar to a repurchase agreement of a security. In case of the most common type of swap transactions, two days after the contract the currencies are delivered, and the next day the currencies are delivered in the reverse direction. If a trader wants to sell forward US dollars, then he makes a swap contract which consists of his buying spot dollars and his selling forward dollars.

A trader who has a swap transaction has to have an extra spot transaction. What accompanies the above example of the swap transaction is that the trader sells the spot dollars which he bought as the spot part of the swap transaction. Because the swap transactions are accompanied by the extra spot transactions, demand and supply of forward exchanges appear in the spot market as the same quantities of demand and supply. We can concentrate on the spot market without losing the effects of forward exchanges.

Brokerage fees are parts of the transaction costs. It does not seem that the brokerage fee influences the agent's speculative decision. Brokerage fees in Tokyo are as follows (Yamamoto, 1988).

Spot US Dollars

Size of Transaction	Fees per Million Dollars
1. one million dollars or greater	4,500 yen
2. smaller than one million dollars, greater than half a million dollars	7,500 yen
3. smaller than half a million dollars	10,000 yen

The unit of the quotation is 0.01 yen per US dollar. If the exchange rate changes by the minimum unit, then the change of the value of US dollar in terms of Japanese yen is 10,000 yen for one million dollars and 5,000 yen for a half million dollars. Meanwhile, the brokerage fee is 4,500 yen and 3,750 yen respectively. In the case of one million dollars, the brokerage fee is compensated if the difference of the transaction prices is one unit of the exchange rate in favor of the agent. In our model we do not consider the effect of the brokerage fee on the agent's transaction decisions.

3. CONNECTIONS WITH EXISTING LITERATURE

3.1. RELEVANCE OF THE MICROSTRUCTURE

We now discuss how our model of the microstructure of the foreign exchange market relates to the existing literature. Key words are random walk, microstructure, Poisson process, bid-ask spread, heterogeneous agents, and relationship between price volatility and trading volume. First of all, exchange rates seem to follows the random walk. The foreign exchange rate has not behaved in conformity with open macroeconomics. Frankel and Meese (1987) summarized the inability of macroeconomic approaches to explain the exchange rate determination:

> No set of macroeconomic variables that have been proposed is capable of explaining a very high percentage of the variation in the exchange rate (p. 128).... A variety of different econometric approaches seem to end up at the same conclusion, that the exchange rate follows a random walk (p. 122).

Although the movement of the exchange rate can be approximated by a random walk or, in the continuous case, by a Brownian motion, the movements are not completely random to those who trade in the foreign exchange market. A particle in the water shows a Brownian motion as molecules of the water collide with the particle. In any market, on the other hand, individual traders are decision makers. Although their transactions look random to econometricians, actions taken by traders are the results of optimizations. Traders collect information, form expectations and, for a given trading mechanism, make transaction decisions. However, academicians have not investigated whether an individual trader's optimizations in the foreign exchange market are consistent with the macroeconomic models. It may be the case that the traders are responding to some unknown variables as well as to a set of variables included in the macroeconomic models and that there is no stabilizing mechanism in the market forcing exchange rates back to a level determined by macroeconomic factors. There has not been any formal analysis to answer these questions. In order to identify sources of the randomness of the exchange rate, we need to understand the actual trading mechanism.

The details of the trading process are studied as a microstructure analysis which has been developed mostly for the equity market. Schwartz (1988) states what is studied as the microstructure:

> The major analytical issues ... can be classified under the following headings: (1) decisions of individual participants in the trading process; (2) advent, dissemination, and impact of information; (3) returns generation and price behavior of securities; (4) measure of market performance (price volatility, size of bid-ask spreads, and correlation patterns in a security's returns; (5) design features of a trading system; (6) regulation of the market.

Here we address (1), (2) and (4).

The foreign exchange market is a wholesale market. On the other hand, in the equity market, *a specialist* who can be a monopolist faces retail customers. Models developed for the equity market are not directly applicable to the foreign exchange market. Boothe (1988, p. 486) states:

> The stock market is essentially a retail market where individual agents confront stock specialist with final supplies and demand

for equities. The exchange market, however, is more of a wholesale market where currency traders (usually banks) deal with each other primarily to satisfy the demands of their customers but also for their own account. Thus, rather than modeling the trading between customer and specialist as in the stock market, in the exchange market it is necessary to consider trading among themselves. The specialist in the stock market often is a monopolist of the stock which he is dealing.

The quantity traded in the market is much bigger than the aggregate retail demand and supply. The differences are generated by the agents within the foreign exchange market. Only 5 percent of the trading volume of foreign exchanges all over the world correspond to international trades and capital transactions (See Ruck, 1981). In our model, as suggested in Boothe (1988), agents' transaction decisions on their profit motivation, and not just the balancing the retail transactions, influence the exchange rate.

We presuppose that individual traders are concerned with daily profit maximizations. The current account of a nation responds to the exchange rate but the response is too slow to have any recognizable effect on the daily demand and supply of the exchange rate. With these presuppositions, our model shows that in intra-day price formation there is no mechanism to put the exchange rate to the level which is consistent with macroeconomic models. In this regard, a metaphor with the equity price which appears in Malkiel (1985, p. 98) is shared by exchange rates:

> ...stock prices are in a sense anchored to certain "fundamentals" but the anchor is easily pulled up and then dropped in another place.

3.2. DOUBLE-SIDED AUCTION

The literature on auction theories is mainly concerned with cases where either one seller faces many buyers or one buyer faces many sellers. The double-sided auction has not been investigated in the context of exchange rate determination.

With regard to difficulties of modeling a sealed and an oral double-sided auction, McAfee and McMillan (1987, p. 726) stated:

> Few results on the double auction exist, because of the difficulties of modeling strategic behavior on both sides of the market. ...The oral double auction, with the bids and offers openly called, is still more difficult to model because the process takes place over time and agents do not know what prices will be available if they wait instead of trading now.

This dissertation cuts across the difficulties mentioned above. Our main theoretical features, the arrival intensities of the buyers and sellers and the assigning of an FLE distribution function, make it possible to apply the double-sided auction's framework.

3.3. POISSON PROCESS

Auctions in foreign exchange and equity markets are continuous. Buyers and sellers arrive asynchronously. Garman (1976) was the first to model the asynchronous arrivals of buyers and sellers as a Poisson process. In his words (p. 257), "It is assumed that a collection of market agents can be treated as a statistical ensemble." He specified the arrivals of buy and sell orders of a given stock as a Poisson processes and he presented two models of price determination. The first one is about the determination of a monopoly dealer's quotes in the equity market. Garman's dealer firm sets quotes to maximize its expected profits from trading per period time. This second model is a double-sided auction model where there is no market maker quoting buying and selling prices concurrently. Garman's second model is Markov process where a sample of the quotations and the number of orders associated with these sample quotations constitute *a state space.* Using Kolmogorov's backward equation, Garman tried to derive a stationary distribution of the states which are defined as above. However the stationary distribution was intractable.

The foreign exchange market falls somewhere between Garman's two models. There is no monopolist market maker as Garman assumed for the equity market. Instead, there is a possibility that some of the agents will become the market makers. In our model, the agents who are neither bullish nor bearish may quote both buying and selling prices concurrently as their optimizing actions. Agents become market makers in order to take advantage of differences in the expectations among agents. The introduction of the market makers into the bidding process reduces the fluctuation of the transaction prices to a narrower range than the one in Garman's auction market model. The reason is that as long as the market maker keeps quoting both buying and selling prices, the transaction price will never go outside of an interval given by the market maker's quotations.

Garman intended to derive the stationary distribution. But the stationary distribution is ephemeral. It will disappear when someone finds it and tries to take advantage of it. An agent who figured out the stationary distribution can have positive profit by acting as a market maker. Orders which are less competitive than the market maker's quotations will not be executed as long as the market maker maintains his quotations. These orders will not disappear unless they are canceled. The process from which the stationary distribution has been derived no longer exists.

In our model, the number of arrivals has a Poisson distribution and the quantity associated with each arrival is also a random variable. What we have is a compound Poisson process (see Appendix B). This specification is applied in two places. First, the arrivals of retail customers at an agent are assumed to constitute a stationary compound Poisson process. Secondly, the arrivals of buyers and sellers at the market are modeled as nonstationary compound Poisson processes. Here nonstationarity means that for a given quotation, the arrival intensities of agents who would hit that quotation may change from time to time. This nonstationarity is caused by shifts in the FLE distribution. The heterogeneity of quotations in Garman's model is generated by the heterogeneity of the individual traders' reservation prices whose determinations were left unexplained. Our model's counterpart of the reservation price is each agent's expected FLE value.

Strictly speaking, the number of retail customers and agents is finite. The arrival process is a collection of binomial decision processes of individual customers and agents who are finite in number. The Poisson distribution is a limit of binomial distributions as the number of trials goes to infinity. Since the number of retail customers per agent and the number of agents in the market is typically more than 100, we can assume for our analysis that the arrival process constitutes a Poisson process.

Specifying asynchronous arrivals of the buyers and sellers as Poisson process has wide applicability. An example is Tinic (1972, p. 81), who describes *a specialist* in the equity market:

> The level of trading activity in a stock influences the size of inventories carried by the specialists. In general, the higher the level of trading, the greater the chance that buy and sell orders will tend to balance during a trading period. Beyond this, the larger the turnover, the easier it is for the specialist to make adjustments in his position, because sizable trading activity reflects traders' interest in a stock. To the extent that active markets tend to self-equate, the need for specialist's inventory participation is reduced, in terms of both the average size of positions and the average holding period.

If the arrival processes of buy and sell orders in the above statement are assumed to be Poisson processes with equal arrival intensities, the expected number of times in which the specialist's position becomes square increases as the orders' arrival intensities increase. However, at the same time, the probability that the specialist's position is square at a given epoch decreases. The statement in Tinic (1972) is not straightforwardly justified mathematically.

3.4. LIQUIDITY

As an application of a Poisson process, the liquidity of an asset can be defined in terms of Poisson process. The liquidity can be defined using the arrival intensities of the buyers and sellers and the heterogeneity of their quotations. Antiques and art works have very low arrival intensities of buyers and sellers, but a distribution of the heterogeneous quotations is stable over a long time. Another example of the application of our approach is the price of real estate. Real estate has the low arrival intensities of buyers and sellers. Each real estate transaction has specific characteristics. An individual buyer is looking for real estate with ideal characteristics. For a given piece of real estate, the buyer's quotation is determined, depending on its distance from these ideal characteristics. If we measure the distributions of ideal characters among the buyers, then for a given piece of real estate, we can define the distribution of the heterogeneous quotations. And if, in addition, we define the arrival rate of the buyers at the market from the general public, then we can derive the arrival intensities of the buyers who decide to buy the given real estate. The liquidity of the real estate is defined as the inverse of an expected waiting time until sold at a mean value of the buyer's quotation. If the seller cannot wait, it becomes *a fire sale.*

According to Schwartz (1988),

> Liquid markets are characterized by depth, breadth, and resiliency:
>
> Depth: A market has depth if a sufficient number of orders exists at prices above and below the price at which shares are currently trading.
>
> Breadth: A market has breadth, if these orders exist in substantial volume.
>
> Resiliency: A market has resiliency if temporary price changes due to temporary order imbalances quickly attract new orders to the market.

Our model can be described as a market with depth of varying degree. As the expected number of agents who are quoting (*i.e.,* in the crystal-glass state) increases or as the variance of FLE values decreases, the market has more depth and the more agents are quoting around the current price. In this case, the batched arrivals cause small disturbances in the sequence of the transaction prices. The quantity which is associated with a given agent's quotation is always equal to the agent's daylight limit, or to twice as much as the daylight limit, if the agent is waiting on his quotations. Since the quantity associated with the quotation does not vary except for the daylight limit and its double amount,[4] we cannot describe the breadth of the market. Resiliency is only partly described in our model at this stage. Unless we have a mechanism to allow the agents to gradually adjust their position while they have bullish or bearish expectations, we cannot attribute resiliency to the agents' decision making. In our model, the arrival intensities of buyers and sellers do not respond to temporary price changes. However resiliency can be attributed to the nonstationarity of the aggregate retail demand and supply.

3.5. DETERMINATION OF BID-ASK SPREAD

Modeling the determination of bid-ask spread is a nontrivial problem. Amihud and Mendelson (1980) considered a monopolist market maker's optimization problem. They depicted stochastic demand and supply as a price-dependent Poisson process. They showed the dependence of the bid-ask prices on the market maker's stock inventory position.

Allen (1977) modeled the behavior of a risk-averse bank trader who buys or sells foreign exchange "with a view to profitably reversing the transaction in the future". Allen's model is the first microstructure model of the foreign exchange market. He showed that the increased variance of the expected future

[4]If the position is square, the quantity is equal to the daylight limit. If the agent has a short (long) position, the quantity to buy (sell) is twice as much as the daylight limit.

prices will cause the profit-maximizing, risk-averse trader to widen his bid-ask spread. Widening the spread can be interpreted as raising the price charged by the bank for its liquidity service.

Allen treats a bank as a monopolist and do not consider interactions between the market makers. In the equity markets, market makers are monopolist but this is not the case in foreign exchange markets. In our model, we do not assume that the market maker is a monopolist. A wider spread is the result of a higher degree of heterogeneity of expectations among the agents.

Garman's model of an auction with a broker (1976) covers the determination of the bid-ask spread. However, it did not have a clear result because the multivariate stationary distribution of the quotations and quantities was intractable. Cohen et al. (1986) analyzed the bid-ask spread in a *limit order* market of an equity which is equivalent to Garman's auction market. First, they focused on stochastic characteristics of the ask price. They modeled the market-ask-price generation process as a compound Poisson process (See Appendix B). The ask price evolves as it jumps randomly. Then they assumed that each such jump is a random variable that is independently and identically distributed over time with mean zero and some variance. The assumption cannot be reconciled with the *resiliency* of the market (See Section 3.4). Market resiliency implies that the majority of agents have similar expectations about the transaction price, and that if transaction prices deviate from their expectations, they respond in a manner that stabilizes the transaction price. As a characteristic of the stochastic process, the resiliency implies that the distribution of the jump depends on where the current price is. Our model describes a limited case of the resiliency where the heterogeneity of the expectation is not stationary. If it is stationary, the jump in the transaction price can be approximated by *i.i.d.* random variables with mean 0. If it is non-stationary, the distribution function of the jump depends on the current transaction price. Regarding the distribution of FLE values among the agents, we can distinguish two cases. The first case is when the heterogeneity of the expectation is stationary[5] and the jump in the transaction price can be approximated by an *i.i.d.* random variable with mean 0. The second case is when the heterogeneity is nonstationary with mean different from 0.

Cohen et al. (1986) showed why a bid-ask spread exists in an auction market. They assumed that investors make transaction decisions for a given trading period and that the price is a continuous variable. Summarizing their work, Schwartz (1988, p. 336) writes:

> Can a buyer make the probability of execution infinitesimally close to unity by writing the buy order at a price infinitesimally close to, but still below, the market ask? No, he or she cannot; a non-infinitesimal probability will remain that the ask price will increase, and that the buy limit that had been infinitesimally close to it will not be hit in the trading period.

This statement says that a trader who is afraid of missing the current ask price hits the ask price rather than putting his quote close to the ask price. Their analysis has three problems. First, that they cannot explain how someone chose a specific value for the existing ask price. Second, their model is limited to a given period whose length is left unexplained, and that it does not consider the possibility that the buyer may optimize over the periods

[5]The FLE distribution function stays at the same location.

using dynamic programming method. Third, since the price is not actually a continuous variable, their argument is not applicable without explanation of why the bid-ask spread remains more than a unit of measurement of the price.

The basic source of the difficulties in Cohen et al. in modeling the bid-ask spread is the lack a model for continuous double-sided auctions.

We presuppose that the fluctuations in the transaction price are generated by two types of the fluctuations: fluctuations within a given FLE distribution and fluctuations due to shifts of the FLE distribution. As long as the FLE distribution stays the same (stationary heterogeneity), the agent can profit by becoming a market maker. Any price in a support of the distribution can be reached although the expected waiting time may be infinity. The agent will wait on his quotation rather than hitting the available price in the market. If other agents recognize what stationary heterogeneity brings, then the bid and ask spread will converge to the minimum unit as more of the agents quote their prices and wait on them. However, we will not observe such a situation. It is possible that the FLE distribution itself shifts. Once this happens, the price which the agent is waiting on may be outside of the support of the distribution function from which the new quotations are chosen. Or the arrival intensity of agents who would hit the agent's quotation may be greatly decreased. When the agent observes a price fluctuation, the agent cannot identify its reason. The agent will hit the available price in the market rather than waiting on his own quotation.

In our model, since the agents are heterogeneous with respect to FLE value and their confidence level about their expectations, transactions take place before the competition among buyers or sellers (*i.e.,* additional arrivals of the buyers or sellers) will narrow the spread.

3.6. HETEROGENEITY OF AGENTS

Mendelson (1985) introduced heterogeneity of traders into the microstructure models. He analyzed a case where the preferences and the endowments of the traders are heterogeneous. He proved that increasing the number of traders always reduces the variability of market clearing prices, and that increasing the variability of the traders' valuations brings about an increase in price variability. Heterogeneity of preferences is described as a difference in reservation prices. Randomly the traders have allocation of one unit of an asset or none at all. Their reservation prices vary according to a underlying distribution. A market clearing price fluctuates less as the variance of the reservation prices becomes smaller and as the number of the traders increases. Our model is similar to Mendelson's with regard to the randomness of the endowments of the asset and the heterogeneity of the reservation price. The difference is that while Mendelson does not allow speculation across periods, in our model the speculation based on the expected time path of the prices plays an important role. Mendelson's model is a multiperiod model without speculation. Therefore, his model explains only the relationship between the heterogeneity of the traders and the price fluctuation. In the foreign exchange market, however, the traders can hold inventory. Mendelson (1985, p. 256) stated that:

> The "market-microstructure" literature typically follows Garman (1976). ...Correspondingly, the elementary building blocks of the resulting models are stochastic "order generating processes"

which represent aggregate market behavior, rather than the characteristics of individual market agents. This makes it difficult, if not impossible, to examine how various traders' characteristics affect the resulting market outcomes, and to perform meaningful comparisons between models with different values of the relevant parameters (or different models of exchange).

In our model, at random epochs, the agents are randomly characterized by their expectations (*i.e.,* the states of glass and FLE) and retail transactions. Individual agents' characteristics affect market outcomes as in Mendelson. It is not clear that a sequence of competitive equilibrium converges to the continuous auction.

Much formal analysis of the security market assumes that different investors have homogeneous expectations. For example, the standard capital asset pricing model assumes homogeneous expectations. However, if we assume homogeneity of expectations in the foreign exchange market, we lose an essential part of the intra-day price formation process. Takahashi (1989, p. 109) states a businessman's understanding that traders in foreign exchange markets have heterogeneous expectations:

> If US dollar is traded at 125 yen, for example, it implies that those who want to sell US dollar at 125 yen exist on one hand and those who want to buy at 125 yen exist on the other hand and that the exchange rate is equilibrated at it. It is difficult to forecast a direction for which the exchange rate is heading for. In short, reasons to buy US dollars are equilibrated with reasons to sell.

Schwartz (1988, p. 272) justifies the assumption of the heterogeneous expectations as follows.

> Expectations are heterogeneous because information is costly and investors do not have perfect information. Each investor obtains that quantity of information that is deemed to be optimal, given his or her cost of acquiring it and efficiency as an information processor. The cost of obtaining and the benefit of having information differ appreciably across investors.

It is not clear what the cost of obtaining information is in the foreign exchange market. What is meant by "perfect information" or "information processor" depends on the underlying model. In the capital asset pricing model for securities, perfect information implies knowledge about the mean and variance of security returns. For our intra-day price formation process in the foreign exchange market, many of the variables which influence the transaction price are not measured daily or are not available to many of the agents. For example, retail transactions are decisive factors in the agent's decision making, but, in general, the agent cannot know the retail transactions of other agents. Perfect information cannot be obtained in our analysis. The intra-day price formation process is a process by which information bits are being disseminated among the agents.

In Schwartz (1988), an information processor seems to mean an econometric model or technique of security analysis, for example. However, such knowledge can be called information rather than information processor. The definition of information needs further clarification. As will be discussed in Chapter 5, information is a σ-field of a family of subsets and, these subsets are defined in a space which has infinite elements. News and an econometric model are those subsets. Obtaining an information bit is equivalent to a refinement of the σ-field. Perfect information means having a σ-field generated by all the subsets which are possessed by the individual agents. It is impossible to have such a σ-field, because there is no mechanism for transferring some of the information bits between the agents. Thus, agents have heterogeneous information and expectations.

Figlewski (1978) assumes that market participants possess heterogeneous information, price expectations and different wealth endowments. Few of the market participants can predict consistently over time, while some others may be richly rewarded by chance. In a large population of investors, some may win often only by chance. We cannot tell who is the most accurate forecaster. Thus, expectations remain heterogeneous and the market does not completely achieve accurate prices. This is Figlewski's rationale to assume persistence of the heterogeneity of information. Names of traders of individual transactions are only partially known in the foreign exchange market. The agents in our model cannot know who was a winner. Their expectations remain heterogeneous.

The market described in Figlewski (1978) can be compared to a bet on a horse race. There exists an underlying mechanism for determining the outcome of the horse race. Participants in the betting have forecast formulas of various degree of accuracy. A distribution of the bet on the individual horses determines the winning return. If you choose a favorite, your rate of return will be smaller than otherwise, when your forecast turns out to be correct. However, the forecast and bets do not influence the factors which determine a winner of the race. But in the foreign exchange market, on the contrary, participants' forecasts influence the outcome of the race. In this regard our model is essentially different from Figlewski's.

3.7. VOLATILITY AND TRADING VOLUME

A positive relationship between price volatility and trading volume in futures and equity markets is found in empirical studies. Two kinds of stochastic models have been suggested for the relationship between volatility and trading volume. One is by Clark (1973) and the other is by Epps and Epps (1976) and Tauchen and Pitts (1983).

In Clark (1973), the price evolves according to the event time, not by calender time. An arrival of news measures the time. The news induces transactions and jumps in price. The jump in the price is assumed to be an *i.i.d.* random variable. The daily price change is the sum of these random jumps. Then the variance of the daily price change is the sum of the variances of the individual jumps and hence, a random variable whose mean is proportional to an expected number of daily transactions. Thus the variance of the daily price change tends to be larger when the trading volume is larger. The positive relationship between price volatility and trading volume is an immediate consequence of such a specification of the price evolution. If we assume that the number of jumps in the price follows a Poisson distribution, then Clark's specification implies that the price evolves as it jumps and that the sum of the jumps constitutes a compound Poisson process. A problem with Clark's model is that it deals with a call market,[6] instead of a continuous auction market, although he analyzes the cotton future market where the auction process is

[6]The essence of a call is that orders that have been accumulated over a period of time are batched for simultaneous execution, and all the crossing orders are executed at the same price (Schwartz, 1988, p. 20).

continuous. It is not clear that a sequence of the transactions in the call market can approximate the continuous auction. His model has another problem. Information cannot disseminate throughout the market instantly. We cannot assume the jumps in price constitute an *i.i.d.* random variable during the dissemination of the news as we shorten interval of the period for which the orders are batched together for the call market.

The second type of model of volatility and trading volume is presented by Epps and Epps (1976) and Tauchen and Pitts (1983). Their specification is that the change in the market price on each within-day transaction or market clearing is the average of the changes in all of the traders' reservation prices. Tauchen and Pitts (1983) state:

> Epps and Epps assume there is a positive relationship between the extent to which traders disagree when they revise their reservation prices and the absolute value of the change in the market price. That is, an increase in the extent to which traders disagree is associated with a larger absolute price change. The price variability-volume relationship arises, then, because the volume of trading is positively related to the extent to which traders disagree when they revise their reservation prices (p. 485). ...Epps and Epps's key assumption gives them a nearly exact positive relationship between the absolute value of the change in the market price and the trading volume on each within-day market clearing (p. 487).

The price may not jump with a proportionately large trading volume as assumed in Epps and Epps. In order to overcome this problem, Tauchen and Pitts (1983) introduced a scheme of variance components into the model. A change in the reservation price consists of two parts. One part is common to all traders and the other part is specific to one trader. If the change in the reservation price is due to the common part, the marker clearing price jumps without being accompanied by a large trading volume.

Their model assumes a call market. They never specify how we can define a Walrasian equilibrium price from the sequence of transaction prices observed in the continuous auction. Another problem of their model is that there is no explanation of why the trader's desired position is given by a constant multiple of the distance between the transaction price and the trader's reservation price.

In our model, the jumps in the transaction price are not always *i.i.d.*. We show that if the heterogeneity of the expectations is stationary, then the jump in the transaction price can be approximated by an *i.i.d.* random variable. If the heterogeneity of FLE values is not stationary, then the jumps are not identically distributed. If we increase the expected number of transitions of expectations, then the expected trading volume increases. If the FLE distribution shifts, then the transaction price will shift without an accompanying increased trading volume. Our model specifies the process by which the arrivals of buyers and sellers are generated. Also, in our model, information is translated into the arrival intensities. We can analyze the respective effects of buyers and sellers who are motivated by heterogeneous expectations or just by liquidity purposes.

PART 2.

4. THE AGENT'S OPTIMIZATION PROBLEM

4.1. CONSTRUCTION OF OPTIMIZATION PROBLEM

In this chapter we specify an agent's optimization problem. For a given expected time path of the exchange rate, agent are assumed to maximize expected profit over a given time period. We show that an action the agent takes depends on his expected FLE value. Assuming that the exchange-rate process is piece-wise stationary, we show that sequences of optimizations using most recently updated FLE values lead to over all optimization. This implies that the *optimal policy*[1] is to concentrate on profit maximization in each interval where the expected time path is monotone. By showing the role of FLE values, we obtain a rationale for assigning a distribution function to the FLE values held by agents. The solution to the optimization problem will lead to the derivation of arrival intensities in Chapter 6.

The agent trades in a wholesale market on the one hand and trades with his retail customers on the other. The difference between selling and buying prices and the fixed rate of commission for a retail transaction become the agent's profit. The agent's optimization problem is essentially an inventory

[1]A policy is a contingency plan for choosing actions (Heyman and Sobel, 1984).

control problem. However, since the auction is double-sided, agents have two kinds of feasible actions: when the agent wants to trade in the market, the agent can choose either a price (*i.e.*, waiting on his quotation) or his inventory level *i.e.*, hitting one of the market rates.[2] If the agent chooses the price, then his position becomes a random variable. If he chooses his position, the price becomes exogenous. Although the agent knows the price when he makes a transaction decision, this given value of the price is a realization of a random process. Since the agent can choose either his price or his position, a set of feasible actions is the product of two subsets: one for choosing the price; and the other for choosing the position.

In Section 4.1, we will define variables and write out an equation of the agent's profit. A closed interval between epoch t_1 and t_2 is denoted by $[t_1, t_2]$. The exchange rate is the price of US dollars in terms of the local currency. For the sake of simplicity, the actual exchange rates with decimal points are redefined to positive integers. Let Ω_p be a finite subset of positive integers whose elements are values which the exchange rate may take (for example, $\Omega_p = \{1, 2, ..., 200\}$). We define random variables $A(t) \in \Omega_p$, $B(t) \in \Omega_p$ and $S(t) \in \Omega_p$ as

$A(t)$: minimum selling price quoted in the market at epoch t. This is called the *offered rate* or *asked rate.*

$B(t)$: maximum buying price quoted in the market at epoch t. This is called the *bid rate.*

$S(t)$: price of the most recent transaction up to and including epoch t.

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[2]see the following paragraph for a definition of the latter

The bid and offered rates together are called the *market rates* at epoch t. At any epoch when a transaction takes place, $S(t)$ coincides with $A(t)$ or $B(t)$. We assume that the agent's retail prices are determined by the current market rates, $A(t)$ and $B(t)$, and a constant margin c, with constant c being exogenously given.

A$(4-1)$: At epoch t, the retail selling price is $A(t)+c$ and the retail buying price is $B(t)-c$.

The agent trades any quantity with retail customers whenever they want to. The quantity and the epoch of each retail transaction are random. The constant margin c becomes a sure profit if the agent buys from (sells to) the market right away when he sells to (buys from) the customer. Let Ω_q be a subset of non-negative integers, including 0. Random variables for the retail transactions, $R_1(t) \in \Omega_q$ and $R_2(t) \in \Omega_q$, are defined as cumulative quantities.

$R_1(t)$: cumulative quantity purchased from customers during $[0,t]$.

$R_2(t)$: cumulative quantity sold to customers during $[0,t]$.

Since $R_1(t)$ and $R_2(t)$ are cumulative quantities, $dR_1(t)>0$ and $dR_2(t)>0$ signify occurrences of retail transactions at epoch t. We use the following convention for subscripts of the variables. Variables with subscript 1 give rise to increases in the agent's US dollar position, and variables with subscript 2 lead to decreases in the agent's position.

A$(4-2)$: the arrivals of retail buyers and sellers constitute *compound Poisson processes* (see Appendix B).

The compound Poisson process implies that the number of arrivals is Poisson distributed and that the quantity of each transaction is a random variable with some distribution. Let α_1 (α_2) be an arrival intensity of retail sellers (buyers) and let C be a quantity traded at each retail transaction. Since R_1 and R_2 are the cumulative quantities generated by the compound Poisson processes, the expected values of R_1 and R_2 for epoch t_0 are proportionate to the length of time. Assuming $R_1(0) = R_2(0) = 0$, the expected values are given by $E[R_1(t_0)] = \alpha_1 t_0 E[C]$ and $E[R_2(t_0)] = \alpha_2 t_0 E[C]$.

Besides retail transactions, the agent has wholesale transactions in the market. In order to describe his wholesale transactions, we define random variables $Z_1 \in \Omega_q$ and $Z_2 \in \Omega_q$ as follows.

$Z_1(t)$: cumulative quantity purchased from the market during $[0, t]$

$Z_2(t)$: cumulative quantity sold to the market during $[0, t]$

If $dZ_1(t)$ and $dZ_2(t)$ are positive, they are quantities which the agent bought and sold in the market at epoch t. We define the prices applied to $dZ_i(t)$ as follows.

$S_1(t)$: a price applied to $dZ_1(t)$

$S_2(t)$: a price applied to $dZ_2(t)$

$S_1(t) \in \Omega_p$ and $S_2(t) \in \Omega_p$ can be random variables, depending on actions which the agent takes. [3] $S_1(t)$ and $S_2(t)$ are specific to the agent, while $S(t)$ is common to all agents.

The Stieltjes integral can be defined for a monotonically increasing, but not continuous, function like $R_i(t)$ and $Z_i(t)$. By definition, we have $\int_0^{t_0} dR_i(t) = R_i(t_0)$ and $\int_0^{t_0} dZ_i(t) = Z_i(t_0)$. Using the Stieltjes integral, wholesale revenue and cost during interval $[0, t_0]$ are expressed as $\int_0^{t_0} S_2(t)\, dZ_2(t)$ and $\int_0^{t_0} S_1(t)\, dZ_1(t)$,

[3]This is explained in Section 4.2.

respectively and revenue and cost of the retail transactions are $\int_0^{t_o}(A(t)+c)\,dR_2(t)$ and $\int_0^{t_o}(B(t)-c)\,dR_1(t)$, respectively.

Let $\Pi[0,T]$ be the agent's profit over interval $[0,T]$. Using the above expressions for revenues and costs, the agent's profit during $[0,T]$ is

$$\begin{aligned}\Pi[0,T] &= \int_0^T S_2(t)\,dZ_2(t) - \int_0^T S_1(t)\,dZ_1(t) + \int_0^T (A(t)+c)\,dR_2(t) - \int_0^T (B(t)-c)\,dR_1(t)\\ &= \int_0^T S_2(t)\,dZ_2(t) - \int_0^T S_1(t)\,dZ_1(t) + \int_0^T A(t)\,dR_2(t) - \int_0^T B(t)\,dR_1(t)\\ &\quad + c\Big(R_2(T)+R_1(T)\Big). \qquad (4-1)\end{aligned}$$

4.2. CONSTRAINTS

Next, we specify the constraints for the profit maximization problem. Let a random variable $Z(t)$ denote the agent's position at epoch t, *i.e.,*

$$Z(t) \equiv Z_1(t) - Z_2(t) + z_0 + R_1(t) - R_2(t), \qquad (4-2)$$

where z_0 is an initial value of $Z(t)$ at epoch 0. $Z(t)$ takes values from a finite subset of integers; $Z(t) \in \{y \mid y = x_1 - x_2 + x_3 - x_4,\ x_i \in \Omega_q, i = 1,\ldots,4\}$. $Z \neq 0$ means an open position and $Z > 0$ ($Z < 0$) means a long (short) position. The last two terms of $(4-2)$, the cumulative retail transactions, are random and exogenously given to the agent. The agent can control $Z(t)$ by increasing $Z_1(t)$ or $Z_2(t)$, but an instantaneous adjustment of $Z_1(t)$ or $Z_2(t)$ is not always possible. We distinguish desired values of $Z(t)$, $Z_1(t)$ and $Z_2(t)$ from their actual values. This is a variation of the inventory control problem. If the agent wants to control $Z(t)$, $Z_1(t)$ and $Z_2(t)$, then his decision is made on the desired values of these variables. Define,

$Z_1^*(t)$: desired value of $Z_1(t)$,

$Z_2^*(t)$: desired value of $Z_2(t)$,

$Z^*(t)$: desired value of $Z(t)$.

In order to prevent catastrophic losses, the management does not allow an agent to assume an infinite open position, no matter what expectations the agent has. The management of a bank for which the agent works imposes restrictions on the magnitude of the agent's open position. These restrictions are imposed on the desired value of $Z(t)$ instead of the actual value. This is because the actual value $Z(t)$ jumps from time to time due to the randomly arriving retail transactions and it is impossible to control $Z(t)$ completely. If $Z(t)$ violates the constraint at a certain epoch, the agent tries to adjust $Z(t)$ to $Z^*(t)$ by taking an action which is feasible at that epoch.

The restrictions on the open position are applied during and at the end of the business day. The restrictions are exogenously given to the agent by his bank. The maximum magnitude of the open position which is allowed during the business day is called *the daylight limit.* Let L denote the daylight limit. Its value is a positive integer.

A$(4-3)$: Let L be the daylight limit, then $|Z^*(t)| \leq L$ for $t \in [0,T]$.

The restriction on the open position at the end of the business day is called *the overnight limit.* The overnight limit is stricter than the daylight limit.

A$(4-4)$: The overnight limit is 0.

If the overnight limit is not zero, then the agent faces an overnight profit maximization. If the agent has a short (long) position at the end of the business day, the agent may borrow (lend) overnight. We would like to avoid the complication of having a loan market in our model at this stage. However, if all of the market makers' overnight limits are zero, this constraint introduces a complicated dynamic programming problem. Also, almost surely not all agents will have square positions at the end of the business day, because the aggregate retail demand and supply will almost surely differ, even if their expected values are equal. We need market makers[4] who can absorb the excess of the aggregate retail demand until the very last epoch of the business day.

A$(4-5)$: Individual local markets on the globe have their specific business hours. Each market has one broker. The start and the end of business hours of the local market overlap with neighboring local markets. Some of the agents have branches in the neighboring local markets. At the end of the business day, some of the agents who have overseas branches remain as market makers. If their positions are open when the transactions in our local market are completed, these market makers continue to make transactions with their branches. The prices applied for these inter-branch transactions are the same as the market rates at the last epoch. If the market maker ends with the short (long) position, he buys from (sells to) the overseas branch at the last offered (bid) rate of local market.

4.3. CHOICE OF FEASIBLE ACTIONS

Expected values for $S(t)$, $B(t)$ and $A(t)$ have to be derived. Let $Y(t)$ be the mid point of the market rates at epoch t. The price at which the agent will trade at epoch t in the future can be either $B(t)$ or $A(t)$, depending on the actions the agent takes. Let $2u$ be an expected bid-ask spread and $\mathcal{F}_{t_0}$ be

[4]Agents who quote both buying and selling prices at the same time.

the information which the agent has at epoch t_0. For $t \geq t_0$, the expectation is conditional on $\mathcal{F}_{t_0}$. The expected values of $B(t)$ and $A(t)$ are denoted by:

$$\xi(t) \equiv \frac{1}{2}E\big[A(t) + B(t) \mid \mathcal{F}_{t_0}\big] = E\big[Y(t) \mid \mathcal{F}_{t_0}\big],$$

$$\xi_a(t) \equiv E[\,A(t) \mid \mathcal{F}_{t_0}] = \xi(t) + u, \quad \xi_b(t) \equiv E[\,B(t) \mid \mathcal{F}_{t_0}] = \xi(t) - u.$$

An agent is called *bearish* (*bullish*) at epoch t_0, if $\xi(t)$ has an interval $[t_0, T_0]$ such that $\frac{d\xi(t)}{dt} < 0$ ($\frac{d\xi(t)}{dt} > 0$), for $t \in [t_0, T_0]$.

The expectation of $\Pi[t_0, T]$, the profit as defined in $(4-1)$, is conditional on $\mathcal{F}_{t_0}$. The expected profit depends on the agent's choice of actions as well as the arrival processes of buyers and sellers in the market. We need to specify a set of feasible actions. Let Γ be a set of actions which is feasible at epoch t. Then, Γ is a collection of four coordinate vectors such that $\Gamma \subseteq \Omega_p \times \Omega_p \times \Omega_q \times \Omega_q$ and $\Gamma \equiv \{S_1^*(t), S_2^*(t), Z_1^*(t), Z_2^*(t)\}$. The action is defined for an epoch. We define Γ_h and Γ_w to be subsets of Γ.

$S_1^*(t)$: Agent's own quotation at epoch t to have $dZ_1(t) > 0$.

$S_2^*(t)$: Agent's own quotation at epoch t to have $dZ_2(t) > 0$.

Γ_h: A set of actions such that the agent chooses values for $Z_1^*(t)$ and $Z_2^*(t)$ and that he adjusts $Z_1(t)$ and $Z_2(t)$ to $Z_1^*(t)$ and $Z_2^*(t)$ immediately. $dZ_1^*(t)$ and $dZ_2^*(t)$ are not positive at the same epoch. $S_1^*(t)$ and $S_2^*(t)$ are equal to the available quotations in the market.

Γ_w: A set of actions such that the agent chooses $S_1^*(t)$ and $S_2^*(t)$, quotes one or both of them, and waits for having his quotation hit. $dZ_1^*(t)$ and $dZ_2^*(t)$ are realized randomly.

Γ_h means that the agent hits one of the market rates. It is possible that $dZ_i^*(t)$ is larger than the quantity available at the existing market rate. Should this occur, less competitive quotations may be hit also at the same epoch. Or the agent may choose to set $dZ_i^*(t)$ equal to the available quantity at the current market rate. If the agent chooses $Z_i^*(t)$, then the price is random.

If the agent chooses Γ_w, then $dZ_1^*(t)$ and $Z_2^*(t)$ become random. However, when the agent quotes his prices to the broker, the agent can specify the maximum quantities which he will trade at his prices: $dZ_1^*(t)$ and $Z_2^*(t)$ are bounded from above. A pair of variables $\{S_1^*(t), S_2^*(t)\}$ or $\{Z_1^*(t), Z_2^*(t)\}$ become random variables, depending on what action the agent takes.

Since the transition probabilities of the market rates are intractable with our arrival model, we have only the expected time paths of the market rates available. Because of this limitation, when we calculate the expected profit, we restrict feasible actions of the future to the smaller set and find an optimal action from this smaller set.

A$(4-6)$: When the agent calculates the expected profit, the set of values which $S_1^*(t)$ and $S_2^*(t)$ can take consists only of $\xi_a(t)$ and $\xi_b(t)$.

The decision variables for the optimization are the desired values. In order to derive the expected profit, we replace $S_1(t)$, $S_2(t)$, $Z_1(t)$, and $Z_2(t)$ in equation $(4-1)$ by their desired values. A$(4-2)$ determines the expected values of $R_i(t)$, for $i = 1, 2$. With given $\xi(t)$, $\gamma \in \Gamma$ and A$(4-2)$, and using $(4-1)$, at epoch t_0, the conditional expectation of the profit over interval $[t_0, T]$ is taken with regard to $R_1(t)$ and $R_2(t)$ and is given by

$$\begin{aligned} E\Big[\ \Pi[t_0,T] \mid \gamma, \mathcal{F}_{t_0}\Big] &= E\Big[\ \int_{t_0}^{T} S_2^*(t)\, dZ_2^*(t) - \int_{t_0}^{T} S_1^*(t)\, dZ_1^*(t) \ \ \mid \gamma, \mathcal{F}_{t_0}\ \Big] \\ &\quad + \int_{t_0}^{T} \xi_a(t)\, d(\ E[R_2(t)]\,) - \int_{t_0}^{T} \xi_b(t)\, d(\ E[R_1(t)]\,) \\ &\quad + c\Big\{\beta_2(T-t_0\,) + \beta_1(T-t_0\,)\Big\} \\ &= E\Big[\ \int_{t_0}^{T} S_2^*(t)\, dZ_2^*(t) - \int_{t_0}^{T} S_1^*(t)\, dZ_1^*(t) \ \ \mid \gamma, \mathcal{F}_{t_0}\Big] \end{aligned}$$

$$+\beta_2 \int_{t_0}^{T} \xi_a(t)\,dt - \beta_1 \int_{t_0}^{T} \xi_b(t)\,dt$$
$$+ c\Big\{\beta_2(T - t_0) + \beta_1(T - t_0)\Big\}.$$

R_1 and R_2 are random measures. Derivations of their expected values use Campbell's Theorem.[5] In addition, the derivation of the last three terms uses properties of the compound Poisson process.

$$E\Big[\Pi[t_0, T] \mid \gamma, \mathcal{F}_{t_0}\Big] = E\Big[\int_{t_0}^{T} S_2^*(t)\,dZ_2^*(t) - \int_{t_0}^{T} S_1^*(t)\,dZ_1^*(t) \mid \gamma, \mathcal{F}_{t_0}\Big]$$
$$+ (\beta_2 - \beta_1) \int_{t_0}^{T} \xi(t)\,dt + (c + u)(\beta_2 + \beta_1)(T - t_0), \quad (4-3)$$

where $\xi(t) = E[Y(t) \mid \mathcal{F}_{t_0}]$ and $\beta_i = \alpha_i E[C] = \int_0^1 R_i(t)\,dt$, for $i = 1, 2$.

4.4. MAXIMIZATION OF EXPECTED PROFIT

The agent's action may influence $\xi(t)$, depending on the quantity the agent wants to trade. We rule this out by introducing an analogy of perfect competition by assuming

A$(4-7)$: $\xi(t)$ is not influenced by Γ.

Moreover, we assume

A$(4-8)$: The agent is risk neutral.

Since the agent does not have controls over the last two terms of equation $(4-3)$, profit maximization is equivalent to maximizing the first term. We use assumptions from A$(4-1)$ to A$(4-8)$. The purpose of assuming A$(4-6)$ is to narrow the set of feasible actions. With given daylight and overnight limits, a given expected time path and a given set of feasible actions, at each epoch, the agent wants to maximize the expected profit of the rest of the day by choosing

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[5]See Daley and Vere-Jones, 1989, p. 188, the expected value of the random integral.

the best actions either from Γ_h or Γ_w, *i.e.*, by choosing the desired values for Z_1 and Z_2, or by choosing his quotations $S_1^*(t)$ and $S_2^*(t)$.

$$\max_{S_1^*,S_2^*,Z_1^*,Z_2^*\in\Gamma} E\Big[\ \Pi[t_0,T]\ \mid \mathcal{F}_{t_0}\Big]$$
$$= \max_{S_1^*,S_2^*,Z_1^*,Z_2^*\in\Gamma} E\Big[\int_{t_0}^{T} S_2^*(t)\,dZ_2^*(t) - \int_{t_0}^{T} S_1^*(t)\,dZ_1^*(t)\ \mid \mathcal{F}_{t_0}\Big] + \text{constant} \qquad (4-4)$$

subject to

$$-L \le Z_1^*(t) - Z_2^*(t) + z_0 + R_1(t) - R_2(t) \le L \qquad \text{for} \quad t \in [t_0, T] \qquad \text{(daylight limit)}$$

$$Z_1^*(T) - Z_2^*(T) + z_0 + R_1(T) - R_2(T) = 0, \qquad \text{(overnight limit)}$$

where $z_0 = Z(t_0)$.

First we consider the case where an expected time path $\xi(t)$ is monotonically increasing and rises by more than the bid-ask spread by the end of the day. Second we consider the case where $\xi(t)$ does not increase or decrease by more than the bid-ask spread. The results obtained in these examples of the shape of $\xi(t)$ will be applied to the cases where $\xi(t)$ has more complicated shapes.

A$(4-9)$: The agent expects that the exchange rate will go up by more than the bid-ask spread: At epoch t_0, there is an interval $[t_0, T_0]$ such that $\frac{d\xi(t)}{dt} > 0$, for $t_0 < t < T_0$ and $\xi_b(T) > \xi_a(t_0)$.

We want to solve the maximization problem $(4-4)$, assuming A$(4-9)$. We solve it in three steps. First, we limit the action to the set Γ_h and find the optimal action from Γ_h. Second, we solve the problem by using actions from the set Γ_w. Third, we compare the results of the first and the second steps and find the over-all optimal action.

Using actions of the subset Γ_h implies that we choose $Z_1^*(t)$ and $Z_2^*(t)$, and that $S_1^*(t)$ and $S_2^*(t)$ become random variables whose expected values are exogenously given. By shifting the starting epoch, the maximization problem $(4-4)$ is equivalent to the following problem.

Optimization over Γ_h: Under A$(4-9)$ and with $t_0 = 0$,

$$\max_{\gamma\in\Gamma_h} E\Big[\int_0^T S_2^*(t)\,dZ_2^*(t) - \int_0^T S_1^*(t)\,dZ_1^*(t) \mid \mathcal{F}_{t_0}\Big]$$
$$= \max_{\gamma\in\Gamma_h}\Big\{\int_0^T \xi_b(t)\,dZ_2^*(t) - \int_0^T \xi_a(t)\,dZ_1^*(t)\Big\} \qquad (4-5)$$

subject to

$$\begin{cases} Z(0) = z_0 \\ |\,Z^*(t)\,| \le L, & \text{for} \quad 0 \le t \le T \\ Z^*(T) = 0 \end{cases}$$

Solution : If we express the solution in the form of the four coordinate vectors, $\{S_1^*(t), S_2^*(t), Z_1^*(t), Z_2^*(t)\}$, then it is given by

$$\begin{aligned} &\Big\{\ \xi_a(t)\ ,\ \xi_b(t)\ ,(\ L - Z(t)\,)^+ + Z_1(t),\ (-L + Z(t)\,)^+ + Z_2(t)\ \Big\}, \text{ for } t \in [\,0, T), \\ &\qquad\qquad \Big\{\ \xi_a(t)\ ,\ \xi_b(t)\ ,\quad Z_1(t) \quad ,\quad Z(t) \quad \Big\}, \text{ for } \ t = T, \end{aligned} \qquad (4-7)$$

where $(\ x\ )^+ \equiv \max\{\,0, x\,\}$. The solution implies maintaining the longest position and selling at the last epoch. The solution means that quantities to trade, depending on the actual position, are as follow.

$$dZ_1^*(t) = \begin{cases} L - z_0 & \text{for } t = 0; \\ (\,L - Z(t)\,)^+ & \text{for } 0 < t \le T; \end{cases}$$

$$(4-6)$$

$$dZ_2^*(t) = \begin{cases} (\,Z(t) - L\,)^+ & \text{for } 0 \le t < T\ ; \\ Z(t) & \text{for } t = T; \end{cases}$$

In $(4-6)$, $dZ_1^*(0) = L - z_0$ means to make the position longest by hitting the offered rate. $dZ_1^*(t) = (\,L - Z(t)\,)^+$, for $t \in (\,0, T]$, means that if the daylight limit on the long position becomes lax due to retail selling, then the agent hits the offered rate and puts the position back to the longest. $dZ_2^*(t) = (\,Z(t) - L\,)^+$ means that if the position jumps out above the daylight limit due to retail buying, then the agent hits the bid rate to sell and bring the position back to the daylight limit. $dZ_2^*(T) = Z(T)$ means that the agent sells all he has by hitting the bid rate.

If we use Γ_w, then we have to specify what happens if the agent's quotation is not hit by the next arrivals. One policy is that the agent keeps quoting the market rate whenever the arrivals of the other agents change the market rate. Another policy is that the agent waits on a given market rate until the arrivals change the market rates and then the agent hits someone else's quotation. In order to choose a policy among those possible policies, we need transition probabilities of bid and offered rates. The transition probabilities are intractable with our model for the arrival processes which is developed in a later chapter.[6] We introduce an assumption about how the agent organizes his idea:

> $\mathbf{A}(4-10)$: When the offered rate is hit, bid and offered rates jump upward by 0, v or $2v$ with equal probabilities, while maintaining the bid-ask spread at v.

With this assumption, we compare the expected profits of actions which are elements of Γ_h and Γ_w. Let γ_1 be the solution $(4-6)$ for epoch 0; $\gamma_1 \equiv \{dZ_1^*(t) = L - z_0\}$ be a policy. Let γ_2 be an action such that $\{S_1^*(t) = B(t)\}$ and refer to $\{\gamma_2, \gamma_1\}$ as a policy. This means that in order to buy the agent waits once and if his quotation is not hit, then he hits the offered rate. We want to compare policy $\{\gamma_2, \gamma_1\}$ with policy $\{\gamma_1\}$. The latter means to hit the offered rate immediately. If the agent chooses γ_2 and his quotation is hit, then his profit is larger than the action γ_1 by amount $v(\equiv a - b)$. If his quotation is not hit, then market rates shift upward and he hits the new offered rate. Then his

[6]For details see in Section 6.2.

profit is larger than $\{\gamma_1\}$, by $0, -v$ and $-2v$ with equal probabilities. Let δ be the probability that the bid rate is hit by the next arrival. $1-\delta$ is the probability that the offered rate is hit by the next arrival. To simplify the argument, we do not consider the case where the next arrival renews, instead of hitting, one of the market rates. The expected difference of the profit from $\{\gamma_2, \gamma_1\}$, compared with $\{\gamma_1\}$, is given by $\delta v + (1-\delta)\{\frac{1}{3}0 + \frac{1}{3}(-v) + \frac{1}{3}(-2v)\} = \delta v + (1-\delta)v = (2\delta - 1) < 0$, because being bullish means $\delta < \frac{1}{2}$. This means that $\{\gamma_1\}$ is better than $\{\gamma_2, \gamma_1\}$.

Next, consider policy $\{\gamma_2, \gamma_2, \gamma_1\}$. As before we use assumption $A(4-10)$ and assume that the transition probabilities remain the same at the first and second transitions. We extend our analysis backward. We have already found that $\{\gamma_1\}$ is better than $\{\gamma_2, \gamma_1\}$. We apply this result on the last two actions of $\{\gamma_2, \gamma_2, \gamma_1\}$. Then $\{\gamma_2, \gamma_1\}$ is better than $\{\gamma_2, \gamma_2, \gamma_1\}$. When we assume the transition probabilities as in $\mathbf{A}(4-10)$, any policy using actions from Γ_w cannot be better than $\{\gamma_1\}$.

Depending on the transition probabilities, it may be possible that a policy which consists of some actions from Γ_w (quoting both market rates most of the time, and quoting only one of the rates to square a position sometimes) has a higher expected profit than $\{\gamma_1\}$ in the example of bullish expectation $A(4-9)$. Since we can not derive the transition probabilities of the market rates using our model, we do not pursue an optimal policy of using Γ_w when the agent has bullish expectation $A(4-9)$. To conclude the comparison of the expected profits by using actions either from Γ_h or Γ_w, we present the solution for $(4-5)$ as an assumption.

$\mathbf{A}(4-11)$: If the agent has a bullish expectation such that, for $0 < t < T$, $\frac{d\xi(t)}{dt} > 0$ and $|\xi(T) - \xi(0)| > 2u$, then the agent chooses $\{\Gamma_h\}$ to solve optimization problem of $(4-5)$.

So far assumption A$(4-9)$ was used. Next we will replace A$(4-9)$ with the assumption that the agent's expectation is neither bullish nor bearish and will solve the optimization problem $(4-4)$. As in the bullish case, we solve the optimization by using actions first from Γ_h and second from $\Gamma_w$and compare the result to choose the overall solution.

A$(4-12)$: The agent does not expect the exchange rate to move more than the bid-ask spread for a while. Namely at epoch t_0, there exists an interval $[t_0, T_0]$ such that $\frac{d\xi(t)}{dt} = 0$ or $\frac{d\xi(t)}{dt} \neq 0$ but $|\xi(T_0) - \xi(t_0)| < 2u$.

First, we apply actions from Γ_h. The use of Γ_h implies that transaction prices are set equal to expected values of the bid and offered rates: $S_1(t) = \xi_a(t)$ and $S_2(t) = \xi_b(t)$. Then, we have $\int_{t_0}^{T} \xi_b(t)\, dZ_2^*(t) - \int_{t_0}^{T} \xi_a(t)\, dZ_1^*(t) < 0$, for any $dZ_2^*(t) > 0$ and $dZ_1^*(t) > 0$, because by A$(4-12)$, $\xi_b(t_1) < \xi_a(t_2)$, for any $t_1, t_2 \in [t_0, T]$. The agent does not have positive profit if he uses actions from Γ_h.

Next, we consider using Γ_w. Actions from Γ_w imply choosing values for $S_1^*(t)$ and $S_2^*(t)$. $Z_1^*(t)$ and $Z_2^*(t)$ are now random variables: $Z_i^*(t) = Z_i(t)$, for $i = 1, 2$. When we use Γ_w, $Z_1^*(t)$ and $Z_2^*(t)$ become random measures. Besides of making different variables random, there is another difference between Γ_h and Γ_w. If we use Γ_h , the constraint on the position at the end of the day $Z^*(T) = 0$ can be almost surely met. On the other hand, $Z^*(T) = 0$ can be achieved only randomly, if we use Γ_w. In this case, instead of the constraint $Z^*(T) = 0$, we introduce a negative final reward or penalty for the open position. The penalty depends on $|Z(T)|$ and is zero, if $Z(T) = 0$. Let the $(2L+1) \times 1$ vector W_T denote a final reward at epoch T such that $W_{T,z}$

means the zth element of the W_T and that

$$W_{T,z} = W_{T,-z} < 0, \quad \text{for} \quad z = 1, \ldots, L,$$

$$W_{T,0} = 0.$$

By shifting the origin of the time axis by t_0 and setting $z_0 = 0$, we solve the following problem:

Optimization over Γ_w: With assumption $A(4-12)$,

$$\begin{aligned}
&\max_{S_1^*, S_2^*} E\Big[\int_0^T S_2^*(t)\, dZ_2^*(t) - \int_0^T S_1^*(t)\, dZ_1^*(t) + W_T(\ Z(t)\ ) \mid \mathcal{F}_{t_0}\Big] \\
&= \max_{S_1^*, S_2^*} \Big\{ \int_0^T S_2^*(t)\, dE[\, Z_2^*(t)\,] - \int_0^T S_1^*(t)\, dE[\, Z_1^*(t)\,] \\
&\quad + E\Big[W_T(\ Z(t)\ ) \mid \mathcal{F}_{t_0}\Big]\Big\}. \qquad (4-8)
\end{aligned}$$

subject to

$$|\, Z^*(t)\,| \leq L, \quad \text{for} \quad 0 \leq t \leq T.$$

Solution: This is a model of a controlled, continuous time Markov process with finite states. We look for an optimal policy among the time invariant policies. The position $Z(t)$ is the state in this model. Each jump from one state to another which signifies a transaction has an associated reward which signifies profit. A model presented in Yushkevich (1977) is applicable. $\{-L, \ldots, 0, \ldots, L\}$ is the state space. State $Z(t)$ jumps when the arrival of a buyer or seller occurs. The arrival process is a Poisson process and hence the inter-arrival time follows an exponential distribution. We have to construct an infinitesimal state transition matrix. Since we work in continuous time, we have an infinitesimal matrix instead of a matrix of transition probabilities. Let Q be the infinitesimal matrix. Assumption $A(4-12)$, $\frac{d\xi(t)}{dt} = 0$, implies that the arrival intensities of buyers and sellers are the same and we limit the values of S_1^* and S_2^* such that $\frac{d\xi(t)}{dt} = 0$. ' Let q be the arrival intensity of the seller or buyer. This is the intensity that $Z(t)$ moves to another state.

A$(4-13)$: Each arrival trades either one or two units.[7]

Let v_1 and v_2 be the probabilities that a buyer or seller wants to trade one unit and two units, conditional on the arrival. Let Q be the infinitesimal transition matrix. Using the notation and assumptions given above, Q is written as

$$Q = qA \equiv q \begin{pmatrix} -1 & v_1 & v_2 & 0 & 0 \\ 1 & -2 & v_1 & v_2 & 0 \\ v_2 & v_1 & -2 & v_1 & v_2 \\ 0 & v_2 & v_1 & -2 & 1 \\ 0 & 0 & v_2 & v_1 & -1 \end{pmatrix} \qquad (4-9)$$

Among the elements of Q, Q_{21} and Q_{45} equal 1. Because of the daylight limit $L = 2$, the agent trades only one unit, even if the arrival wants to trade two units. The first row of Q means that $Z(t) = 2$ originally and the state $Z(t)$ jumps to $Z(t) = 1$ with conditional probability v_1 or to $Z(t) = 0$ with v_2 .

Each jump is associated with a reward. A jump from $Z = 2$ to $Z = 1$ ($Z = 0$) means one unit (two units) of sale. The expected reward when $Z = 2$ is given by v_1+2v_2, using the bid-ask spread as the unit for measuring revenue. An upward jump from $Z = -2$ to, say, $Z = -1$ is one unit of purchase which incurs cost. We do not have to assign a negative reward to the purchase jump. Except for the last downward jump or the last consecutive jumps downward before the end of the day, the downward jump of Z already signifies a profit, not just revenue. Let the 5×1 vector R be the vector of the expected rewards of jumps, *i.e.*,

$$R \equiv 2uq \begin{pmatrix} v_1 + 2v_2 \\ v_1 + 2v_2 \\ v_1 + 2v_2 \\ 1 \\ 0 \end{pmatrix} . \qquad (4-10)$$

Let the 5×1 vector $W(t)$ denote the expected profit between epoch t and the end of the day T . The i-th element of $W(t)$ is the expected profit

[7]This automatically implies that for an individual agent's daylight limit $L \leq 2$ for every agent.

the agent has if the state is at the i-th state at epoch t. In our model, the first element of $W(t)$ corresponds to $Z = 2$. The expected profit satisfies the following differential equation and is obtained as the solution of the differential equation (Yushkevich 1977, Corollary 3.2 and Supplementary Remarks 5.),

$$W'(t) = -R - QW(t) \qquad (4-11)$$

with the boundary condition $W(T) = W_T$. $(4-11)$ is an example of Bellman's equation. The elements of R and Q are constant. Since the sum of the elements in each row of Q is zero, Q is singular. The solution of $(4-11)$ is given by

$$W(0) = \Big(\int_0^T e^{sQ} ds\Big) R + W_T \qquad (4-12)$$

where $e^{sQ} \equiv I + \frac{sQ}{1!} + \frac{(sQ)^2}{2!} + \frac{(sQ)^3}{3!} + \ldots + \frac{(sQ)^k}{k!} + \ldots$. Since Q is not invertible, the expression for $\int_0^T e^{sQ} ds$ cannot be simplified. $\int_0^T e^{sQ} ds = T\Big(I + \frac{TQ}{2!} + \frac{(TQ)^2}{3!} + \frac{(TQ)^3}{4!} + \ldots + \frac{(TQ)^{k-1}}{k!} + \ldots\Big)$.

Next we find optimal quotations S_1^* and S_2^*. Since we have already limited our argument to the case where $\frac{d\xi(t)}{dt} = 0$, finding S_1^* and S_2^* is the same as finding the optimal bid-ask spread $2u^*$. The arrival intensity of a buyer and that of a seller is q. The value of q depends on the spread. As shown in $(4-10)$, R is also a function of u. The value of u which maximizes $(4-12)$ is the optimal spread. Substitute $q = f(u)$ and $(4-10)$ into $(4-12)$ and differentiate with respect to u. Since $\Big(\int_0^T e^{sQ} ds\Big) R = T\Big(qI + \frac{q(TqA)}{2!} + \frac{q(TqA)^2}{3!} + \ldots + \frac{q(TqA)^{k-1}}{k!} + \ldots\Big) u2V$, where $V \equiv \frac{1}{2uq} R$ and $Q = qA$,

$$\begin{aligned} &\frac{d}{du}\Big[\int_0^T e^{sQ} ds R\Big] \\ &= T\Big\{ Iq' + (TqA)q' + \frac{(TqA)^2 q'}{2!} + \frac{(TqA)^3 q'}{3!} + \ldots + \frac{(TqA)^{k-1} q'}{(k-1)!} + \ldots \Big\} u2V \\ &\quad + T\Big\{ +qI + \frac{q(TqA)}{2!} + \frac{q(TqA)^2}{3!} + \ldots + \frac{q(TqA)^{k-1}}{k!} + \ldots \Big\} 2V \\ &= \Big\{ Te^{TQ} q' u + q\Big(\int_0^T e^{sQ} ds\Big)\Big\} 2V. \qquad (4-13) \end{aligned}$$

The optimal value of u makes $(4-13)$ equal to zero. Te^{TQ} and $\int_0^T e^{sQ}ds$ become proportional to T, as $T \to \infty$. $(4-13)$ can be solved for $T = \infty$.

e^{tQ} is the probability distribution of the state in which Z would stay at epoch t, starting at epoch 0 from one of the states. The row gives the starting state and the column gives the state at epoch t. e^{tQ} converges to a stationary distribution, as $t \to \infty$. Let C be the stationary distribution which is associated with Q , *i.e.,* $C = \lim_{t\to\infty} e^{tQ}$. We can show that $\frac{1}{t}\int_0^T e^{sQ}ds$ also converges to C. For a given $\epsilon > 0$, , there exists τ such that $\|e^{tQ} - C\| < \epsilon$ for $t > \tau$. $\|\cdot\|$ represents the maximum absolute value of the individual elements of a matrix. Using the facts that $\frac{1}{t}\int_0^t e^{sQ}ds = \frac{1}{t}\int_0^\tau e^{sQ}ds + \frac{1}{t}\int_\tau^t e^{sQ}ds$ and $\frac{1}{t}\int_0^t e^{sQ}ds - C = \frac{1}{t}\int_0^t (e^{sQ} - C)ds$, we have $\lim_{t\to\infty} \frac{1}{t}\int_0^t (e^{sQ} - C)ds = \lim_{t\to\infty} \frac{1}{t}\int_0^\tau (e^{sQ} - C)ds + \lim_{t\to\infty} \frac{1}{t}\int_\tau^t (e^{sQ} - C)ds$. Because $\lim_{t\to\infty} \|\frac{1}{t}\int_0^\tau (e^{sQ} - C)ds\| = 0$ and $\lim_{t\to\infty} \|\frac{1}{t}\int_\tau^t (e^{sQ} - C)ds\| \le \lim_{t\to\infty} \frac{1}{t}\int_\tau^t \|(e^{sQ} - C)\|ds \le \epsilon$, we have $\|\frac{1}{t}\int_0^t e^{sQ}ds - C\| \le \epsilon$ and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{sQ}ds = C = \lim_{t\to\infty} e^{tQ}ds \qquad (4-14)$$

Divide $(4-13)$ by T, let $T \to \infty$, and in $(4-14)$ replace t by T, then $(4-13)$ becomes zero, if $q'u + q = 0$. Let $q = f(u)$.[8] Then the necessary condition for the optimal spread u^* for the asymptotic case is given by

$$f'u + f = 0. \qquad (4-15)$$

The optimal quotations are given by $S_1^* = \xi(t) - u^*$ and $S_2^* = \xi(t) + u^*$, where $\xi(t)$ is assumed to satisfy A$(4-12)$.

[8]The explicit form of $f(u)$ will be derived in Section 6.5.

The solution to the optimization problem when we assume A(4−12) belong to Γ_w and the solution is as follows. If $|Z(t)| \leq L$, then

$$\left\{ \xi(t) - u^*,\ \xi(t) + u^*,\ (\ L - Z(t)\ )^+ + Z_1(t),\ (\ L + Z(t)\ )^+ + Z_2(t)\ \right\}. \quad (4-16-a)$$

If $|Z(t)| > L$, then

$$\left\{ \xi(t) + u^*,\ \xi(t) - u^*,\ (-L - Z(t))^+ + Z_1(t),\ (-L + Z(t))^+ + Z_2(t)\ \right\}. \quad (4-16-b)$$

$(4-16-a)$ implies that the agent waits on the market rates and that the associated quantities are the maximum quantities within the daylight limit. $(4-16-b)$ corresponds to a situation such that the daylight limit is violated by retail transactions. $(4-16-b)$ implies that if the position moves beyond the daylight limit due to retail transactions, the agent hits one of the market rates to put the position back to the level of the daylight limit. In order for $(4-16-b)$ to be implemented, we need the existence of market makers who quote their prices until the last epoch as is assumed in $A(4-5)$.

So far the magnitude of the maximum open position is exogenously given as the daylight limit. If the size of the limit on the open position is also a choice variable, it is possible that *a voluntary limit* which the agent chooses is smaller than the exogenously given daylight limit. Depending on W_T, this can happen. The expected profit $(4-12)$ consists of two terms. It can be proved that with infinitesimal matrix Q as in $(4-9)$, $\left(\int_0^T e^{sQ} ds\right) R$ increases as the daylight limit L increases. The negative elements of W_T, which are penalties on the open position at T, may decrease more than proportionately as L increases; for example, $W_T = \left(-2^2, -1, 0, -1, -2^2\right)^T$. [9] Had this been the case for finite T, we will have a trade-off between $\left(\int_0^T e^{sQ} ds\right) R$ and W_T as

[9] Superscript T means transpose.

L increases. So the voluntary limit may be smaller than the daylight limit L. This situation matches empirical observations that traders rarely assume an open position up to the daylight limits. We will not pursue this possibility here.

4.5. OPTIMAL POLICY

In the preceding section, we solved the optimization problem with the assumptions A$(4-9)$ and A$(4-12)$. $\xi(t)$ was assumed to be monotonically increasing A$(4-9)$, or more or less horizontal, compared with the bid-ask spread A$(4-12)$. Next, we want to consider the case where $\xi(t)$ has local maxima or minima. Before we do that, we need to provide the notation for the sets of actions including the solutions of $(4-4)$ and $(4-8)$;

$$\Gamma_{hl} \equiv \Big\{ \xi_a(t),\ \xi_b(t),\ (L - Z(t))^+ + Z_1(t),\ (-L + Z(t))^+ + Z_2(t) \Big\} \quad (4-17-a)$$

$$\Gamma_{hs} \equiv \Big\{ \xi_a(t),\ \xi_b(t),\ (-L - Z(t))^+ + Z_1(t),\ (L + Z(t))^+ + Z_2(t) \Big\} \quad (4-17-b)$$

$$\Gamma_{hq} \equiv \Big\{ \xi_a(t),\ \xi_b(t),\ (-Z(t))^+ + Z_1(t)\ ,\ (Z(t))^+ + Z_2(t) \Big\} \quad (4-17-c)$$

$$\Gamma_{h12} \equiv \Big\{ \xi_a(t),\ \xi_b(t),\ (-L - Z(t))^+ + Z_1(t),\ (-L + Z(t))^+ + Z_2(t) \Big\} \quad (4-17-d)$$

$$\Gamma_{w1} \equiv \Big\{ \xi_b(t)\ ,\ .\ ,\ (L - Z(t))^+ + Z_1(t)\ ,\ ----- \Big\} \quad (4-17-e)$$

$$\Gamma_{w2} \equiv \Big\{ \ .\ ,\ \xi_a(t)\ ,\ ----\ ,\ \Big(L + Z(t)\Big)^+ + Z_2(t) \Big\} \quad (4-17-f)$$

$$\Gamma_{w12} \equiv \Big\{ \xi_b(t),\ \xi_a(t),\ \Big(L - Z(t)\Big)^+ + Z_1(t),\ \Big(L + Z(t)\Big)^+ + Z_2(t) \Big\} \quad (4-17-g)$$

Γ_{hl} means that the agent tries to maintain the longest position by hitting the market rates. Γ_{hs} means doing the same to maintain the shortest position. For both Γ_{hl} and Γ_{hs}, the quantities in the coordinates of $Z_1^*(t)$ and $Z_2^*(t)$ are the quantities about which the agent notifies the broker together with his quotations. These quantities are the maximum quantities which the agent trades at his quotations. $dZ_1^*(t) = \ (L - Z(t))^+$ of $(4-17-a)$ and $dZ_2^*(t) = \ (L+$

$Z(t))^+$ of $(4-17-b)$ are the maximum changes of the position while the daylight limit is met. $dZ_2^*(t) = (-L+Z(t))^+$ of $(4-17-a)$ and $dZ_1^*(t) = (-L-Z(t))^+$ of $(4-17-b)$ are the minimum quantities needed to put the position back to the daylight limit. Γ_{hq} means maintaining a square position by hitting the market rate, if the position deviates from 0. Γ_{h12} means keeping the position at the daylight limit by hitting the market rate, if the position deviates outside the daylight limit.

Γ_{w1} and Γ_{w2} mean waiting on one of the market rates until the daylight limit becomes binding; Γ_{w1} to have the long position and Γ_{w1} to have the short position. Γ_{w12} means to wait on the both market rates. The expressions in the coordinates for $Z_1^*(t)$ and $Z_2^*(t)$ in $(4-17-g)$ are the maximum quantities which the agent trades at his quotations which are equal to the expected bid and offered rates by assumption $A(4-6)$. If $Z(t) = L$, then Γ_{w12} becomes $dZ_1^*(t) = 0$ and $dZ_2^*(t) = 2L$ and the agent quotes $\xi_a(t)$ only. If $Z(t) = -L$, then Γ_{w12} becomes $dZ_1^*(t) = 2L$ and $dZ_2^*(t) = 0$, which implies that the agent quotes $\xi_b(t)$ only.

Solution $(4-7)$ consists of Γ_{hl}, for $t \in [0,T)$, and Γ_{hq}, for $t = T$. Solution $(4-9)$ consists of Γ_{w12} and Γ_{h12}, for $t \in [0,T)$. Since solutions $(4-9)$ assign an action to each epoch, contingent on the actual position, solutions $(4-9)$ represent a policy. A summary of the agent's choices of action is as follows.

Bullish ($A(4-9)$ is the case): The agent hits the offered rate, buys until the daylight limit becomes binding and quotes an expected offered rate for the end of the monotone period, $\xi_a(T)$, as his selling price, *i.e.,* Γ_{hl}.

$(4-18)$

Bearish (counterpart of A$(4-9)$): The agent hits the asked rate, sells the maximum quantity which the daylight limit allows and now quotes $\xi_b(T)$ as his buying price, *i.e.*, Γ_{hs}.

$$(4-19)$$

Otherwise : The agent quotes both the bid and the asked rates at the same time unless the daylight limit is binding. If it is binding, then the agent quotes only one of the rates, Γ_{w12}, Γ_{w1} or Γ_{w2}.

$$(4-20)$$

We want to consider more variations of the shape of the expected time path besides a monotonically increasing one ($A(4-9)$) or a flat one (A$(4-12)$). Suppose that $\xi(t)$ is bullish in the sense of A$(4-9)$ until epoch τ and that after τ, $\xi(t)$ becomes horizontal. For interval $[0,\tau)$, the optimization problem $(4-5)$ is applicable. At epoch τ, the expected profit increases by choosing to wait on $\xi_a(t)$ rather than hitting the bid rate $\xi_b(t)$. Using the notations defined in $(4-17)$, the solution policy is given by $\left\{\Gamma_{hl},\ \text{for } t \in [0,T)\ ;\ \ \Gamma_{w2},\ \text{for } t = T\right\}$.

Next, suppose that $\xi(t)$ is bullish before τ and bearish after τ in the sense of A$(4-9)$. The horizons for the optimization are divided into three periods, namely $[0,\tau)$, τ and $(\tau,T]$. Then, we have solution

$$\left\{\Gamma_{hl},\ \text{for}\ \ t \in [0,\tau)\ ;\ \Gamma_{w2},\ \text{for}\ \ t = \tau\ ;\ \Gamma_{hs},\ \text{for}\ \ t \in (\tau,T)\ ;\ \Gamma_{hq},\ \text{for}\ \ t = T\right\}.$$

An example is shown in Figure 2. $\xi(t)$ has a local extremum at t_1, a local minimum at t_2 and the global maximum at epoch T. Buying at $t=0$ and selling at T is not optimal. If the agent divides the horizons into intervals where $\xi(t)$ is monotone and chooses the optimal action in each interval, then the agent has a larger expected profit than by buying at $t = 0$ and selling at T. This is because by the successive optimizations of the individual intervals, the agent takes advantage of all of $| \xi(t_i) - \xi(t_{i+1}) |$ rather than just $| \xi(T) - \xi(t_0) |$. The solution for the case of Figure 2 is given by $\Big\{ \Gamma_{hl}$, for $t \in [0, t_1)$; Γ_{w2}, for $t = t_1$; Γ_{hs}, for $t \in (t_1, t_2)$; Γ_{w1}, for $t = t_2$; Γ_{hl}, for $t \in (t_2, T)$; Γ_{hq}, for $t = T \Big\}$.

So far the expected time path $\xi(t)$ has been given. However, the expected time path $\xi(t)$ may change its slope from time to time, since the system which determines the exchange rate in the market is non-stationary. Also, as the agent continuously updates and refines the information $\mathcal{F}_t$, he may recognize new local maxima or minima as FLE while the previously known extremum still remains unchanged. As the preceding example showed, the agent can increase his profit by taking advantage of local minima or maxima. An optimal policy is that, as the agent recognizes local minima and maxima of $\xi(t)$, he should take advantage of such updated FLE values and concentrate on the optimization of the current interval.

The optimal policy implies that, for a given epoch, the agent's action depends on FLE values and not on any other extrema of the expected time path $\xi(t)$. The fact that the action depends on FLE values gives a rationale why, in Chapter 6, a distribution function is assigned to the agents' FLE values for a given epoch. Since the FLE value determines an individual agent's action, by specifying an FLE distribution function, we can obtain a distribution of the agents' actions at a given epoch.

The optimal policy derived here coincides with the rules of thumb in the foreign exchange business. Judging whether the price has reached the bottom or the ceiling based on observations is an important skill for traders who work in the actual foreign exchange markets.

4.6. TWO STATES OF EXPECTATION

So far the agent has made decisions using expected values. However, an agent does not always have a clear idea about which way the transaction price will go. Here we redefine two states of an agent's level of confidence about his expected FLE value.

State of the frosted glass: Based on the agent's information, the transaction price may jump in either direction. The agent does not assume an open position.

State of the crystal glass: The agent thinks that his predicted FLE value is accurate enough to assume an open position.

As is shown in Figure 3, the agent moves between states of frosted and crystal glass as he receives news and observes the arrival process.

Our model uses only the expected values, not variances, and assumes the agent is risk neutral. We need an additional assumption, if we want to explain why the agents do not always assume the open positions. The criterion of bankruptcy avoidance is introduced.

A$(4-13)$(*bankruptcy avoidance*): The agent always maintains the probability of bankruptcy below a given level.

In other words, the agent does not speculate, if the probability of catastrophic loss exceeds a given level. The rationale of this criterion is that once bankrupt, an agent cannot recover, and that even a hint of risky operations by the bank for which the agent works can cause a run on the bank by its customers.

If the agent's expectation about the exchange rate is associated with a large variance, then the agent will not assume an open position, even if the action he is considering has a positive expected profit. In this case, the probability of catastrophic loss from an open position of even one transaction unit exceeds the given level. The criterion of bankruptcy avoidance is binding even with an open position of one transaction unit.

Bankruptcy avoidance is also applied to the management of the agent's bank. This results in daylight and overnight limits which are exogenously given to the agent from the management of his bank.

4.7. LIMITATION OF THE MODEL

When the agent assumes an open position based on his bullish or bearish expectations, the optimal position is at the maximum magnitude within the daylight limit. This is not what is observed in reality. By assuming risk neutrality, our model cannot make an intermediate open position the optimal position. Introducing a voluntary limit while maintaining the assumption of risk neutrality remains to be done.

5. HETEROGENEOUS INFORMATION

Billions of activities in the economy all over the world generate the demand and supply of the foreign exchange. Knowledge about individual economic activities of the past and the present is summarized. The GNP measure is an example. Pieces of knowledge may be summarized and analyzed and come out as econometric forecasts. We interpret information as an σ-field generated by a collection of pieces of knowledge. Knowledge about each individual activity constitutes an infinite set. This infinite set of pieces of knowledge is covered by a finite number of subsets. Each subset and the family of the unions and intersections of these subsets is called information. Information is a σ-field of informational subsets. Knowing a specific informational subset does not mean knowing an element of the subset unless the element itself is the informational subset. For example, the agent may know the GNP value for a quarter, but he may not know the exact figures of each component. Some of the informational subsets are specific to individual agents. In this regard, an example in our model is the arrival of retail customers to the individual agents. Since arrival processes in the market depend on the degree of disagreement among agents with respect to expectations, agents try to estimate what others are expecting. Agents may disagree in their estimates. They may have equally refined information but the information is heterogeneous. It is not meaningful to state that the agent has all the information to estimate the intra-day movement of the exchange rate. We use $\mathcal{F}_{t_0}$ to denote the agent's information at epoch t_0.

The information which enters the agent's decision making is grouped in the following four types:[10]

(1) Statistical data which are released by public and private institutions.

(2) Aggregate retail demand and supply of the day.

(3) Identity of buyers and sellers. Agents may take different actions to speculate, depending on their daylight limits and their retail customers.

[10]This list is drawn from Oguchi (1983).

(4) General news, such as the Fed fund rate, Euro dollar interest rates or political news.

The agents have heterogeneous information (in the sense of the informational subsets) of the second and third type of information. In general, agents do not know other agents' retail transactions, which can be substantial and variable. Also, some agents may have more refined information for types two and three than other agents. Refined information of these types helps agents to form more accurate forecasts about the arrivals of buyers and sellers. We can interpret this kind of the heterogeneity of information as a difference in the number of variables which an individual agent's econometric models uses. Suppose that the first agent's econometric model contains variables $X_1, \ldots, X_m$ and Y_1. The second agent's model contains $X_1, \ldots, X_m$ and Y_2 and so forth. A model for an overall market contains $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$. Individual agents' forecasts can be different but equally accurate on average.

As for the first and fourth types of information, the agents may have heterogeneous interpretations for given news. Agents try to analyze how other agents interpret news with regard to foreign exchange rates. Since an agent can take advantage of fluctuations in the exchange rate, whatever the cause, analyzing other agents' responses to the news is as important as judging whether those responses are consistent with the agent's econometric analysis.

There are examples to tell how the agents' responses influence the exchange rates (see Oguchi, 1983).

(1) Economic indicators which influence the exchange rates when their data are released vary.

(2) If the newly released data of the indicators coincide with the agents' expectations, there is little response in the market.

(3) Even if the release of the data concerning some economic indicators does not cause reactions in the market immediately, it is not unusual that later exchange rate moves in a direction which coincides with what the agent's econometric model predicts.

The first and third types of information seem to indicate either that agents are not rational or that a course of events depends on history.[11] Whatever the case, for given economic indicators which are currently influential in the market, the agents are rational with regard to maximizing daily profits.

6. ARRIVAL PROCESSES OF BUYERS AND SELLERS

6.1. DISTRIBUTIONS FOR THE ENSEMBLE

It was shown in Chapter 4 how FLE values and daylight limit determine an individual agent's optimal policy. Here in Chapter 6, we consider the collective actions of a large number of agents. The agents are heterogeneous with respect to expected FLE values and daylight limits as well as retail transactions. The transitions of expectations take place intermittently among the agents and so do arrivals of agents. Each arrival has his expected FLE value

[11] In the sense that who responds first and how may determine the actions of the other agents.

and we assign a distribution function over all these values. Also, we assign distribution functions to the lengths of time during which agents stay in each state of expectation. With these distribution functions we can derive the distribution of the actions which the agents may take at a given epoch. The agents represent *a statistical ensemble.*

We assume that when an agent determines the FLE value, it is a random drawing from a distribution which exists at that epoch. Let X_t be a FLE value which is chosen by an agent who revised his expectation at epoch t. Let $G_t(x)$ be the distribution function from which X_t is drawn. At any epoch, there is a sample of X_t's of those who have already arrived but have not yet left. Their quotations are distributed over some range. Let X_{ti}, for $i = 1, \ldots N_c$,[12] be such FLE values at epoch t. Let $H_t(x)$ be a sample distribution of X_{ti}'s at epoch t. The agents who are approximated by $H_t(x)$ are the ones that are waiting on their quotations. For a given t, $G_t(x)$ is the distribution function from which X_t of a new arrival is drawn, while $H_t(x)$ is the sample distribution of X_t of those who have already arrived and are quoting in the market. $G_t(x)$ and $H_t(x)$ shift from time to time and have supports which are subsets of Ω_p. Although X_t takes only positive integer values, we approximate the true $G_t(x)$ and $H_t(x)$ by absolutely continuous functions.

Besides the degree of heterogeneity for FLE values, agents, as a statistical ensemble, are also characterized by the frequency of revisions of their expectations. Another expression of the frequency of the revision is length of time during which an agent maintains the same expectations. We assume that the

[12]Definition of N_c is given below.

lengths of time during which individual agents stay in the frosted and crystal states follow exponential distributions.

A$(6-1)$: The transition between the two states is an alternating renewal process whose renewal epochs follow exponential distributions.

Let $f_1(t) = \theta_1 \exp(-\theta_1 t)$ and $f_2(t) = \theta_2 \exp(-\theta_2 t)$ be the density functions for the length of time t during which individual agents stay in the frosted and crystal state, respectively. Then, the expected lengths of the stays are given by $\frac{1}{\theta_1}$ and $\frac{1}{\theta_2}$. Let N be the total number of agents in this economy which is exogenously given and let N_f and N_c denote the numbers of agents who stay in the frosted state and in the crystal state for a given epoch. Then, their expected numbers are given by $E[N_f] = \frac{\theta_2}{\theta_1+\theta_2}N$ and $E[N_c] = \frac{\theta_1}{\theta_1+\theta_2}N$, respectively.

For given constants θ_1 and θ_2, the expected number of agents in each state at a given epoch is constant. However, the agents may change values. Some of the agents who have been quoting leave, and new arrivals join the existing agents. If $G_t(x) = H_t(x)$, we say that *the heterogeneity is stationary.* Otherwise, we say that the heterogeneity is *non-stationary.* If $G_0(x) \neq H_0(x)$, then $H_t(x)$ shifts so that $|H_t(x) - G_t(x)| \to 0$, for any $x \in \Omega_p$, as $t \to \infty$, because of the agents' changing values.

6.2. BIDDING AT THE BROKER

Only one broker exists in a local foreign exchange market. Suppose that at a given epoch, the quotations of buying and selling prices exist as in Figure 4. Among the existing quotations, the broker announces the maximum buying price and the minimum selling price to all the agents. Although $A(t)$ and $B(t)$ are known to every agent, agents do not know how the other quotations are distributed. The agent arrives at the market with his expected FLE value. Depending on his value, he chooses an action from Γ_{hl}, Γ_{hs} or Γ_{w12}.

Suppose that the next arrival hits $B(t)$ of Figure 1. $B(t)$ may remain at B_1, jump to B_2, or even to B_3, depending on the quantities associated with the arrival and B_1 and B_2. If the agent who quoted B_1 was using Γ_{w1} when hit, then $A(t)$ may jump up as he switches to Γ_{w12}, depending on which is the more competitive, A_1 or the agent's selling price. If the agent who quoted B_1 was using Γ_{w12}, then $A(t)$ will not jump up. Besides this complication of transitions of the market rates, quotations are cancelled as agents move from the crystal to the frosted-glass state from time to time. While we can derive the expected time path of the market rates, the transition probabilities of the market rates are intractable.

6.3. ARRIVAL INTENSITIES

When agents move from one state to another state, they want to adjust their positions. These represent arrivals of agents. Arrivals are also due to retail transactions. The arrivals of the agents are generated from four situations. In the following paragraphs we discuss in detail how the arrivals are generated in each situation. Suppose that $G_t(x)$ shifted at epoch 0 and that $G_t(x) \neq H_t(x)$, *i.e.,* non-stationary heterogeneity. For $t > 0$, the sample of FLE values consists of two groups; those drawn from before epoch 0 and those drawn after epoch 0. We use $H(x)$ to denote a distribution function for a sample of X_t which arrived before epoch 0; and use $G(x)$ to denote a distribution function for the other sample of X_t which is drawn from the new $G_t(x)$ after epoch 0.

Frosted glass: When the agent is in the state of frosted glass, he does not assume an open position. So each time the agent has a retail transaction, he carries out a reverse transaction in the market. The agent is expected to sell β_2 (buy β_1) US dollars per epoch to (from) his retail customers. So he is expected to buy β_2 (sell β_1) US dollars per epoch in the market. There are N_f agents who are in the frosted state at a given epoch.

Transition 1 : In this transition, the agent leaves the frosted state and chooses his X_t from $G_t(x)$ as shown in Figure 4. For a given set of the market rates $B(t)$ and $A(t)$, if the agent's X_t is greater than $A(t)+u$, where u is a half of the expected spread, then the agent becomes a buyer and hits $A(t)$. Now he has long position and quotes his selling price; taking actions $(4-18)$. If the agent's X_t is smaller than $B(t)-u$, the agent becomes a buyer and hits $B(t)$. He has a short position and quotes his buying price; taking actions $(4-19)$. If the agent's X_t falls between $B(t)-u$ and $A(t)+u$, then the agent quotes both the buying and selling prices; taking actions $(4-20)$. The expected number of agents per epoch who have Transaction 1 is given by $\theta_1 E[N_f]$. $1-G_t(a)$ of them hit $A(t)$ and $G_t(b)$ of them hit $B_t(x)$, where $a \equiv A(t)+u$ and $b \equiv B(t)-u$.

Crystal glass: The agent has been quoting his price or prices. The agent must have adjusted his position according to his X_t. Unless X_t falls in the interval between b and a, the desired position is the maximum open position. The values of the X_t's of the agents who are in the crystal glass have distribution function $H_t(x)$. Among the agents in the crystal glass, $H_t(b)$ of them have the short positions and $1-H_t(a)$ of them have long positions. Their daylight limits are binding. Each one of them is expected to sell β_2 (buy β_1) US dollars per epoch to (from) the retail customers. If his actual position deviates from the desired position due to the arrival of retail customers, he hits one of the market rates in order to adjust his position promptly. In this case, the agent's policy is either Γ_{hl} $(4-17-a)$ or Γ_{hs} $(4-17-b)$. If the agent's X_t falls between b and a, his desired position is equal to his actual position most of time. The agent's policy is Γ_{w12} $(4-17-g)$, together with Γ_{h12} $(4-17-d)$ which is applied when the daylight limit is violated. So if $b < x_t < a$, then he does not trade in the market each time he has the retail transaction, unless the daylight limit happens to be binding at that epoch.

Transition 2: Transition 2 means that the agent abandons his expectation and wants to square his position. If he has a long (short) position when Transition 2 occurs, he becomes a seller (buyer) and hits $B(t)$ ($A(t)$). The expected number of agents per epoch who have Transition 2 is given by $\theta_2 E[N_c]$. As is shown in Figure 5, among the agents who are quoting their prices, $H_t(b)$ of them have short positions and will become buyers when Transition 2 takes place. And $1-H_t(a)$ of them have long positions and will become sellers. If $G_t(x)$ and $H_t(x)$ are not the same, the sample of the agents in the crystal state consists of two groups of x_t's; those drawn from the old $G_t(x)$, for $t < 0$, and those from new $G_t(x)$, for $t > 0$. The agents in both sample groups may have Transition 2. We have to derive the number of agents whose X_t had come from the new $G_t(x)$ and who already have had Transition 2. The expected number of such agents is given by $\frac{\theta_1\theta_2 N}{\theta_1+\theta_2}(1-e^{-\theta_2 t})$. The expected number of the agents drawn from the old $G_t(x)$ is given by $\frac{\theta_1\theta_2 N}{\theta_1+\theta_2}e^{-\theta_2 t}$. $1-G(a)$ of the respective sample group become sellers and $G(b)$ of them become buyers.

We want to derive the arrival intensities of buyers and sellers who hit the given market rates. To simplify the calculation, make the following assumption:

A$(6-2)$: The quantity of each arrival is unity.

Let $H(x)$ denote $H_t(x)$ of before epoch 0, and let $G(x)$ denote the new distribution function of $G_t(x)$ after epoch 0. Then, using the results of the preceding paragraphs, the arrival intensity of buyers who hit the offered rate is given as a sum of the following terms.

Frosted glass: $\beta_2 E[N_f]$ $\qquad (6-1-a)$

Transition 1: $\theta_1 E[N_f](1-G(a)) = \frac{\theta_1\theta_2 N}{\theta_1+\theta_2}(1-G(a)) = l(1-G(a))$ $\qquad (6-1-b)$

Transition 2: $\theta_2 E[N_c]\Big\{e^{-\theta_2 t}H(b) + (1-e^{-\theta_2 t})G(b)\Big\}$ $\qquad (6-1-c)$

Crystal glass: $\beta_2 E[N_c]$, $\qquad (6-1-d)$

where $b \equiv B(t) - u$, $a \equiv A(t) + u$ and $l \equiv \frac{\theta_1\theta_2 N}{\theta_1+\theta_2}$.

The arrival intensity of sellers who hit the bid rate is given as a sum of the following terms:

Frosted glass: $\beta_1 E[N_f]$ $\qquad (6-2-a)$

Transition 1: $\theta_1 E[N_f]G(b) = \frac{\theta_1\theta_2 N}{\theta_1+\theta_2}G(b) = lG(b)$ $\qquad (6-2-b)$

Transition 2: $\theta_2 E[N_c]\Big\{e^{-\theta_2 t}(1-H(a)) + (1-e^{-\theta_2 t})(\ 1-G(a)\ )\Big\}$ $\qquad (6-2-c)$

Crystal glass: $\beta_1 E[N_c]$. $\qquad (6-2-d)$

Let $\lambda_a(\lambda_b)$ be the arrival intensity of the agents who hit the offered (bid) rate. Each arrival intensity consists of two parts,

$$\lambda_a \equiv \lambda_{a1} + \lambda_{a2} \quad \text{and} \quad \lambda_b \equiv \lambda_{b1} + \lambda_{b2},$$

Here, λ_{a1} represents the arrivals of buyers who hit the offered rate in order to counteract their retail selling which changed their position from the desired level; λ_{a1} is the sum of $(6-1-a)$ and $(6-1-d)$. λ_{a2} represents the arrivals of buyers who hit the offered rate due to heterogeneous expectations; λ_{a2} is the sum of $(6-1-b)$ and $(6-1-c)$. λ_b represents the arrival intensity of sellers who hit the bid rate. λ_{b1} represents the arrivals of sellers who hit the bid rate in order to counteract their retail buying which deviated their position from the desired level; λ_{b1} is the sum of $(6-2-a)$ and $(6-2-d)$. λ_{b2} represents the arrivals of sellers who hit the bid rate due to the heterogeneity of expectations. λ_{b2} is the sum of $(6-2-b)$ and $(6-2-c)$.

Substituting b, c and d of $(6-1)$ and $(6-2)$, and using $\theta_2 E[N_c] = l$, we obtain

$$\begin{aligned} \lambda_a &= \lambda_{a1} + l\Big\{1 - G(a) + e^{-\theta_2 t} H(b) + \big(1 - e^{-\theta_2 t}\big) G(b)\Big\}, \\ \lambda_b &= \lambda_{b1} + l\Big\{G(b) + e^{-\theta_2 t}\big(1 - H(a)\big) + \big(1 - e^{-\theta_2 t}\big)\big(1 - G(a)\big)\Big\}. \end{aligned} \qquad (6-3)$$

6.4. DERIVATION OF THE EXPECTED TIME PATH

To derive the expected time path of the exchange rate, we need arrival intensities of the agents who hit the bid and the offered rates. When the arrival intensities are not equal, the exchange rate is expected to shift. There are two sources shifting the exchange rate. The first is unmatched arrivals of aggregate retail demand and supply. The second source is shifts of $G_t(x)$ which may be caused by the arrival of news or by adaptive expectations as a response to a trend in the transaction price. The first cause may give rise to the second cause. Here we do not consider this interaction of the two sources. We consider the expected time path, depending on whether the heterogeneous expectations are stationary, or whether aggregate retail demand meets supply.

A$(6-3)$: Each agent's daylight limit is equal to one transaction unit.

A$(6-4)$: $H_t(x)$ and $G_t(x)$ are not equal. (Heterogeneous expectations are non-stationary.)

We continue the example discussed in Section 6.3. We neglect the bid-ask spread. So let $a = b$ and let $x = a$. Then,

$$\lambda_a = \lambda_{a1} + l\Big\{1 - G(x) + e^{-\theta_2 t} H(x) + \big(1 - e^{-\theta_2 t}\big) G(x)\Big\}$$

$$\lambda_b = \lambda_{b1} + l\Big\{G(x) + e^{-\theta_2 t}\big(1 - H(x)\big) + \big(1 - e^{-\theta_2 t}\big)\big(1 - G(x)\big)\Big\}$$

Suppose that during Δt, the exchange rate changes by Δx. The agents whose expected FLE values fall in $[\ x, x+\Delta x\ ]$ switch from bullish to bearish or vice versa as the transaction price changes by Δx. These agents' positions switch from long to short or vice versa and absorb 2 units of arrivals. There are $h(x)\Delta x$ of the agents who switch positions. It was assumed that $H_t(x) = G_t(x)$ until epoch 0 and that $G_t(x)$ shifted at epoch 0. During $[0,t]$, the expected number of agents who constitute a sample of $H(t)$ decreased from $E[N_c]$ to $e^{-\theta_2 t}E[N_c]$. Meanwhile the expected number of agents who constitute a sample drawn from $G(t)$ increases from 0 to $(1-e^{-\theta_2 t})E[N_c]$. The expected excess arrivals of buyers in $[0,t]$ is $(\lambda_a - \lambda_b)\Delta t$. This excess of buyers is matched by a change of positions by the quoting agents. These agents consist of those from the sample of $H(x)$ and those from $G(x)$. Let $h(x)$ and $g(x)$ be density functions of $H(x)$ and $G(x)$. Then, the expected number of these agents is $h(x)\Delta x$ and $g(x)\Delta x$, respectively. The quantity which each agent absorbs by switching from long to short positions is 2 units. Using equalities $\frac{l}{\theta_2} = \frac{\theta_1 N}{\theta_1+\theta_2} = E[N_c]$, we obtain

$$\begin{aligned}(\lambda_a - \lambda_b)\Delta t &= \frac{l}{\theta_2}e^{-\theta_2 t}2h(x)\Delta x + \frac{l}{\theta_2}(1-e^{-\theta_2 t})2g(x)\Delta x \\ &= \frac{2l}{\theta_2}\Big\{e^{-\theta_2 t}h(x) + \big(1-e^{-\theta_2 t}\big)g(x)\Big\}\Delta x.\end{aligned}$$

Therefore,

$$\frac{\Delta x}{\Delta t} = \frac{\lambda_a - \lambda_b}{\frac{2l}{\theta_2}\Big\{e^{-\theta_2 t}h(x) + \big(1-e^{-\theta_2 t}\big)g(x)\Big\}}. \qquad (6-4)$$

A$(6-5)$: $G_t(x)$ and $H_t(x)$ are uniform distributions.

Combining $A(6-4)$ and $A(6-5)$, let $G(x) = \frac{x}{k}$, for $x \in [0,k]$, and $H(x) = \frac{x-m_1}{k}$, for $x \in [m_1, m_1+k]$. Substituting $G(x)$ and $H(x)$ into the formula for λ_a and λ_b, we obtain

$$\begin{aligned}\lambda_a &= \lambda_{a1} + l\Big\{1 - \frac{x}{k} + e^{-\theta_2 t}\frac{x-m_1}{k} + \big(1-e^{-\theta_2 t}\big)\frac{x}{k}\Big\} \\ &= \lambda_{a1} + l\Big(1 - \frac{m_1}{k}e^{-\theta_2 t}\Big)\end{aligned}$$

and

$$\lambda_b = \lambda_{b1} + l\Big\{\frac{x}{k} + e^{-\theta_2 t}\Big(1 - \frac{x - m_1}{k}\Big) + \big(1 - e^{-\theta_2 t}\big)\frac{x}{k}\Big\}$$
$$= \lambda_{b1} + l\Big(1 + \frac{m_1}{k}e^{-\theta_2 t}\Big)$$

Hence, $\lambda_a - \lambda_b = \lambda_{a1} - \lambda_{b1} - 2l\frac{m_1}{k}e^{-\theta_2 t}$. Substituting this into $(6-4)$ yields

$$\frac{\Delta x}{\Delta t} = k\Big(\lambda_{a1} - \lambda_{b1} - 2l\frac{m_1}{k}e^{-\theta_2 t}\Big)\frac{\theta_2}{2l}$$
$$= \frac{k\theta_2}{2l}\Big(\lambda_{a1} - \lambda_{b1}\Big) - m_1\theta_2 e^{-\theta_2 t}.$$

We want to solve this differential equation. The answer is given by

$$x(t) = \frac{k\theta_2}{2l}\Big(\lambda_{a1} - \lambda_{b1}\Big)t + m_1 e^{-\theta_2 t} + c_0.$$

Let $x(0) = x_0$ be an initial condition. Then, $c_0 = x_0 - m_1$ and

$$x(t) = \frac{k\theta_2}{2l}\Big(\lambda_{a1} - \lambda_{b1}\Big)t + m_1 e^{-\theta_2 t} + x_0 - m_1, \qquad (6-5)$$

where x_0 is the value of x when the shift of $G_t(x)$ happened. As $t \to \infty$, $x(t) \to x_0 - m_1$, provided that $\lambda_{a1} = \lambda_{b1}$.

A$(6-4)'$: $G_t(x)$ and $H_t(x)$ are the same distributions. (Expectations are heterogeneous but stationary.)

Substitute $m_1 = 0$ into the above equation. Then,

$$x(t) = \frac{k\theta_2}{2l}\Big(\lambda_{a1} - \lambda_{b1}\Big)t + x_0. \qquad (6-6)$$

This holds only while $0 < x(t) < k$. When we assume $A(6-4)'$, a slope of the expected time path is determined only by the arrivals which are due to retail transactions. The arrivals due to heterogeneous expectations do not have an influence in $(6-6)$.

If the heterogeneity of the expectations is stationary $\big(i.e., G_t(x) = H_t(x)\big)$, then, according to $(6-3)$, $\lambda_a - \lambda_b$ depends entirely on $\lambda_{a1} - \lambda_{b1}$, since $\lambda_{a2} = \lambda_{b2} = 1 - G(a) + G(b)$. This means that as long as aggregate retail demand and supply are expected to be equal, *i.e.,* $\lambda_{a1} = \lambda_{b1}$, the transaction price will not move to the mean value taken with respect to $G(x)$. In other words, the agents' expectations do not determine a kind of stable equilibrium point. If $\lambda_{a1} = \lambda_{b1}$, anywhere within the support of $G(x)$ and $H(x)$, the transaction price is expected to stay at the level at which the random arrivals put it. It is similar to a Martingale process for the following reason. When heterogeneous expectations are stationary, the expected number of entries into the crystal state by Transition 1 which hits the offered (bid) rate is $l(1-G(a))$ $\left(lG(b)\right)$ and the expected number of exits from the crystal glass by Transition 2 which hits the bid (offered) rate is also $l(1-G(a))$ $\left(lG(b)\right)$. Forces of the same magnitude affect bid and offered rates.

6.5. THE MARKET MAKER'S OPTIMAL QUOTATIONS

Agents who quote both buying and selling prices at the same time in order to take advantage of the heterogeneous expectations of other agents are called *market makers.* Their policy is $\left\{\Gamma_{w12} \text{ if } |Z| \leq L \ ; \ \Gamma_{h12} \text{ if } |Z| > L\right\}$ $\left((4-17-g) \text{ and } (4-17-d)\right)$. if for a given expected FLE value x_t, An agent becomes a market maker, interval $[x_t - u, x_t + u]$ overlaps with $[B(t), A(t)]$. The market maker wants both of his quotations to be hit by the same arrival intensity. Among the combinations of such quotations, he chooses the ones which maximize his expected profit.

The value of u^*, which solves $(4-15)$, maximizes the long run average expected profit. To derive $(4-15)$, we treated the arrival intensity q of buyers and also of sellers as a function of u which is half of the bid-ask spread. In this section, we derive an explicit form of $q = f(u)$, assuming uniform distribution for the expected FLE value. It is possible that the agent will choose Γ_{w12} and become a market maker even if he is bullish or bearish in the sense of A$(4-9)$. Such action may be profitable, if the arrival intensity is large enough compare to the slope of $\xi(t)$. However, we do not consider such a situation here.

In order to derive the arrival intensity, we consider which actions agents will take given their expected FLE values. Suppose Agent 0 wants to determine the arrival intensity for his quotations. Let x_0 be his expected FLE value and $2u$ be the bid-ask spread. Let A and B be Agent 0's selling and buying prices such that $A = x_0 + u$ and $B = x_0 - u$. As u becomes larger, the number of competitors whose quotations fall in interval $[B, A]$ will increase. In order to calculate the expected number of competitors we assume the following.

A$(6-6)$: All agents have the same bid-ask spread.

Besides Agent 0, any agents whose FLE values happen to fall around the current market rates would quote the buying and selling prices with the same spread, provided that their daylight limits are not binding. In addition to $A(6-6)$, we assume A$(6-3)$, A$(6-4)'$ and A$(6-5)$ in the following.

Let $a \equiv A + u$ and $b \equiv B - u$. The supports of G and H are divided into six regions as is shown in Figure 6. Let x be the expected FLE value of a given agent. Since the bid-ask spread is $2u$ for every agent, $x = a$ is the minimum value of x, such that the buying price $x - u$ is greater or equal to Agent 0's selling price, A. Among the agents who are arriving with x drawn from $G(x)$, those in region S_6 which constitute $1 - G(a)$ of the arrivals hitting A. If $x \in S_5$, then the arrivals quote the buying prices which fall into S_4. If $x \in S_4$, then the buying prices fall into S_3. If $x \in S_3$, then the selling prices fall into S_4. If $x \in S_2$, then the selling prices fall into S_3 . Selling prices $x \in S_2$ are lower than buying prices $x \in S_5$, cancelling each other. If $x \in S_1$, then the arrivals hit B. The expected number of arrivals of those drawn from

G is $\theta_1 E[N_f]$. This is the per-epoch expected number of agents who have Transition 1 and it can be written as $\theta_1 E[N_f] = \frac{\theta_1\theta_2 N}{\theta_1+\theta_2} = l$. Among the arrivals, $(1-G(a))l$ of them hit A and $(G(x_0)-G(B))l$ block A.

Next we consider effects of prices which have been quoted. Their associated FLE values are distributed according to H. If $x \in S_6$, then agents have long positions, and their selling prices are less competitive than A. If $x \in S_5$, then the buying prices fall into S_4. Meanwhile, if $x \in S_2$, then selling prices fall into S_3. It is impossible to have a sample of quotations such that the quoted buying prices quoted are higher than the selling prices. Therefore, selling prices $x \in S_2$ must have cancelled buying prices $x \in S_5$ and thus are not blocking A. If $x \in S_4$, then buying prices fall into S_3. If $x \in S_3$, then selling prices fall into S_4 and do not cancel the buying prices $x \in S_4$. Buying prices $x \in S_3$ block A.

Agents who are quoting prices are in the state of the crystal glass. Their expected number is $E[\,N_c\,] = \frac{\theta_1 N}{\theta_1+\theta_2} = \frac{l}{\theta_2}$. The expected number of agents with $x \in S_3$, blocking A, is given by $(H(x_0)-H(B))\frac{l}{\theta_2}$. Meanwhile, $\theta_2 E[N_c]$ of the agents are expected to have Transition 2, discarding their expectations, and to square their positions. Among these agents, $H(x_0)$ of them have short positions and become buyers when they have Transition 2. The expected number of buyers who hit A is given by $H(x_0)\theta_2 E[N_c] = H(x_0)l$.

The daylight limits are binding for the agents who constitute the sample for H. If due to retail transactions their positions deviate from the daylight limits, they bring the position to the original level by hitting market rates. Aggregate retail transactions become the arrivals at the market right away. The arrival intensity in this case is λ_{a1}.

Combining the preceding discussion, $\Big(1-G(a)\Big)l+H(x_0)l+\lambda_{a1}$ hit A and $\Big(G(x_0)-G(B)\Big)l+\Big(H(x_0)-H(B)\Big)\frac{l}{\theta_2}$ block A. Let $q=f(u)$ be the arrival intensity of agents who hit A as a function of u.

$$f(u)=\Big(1-G(a)\Big)l+H(x_0)l+\lambda_{a1}-\Big(G(x_0)-G(B)\Big)l-\Big(H(x_0)-H(B)\Big)\frac{l}{\theta_2}$$

Let $G(x)=H(x)=\frac{x}{k}$. Then, $1-G(a)=1-\frac{x_0+2u}{k}$, $H(x_0)=\frac{x_0}{k}$ and $G(x_0)-G(B)=H(x_0)-H(B)=\frac{u}{k}$. By substituting these into $f(u)$, we obtain

$$f(u)=l+\lambda_{a1}-\frac{l}{k}\Big(3+\frac{1}{\theta_2}\Big)u \qquad (6-7)$$

The necessary condition for the long run average expected profit maximization according to $(4-15)$ is $f'u+f=0$. Substituting $(6-7)$ into $(4-15)$, we obtain

$$u^*=\frac{k}{2}\frac{\theta_2}{3\theta_2+1}\Big(1+\frac{\lambda_{a1}}{l}\Big). \qquad (6-8)$$

The optimal spread is $2u^*$, and Agent 0, whose expected FLE value is x_0, will quote $A=x_0+u^*$ and $B=x_0-u^*$.

6.6. AGENT'S INFORMATION

The exchange rate is determined through the arrival process of agents. The parameters of this process are determined by aggregate retail demand and supply, $ARD(t)$ and $ARS(t)$, and the FLE distribution functions,$H_t(x)$ and $G_t(x)$. Agents estimate $ARD(t)$, $ARS(t)$, $H_t(x)$, and $G_t(x)$. which are all nonstationary. The agents estimate these parameters with varying degrees of accuracy. Since $ARD(t)$ and $ARS(t)$ depend on the individual agents' retail customers, no agent can estimate $ARD(t)$ and $ARS(t)$ accurately all the time. The market is not efficient as is discussed in Levich (1985).

7. PROPOSITIONS ABOUT VOLATILITY AND TRADING VOLUME

Proposition 1.

$$\text{As } \mathrm{Var}[X(t)] \text{increases, } E[A(t)-B(t)] \text{increases.}$$

(The more people disagree, the wider becomes the bid and ask spread.)

Proof: We prove the proposition with the following assumptions: **A**$(6-4)'$, **A**$(6-5)$ and **A**$(6-6)$. Also, we need one more assumption:

A$(7-1)$: All agents make the same estimate about the variance of $G_t(x)$ and $H_t(x)$ and $E[N_c]$.

In Chapter 6, the optimal spread $2u^*$ is derived with the assumption that the agents would have the same spread when they quote both buying and selling prices. We need $A(7-1)$ in order for $(6-8)$, the formula for u^*, to be consistent with $\mathrm{A}(6-6)$ with which the derivation of $(6-8)$ is started. Let $G(x) = H(x) = \frac{x}{k}$ and σ^2 be a variance of X_t. Then, $\sigma^2 = \frac{k^2}{12}$. For the sake of simplifying the calculation, we shifted the support of the distribution functions to $[0, k]$. We consider three types of market-rates quotes. The first is when only one market maker quotes both bid and offered rates. The second is when more than one market maker is quoting. Market rates consist of quotations of the different market makers. The third is when no agent is quoting two rates. We want to show in all three cases that $E[A(t)-B(t)]$ increases as σ^2 increases. Instead of writing the optimal spread as $2u^*$, let $v^* \equiv 2u^*$. From $(6-8)$,

$$v^* = k\frac{\theta_2}{3\theta_2+1}\left(1+\frac{\lambda_{a1}}{l}\right).$$

Case 1: We want to show that $\frac{\partial v^*}{\partial \sigma^2} > 0$. Since $\frac{\partial v^*}{\partial k} = \frac{\theta_2}{3\theta_2+1}\left(1 + \frac{\lambda_{a1}}{l}\right) > 0$ and $\frac{\partial k}{\partial \sigma^2} = 12 > 0$,

$$\frac{\partial v^*}{\partial \sigma^2} = \frac{\partial v^*}{\partial k}\frac{\partial k}{\partial \sigma^2} > 0. \qquad (7-1)$$

Case 2: Suppose that the first market maker quotes $(B_1, A_1) = (x - \frac{v^*}{2}, x + \frac{v^*}{2})$, as his buying and selling prices, that the second market maker joins the first market maker quoting $(B_2, A_2) = (y - \frac{v^*}{2}, y + \frac{v^*}{2})$, and that their quotations overlap. If their quotations overlap, then $B_1 \le A_2$ and $B_2 \le A_1$. Hence, $x - \frac{v^*}{2} = B_1 \le A_2 = y + \frac{v^*}{2}$ and $y - \frac{v^*}{2} = B_2 \le A_1 = x + \frac{v^*}{2}$. Fixing x, the density function of y, conditional on that $[B_1, A_1]$ and $[B_2, A_2]$ overlap, is given by $\frac{1}{2v^*}$, provided $v^* \le x \le k - v^*$, because y can take value from $x - v^*$ to $x + v^*$. Within this range, if the value of y is such that $x - v^* \le y \le x$, then $A(t) = A_2$ and $B(t) = B_1$. The offered rate is quoted by the second market maker, the bid rate is quoted by the first market maker, and $A(t) - B(t) = A_2 - B_1 = y - x + v^*$. If $x \le y \le x + v^*$, then $A(t) = A_1$, $B(t) = B_2$ and $A(t) - B(t) = A_1 - B_2 = -y + x + v^*$. Provided that $v^* \le x \le k - v^*$, the expected spread is given by

$$\begin{aligned} E[A(t) - B(t)] &= \int_{x-v^*}^{x} (y - x + v^*)\frac{1}{2v^*}dy + \int_{x}^{x+v^*} (-y + x + v^*)\frac{1}{2v^*}dy \\ &= \frac{v^*}{2}. \end{aligned}$$

If $x \le v^*$, and if the quotations overlap, it must be the case that $0 \le y \le x + v^*$, because $y - \frac{v^*}{2} = B_2 \le A_1 = x + \frac{v^*}{2}$ must hold. If $0 \le y \le x$, then $A(t) = A_2$ and $B(t) = B_1$. If $x \le y \le x + v^*$, then $A(t) = A_1$ and $B(t) = B_2$. The density function of y, conditional on their quotation overlap, is given by $\frac{1}{x+v^*}$. The expected bid-ask spread, for $x \le v^*$, is given by

$$\begin{aligned} E[A(t) - B(t)] &= \int_{0}^{x} (y - x + v^*)\frac{1}{x + v^*}dy + \int_{x}^{x+v^*} (-y + x + v^*)\frac{1}{x + v^*}dy \\ &= \frac{x + v^*}{2} - \frac{x}{x + v^*} \end{aligned}$$

Using the above result, we have $\frac{\partial}{\partial v^*}(\frac{x+v^*}{2} - \frac{x}{x+v^*}) = \frac{1}{2} + \frac{x}{(x+v^*)^2} > 0$. Hence, for two cases of $v^* \leq x \leq k - v^*$ and $x \leq v^*$, using $(7-1)$, $\frac{\partial}{\partial \sigma^2}\Big(E[A(t) - B(t)]\Big) = \frac{\partial}{\partial v^*}\Big(E[A(t) - B(t)]\Big)\frac{\partial v^*}{\partial \sigma^2} > 0$. If $x \geq k - v^*$ and the quotations overlap, then, similarly, $\frac{\partial}{\partial \sigma^2}(E[A(t) - B(t)]) > 0$.

If there are more than two market makers who are quoting, then we have the same result by the similar derivation.

Case 3: No agent is quoting both bid and offered rates at the same time. And the bid and ask spread in the market is wider than v^*. If an agent's FLE is x, then his bid rate is $x - \frac{v^*}{2}$ and his offered rate is $x + \frac{v^*}{2}$. Since a support of $G(x)$ and $H(x)$ is $[0, k]$, the bid (offered) rate is distributed over $[-\frac{v^*}{2}, k - \frac{v^*}{2}]$ $\Big([\frac{v^*}{2}, k + \frac{v^*}{2}]\Big)$. If there is no quotation in a given interval, $[x_0, x_1]$, for $-\frac{v^*}{2} \leq x_0 \leq k - \frac{v^*}{2}$ and $\frac{v^*}{2} \leq x_1 \leq k + \frac{v^*}{2}$, any quoted price must fall outside of $[x_0, x_1]$. Together with $A(6-5)$, it implies that if any agent is quoting price, his expected FLE value x must satisfy the following inequality. $x - \frac{v^*}{2} \leq x_0$ or $x_1 \leq x + \frac{v^*}{2}$. No agent's expected FLE value x stays in an interval $[x_0 + \frac{v^*}{2}, x_1 - \frac{v^*}{2}]$. Let $w \equiv x_1 - x_0 - v^*$; the length of the interval where nobody's expected FLE value x is located, when no price is quoted in $[x_0, x_1]$. Then the length of the interval where nobody's quotation is located is $w + v^*$. Let ω_0 be an event which is expressed as intersections of three events, $\{$ No agent is quoting both rates. $\} \cap$ $\{m$ agents are quoting their prices. $\} \cap \{A(t) - B(t) = w + v^*\}$. Event ω_0 is equivalent to $\Big\{$ No one's x is located in the interval of length w when m agents are quoting their prices. $\Big\}$. Hence, the probability of event ω_0 is given by

$$\Pr(\omega_0) = \Big(1 - \frac{w}{k}\Big)^m. \qquad (7-2)$$

We approximate $(7-2)$ by Taylor's expansion. Let $f(\frac{w}{k}) = (1-\frac{w}{k})^m$. Then,

$$\begin{aligned} f(\frac{w}{k}) &\approx f(0) + f'(0)\frac{w}{k} + \frac{f''(0)}{2}(\frac{w}{k})^2 \\ &= 1 + m\frac{w}{k} + \frac{m(m-1)}{2}(\frac{w}{k})^2 \end{aligned}$$

Let $y \equiv \frac{w}{k}$. Then $0 \le y \le 1$ and $w = ky$. Given that m agents are quoting, but that none of them is a market maker, the expected value of the bid-ask spread is given by

$$\begin{aligned} E[A(t) - B(t)] &= \int_0^1 (ky + v^*) f(y)\, dy \\ &= \int_0^1 (ky + v^*)\Big\{1 + my + \frac{m(m-1)}{2}y^2\Big\}\, dy \\ &= \frac{k}{24}(3m^2 + 5m + 12) + \frac{v^*}{6}(m^2 + 2m + 6). \end{aligned}$$

For any m, $3m^2 + 5m + 12 > 0$ and $m^2 + 2m + 6 > 0$. ¿From the result of the first case, $\frac{\partial v^*}{\partial k} > 0$. We obtain $\frac{\partial}{\partial k}E[A(t) - B(t)] > 0$. Therefore in all three cases, $\frac{\partial}{\partial \sigma^2}E[A(t) - B(t)] > 0$. Q.E.D.

Proposition 2.

As $\mathrm{Var}[X(t)]$increases, $\mathrm{Var}[S(t)]$ increases.

Proof: Arrivals of agents are generated by two causes; heterogeneity of expectations and retail transactions. In order to prove the Proposition 2, we have to distinguish arrivals due to two causes. We define three variables, aggregate heterogeneity transactions, aggregate retail transactions and aggregate excess demand. A definition of *aggregate heterogeneity transactions* is cumulative quantity which the offered rate was hit minus cumulative quantity which the bid rate was hit by those who arrived due to heterogeneous expectations. Let $AH(t)$ denote aggregate heterogeneity transactions at epoch t. Its expected value is given by $E[AH(t)] = (\lambda_{a2} - \lambda_{b2})t$. If A$(6-4)$ is assumed, then by $(6-7)$, $\lambda_{a2} = \lambda_{b2}$, and hence, $E[AH(t)] = 0$. Second, we define aggregate retail transactions. *Aggregate retail transactions* are excess of aggregate cumulative retail selling over aggregate cumulative retail buying. Let $ARD(t)$, $ARS(t)$ and $AR(t)$ denote aggregate retail selling, aggregate retail buying and aggregate retail transactions at epoch t, respectively. By definition, $AR(t) \equiv ARD(t) - ARS(t)$ and $E[AR(t)] = (\lambda_{a1} - \lambda_{b1})t$. Thirdly, we define *excess demand* at epoch t, $ED(t)$, such that $ED(t) \equiv AR(t) + AH(t)$.

We use the same assumptions as in Proposition 1 and one more assumption.

A$(7-2)$: Aggregate retail demand and supply have the same arrival rates, $\lambda_{a1} = \lambda_{b1}$.

Suppose that the aggregate excess demand equals q at epoch t, $ED(t) = q$ and that, initially, $AR(0) = 0$ and $S(0) = s_0$. Let $W \equiv S(t) - E[S(t)]$, then $\text{Var}[W] = \text{Var}[S(t)]$. The difference in the arrivals of buyers and sellers is absorbed by agents who switch their positions. The FLE values of those agents have distribution $H_t(x)$. To simplify the proof, let us suppose that the agents' FLE values are deterministically distributed according to $H_t(x)$. For given q, and m the number of agents quoting their prices, there is a value for W, denoted by w, which satisfies

$$q = 2\int_{s_0}^{w+s_0} h(u)du\, m. \qquad (7-3)$$

$(7-3)$ implies that excess aggregate retail demand is absorbed by the agents who switched their positions, for example, from short to long if $q > 0$, and that the number of those agents are $\int_{s_0}^{w+s_0} h(u)du$ percent of m. With **A**$(6-5)$, $(7-3)$ becomes $q = 2(H(w+s_0) - H(s_0))m = 2\frac{wm}{k}$. Hence, $w = \frac{k}{2m}q$ holds and

$$W = \frac{k}{2m}ED(t).$$

$ED(t)$ is a sum of the four compound Poisson processes. We assume all four arrival processes are Poisson processes in order to simplify the proof,[13] instead of assuming compound Poisson.[14] Then,

$$E[ED(t)] = (\lambda_a - \lambda_b)t \quad \text{and} \quad \mathrm{Var}[ED(t)] = (\lambda_a + \lambda_b)t.$$

Since $W = \frac{k}{2m}ED(t)$,

$$\mathrm{Var}[W] = \left(\frac{k}{2m}\right)^2 \mathrm{Var}[ED(t)]. \tag{7-4}$$

Neglecting the bid-ask spread, from $(6-3)$, we have

$$\lambda_b = \lambda_{b1} + l \quad \text{and} \quad \lambda_a = \lambda_{a1} + l, \tag{7-5}$$

where $l = \frac{\theta_1\theta_2 N}{\theta_1+\theta_2}$. Hence, $\mathrm{Var}[AR(t)]$ does not depend on the heterogeneity of expectations. *i.e.,* not depending k, the parameter of the FLE distribution. Hence, $\frac{\partial \mathrm{Var}[W]}{\partial k} = \frac{k}{2m^2}\mathrm{Var}[ED(t)] > 0$. Therefore, $\frac{\partial \mathrm{Var}[S(t)]}{\partial \sigma^2} = \frac{\partial \mathrm{Var}[W]}{\partial k}\frac{\partial k}{\partial \sigma^2} > 0$. Q.E.D.

According to existing theories, during a so-called "turbulent era," when $\mathrm{Var}[S(t)]$ is larger, the bid-ask spread is widened in order to compensate market makers who still stand ready to trade with other agents. Contrary to these theories, in our model the wider bid-ask spread during a volatile period is not due to compensations for market makers to take additional risks. What makes the spread wider is the degree of disagreement among the agents.

Proposition 3.

$$\mathrm{Var}[S(t)]\text{increases}, \quad \text{as } N \downarrow 0 \ \text{(effect of thin market)}.$$

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[13]To obtain the expected values of the compound Poisson process, multiply λ_b and λ_a by the expected value of the individual retail transaction, $E[C]$. The variance is given by $\left\{\mathrm{Var}[C] + E[C]^2\right\}(\lambda_a + \lambda_b)$.

[14]They are all uncorrelated by assumption.

Proof: We use the same assumptions as Propositions 1 and 2. So far m is an arbitrary positive integer. Here, let m denote an expected number of agents who are quoting. From (7−5), $\lambda_b = \lambda_{b1}+l = \lambda_{b1}+\theta_2 m$ and $\lambda_a = \lambda_{a1}+l = \lambda_{a1}+\theta_2 m$, where $m \equiv E[N_c] = \frac{\theta_1}{\theta_1+\theta_2}N$. And $\mathrm{Var}[AR(t)] = \lambda_b + \lambda_a = \lambda_{b1} + \lambda_{a1} + 2\theta_2 m$. We differentiate $(7-4)$ with respect to m, to obtain

$$\begin{aligned}\frac{\partial \mathrm{Var}[\,W]}{\partial m} &= -\frac{k^2}{2m^3}\mathrm{Var}[AR(t)] + \Big(\frac{k}{2m}\Big)^2\frac{\partial \mathrm{Var}[AR(t)]}{\partial m}\\ &= -\frac{k^2}{2m^3}(\lambda_{a1}+\lambda_{b1}) + \frac{k^2\theta_2}{2m^2}\\ &= \frac{k^2}{2m^3}\Big\{-(\lambda_{a1}+\lambda_{b1}) + m\theta_2\Big\}.\end{aligned}$$

If $\lambda_{a1}+\lambda_{b1} > m\theta_2$, or equivalently, if $\lambda_{a1}+\lambda_{b1} > l = \frac{\theta_1\theta_2 N}{\theta_1+\theta_2}$, then $\frac{\partial \mathrm{Var}[\,W]}{\partial m} < 0$.

Because $\frac{\partial \mathrm{Var}[S(t)]}{\partial N} = \frac{\partial \mathrm{Var}[W]}{\partial m}\frac{\partial m}{\partial N}$ and $\frac{\partial m}{\partial N} > 0$, the variance of the transaction price increases, as the number of the agents decreases from $m = \frac{1}{\theta_2}(\lambda_{a1}+\lambda_{b1})$. If $m > \frac{1}{\theta_2}(\lambda_{a1}+\lambda_{b1})$, then the variance of the transaction price increases as the number of the agents increases. Q. E. D.

In the first case above, as the number of market participants becomes smaller, the variance of the transaction prices increases. This is an effect of thin markets. For a given $H_t(x)$, as N increases from 0, a larger number of agents are waiting behind the market rates $A(t)$ and $B(t)$. Therefore, jumps of $S(t)$, caused by batch arrivals of buyers and sellers, tend to be smaller. Hence the fluctuation of $S(t)$ becomes smaller. However, after N reaches $(\lambda_{a1}+\lambda_{b1})\frac{(\theta_1+\theta_2)}{\theta_1\theta_2}$, an increase in N results in further fluctuation of $S(t)$. An increase in N increases λ_{a2} and λ_{b2}. Since the variance of the number of arrivals of the Poisson Process is equal to its expected value, the variance increases as the arrival intensities increase. As the variance of the arrival number increases, $S(t)$ fluctuates more. This is a destabilizing effect of an increase in the number of agents. For $N > (\lambda_{a1}+\lambda_{b1})\frac{(\theta_1+\theta_2)}{\theta_1\theta_2}$, the destabilizing effect of ΔN dominates the stabilizing effect which absorbs more of the variation in $AR(t)$.

Proposition 4. When $\mathrm{Var}[S(t)]$ is large, the expected value of trading volume is also large.

Proof: Let $TV(T)$ be the trading volume of the day. The expected trading volume is equal to $\max\left\{\lambda_a T\ ,\lambda_b T\right\}$.[15] Since the arrival processes are piece-wise stationary, the arrival intensities defined for an entire day are weighted averages of the arrival intensities which constitute piece-wise stationarity. The weights are the lengths of each stationary period divided by T. Excess demand was defined as $ED(t) \equiv AR(t) + AH(t)$. We interpret $ED(t)$ as $ED(t) \equiv AD(t) - AS(t)$. Then with A(6 − 4)′ and A(7 − 2), $E[AD(t)] = E[AS(t)] = \lambda_a t$. From (7 − 5), $\lambda_i = \lambda_{i1} + l$ for $i = a, b$, where $l = \frac{\theta_1\theta_2}{\theta_1+\theta_2}N$. Since $\frac{\partial l}{\partial\theta_1} = \frac{(\theta_2)^2 N}{(\theta_1+\theta_2)^2} > 0$ and $\frac{\partial l}{\partial\theta_2} > 0$, we have $\frac{\partial\lambda_i}{\partial\theta_1} > 0$ and $\frac{\partial\lambda_i}{\partial\theta_2} > 0$, for $i = a, b$. Then,

$$\frac{\partial E[TV(t)]}{\partial\theta_1} = \frac{\partial E[TV(t)]}{\partial\lambda_j}\frac{\partial\lambda_j}{\partial l}\frac{\partial l}{\partial\theta_1} = \frac{(\theta_2)^2 N}{(\theta_1+\theta_2)^2}t > 0,$$

where $\lambda_j = \max\{\lambda_a, \lambda_b\}$. The expected trading volume increases as θ_1 and θ_2 increase. By (7 − 5) and (7 − 4), the variance of $S(t)$ increases as λ_i, $i = a, b$, increases. And we know $\frac{\partial\lambda_i}{\partial\theta_1} > 0$, for $i = a, b$. Therefore, $\mathrm{Var}[S(t)]$ and the trading volume increase together as θ_1 and θ_2, or both increase. Q.E.D.

When the variance of X_t is larger, the agents revise their expectation more frequently. This frequent revision means that θ_1 and θ_2 are larger. It implies that the number of times when the transitions between the two states of expectations occur is larger. For a given $H_t(x)$, the arrival intensities λ_b and λ_s become larger and the trading volume becomes larger.

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[15] Difference $(\lambda_a - \lambda_b)T$ is filled by net aggregate open positions during the business day and either by flow to or from the markets abroad or by the loan market at the end of the business day.

If we introduce the additional assumption that $\mathrm{Var}[X(t)]$ is larger when θ_1 and θ_2 are larger, then Proposition 4 has an additional supportive argument: When big news hits the market, not only the mean value of X_t shifts, but also $\mathrm{Var}(X_t)$ increases. This is because even if all the agents agree on the direction of the price shift, they do not agree on the exact quantitative effect of the news. When the degree of disagreement increases due to the big news, agents tend to revise their expectations more frequently. As agents revise their expectations more often, they adjust their position more frequently. The trading volume increases.

Another argument to support Proposition 4 is possible if we introduce an adaptive expectation which associates shifts of $G_t(x)$ with the movements of the transaction price. Suppose that if the agents are more uncertain about the trend of the exchange rate, individual agents tend to revise their expectations more frequently, *i.e.,* larger θ_1 and θ_2, and that in such periods, the variance of X_t is larger. When $S(t)$ moves due to the unmatched arrival of agents, the change in $S(t)$ will be faster as θ_1 becomes larger, because the number of Transition 1 of expectations for a given interval of time is larger for larger θ_1. For a given τ, which is a length of time during which the moving average is taken, a given change in $S(t)$ has more effect on the autoregressive terms, which cause the shift of $G_t(x)$. Therefore, if expectations are revised more frequently, $G_t(x)$ shifts more frequently. Due to the more frequent shifts of $G_t(x)$ when the variance of X_t is larger, the exchange rate is more volatile. Thus, the increased volatility of the exchange rate and a wider bid-ask spread can occur at the same time.

However, Propositions 1 to 4 together will explain the reason why, in contrast to the equity market, empirical researches fail to show a clear positive relationship between price volatility and trading volume in the foreign exchange market.[16] Distribution functions $H_t(x)$ have different values for their variances. The trading volume increases by increases in λ_a, λ_b or both. λ_a and λ_b consist of two parts. One represents arrivals originated in the aggregate retail transactions and the other represents arrivals generated by the heterogeneity of expectations. The aggregate retail transactions may vary, while $\text{Var}[X_t]$ is determined separately. It is possible that on a given day, λ_{a1} and λ_{b1} are larger than their daily averages while $\text{Var}[X_t]$ is smaller than its daily average. Had this situation occurred, then $\text{Var}[S(t)]$ would be smaller than its daily average, while the trading volume would be higher than its daily average. If $\text{Var}[X_t]$ is smaller while λ_{a1} and λ_{b1} are larger, $\text{Var}[S(t)]$ may stay the same or even smaller, while the trading volume is higher.

In the equity market,[17] on the contrary if λ_a, or λ_b increase, then almost all of increments consist of arrivals due to the heterogeneity of the expectations. θ_1 and θ_2 are larger and $\text{Var}[X_t]$ is larger at the same time. Therefore, in the equity market an increase in the trading volume is always accompanied by an increase in $\text{Var}[S(t)]$.

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[16]Details are discussed in the introduction.

[17]The equity market is a retail market, while the foreign exchange market is a wholesale market. Interpretation of λ_{a1} and λ_{b1} for the equity market is arrivals of orders generated by liquidity purpose by the public. λ_{a2} and λ_{b2} are generated by revisions of the expected stock prices by the public.

PART 3.

APPENDIX

A. SWAP TRANSACTIONS

Swap in foreign exchange markets means you trade all at once the foreign currency of two different delivery dates with trading of one delivery date reversing the other. Most swap transactions consist of either buying the spot and selling the forward or selling the spot and buying the forward. Some swap transactions consist of other combinations of delivery dates, including today-tomorrow, tomorrow-spot,[1] and forward-forward.

As an example of a swap transaction, you may sell spot US dollars against other currency and buy forward US dollars at the same time. This transaction can be thought to be a repurchase agreement of a currency or a loan of one currency with the other currency as a collateral. You have the other currency instead of US dollars until the due date of the forward delivery. You may make a loan in the other currency. On the delivery date of the forward, you receive the principal and the interest on the loan. You use that principal to complete the swap transaction. You deliver the other currency and receive US dollars. The forward rate and the relevant interest rate are known when you

[1]Delivery of the spot currency is two business days after a contract.

make the swap transaction, except for the spot rate which may be applied on the interest from the loan. The market makers' profits are determined today and the profits are realized on the delivery dates of the forward. The market makers of swap transactions are not exposed to any exchange risks. They only take default risks. The default risk of banks is practically negligible. The market makers stand ready to trade at their quotations without taking into account of the expectation of the future spot rate.

B. POISSON AND COMPOUND POISSON PROCESSES

B.1. POISSON PROCESSES

A stochastic process $\{N(t), t > 0\}$ is said to be a counting process if $N(t)$ represents the total number of events that have occurred up to epoch t. A counting process is said to possess independent increments if the number of events that occur in disjoint time intervals are independent. A counting process is said to possess stationary increments if the distribution of the number of events that occur during any interval of time depends only on the length of the time interval. The counting process $\{N(t), t > 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

(i) $N(0) = 0$.

(ii) The process has independent increments.

(iii) The number of events in any interval of length t is Poisson distributed with mean λt. In this paper, we call this λ an arrival intensity as in queueing theory.

B.2. COMPOUND POISSON PROCESSES

A compound Poisson process means that the number of the arrival is a Poisson process and that each arrival has its quantity. An example of a compound Poisson process is cumulative insurance payments. Let $N(t_0)$ be the number of insurance claims during an $[0, t_0]$. $N(t_0)$ is a random variable and $N(t_0)$ follows a Poisson distribution with an arrival intensity α. It implies $E[N(t_0)] = \alpha t_0$. The expected number of arrivals is proportionate to the length of time. Let X_i be an amount of the ith insurance claim which is filed with an insurance company during the interval $[0, t_0]$. A sequence of random variables $\{X_i\}$ are assumed to be independently and identically distributed. Let Y be the cumulative amount of the insurance claims for the period of $[0, t_0]$; $Y = X_1 + X_2 + X_3 + \cdots + X_N$. If we assume X_i and $N(t_0)$ are independent, Y follows the compound Poisson process. Since $E[N(t_0)] = \alpha t_0$, the $E[Y]$ is given by

$$\begin{aligned} E[Y] &= E[X_1]E[N(t_0)] \\ &= \alpha t_0 E[X_1] \end{aligned}$$

The expected values of cumulative retail transactions in our model are derived in an identical manner. Since the arrival process is Poisson, the expected values of R_1 and R_2 are proportionate to the length of the interval. Let's take an example of interval $[0, t_0]$. Assuming $R_1(0) = R_2(0) = 0$,

$$E[R_1(t_0)] = \alpha_1 t_0 E[X] \qquad \text{and} \qquad E[R_2(t_0)] = \alpha_2 t_0 E[X]$$

where α_1 is an arrival intensity of retail sellers, α_2 is an arrival intensity of retail buyers, and X is a quantity demanded or supplied by each retail arrival.

The retail transactions between epoch t_0 and T are given by $R_2(T) - R_2(t_0)$ and $R_1(T) - R_1(t_0)$. Their expected values are proportionate to the length of time of this interval, $T - t_0$. Let $\beta_i = \alpha_i E[X]$, for $i = 1, 2$. Then the expected value is written as

$$E[R_i(T) - R_i(t_0)] = \beta_i(T - t_0) \quad \text{for} \quad i = 1, 2.$$

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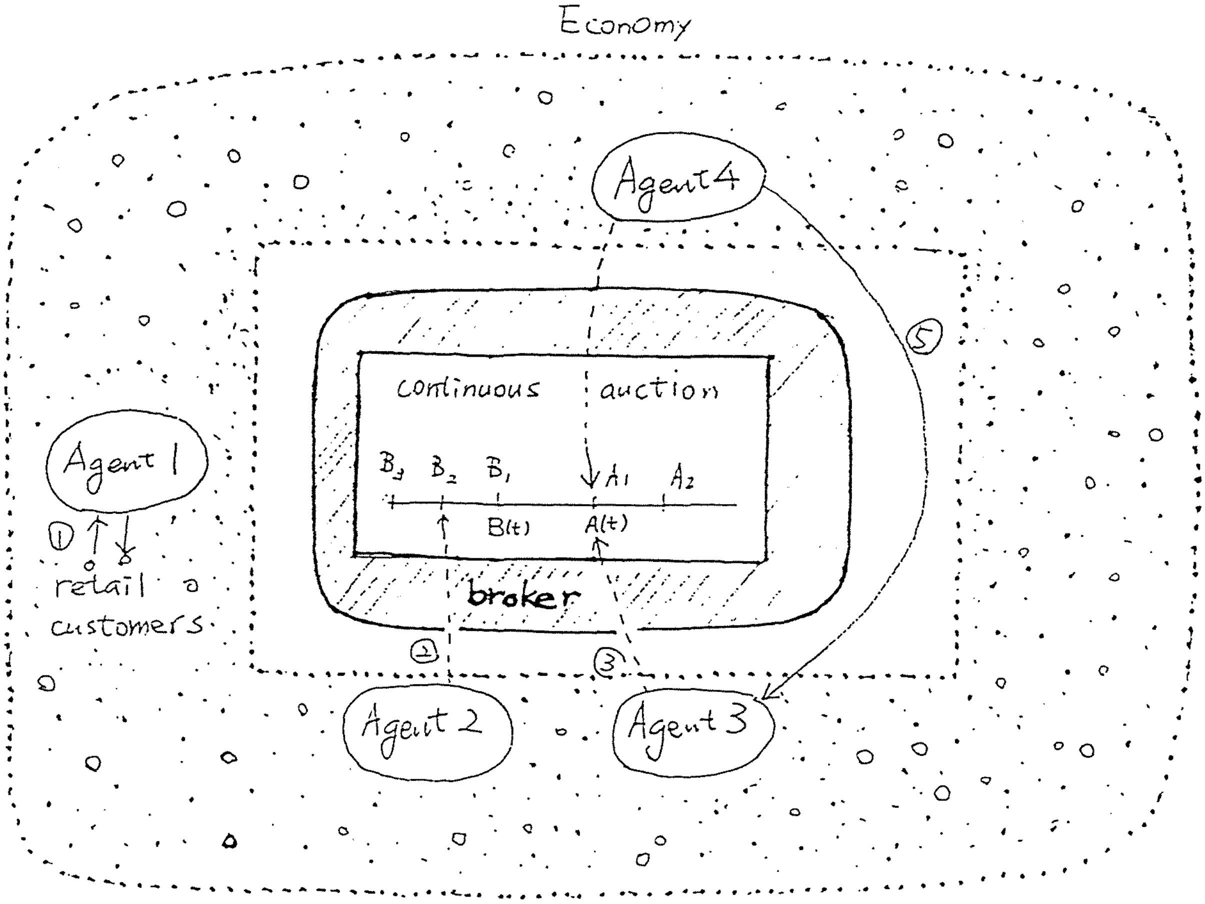

Figure 1. a Glance at the Market

1. Agent1 trades US dollars with his retail customers whenever they want.

2. Agent2 declared his buying price B_2 to the broker. The broker keeps B_2 on the list but he does not announce B_2 to the agents because B_2 is not the maximum buying price among those being quoted.

Figure 1. a Glance at the Market (Continued)

3. Agent3 declared his selling price A_1 to the broker. The broker announce A_1 which is the minimum of the quoted selling price to all the agents. The quotation is anonymous.
4. Agent4 shouts "A_1 is taken." He buys at A_1 which has been quoted for a while.

5. Agent4 finds an identity of the seller who is Agent3. US dollars are delivered by Agent3 to Agent4 two days later.

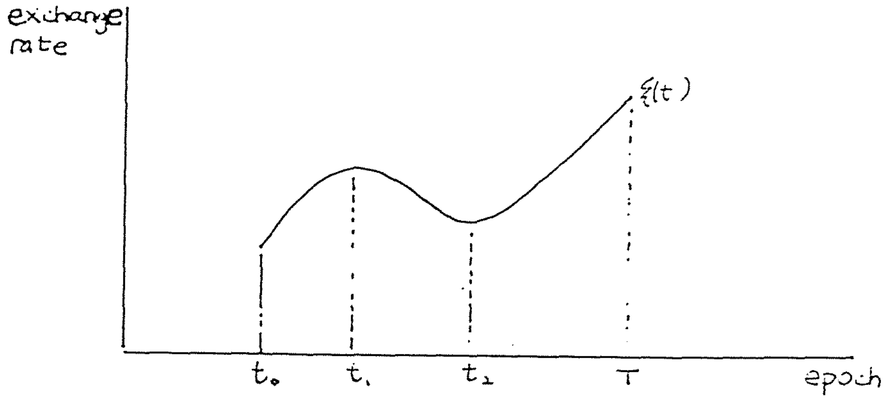

Figure 2. Successive maximizations

An agent maximizes the expected profit of each monotone subperiod: 1. buy at $S(t_0)$; 2. sell at $\xi(t_1)$; 3. buy at $\xi(t_2)$; 4. sell at $\xi(T)$. The maximization over an entire interval $[t_1, T]$ is obtained by such successive maximizations in the monotone subperiods.

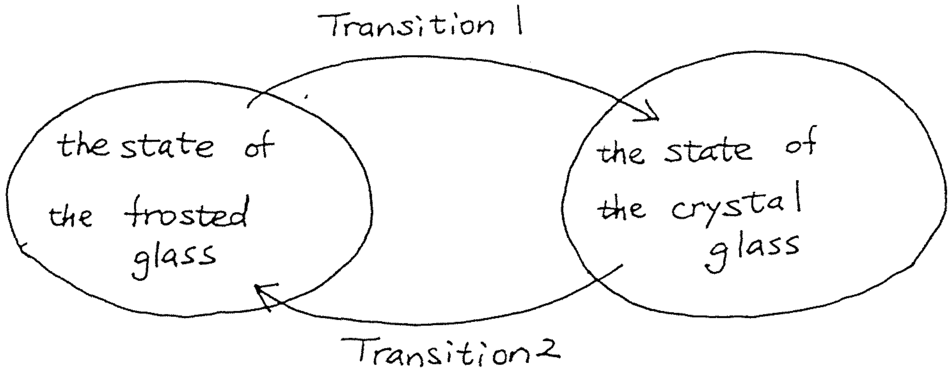

Figure 3. Transitions of Expectations

The length of time an agent stays in the state of the frosted glass follow an exponential distribution $\theta_1 \exp(-\theta_1 t)$. And $\theta_2 \exp(-\theta_2 t)$ for the state of the crystal glass. The expected length of stay is $\frac{1}{\theta_1}$ for the state of the frosted glass and $\frac{1}{\theta_2}$ for the state of the crystal glass.

For a total number N of agents, $E[N_c]$, the expected number of agents who stay in the state of the crystal glass for a given epoch is given by $E[N_c] = \frac{1/\theta_1}{1/\theta_1 + 1/\theta_2} N = \frac{\theta_2}{\theta_1 + \theta_2} N$. When transition 1 occurs, the agent's x_t follows $G_t(x)$. $H_t(x)$ is the sample distribution of x_t for the agents who have already arrived and are quoting their prices. If $G_t(x) = H_t(x)$, then the heterogeneity of the expectations is stationary. When transition 2 occurs, the agent's squares his position, *i.e.*, eliminating the open position.

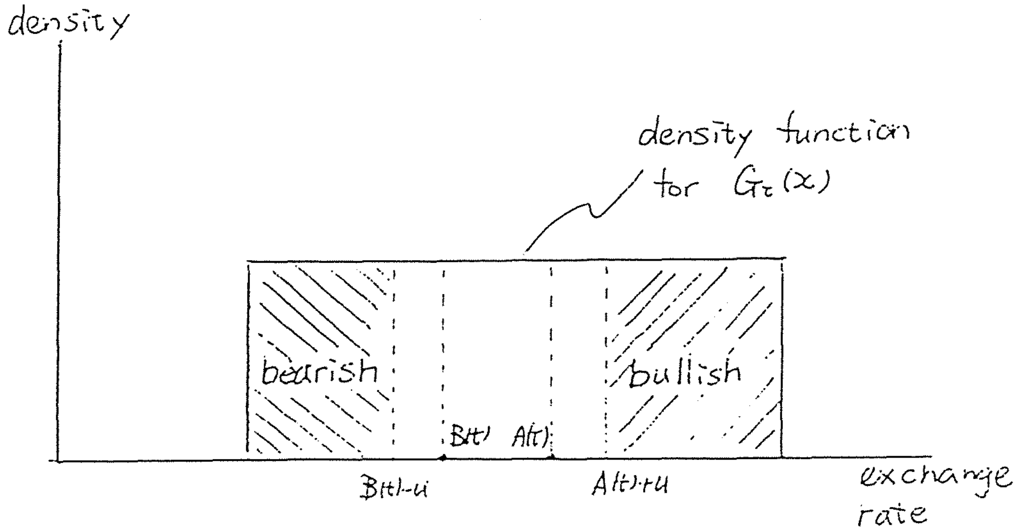

Figure 4. Distribution of x_t of New Arrivals

$G_t(x)$ is a distributions function such that when transition 1 occurs with a given agent, the value of x_t is random drawing with this distribution. Let $a \equiv A(t) + u$, $b \equiv B(t) - u$ where $2u$ is the expected bid-ask spread. If $X < b$, then an agent will hit the bid rate $B(t)$ upon arrival, since the agent is bearish in a similar sense of $A(4-9)$. Then the agent quotes $x_{ti} - u$ as his buying price. If $X > a$, then the agent will hit the offered rate $A(t)$ upon arrival, since the agent is bullish in a sense of $A(4-9)$. Then the agent quotes $x_{tj} + u$ as his selling price.

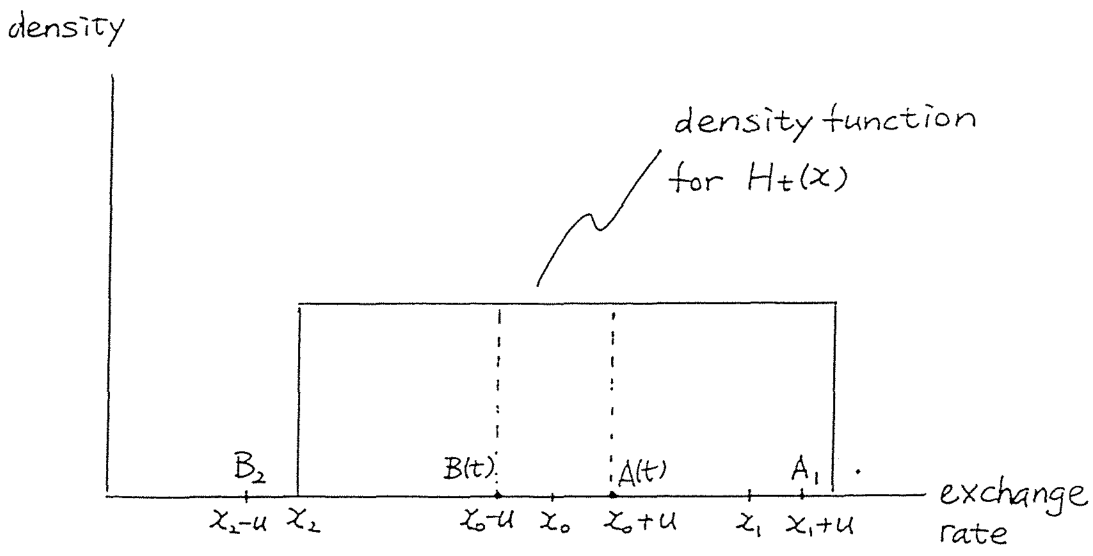

Figure 5. Distribution of Sample x_t

$H_t(x)$ is a sample distribution of x_t . A_i and B_j are selling and buying prices being quoted by the agents whose FLE are x_{ti} and x_{tj}. Let $a \equiv A(t) + u$, $b \equiv B(t) - u$ where $2u$ is the expected bid and ask spread. Since $A_i = x_{ti} + u$ and $B_i = x_{tj} - u$, the quotations are distributed over the wider range than the support of $H_t(x)$. If $X < b$, then an agent is bearish in a similar sense of A(4 – 9). He must have sold US dollars upon arrival and he is currently assuming a short position. The agent will become a buyer when transition 2 of his expectation occurs. If $X > a$, then the agent is bullish in a sense of A(4 – 9). He must have bought US dollars upon arrival and currently he is assume a long position. He will become a seller when transition 2 occurs.

Figure 5. Distribution of Sample x_t (continued)

If the heterogeneous expectations are stationary, i.e., $(G_t(x) = H_t(x))$, then the arrival intensity of the new arrivals who hit the offered (bid) rate is equal to the intensity of the agents who are leaving the crystal glass and hit the bid (offered) rate. As a result, like a Martingale process, the transaction price is expected to stay at the location where the random arrivals put it. The transaction price does not converge to the mean value of FLE taken with respect to $H_t(x)$ and $G_t(x)$. This result does not depend on the shape of $H_t(x)$ and $G_t(x)$.

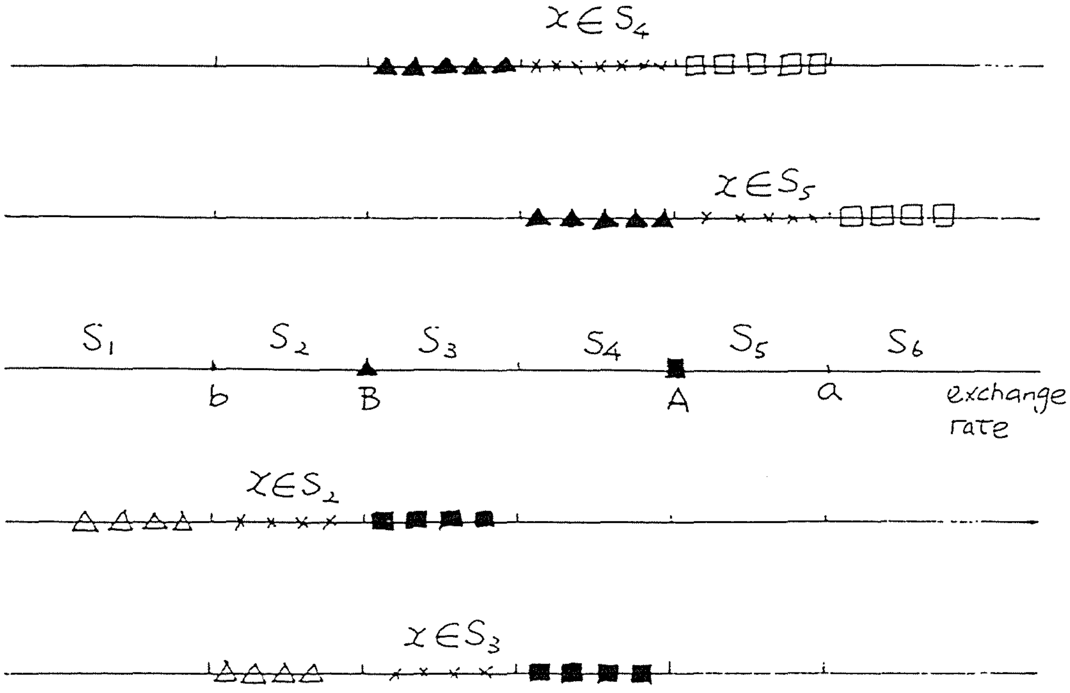

Figure 6. Actions Depending on Expected FLE Values

▲ : buying price being quoted.

■ : selling price being quoted.

△ : buying price which would be quoted unless the daylight limit is binding.

□ : selling price which would be quoted unless the daylight limit is binding.

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