No.7

The Existence of Ramsey Equilibrium with Consumption Externality

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September 1993

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Preliminary Version

Comments Welcome

The Existence of Ramsey Equilibrium with Consumption Externality (*)

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May 1993

Abstract: Consumption externality is introduced into the Ramsey equilibrium model. Existence of Ramsey equilibrium is demonstrated.

(*) We are grateful to John H. Boyd and Hideo Konishi for stimulating discussions and comments.

1. Introduction

We introduce consumption externality into a Ramsey equilibrium model studied by Becker (1980), Becker and Foias (1987), and Becker, Boyd, and Foias (1991). We prove the existence of a Ramsey equilibrium with consumption externality by modifying the arguments employed in Becker, Boyd, and Foias (1991) .

2. The Ramsey Economy with Consumption Externality

We start with a description of a <u>Ramsey economy with consumption externality</u> $(u^1,\ldots,u^H, k^1,\ldots,k^H,f)$.

Time is dealt with as a discrete variable and is denoted t. We assume that there is a single commodity which serves both as capital and as consumption good. There are H households indexed by $h = 1,\ldots, H$. Each household gets utility from consumption in each period. But the utility level also depends on consumption of other households. Let $C^h = \{c_t^h\}_{t=1}^{\infty}$ be a consumption stream of household h and let $\overline{C}^{-h} = (\overline{C}^1,\ldots, \overline{C}^{h-1},\overline{C}^{h+1},\ldots,\overline{C}^H)$ be an array of consumption stream of other households. Then the utility of household h is denoted by $u^h(C^h, \overline{C}^{-h})$. At time zero each household is endowed with capital stocks $k_0^h > 0$. Also, each household supplies one unit of labor in each period. Each household receives capital and labor income in each period. If w_t is the wage rate in period t, then the labor income of household h in period t equal to w_t. If h rents x_{t-1}^h units of capital at the rental rate equal to $1 + r_t$, then the capital income of h in period t will be $(1 + r_t)x^h_{t-1}$. The household faces a budget constraint in each period:

$$(2.1) \qquad c_t + x^h_t = w_t + (1 + r_t)x^h_t,\ x^h_o = k^h_o,\ c^h_t \geq 0,\ x^h_t \geq 0 \qquad (t = 1, 2, \ldots)$$

That is, each household is allowed to consume up to its income in each period, but not beyond that level. Unspent part of income will be carried over to the next period as capital stocks. We assume that consumers perfectly foresee the sequence of factor returns $\{1 + r_t, w_t\}_{t=1}^{\infty}$ and they are competitive. The household's objective is to maximize its utility subject to a sequence of budget constraints, given the anticipated prices $\{1 + r_t, w_t\}_{t=1}^{\infty}$ and the consumption streams of other households $\overline{C}^{-h}$:

$P(\{1 + r_t, w_t\}_{t=1}^{\infty}, \overline{C}^{-h})$: maximize $u^h(C^h, \overline{C}^{-h})$ subject to (2.1).

Let us state the properties of the utility functions of all households. Let s_+ be the set of all sequences of nonnegative real numbers. The domain of u^h is assumed to be H -fold copies of s_+, which is denoted by s^H_+. The first component in u^h signifies lifetime consumption stream of h, and other components represent lifetime consumption of other households. The set s_+ is endowed with the topology of coordinatewise convergence and s^H_+ is given the product topology. We assume the following properties of u^h:

(U 1): u^h is continuous.

(U 2): $u^h(\cdot, \overline{C}^{-h})$ is concave for each $\overline{C}^{-h}$.

By (U 2), the left- and right-hand directional derivatives of $u^h(\cdot, \overline{C}^{-h})$

exist. Let E_t =(0, 0,...,0, 1, 0,...) where 1 is in the t th place and define $u_t^{h+}(C, \bar{C}^{-h})$ by

(2.2) $$u_t^{h+}(C, \bar{C}^{-h}) = \lim_{\varepsilon \to 0+} [u^h(C + \varepsilon E_t, \bar{C}^{-h}) - u^h(C, \bar{C}^{-h})]/\varepsilon .$$

The left-hand partial derivative $u_t^{h-}(C, \bar{C}^{-h})$ is defined by letting $\varepsilon \to 0-$ in the limit. We say that t th partial derivative of $u^h(\cdot, \bar{C}^{-h})$ at C exists if $u_t^{h+}(C, \bar{C}^{-h}) = u_t^{h-}(C, \bar{C}^{-h})$, which is denoted by $u_t^h(C, \bar{C}^{-h})$. The following assumption is of a purely technical nature:

(U.3) $u_t^h(C, \bar{C}^{-h})$ exists for every $(C, \bar{C}^{-h})$ and for every t

The production technology is represented by a constant returns to scale production function F. Given H units of labor and x units of capital, $F(x, H) + (1 - \gamma)x$ is the total output available in a single period, where $\gamma \in (0, 1)$ is the depreciation rate of capital. Under the assumption of inelastic labor supply, it is convenient to deal with the total output as a function of capital alone and it is denoted by $f(x) = F(x, H) + (1 - \gamma)x$. Call f the total output function. The following properties of the output function will be assumed:

(F1) The total output function $f: \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $f(0) = 0$. It is of C^2 on $\mathbb{R}_{++}$ with $f' > 0$, $f'' > 0$, and $1 < f'(0_+) \leq \infty$.

The firm's objective is to maximize profit in each period, given the competitive rental price $(1 + r_t)$:

$P(1 + r_t)$: maximize $f(k_{t-1}) - (1 + r_t)k_{t-1}$.

Since point-input point-output production is assumed here, maximization of each period profit is equivalent to that of discounted profit. This completes the description of the Ramsey economy with consumption externality $(u^1,\dots,u^H,\ k_0^1,\dots,k_0^H,f)$.

A <u>Ramsey</u> <u>equilibrium</u> for the economy $(u^1,\dots,u^H,\ k_0^1,\dots,k_0^H,f)$ is a list $\{1+\bar{r}_t,\ \bar{w}_t,\ \bar{k}_{t-1},(\bar{c}_t^h,\ \bar{x}_t^h)_{h=1}^H\}_{t=1}^{\infty}$ satisfying the following conditions:

(2.2) $\bar{C}^h=\{\bar{c}_t^h\}_{t=1}^{\infty}$ solves $P(\{1+\bar{r}_t,\ \bar{w}_t\}_{t=1}^{\infty},\ \bar{C}^{-h})$, where $\bar{C}^{-h} = (\bar{C}^1,\dots,\bar{C}^{h-1},\bar{C}^{h+1},\dots,\bar{C}^H)$ (h = 1, 2,..., H).

(2.3) $\bar{k}_{t-1}$ solves $P(1+\bar{r}_t)$ (t=1, 2,...).

(2.4) $Hw_t = f(\bar{k}_{t-1}) - (1+\bar{r}_t)\bar{k}_{t-1}$, and

(2.5) $\sum_h \bar{x}_t^h = \bar{k}_t$, $\bar{k}_0 = \sum_h k_0^h$.

Three more assumptions need to be stated. Let $k = \sum_h k_0^h$. If there is a maximum sustainable stock b, and b < k, then let a = b. Otherwise, let a be a real number such that a > k. Let f^t be the t th iterate of f. The following assumptions are adapted from Becker, Boyd, and Foias [1991]:

(U 4): If $\{C^n\}$ is a sequence in $\Pi_{t=1}^{\infty}[0,\ f^t(a)]$ with $\lim_{n\to\infty} c_t^n = 0$ for some t, then for any $\bar{C}^{-h}$ we have $\lim_{n\to\infty} u_t^h(C,\ \bar{C}^{-h}) = \infty$.

(U 5): There is a $\delta_h \in (0,\ 1)$ such that $1/\delta_h = \sup\ \{u_t^h(C,\ \bar{C}^{-h})/u_{t+1}^h(C,\ \bar{C}^{-h}) : C \in \Pi_{\tau=1}^{\infty}[0,\ f^{\tau}(a)],\ c_{t+1} = c_t,\ \bar{C}^{-h} \in s_+^{H-1}\}$.

The assumptions (U 4) and (U 5) are respectively called Inada and Bounded Norm of Marginal Impatience Conditions by Becker, Boyd, and Foias [1991] when there is no consumption externality. We shall provide a few examples of utility functions satisfying (U 1), (U 2), (U 3), (U 4), (U 5). Let $U: \mathbb{R}_+ \to \mathbb{R}$ be a continuous, concave function which is differentiable on $\mathbb{R}_{++}$ and satisfies $U'(0_+) = \infty$. Let $V: s_+^{H-1} \to \mathbb{R}_{++}$ be an arbitrary continuous function. Then, $\Sigma_{t=1}^{\infty} \delta_h^t U(c_t) + V(\overline{C}^{-h})$ and $\Sigma_{t=1}^{\infty} \delta_h^t U(c_t) V(\overline{C}^{-h})$ satisfy all of the assumptions.

The last assumption is concerned with the relationship between the household sector and the production sector:

(UF 1): $f'(0_+) > \max_h (1/\delta_h)$

3. An Existence Theorem

The main result of this paper is stated below.

Existence Theorem: Let $(u^1, \ldots, u^H, k^1, \ldots, k^H, f)$ be a Ramsey economy with consumption externality satisfying assumptions (U 1), (U 2), (U 3), (U 4), (U 5), (F 1), and (UF 1). Then a Ramsey equilibrium exists.

The idea of our existence proof is to incorporate consumption externality into the tâtonnement map used by Becker, Boyd, and Foias [1991]. The intuition is that households would behave continuously with the changes in consumption externality if households' preferences are continuous with respect to consumption externality. The point of this paper is to verify this intuition. Toward this end, let us make a temporary assumption:

(U 2'): $u^h(\cdot, \overline{C}^{-h})$ is strictly concave for each $\overline{C}^{-h}$.

Our fixed point argument will be made based on this assumption first.

In the last stage, we will remove this assumption.

Let ε be a real number such that $0 < \varepsilon < k$ and $f'(\varepsilon) > \max_h(1/\delta_h)$. By (UF 1), such an ε exists. Define sets $\mathbb{K}$ and $\mathbb{D}$ by

$$(3.1) \qquad \mathbb{K} = \{K = \{k_t\}_{t=1}^{\infty} \in \prod_{t=1}^{\infty} [\varepsilon, f^t(a)] : k_o = k\},$$

and

$$(3.2) \qquad \mathbb{D} = \prod_{t=1}^{\infty} [0, A(k, \varepsilon)f'(\varepsilon)^t],$$

where $A(k, \varepsilon) = k + (f(\varepsilon) - \varepsilon f'(\varepsilon))(f'(\varepsilon) - 1)^{-1}$. By Tychonoff's theorem, $\mathbb{K}$ and $\mathbb{D}$ are compact in the product topology. Clearly, they are convex.

Given $K \in \mathbb{K}$, define $B^h(K)$ by

$$(3.3) \qquad B^h(K) = \{C = \{c_t\}_{t=1}^{\infty} \mid \exists X = \{x_t\}_{t=1}^{\infty} \ni c_t + x_t = w_t + (1 + r_t)x_{t-1},\ c_t \geq 0,\ x_t \geq 0\ (t = 1, 2, \ldots)\},$$

where $1 + r_t = f'(k_{t-1})$, and $w_t = [f(k_{t-1}) - f'(k_{t-1})k_{t-1}]/H$.
Then $B^h(K) \subset \mathbb{D}$ for all $X \in \mathbb{K}$ (see Becker, Boyd, and Foias (1991, pg 452)). Hence, $B^h(K)$ is compact since it is a closed subset of $\mathbb{D}$. Also, $B^h(K)$ is convex.

For a positive integer n, let $\mathbb{D}^n$ be the n-fold Cartesian product of $\mathbb{D}$. Given $K \in \mathbb{K}$ and $\overline{C}^{-h} \in \mathbb{D}^{H-1}$, define $C^h(K, \overline{C}^{-h}) = \{c_t^h(K, \overline{C}^{-h})\}_{t=1}^{\infty}$ by

$$(3.2) \quad C^h(K, \overline{C}^{-h}) = \operatorname{argmax} \{u^h(C^h, \overline{C}^{-h}): C^h \in B^h(K)\}.$$

By virtue of (U 1) and compactness of $B^h(K)$, $C^h(K, \bar{C}^{-h})$ is non-empty. Further, $C^h(K, \bar{C}^{-h})$ is a singleton by (U 2'). We call the resulting function the <u>consumption demand function</u> of h. The <u>capital supply function</u> $X^h(K, \bar{C}^{-h}) = \{x_t^h(K, \bar{C}^{-h})\}_{t=1}^{\infty}$ is defined recursively by

$$c_1(K, \bar{C}^{-h}) + x_1(K, \bar{C}^{-h}) = w_1 + (1 + r_1)k_0^h, \text{ and}$$

$$c_t(K, \bar{C}^{-h}) + x_t(K, \bar{C}^{-h}) = w_t + (1 + r_t)x_{t-1}(K, \bar{C}^{-h}) \quad (t = 1, 2, \ldots),$$

where $1 + r_t = f'(k_{t-1})$, and $w_t = [f(k_{t-1}) - f'(k_{t-1})k_{t-1}]/H$. Let $c_t(K, \bar{C}) = \Sigma_h c_t^h(K, \bar{C}^{-h})$ and let $k_t(K, \bar{C}) = \Sigma_h x_t^h(K, \bar{C}^{-h})$.

Define a map $\Phi : \mathbb{K} \times \mathbb{D}^H \to \mathbb{D}$ coordinatewise by

$$\Phi(K, \bar{C}) = \min \{\max\{\varepsilon, k_t(K, \bar{C})\}, f^t(a)\} \ (t = 1, 2, \ldots).$$

Define $\Gamma: \mathbb{K} \times \mathbb{D}^H \to \mathbb{K} \times \mathbb{D}^H$ by

$$\Gamma(K, \bar{C}) = (\Phi(K, \bar{C}), C^1(K, \bar{C}^{-1}), \ldots, C^H(K, \bar{C}^{-H}))$$

for $(K, \bar{C}) \in \mathbb{K} \times \mathbb{D}^H$.

The first component map of Γ is a modification of the tâtonnement map used by Becker, Boyd, and Foias(1991). The interpretation of the map is the same as theirs except that the consumption externality $\bar{C}^{-h}$ enters into the consumption decision problem of each h as a parameter. Other component maps are used as adjustments of consumption externalitites.

Lemma 1: The map Γ has a fixed point $(\bar{K}, \bar{C})$.

Proof: With a slight modification of the arguments in Appendix 2 of Becker, Boyd, and Foias(1991), we can show that Γ is continuous. Also, Γ maps the non-empty compact, convex set $\mathbb{K} \times \mathbb{D}^H$ into itself. Since $\mathbb{K} \times \mathbb{D}^H$ is contained in the (H + 1)-fold Cartesian product of s, which is a locally convex Hausdorff space, the map Γ has a fixed point $(\bar{K}, \bar{C})$ by the

Shauder-Tychonoff Theorem(see Dunford and Schwartz (1957)).

Q.E.D.

Lemma 2: Let $(\bar{K}, \bar{C})$ be the fixed point in Lemma 1. Then, $\bar{k}_t = k_t(\bar{K}, \bar{C})$.

Proof: The arguments in the proof of proposition 4.4 in Becker, Boyd, and Foias (1991) can be easily adapted to our model.

Q.E.D.

Now, we are ready to prove the main result.

Proof of Existence Theorem: Let $(u^1,\dots,u^H, k^1,\dots,k^H,f)$ be a Ramsey economy with consumption externality satisfying assumptions (U 1), (U 2), (U 3), (U 4), (U 5), (F 1), and (UF 1). Let $v^h(C^h, \bar{C}^{-h}) = u^h(C^h, \bar{C}^{-h}) + \eta g^h(C^h)$ where $g^h(C^h) = \sum_{t=1}^{\infty} \delta_h^{t-1}[-\exp(-c_t) + 1]$. Then a Ramsey economy $(v^1,\dots,v^H, k^1,\dots,k^H,f)$ satisfies assumptions (U 1), (U 2'), (U 3), (U 4), (U 5), (F 1), and (UF 1). Thus we can apply Lemmata 1 and 2 to this economy: There exists a fixed point $(\bar{K}^\eta, \bar{C}^\eta) \in \mathcal{K} \times \mathbb{D}^H$ of the map Γ with $\bar{k}_t^\eta = k_t(\bar{K}^\eta, \bar{C}^\eta)$. Let $\eta = 1/n$, n = 1, 2, ... Since $\mathcal{K} \times \mathbb{D}^H$ is compact, $\{(\bar{K}^\eta, \bar{C}^\eta)\}$ has a converging subsequence. Let $(\bar{K}, \bar{C})$ be its limit point. Then, it is easy to construct a Ramsey equilibrium for the original economy from $(\bar{K}, \bar{C})$.

Q.E.D.

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