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Supplement II to the paper "Asymptotic cumulants of ability estimators using fallible item parameters" − Expectations (Corrected version) April, 2014

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This supplement includes Subsection A.6 of the appendix of Ogasawara (2013).

A.6 Expectations

A.6.1 Non-studentized estimator $\hat{\theta}$

(a) Non-studentized estimator $\hat{\theta}$ under Condition A and m.m.: $N = O(n)$ $(\bar{c} = n/N = O(1))$

(a.1) The first asymptotic cumulant

Define $\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \equiv \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0}\right)$, which will be frequently used. In the following results, $m^{(\Delta)} = m^{(\Delta 3)} = m^{(\Delta\Delta b)} = 0$ under m.m.

$$\beta_1^{(\Delta)} = N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(q^{(22)}_{O_p(N^{-1})})$$

$$= N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta b2)}_{\theta_0 O_p(N^{-1})} + \gamma^{(1)}_{\theta_0} l^{(\Delta\Delta b1)}_{\theta_0 O_p(N^{-1})} + \gamma^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})$$

$$= \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\underset{(\mathrm{A})}{} N\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\{m^{(\Delta)}_{O_p(N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})},\ (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\}'$$

$$+ N\gamma^{(1)}_{\theta_0}\left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'(\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0} - N^{-1}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})} \right.$$

$$\left. + \frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0{}')^{<2>}}_{O_p(1)}\right)(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})})^{<2>} \right\}$$

$$+N\frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}{}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}{}'\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\quad]_{(\mathrm{A})}$$

$$=\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\left\{\left(\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right\}'$$

$$+\frac{\gamma_{\theta_0}^{(1)}}{2}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right\}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})$$

$$-\gamma_{\theta_0}^{(1)}\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}+\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

where $\boldsymbol{\Omega}_{\mathrm{T}}=N\,\mathrm{cov}(\mathbf{p})=\mathrm{diag}(\boldsymbol{\pi}_{\mathrm{T}})-\boldsymbol{\pi}_{\mathrm{T}}\boldsymbol{\pi}_{\mathrm{T}}{}'$ is the N times the covariance matrix of the vector $\mathbf{p}$ of the sample proportions of 2^n response patterns with $\mathrm{E}_{\mathrm{T}\alpha_0}(\mathbf{p})=\boldsymbol{\pi}_{\mathrm{T}}$,

$$\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}=N\mathrm{acov}(\hat{\boldsymbol{\alpha}})=\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\boldsymbol{\Gamma}_{\mathbf{G}_0}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}{}',\ \ \boldsymbol{\Gamma}_{\mathbf{G}_0}\equiv N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}{}'),\ \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}\equiv\frac{\partial\overline{l}_{\theta_0}}{\partial\boldsymbol{\alpha}_0},$$

$$\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}=-\left\{\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\left(\frac{\partial^2\overline{l}_{\boldsymbol{\alpha}_0}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\alpha}_0{}'}\right)\right\}^{-1}\frac{\partial^2\overline{l}_{\boldsymbol{\alpha}_0}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\pi}_{\mathrm{T}}{}'},\quad\frac{\partial^2\overline{l}_{\boldsymbol{\alpha}_0}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}=O(1)\right),$$

$$\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}=\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\left(\frac{\partial^2\overline{l}_{\boldsymbol{\alpha}_0}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\alpha}_0{}'}\right),\ \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}=-\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}.$$

The following expressions and similar ones using partial derivatives of $\boldsymbol{\alpha}_0$ with respect to $\boldsymbol{\pi}_{\mathrm{T}}$, in form, will also be used (see Ogasawara, 2009):

$$\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}=-\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\frac{\partial\overline{l}_{\theta_0}}{\partial\boldsymbol{\alpha}_0}=-\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\frac{\partial^2\overline{l}_{\boldsymbol{\alpha}_0}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})=\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}}),$$ where

$\frac{\partial^2\overline{l}_{\boldsymbol{\alpha}_0}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\boldsymbol{\pi}_{\mathrm{T}}=\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial\overline{l}_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right)=\mathrm{E}_{\theta_0}\left(\frac{\partial\overline{l}_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right)=0$ with $\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)=0$ by assumption/construction.

(a.2) The second asymptotic cumulant

(a.2.1) $\beta_2^{(\Delta)} = N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^2\} = \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{N(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\}$

$$= (\gamma^{(1)}_{\theta_0})^2 \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}' \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(N\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}{}')\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} = (\gamma^{(1)}_{\theta_0})^2 \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}.$$

(a.2.2) $\beta_{\mathrm{H}2}^{(\Delta a)}$

$$= Nn \underset{(\mathrm{A})}{[} \; \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{B})}{\{} (q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})^2 + 2q^{(20)}_{O_p(n^{-1})} q^{(22)}_{O_p(N^{-1})} + 2[\, q^{(10)}_{O_p(n^{-1/2})} q^{(32)}_{O_p(n^{-1/2}N^{-1})}$$

$$+ q^{(11)}_{O_p(N^{-1/2})}\{q^{(31)}_{O_p(n^{-1}N^{-1/2})} - (n^{-1}(\lambda^{-1}_{\theta_0}\eta_{\theta_0})^{(\Delta)})_{O_p(n^{-1}N^{-1/2})}\} \;]\; \underset{(\mathrm{B})}{\}} \; \underset{(\mathrm{A})}{]}_{O(n^{-1}N^{-1})}$$

$$-2(\beta_1^{(0)} + \lambda^{-1}_{\theta_0}\eta_{\theta_0})\beta_1^{(\Delta)}$$

$$= Nn\,\mathrm{E}_{\mathrm{T}}\{(\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta a2)}_{\theta_0} + \gamma^{(1)}_{\theta_0} l^{(\Delta\Delta a1)}_{\theta_0} + \gamma^{(\Delta 1)}_{\theta_0} l^{(1)}_{\theta_0})^2\}_{O_p(n^{-1}N^{-1})}$$

$$\underline{+2(\beta_1^{(0)} + \lambda^{-1}_{\theta_0}\eta_{\theta_0})\beta_1^{(\Delta)}}$$

$$+2Nn\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \; (\gamma^{(1)}_{\theta_0} l^{(1)}_{\theta_0})_{O_p(n^{-1/2})} (\boldsymbol{\gamma}^{(3)}_{\theta_0}{}'\mathbf{l}^{(\Delta b3)}_{\theta_0} + \boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta\Delta b2)}_{\theta_0} + \boldsymbol{\gamma}^{(\Delta 2)}_{\theta_0}{}'\mathbf{l}^{(\Delta a2)}_{\theta_0}$$

$$+ \gamma^{(1)}_{\theta_0} l^{(\Delta\Delta\Delta a1)}_{\theta_0} + \gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta\Delta a1)}_{\theta_0} + \gamma^{(\Delta\Delta 1)}_{\theta_0} l^{(1)}_{\theta_0})_{O_p(n^{-1/2}N^{-1})}$$

$$+(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1/2})}(\; \boldsymbol{\gamma}^{(3)}_{\theta_0}{}'\mathbf{l}^{(\Delta a3)}_{\theta_0} + \boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta\Delta a2)}_{\theta_0} + \boldsymbol{\gamma}^{(\Delta 2)}_{\theta_0}{}'\mathbf{l}^{(2)}_{\theta_0}$$

$$-n^{-1}(\partial\lambda^{-1}_{\theta_0}\eta_{\theta_0} / \partial\boldsymbol{\alpha}_0{}')\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0} \;)_{O_p(n^{-1}N^{-1/2})} \; \underset{(\mathrm{A})}{\}}_{O_p(n^{-1}N^{-1})} \; \underline{-2(\beta_1^{(0)} + \lambda^{-1}_{\theta_0}\eta_{\theta_0})\beta_1^{(\Delta)}}$$

(the underscored terms are canceled)

$$= Nn[\; \boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathrm{E}_{\mathrm{T}}(\mathbf{l}^{(\Delta a2)}_{\theta_0}\mathbf{l}^{(\Delta a2)}_{\theta_0}{}')\boldsymbol{\gamma}^{(2)}_{\theta_0} + (\gamma^{(1)}_{\theta_0})^2\mathrm{E}_{\mathrm{T}}\{(l^{(\Delta\Delta a1)}_{\theta_0})^2\}$$

$$+ \mathrm{E}_{\mathrm{T}}\{(\gamma^{(\Delta 1)}_{\theta_0} l^{(1)}_{\theta_0})^2\} + 2\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathrm{E}_{\mathrm{T}}(\mathbf{l}^{(\Delta a2)}_{\theta_0} l^{(\Delta\Delta a1)}_{\theta_0})\gamma^{(1)}_{\theta_0}$$

$$+ 2\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathrm{E}_{\mathrm{T}}(\mathbf{l}^{(\Delta a2)}_{\theta_0}\gamma^{(\Delta 1)}_{\theta_0} l^{(1)}_{\theta_0}) + 2\gamma^{(1)}_{\theta_0}\mathrm{E}_{\mathrm{T}}(l^{(\Delta\Delta a1)}_{\theta_0}\gamma^{(\Delta 1)}_{\theta_0} l^{(1)}_{\theta_0}) \;]_{O_p(n^{-1}N^{-1})}$$

(the above terms are defined as Terms (1) to (6))

$$+2Nn \underset{(A)}{[} \ \gamma_{\theta_0}^{(1)}\{ \mathrm{E_T}(l_{\theta_0}^{(1)}\mathbf{l}_{\theta_0}^{(\Delta b3)}{'})\boldsymbol{\gamma}_{\theta_0}^{(3)} + \mathrm{E_T}(l_{\theta_0}^{(1)}\mathbf{l}_{\theta_0}^{(\Delta\Delta b2)}{'})\boldsymbol{\gamma}_{\theta_0}^{(2)} + \mathrm{E_T}(l_{\theta_0}^{(1)}\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)}{'}\mathbf{l}_{\theta_0}^{(\Delta a2)})$$

$$+ \mathrm{E_T}(l_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta\Delta a1)})\gamma_{\theta_0}^{(1)} + \mathrm{E_T}(l_{\theta_0}^{(1)}\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta\Delta a1)}) + \mathrm{E_T}(l_{\theta_0}^{(1)}\gamma_{\theta_0}^{(\Delta\Delta 1)}l_{\theta_0}^{(1)})\}_{O_p(n^{-1}N^{-1})}$$

$$+\gamma_{\theta_0}^{(1)}\{ \ \mathrm{E_T}(l_{\theta_0}^{(\Delta 1)}\mathbf{l}_{\theta_0}^{(\Delta a3)}{'})\boldsymbol{\gamma}_{\theta_0}^{(3)} + \mathrm{E_T}(l_{\theta_0}^{(\Delta 1)}\mathbf{l}_{\theta_0}^{(\Delta\Delta a2)}{'})\boldsymbol{\gamma}_{\theta_0}^{(2)} + \mathrm{E_T}(l_{\theta_0}^{(\Delta 1)}\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)}{'}\mathbf{l}_{\theta_0}^{(2)})$$

$$- n^{-1}(\partial\lambda_{\theta_0}^{-1}\eta_{\theta_0} / \partial\boldsymbol{\alpha}_0{'})\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)} \, l_{\theta_0}^{(\Delta 1)}) \ \}_{O_p(n^{-1}N^{-1})} \ \underset{(A)}{]}$$

(the above terms are defined as Terms (7) to (16)).

Term (1): $Nn\,\mathrm{E_T}(\mathbf{l}_{\theta_0}^{(\Delta a2)}\mathbf{l}_{\theta_0}^{(\Delta a2)}{'})$ ($m^{(\Delta)} = 0$ under m.m.)

$$= Nn\mathrm{E_T}\{ \ [m_{O_p(n^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta)}_{O_p(N^{-1/2})}l^{(1)}_{\theta_0 O_p(n^{-1/2})},$$

$$2l^{(1)}_{\theta_0 O_p(n^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}]'_{O_p(n^{-1/2}N^{-1/2})}[\ \cdot\ ]_{O_p(n^{-1/2}N^{-1/2})} \ \}$$

$$\equiv \begin{bmatrix} e_{11} & \text{sym.} \\ e_{21} & e_{22} \end{bmatrix} \text{ with}$$

$$e_{11} = n\,\mathrm{E}_{\mathrm{T}\theta_0}\{(m_{O_p(n^{-1/2})})^2\}N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\}$$

$$+ n\,\mathrm{E}_{\mathrm{T}\theta_0}\{(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2\}N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(m^{(\Delta)}_{O_p(N^{-1/2})})^2\}$$

$$+ 2n\mathrm{E}_{\mathrm{T}\theta_0}(m_{O_p(n^{-1/2})}l^{(1)}_{\theta_0 O_p(n^{-1/2})})N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta)}_{O_p(N^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}),$$

$$e_{21} = 2n\mathrm{E}_{\mathrm{T}\theta_0}(l^{(1)}_{\theta_0 O_p(n^{-1/2})}m_{O_p(n^{-1/2})})N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\}$$

$$+ 2n\,\mathrm{E}_{\mathrm{T}\theta_0}\{(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2\}N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta)}_{O_p(N^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}),$$

$$e_{22} = 4n\,\mathrm{E}_{\mathrm{T}\theta_0}\{(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2\}N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\},$$

where the expectations associated with $O_p(n^{-1/2})$ are known. The other expectations are

$$N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\} = \boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}{'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0},$$

$N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(m^{(\Delta)}_{O_p(N^{-1/2})})^2\}$ (this term is 0 under m.m.))

$$=\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\},$$

$$N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta)}_{O_p(N^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})=\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}.$$

Term (2): $Nn\mathrm{E}_{\mathrm{T}}\{(l^{(\Delta\Delta a1)}_{\theta_0})^2\}$

$$=\mathrm{tr}\left[n\,\mathrm{E}_{\mathrm{T}\theta_0}\left\{\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0}-\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)\right)\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}-\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)\right)\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\right].$$

In Term (2),

$$n\mathrm{E}_{\mathrm{T}\theta_0}\{(\cdot)(\cdot)\}=n\,\mathrm{cov}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right)$$

$$=n^{-1}\sum_{k=1}^{n}\begin{pmatrix}-\frac{1}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0'}+\frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\\ -\frac{1}{Q_k^2}\frac{\partial Q_k}{\partial\theta_0}\frac{\partial Q_k}{\partial\boldsymbol{\alpha}_0'}+\frac{1}{Q_k}\frac{\partial^2 Q_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\end{pmatrix}'\begin{bmatrix}P_{\mathrm{T}k}Q_{\mathrm{T}k} & -P_{\mathrm{T}k}Q_{\mathrm{T}k}\\ -P_{\mathrm{T}k}Q_{\mathrm{T}k} & P_{\mathrm{T}k}Q_{\mathrm{T}k}\end{bmatrix}(\cdot)$$

$$=n^{-1}\sum_{k=1}^{n}P_{\mathrm{T}k}Q_{\mathrm{T}k}\sum_{P(Q)}^{2}\left(-\frac{1}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}+\frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right)$$
$$\times\left(\frac{P_k-Q_k}{P_k^2Q_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0'}+\frac{1}{P_kQ_k}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right),$$

where $\sum_{P(Q)}^{2}$ indicates the sum of two terms exchanging *P* and *Q*. The above result is alternatively expressed as

$$=n^{-1}\sum_{k=1}^{n}P_{\mathrm{T}k}Q_{\mathrm{T}k}\left(\frac{P_k-Q_k}{P_k^2Q_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}+\frac{1}{P_kQ_k}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right)(\cdot)'.$$

Term (3): $Nn\,\mathrm{E}_{\mathrm{T}}\{(\gamma^{(\Delta 1)}_{\theta_0}l^{(1)}_{\theta_0})^2\}$

$$N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\gamma_{\theta_0}^{(\Delta 1)})^2\} n\,\mathrm{E}_{\mathrm{T}\theta_0}\{(l_{\theta_0}^{(1)})^2\}$$

$$= \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \lambda_{\theta_0}^{(11)} \quad (\lambda_{\theta_0}^{(11)} \equiv n\mathrm{E}_{\mathrm{T}\theta_0}\{(l_{\theta_0}^{(1)})^2\}).$$

Term (4): $Nn\,\mathrm{E}_{\mathrm{T}}(\mathbf{l}_{\theta_0}^{(\Delta a2)} l_{\theta_0}^{(\Delta\Delta a1)})$ ($m^{(\Delta)} = 0$ under m.m.)

$$= Nn\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \; [m_{O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta)}_{O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})},\; 2 l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}]'$$

$$\times \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial\, \boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0\, \boldsymbol{\alpha}_0}{}' \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \underset{(\mathrm{A})}{\}}$$

$=[e_1,\ e_2]'$, where

$$e_1 = n\mathrm{E}_{\mathrm{T}\theta_0} \left\{ m_{O_p(n^{-1/2})} \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial\, \boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0\, \boldsymbol{\alpha}_0}{}' \right)_{O_p(n^{-1/2})} \right\}$$

$$\times N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}\}$$

$$+ n\mathrm{E}_{\mathrm{T}\theta_0} \left\{ l^{(1)}_{\theta_0 O_p(n^{-1/2})} \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial\, \boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0\, \boldsymbol{\alpha}_0}{}' \right)_{O_p(n^{-1/2})} \right\}$$

$$\times N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} m^{(\Delta)}_{O_p(N^{-1/2})}\} \;\; (\text{the last term is } 0 \text{ under m.m})$$

$$= n \operatorname{cov}\left(m_{O_p(n^{-1/2})},\ \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial\, \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0\, \boldsymbol{\alpha}_0}$$

$$+ n \operatorname{cov}\left(l^{(1)}_{\theta_0 O_p(n^{-1/2})},\ \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial\, \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial\, \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial\, \boldsymbol{\alpha}_0} \right\}$$

(the last term is 0 under m.m.),

$$e_2 = 2n\mathrm{E}_{\mathrm{T}\theta_0}\left\{ l^{(1)}_{\theta_0 O_p(n^{-1/2})} \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \right)_{O_p(n^{-1/2})} \right\}$$

$$\times N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}\}$$

$$= 2n\,\mathrm{cov}\left(l^{(1)}_{\theta_0 O_p(n^{-1/2})}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}.$$

In the above results,

$$n\,\mathrm{cov}\left(m_{O_p(n^{-1/2})}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0} \right) = n^{-1}\sum_{k=1}^{n} P_{\mathrm{T}k} Q_{\mathrm{T}k} \sum_{P(Q)}^{2} \left\{ -\frac{1}{P_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2 + \frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0^2} \right\}$$

$$\times \left(\frac{P_k - Q_k}{P_k^2 Q_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} + \frac{1}{P_k Q_k}\frac{\partial^2 P_k}{\partial\theta_0 \partial\boldsymbol{\alpha}_0} \right)$$

(or alternatively)

$$= n^{-1}\sum_{k=1}^{n} P_{\mathrm{T}k} Q_{\mathrm{T}k} \left\{ \frac{P_k - Q_k}{P_k^2 Q_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2 + \frac{1}{P_k Q_k}\frac{\partial^2 P_k}{\partial\theta_0^2} \right\}$$

$$\times \left(\frac{P_k - Q_k}{P_k^2 Q_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} + \frac{1}{P_k Q_k}\frac{\partial^2 P_k}{\partial\theta_0 \partial\boldsymbol{\alpha}_0} \right),$$

$$n\,\mathrm{cov}\left(l^{(1)}_{\theta_0 O_p(n^{-1/2})}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0} \right)$$

$$= n^{-1}\sum_{k=1}^{n} \frac{P_{\mathrm{T}k} Q_{\mathrm{T}k}}{P_k Q_k} \sum_{P(Q)}^{2} \left(-\frac{1}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} + \frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0 \partial\boldsymbol{\alpha}_0} \right)\frac{\partial P_k}{\partial\theta_0}$$

(or alternatively)

$$= n^{-1}\sum_{k=1}^{n} \frac{P_{\mathrm{T}k} Q_{\mathrm{T}k}}{P_k Q_k}\frac{\partial P_k}{\partial\theta_0} \left(\frac{P_k - Q_k}{P_k^2 Q_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} + \frac{1}{P_k Q_k}\frac{\partial^2 P_k}{\partial\theta_0 \partial\boldsymbol{\alpha}_0} \right).$$

Term (5): $Nn\,\mathrm{E}_{\mathrm{T}}(\mathbf{l}^{(\Delta a 2)}_{\theta_0}\gamma^{(\Delta 1)}_{\theta_0} l^{(1)}_{\theta_0})$ ($m^{(\Delta)} = 0$ under m.m.)

$$= Nn\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \ [m_{O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta)}_{O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})}, \ 2 l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}]'$$

$$\times \left(\frac{\partial \gamma^{(1)}_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0} \right)_{O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})} \underset{(\mathrm{A})}{\}}$$

$= [e_1, \ e_2]'$, where

$$e_1 = n \operatorname{cov}\left(m_{O_p(n^{-1/2})}, l^{(1)}_{\theta_0 O_p(n^{-1/2})} \right) \frac{\partial \gamma^{(1)}_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ \lambda^{(11)}_{\theta_0} \frac{\partial \gamma^{(1)}_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\}$$

(the last term is 0 under m.m.),

$$e_2 = 2 \lambda^{(11)}_{\theta_0} \frac{\partial \gamma^{(1)}_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} .$$

Term (6): $Nn\,\mathrm{E}_{\mathrm{T}}(l^{(\Delta\Delta a 1)}_{\theta_0} \gamma^{(\Delta 1)}_{\theta_0} l^{(1)}_{\theta_0})$

$$= n\mathrm{E}_{\mathrm{T}\theta_0} \left\{ \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \right)_{O_p(n^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})} \right\}$$

$$\times N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} \left\{ (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})} \left(\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0} ' \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma^{(1)}_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right)_{O_p(N^{-1/2})} \right\},$$

$$= n \operatorname{cov}\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '}, l^{(1)}_{\theta_0} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma^{(1)}_{\theta_0}}{\partial \boldsymbol{\alpha}_0}$$, where $n \operatorname{cov}(\cdot)$ was given earlier.

(the second half)

Term (7): $Nn\,\mathrm{E}_{\mathrm{T}}(l^{(1)}_{\theta_0} \mathbf{l}^{(\Delta b 3)}_{\theta_0})$ ($m^{(\Delta)} = m^{(\Delta 3)} = 0$ under m.m.)

$$= Nn\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \ l^{(1)}_{\theta_0 O_p(n^{-1/2})}[2m_{O_p(n^{-1/2})} m^{(\Delta)}_{O_p(N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + (m^{(\Delta)}_{O_p(N^{-1/2})})^2 l^{(1)}_{\theta_0 O_p(n^{-1/2})},$$

$$2m^{(\Delta)}_{O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m_{O_p(n^{-1/2})} (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2,$$

$$2m^{(\Delta 3)}_{O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(3)}_{O_p(n^{-1/2})} (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2,$$

$$3(l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2 l^{(1)}_{\theta_0 O_p(n^{-1/2})}, \ (0, \ 0) \ ]'_{O_p(n^{-1/2}N^{-1})} \ \underset{(\mathrm{A})}{\}}$$

$$= [2n\,\mathrm{E}_{\mathrm{T}\theta_0}(l^{(1)}_{\theta_0} m) \ N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta)} l^{(1)}_{\theta_0}) + \lambda^{(11)}_{\theta_0} N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(m^{(\Delta)})^2\},$$

$$2\lambda^{(11)}_{\theta_0} N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta)} l^{(\Delta 1)}_{\theta_0}) + n\,\mathrm{E}_{\mathrm{T}\theta_0}(l^{(1)}_{\theta_0} m)\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$2\lambda^{(11)}_{\theta_0} N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta 3)} l^{(\Delta 1)}_{\theta_0}) + n\,\mathrm{E}_{\mathrm{T}\theta_0}(l^{(1)}_{\theta_0} m^{(3)})\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$3\lambda^{(11)}_{\theta_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}, (0, 0) \quad]',$$

where

$$N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta)} l^{(\Delta 1)}_{\theta_0}) = \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right\}_{O(1)} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

(0 under m.m.),

$$N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(m^{(\Delta)})^2\} = \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right\}_{O(1)} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\{\cdot\}' \ \text{(0 under m.m.)},$$

$$N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta 3)} l^{(\Delta 1)}_{\theta_0}) = \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial \theta_0^3 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial}{\partial \boldsymbol{\alpha}_0{}'} \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}_{O(1)} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

(0 under m.m.).

For $n\,\mathrm{E}_{\mathrm{T}\theta_0}(l^{(1)}_{\theta_0} m^{(3)})$, see Ogasawara (2012a, Appendix).

Term (8): $Nn\,\mathrm{E}_{\mathrm{T}}(l^{(1)}_{\theta_0}\mathbf{l}^{(\Delta\Delta b 2)}_{\theta_0})$ ($m^{(\Delta)} = m^{(\Delta\Delta b)} = 0$ and $m^{(\Delta\Delta a)}$ is non-zero under m.m.)

$$= Nn\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}\; l^{(1)}_{\theta_0 O_p(n^{-1/2})}[m_{O_p(n^{-1/2})} l^{(\Delta\Delta b1)}_{\theta_0 O_p(N^{-1})} + m^{(\Delta)}_{O_p(N^{-1/2})} l^{(\Delta\Delta a1)}_{\theta_0 O_p(n^{-1/2}N^{-1/2})}$$

$$+ m^{(\Delta\Delta a)}_{O_p(n^{-1/2}N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta\Delta b)}_{O_p(N^{-1})} l^{(1)}_{\theta_0 O_p(n^{-1/2})},$$

$$2 l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta\Delta b1)}_{\theta_0 O_p(N^{-1})} + 2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} l^{(\Delta\Delta a1)}_{\theta_0 O_p(n^{-1/2}N^{-1/2})}]'_{O_p(n^{-1/2}N^{-1})} \underset{(\mathrm{A})}{\}}$$

$$= [\; n\mathrm{E}_{\mathrm{T}\theta_0}(l^{(1)}_{\theta_0} m)\; N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l^{(\Delta\Delta b1)}_{\theta_0}) + Nn\mathrm{E}_{\mathrm{T}}(l^{(1)}_{\theta_0} m^{(\Delta)} l^{(\Delta\Delta a1)}_{\theta_0})$$

$$+ Nn\mathrm{E}_{\mathrm{T}}(l^{(1)}_{\theta_0} m^{(\Delta\Delta a)} l^{(\Delta 1)}_{\theta_0}) + \lambda^{(11)}_{\theta_0}\; N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(m^{(\Delta\Delta b)}),$$

$$2\lambda^{(11)}_{\theta_0}\; N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l^{(\Delta\Delta b1)}_{\theta_0}) + 2Nn\mathrm{E}_{\mathrm{T}}(l^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0} l^{(\Delta\Delta a1)}_{\theta_0})\quad]',$$

where

$$N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l^{(\Delta\Delta b1)}_{\theta_0}) = \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\left(\frac{1}{2}\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}}) - \boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right)$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0\partial(\boldsymbol{\alpha}_0{}')^{<2>}}\right)\mathrm{vec}(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}),$$

$$Nn\mathrm{E}_{\mathrm{T}}(l^{(1)}_{\theta_0} m^{(\Delta)} l^{(\Delta\Delta a1)}_{\theta_0})$$

$$= Nn\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{[}\; l^{(1)}_{\theta_0 O_p(n^{-1/2})}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}_{O(1)} (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})}$$

$$\times\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})}\quad \underset{(\mathrm{A})}{]}$$

$$= n\;\mathrm{cov}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'},\; l^{(1)}_{\theta_0}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\}$$

with $n\;\mathrm{cov}(\cdot\,,\,\cdot)$ given earlier,

$$NnE_{T}(l_{\theta_0}^{(1)} m^{(\Delta\Delta a)} l_{\theta_0}^{(\Delta 1)})$$

$$= NnE_{T}\underset{(A)}{\{}\; l^{(1)}_{\theta_0 O_p(n^{-1/2})}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'} - E_{T\theta_0}(\cdot)\right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})}$$

$$\times \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})}\underset{(A)}{\}}$$

$$= n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$= n^{-1}\sum_{k=1}^{n}\frac{P_{Tk}Q_{Tk}}{P_kQ_k}\sum_{P(Q)}^{2}\left[\left\{\frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2 - \frac{1}{P_k^2}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0'}\right.$$

$$\left. -\frac{2}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0'} + \frac{1}{P_k}\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right]\frac{\partial P_k}{\partial\theta_0}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

$$NE_{T\boldsymbol{\alpha}_0}(m^{(\Delta\Delta b)}) = \left\{E_{T\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}_{O(1)}$$

$$\times\left\{\frac{1}{2}\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_T')^{<2>}}\mathrm{vec}(\boldsymbol{\Omega}_T) - \boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right\}$$

$$+\frac{1}{2}\left\{E_{T\theta_0}\left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0^2\partial(\boldsymbol{\alpha}_0')^{<2>}}\right) - \frac{\partial^2\lambda_{\theta_0}}{(\partial\boldsymbol{\alpha}_0')^{<2>}}\right\}_{O(1)} \mathrm{vec}(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}),$$

$$NnE_{T}(l_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta\Delta a1)}) = NnE_{T}\underset{(A)}{\{}\; l^{(1)}_{\theta_0 O_p(n^{-1/2})}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})}$$

$$\times\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'} - \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})}\underset{(A)}{\}}$$

$$= n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$ with $n\,\mathrm{cov}(\cdot\,,\cdot)$ given earlier.

Term (9): $Nn\,\mathrm{E}_{\mathrm{T}}(l_{\theta_0}^{(1)}\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)}{}'\mathbf{l}_{\theta_0}^{(\Delta a2)})$ ($m^{(\Delta)}=0$ under m.m.)

$$= Nn\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}\; l_{\theta_0 O_p(n^{-1/2})}^{(1)}\left(\frac{\partial\boldsymbol{\gamma}_{\theta_0}^{(2)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}\right)'_{O_p(N^{-1/2})}$$

$$\times[m_{O_p(n^{-1/2})}l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}+m_{O_p(N^{-1/2})}^{(\Delta)}l_{\theta_0 O_p(n^{-1/2})}^{(1)},2l_{\theta_0 O_p(n^{-1/2})}^{(1)}l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}]'_{O_p(n^{-1/2}N^{-1/2})}\underset{(\mathrm{A})}{\}}$$

$$= n\,\mathrm{cov}(l_{\theta_0}^{(1)},m)\frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}+\lambda_{\theta_0}^{(11)}\frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\,\boldsymbol{\alpha}_0}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\,\boldsymbol{\alpha}_0}\right\}_{O(1)}+2\lambda_{\theta_0}^{(11)}\frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_2}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}$$

(the second last term is 0 under m.m.).

Term (10): $Nn\mathrm{E}_{\mathrm{T}}(l_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta\Delta a1)})$

$$= Nn\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{[}\;\; l_{\theta_0 O_p(n^{-1/2})}^{(1)}\underset{(\mathrm{B})}{[}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\,\boldsymbol{\alpha}_0{}'}-\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}{}'\right)_{O_p(n^{-1/2})}$$

$$\times(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)}-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+\frac{1}{2}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\,\boldsymbol{\alpha}_0{}')^{<2>}}-\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)\right)_{O_p(n^{-1/2})}\{(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}\}_{O_p(N^{-1})}\;\underset{(\mathrm{B})}{]}_{O_p(n^{-1/2}N^{-1})}\;\underset{(\mathrm{A})}{]}$$

$$= n\,\mathrm{cov}\left(l_{\theta_0}^{(1)},\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\,\boldsymbol{\alpha}_0{}'}\right)\left\{\frac{1}{2}\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\,\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})-\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right\}$$

$$+\frac{1}{2}n\,\mathrm{cov}\left(l_{\theta_0}^{(1)},\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\,\boldsymbol{\alpha}_0{}')^{<2>}}\right)\mathrm{vec}(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}),$$

where

$$n\,\mathrm{cov}\left(l_{\theta_0}^{(1)},\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<2>}}\right)=n^{-1}\sum_{k=1}^{n}\frac{P_{\mathrm{T}k}Q_{\mathrm{T}k}}{P_kQ_k}\sum_{P(Q)}^{2}\left\{\frac{2}{P_k^3}\frac{\partial P_k}{\partial\theta_0}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right)^{<2>}\right.$$

$$\left.-\frac{1}{P_k^2}\left(\sum_{\otimes}^{2}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}+\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_0)^{<2>}}\right)+\frac{1}{P_k}\frac{\partial^3 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<2>}}\right\}\frac{\partial P_k}{\partial\theta_0}.$$

Term (11): $NnE_{\mathrm{T}}(l_{\theta_0}^{(1)}\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta\Delta a1)})$

$$=NnE_{\mathrm{T}}\underset{(\mathrm{A})}{[}\;l_{\theta_0 O_p(n^{-1/2})}^{(1)}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'}-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\right)_{O_p(n^{-1/2})}$$

$$\times(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})}\underset{(\mathrm{A})}{]}$$

$$=n\,\mathrm{cov}\left(l_{\theta_0}^{(1)},\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}.$$

Term (12): $NnE_{\mathrm{T}}(l_{\theta_0}^{(1)}\gamma_{\theta_0}^{(\Delta\Delta 1)}l_{\theta_0}^{(1)})$

$$=\lambda_{\theta_0}^{(11)}NE_{\mathrm{T}\boldsymbol{\alpha}_0}\left\{\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)}-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})+\frac{1}{2}\frac{\partial^2\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0{}')^{<2>}}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}\right\}$$

$$=\lambda_{\theta_0}^{(11)}\left[\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\left\{\frac{1}{2}\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})-\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right\}+\frac{1}{2}\frac{\partial^2\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\mathrm{vec}(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0})\right].$$

Term (13): $NnE_{\mathrm{T}}(l_{\theta_0}^{(\Delta 1)}\mathbf{l}_{\theta_0}^{(\Delta a3)})$ ($m^{(\Delta)}=m^{(\Delta 3)}=0$ under m.m.)

$$=NnE_{\mathrm{T}}\underset{(\mathrm{A})}{\{}\;\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})}$$

$$\times[\;2m_{O_p(n^{-1/2})}m_{O_p(N^{-1/2})}^{(\Delta)}l_{\theta_0 O_p(n^{-1/2})}^{(1)}+(m^2)_{O_p(n^{-1})}l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)},$$

$$2m_{O_p(n^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta)}_{O_p(N^{-1/2})} (l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2,$$
$$2m^{(3)}_{O_p(n^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta 3)}_{O_p(N^{-1/2})} (l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2,$$
$$3(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}, n^{-1}(m^{(\Delta)},\ l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1/2})}\ ]'_{O_p(n^{-1}N^{-1/2})}\ \}_{(\mathrm{A})}$$
$$= \underset{(\mathrm{A})}{[}\ 2n\mathrm{E}_{\mathrm{T}\theta_0}(ml^{(1)}_{\theta_0})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\}$$
$$+ n\mathrm{E}_{\mathrm{T}\theta_0}(m^2)\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$
$$2n\mathrm{E}_{\mathrm{T}\theta_0}(ml^{(1)}_{\theta_0})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} + \lambda^{(11)}_{\theta_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\},$$
$$2n\,\mathrm{cov}(m^{(3)},l^{(1)}_{\theta_0})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$
$$+\lambda^{(11)}_{\theta_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4\overline{l}_{\theta_0}}{\partial\theta_0^3\partial\boldsymbol{\alpha}_0}\right)-\frac{\partial}{\partial\boldsymbol{\alpha}_0}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^3}\right)\right\},$$
$$3\lambda^{(11)}_{\theta_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$
$$\left(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\},\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)\quad \underset{(\mathrm{A})}{]}{}',$$

where $n\,\mathrm{cov}(m^{(3)},l^{(1)}_{\theta_0}) = n\,\mathrm{cov}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^3},\frac{\partial\overline{l}_{\theta_0}}{\partial\theta_0}\right)$ was mentioned earlier.

Term (14): $Nn\mathrm{E}_{\mathrm{T}}(l^{(\Delta 1)}_{\theta_0}\mathbf{l}^{(\Delta\Delta a2)}_{\theta_0}{}')$

$$= Nn\mathrm{E}_{\mathrm{T}}\{\ l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}[m_{O_p(n^{-1/2})} l^{(\Delta\Delta a1)}_{\theta_0 O_p(n^{-1/2}N^{-1/2})} + m^{(\Delta\Delta a)}_{O_p(n^{-1/2}N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})},$$
$$2l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta\Delta a1)}_{\theta_0 O_p(n^{-1/2}N^{-1/2})}\ ]\ \},$$

where

$$= Nn\mathrm{E}_{\mathrm{T}}(\, l_{\theta_0}^{(\Delta 1)} m l_{\theta_0}^{(\Delta\Delta a1)})$$

$$= Nn\mathrm{E}_{\mathrm{T}}\left\{ l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} m_{O_p(n^{-1/2})} \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\, \boldsymbol{\alpha}_0\, '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}\, ' \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \right\}$$

$$= n\,\mathrm{cov}\left(m, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\, \boldsymbol{\alpha}_0\, '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \quad \text{with}$$

$$n\,\mathrm{cov}\left(m, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\, \boldsymbol{\alpha}_0} \right) = n^{-1} \sum_{k=1}^{n} P_{\mathrm{T}k} Q_{\mathrm{T}k} \sum_{P(Q)}^{2} \left\{ -\frac{1}{P_k^2} \left(\frac{\partial P_k}{\partial\theta_0} \right)^2 + \frac{1}{P_k} \frac{\partial^2 P_k}{\partial\theta_0^2} \right\}$$

$$\times \left(\frac{P_k - Q_k}{P_k^2 Q_k^2} \frac{\partial P_k}{\partial\theta_0} \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} + \frac{1}{P_k Q_k} \frac{\partial^2 P_k}{\partial\theta_0 \partial \boldsymbol{\alpha}_0} \right),$$

$$Nn\mathrm{E}_{\mathrm{T}}(l_{\theta_0}^{(\Delta 1)} m^{(\Delta\Delta a)} l_{\theta_0}^{(1)})$$

$$= Nn\mathrm{E}_{\mathrm{T}}\left\{ l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial\, \boldsymbol{\alpha}_0} - \mathrm{E}_{\mathrm{T}\theta_0}(\cdot) \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} l_{\theta_0 O_p(n^{-1/2})}^{(1)} \right\}$$

$$= n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial\, \boldsymbol{\alpha}_0\, '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$2Nn\mathrm{E}_{\mathrm{T}}(\, l_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta a1)}) = 2Nn\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \; l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_{0 O_p(n^{-1/2})}}^{(1)}$$

$$\times \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\, \boldsymbol{\alpha}_0\, '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}\, ' \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \quad \underset{(\mathrm{A})}{\}}$$

$$= 2n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\, \boldsymbol{\alpha}_0\, '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}\, .$$

Term (15): $Nn\mathrm{E}_{\mathrm{T}}(l_{\theta_0}^{(\Delta 1)} \boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)}\, ' \mathbf{l}_{\theta_0}^{(2)})$

$$= Nn\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{[} \ l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} \left\{ \frac{\partial \boldsymbol{\gamma}^{(2)}_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})} \right\}' \{ml^{(1)}_{\theta_0}, (l^{(1)}_{\theta_0})^2\}'_{O_p(n^{-1})} \ \underset{(\mathrm{A})}{]}$$

$$= \{n\mathrm{E}_{\mathrm{T}\theta_0}(ml^{(1)}_{\theta_0}), \lambda^{(11)}_{\theta_0}\} \frac{\partial \boldsymbol{\gamma}^{(2)}_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}.$$

Term (16):

$$-\gamma^{(1)}_{\theta_0} \frac{\partial \lambda^{-1}_{\theta_0} \eta_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0} l^{(\Delta 1)}_{\theta_0}) = -\gamma^{(1)}_{\theta_0} \frac{\partial \lambda^{-1}_{\theta_0} \eta_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

(a.2.3) $\beta^{(\Delta b)}_{\mathrm{H2}}$

$$= N^2[\ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(22)}_{O_p(N^{-1})})^2 + 2q^{(11)}_{O_p(N^{-1/2})} q^{(22)}_{O_p(N^{-1})}$$

$$+ 2q^{(11)}_{O_p(N^{-1/2})} q^{(33)}_{O_p(N^{-3/2})}\}\ ]_{O(N^{-2})} - (\beta^{(\Delta)}_1)^2$$

$$= N^2 \underset{(\mathrm{A})}{[} \ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} \underset{(\mathrm{B})}{\{} \ ((\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta b2)}_{\theta_0} + \gamma^{(1)}_{\theta_0} l^{(\Delta\Delta b1)}_{\theta_0} + \gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1})})^2$$

$$+ 2(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1/2})} (\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta b2)}_{\theta_0} + \gamma^{(1)}_{\theta_0} l^{(\Delta\Delta b1)}_{\theta_0} + \gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1})}$$

$$+2(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1/2})} (\boldsymbol{\gamma}^{(3)}_{\theta_0}{}'\mathbf{l}^{(\Delta c3)}_{\theta_0} + \boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta\Delta c2)}_{\theta_0} + \boldsymbol{\gamma}^{(\Delta 2)}_{\theta_0}{}'\mathbf{l}^{(\Delta b2)}_{\theta_0}$$

$$+ \gamma^{(1)}_{\theta_0} l^{(\Delta\Delta\Delta b1)}_{\theta_0} + \gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta\Delta b1)}_{\theta_0} + \gamma^{(\Delta\Delta 1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-3/2})} \ \underset{(\mathrm{B})}{\}} \ \underset{(\mathrm{A})}{]}_{O(N^{-2})}$$

$$-(\beta^{(\Delta)}_1)^2$$

$$= N^2 \underset{(\mathrm{A})}{[} \ \boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}^{(\Delta b2)}_{\theta_0} \mathbf{l}^{(\Delta b2)}_{\theta_0}{}')\boldsymbol{\gamma}^{(2)}_{\theta_0} + (\gamma^{(1)}_{\theta_0})^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(l^{(\Delta\Delta b1)}_{\theta_0})^2\}$$

$$+ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})^2\} + 2\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}^{(\Delta b2)}_{\theta_0} l^{(\Delta\Delta b1)}_{\theta_0})\gamma^{(1)}_{\theta_0}$$

$$+2\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}^{(\Delta b2)}_{\theta_0} \gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0}) + 2\gamma^{(1)}_{\theta_0}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l^{(\Delta\Delta b1)}_{\theta_0} \gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0}) \ \underset{(\mathrm{A})}{]}$$

(the above results are defined as Terms (1) to (6))

$$+2N^2[\ \gamma^{(1)}_{\theta_0}\{\ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l^{(\Delta 1)}_{\theta_0} \mathbf{l}^{(\Delta b2)}_{\theta_0}{}')\boldsymbol{\gamma}^{(2)}_{\theta_0} + \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l^{(\Delta 1)}_{\theta_0} l^{(\Delta\Delta b1)}_{\theta_0})\gamma^{(1)}_{\theta_0}$$

$$+ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l^{(\Delta 1)}_{\theta_0} \gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})\ \}_{O(N^{-2})}\ ]$$

$$+2N^2[\ \gamma_{\theta_0}^{(1)}\{\ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)}\mathbf{l}_{\theta_0}^{(\Delta c3)}{}')\boldsymbol{\gamma}_{\theta_0}^{(3)} + \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)}\mathbf{l}_{\theta_0}^{(\Delta\Delta c2)}{}')\boldsymbol{\gamma}_{\theta_0}^{(2)}$$

$$+ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)}\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)}{}'\mathbf{l}_{\theta_0}^{(\Delta b2)}) + \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta\Delta\Delta b1)})\gamma_{\theta_0}^{(1)}$$

$$+ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)}\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta\Delta b1)}) + \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)}\gamma_{\theta_0}^{(\Delta\Delta 1)}l_{\theta_0}^{(\Delta 1)})\ \}_{O(N^{-2})}\ ] - (\beta_1^{(\Delta)})^2$$

(the above results except $-(\beta_1^{(\Delta)})^2$ are defined as Terms (7) to (15)).

Term (1): $N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}_{\theta_0}^{(\Delta b2)}\mathbf{l}_{\theta_0}^{(\Delta b2)}{}')$ ($m^{(\Delta)} = 0$ under m.m.)

$$= N^2\begin{bmatrix} \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(m^{(\Delta)}l_{\theta_0}^{(\Delta 1)})^2\} & \text{sym.} \\ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{m^{(\Delta)}(l_{\theta_0}^{(\Delta 1)})^3\} & \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(l_{\theta_0}^{(\Delta 1)})^4\} \end{bmatrix},$$

where

$$N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(m^{(\Delta)}l_{\theta_0}^{(\Delta 1)})^2\} = \left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+2\left[\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right]^2 + O(N^{-1}),$$

$$N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{m^{(\Delta)}(l_{\theta_0}^{(\Delta 1)})^3\} = 3\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$\times\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} + O(N^{-1}),$$

$$N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(l_{\theta_0}^{(\Delta 1)})^4\} = 3(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2 + O(N^{-1}).$$

Term (2): $N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(l_{\theta_0}^{(\Delta\Delta b1)})^2\}$

$$= N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\left\{\left[\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})\right.\right.$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}\Bigg]^2\ \Bigg\}$$

$$=\frac{1}{4}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\sum_{i^*,j,k,l^*=1}^{2^n}\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}i^*}\partial\pi_{\mathrm{T}j}}\{(\boldsymbol{\Omega}_\mathrm{T})_{i^*j}(\boldsymbol{\Omega}_\mathrm{T})_{kl^*}$$

$$+(\boldsymbol{\Omega}_\mathrm{T})_{i^*k}(\boldsymbol{\Omega}_\mathrm{T})_{jl^*}+(\boldsymbol{\Omega}_\mathrm{T})_{i^*l^*}(\boldsymbol{\Omega}_\mathrm{T})_{jk}\}\frac{\partial^2\boldsymbol{\alpha}_0'}{\partial\pi_{\mathrm{T}k}\partial\pi_{\mathrm{T}l^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_\mathrm{T}')^{<2>}}\mathrm{vec}(\boldsymbol{\Omega}_\mathrm{T})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}+(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})^2$$

$$+\frac{1}{2}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\sum_{i^*,j,k,l^*=1}^{2^n}\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}i^*}\partial\pi_{\mathrm{T}j}}\{(\boldsymbol{\Omega}_\mathrm{T})_{i^*j}(\boldsymbol{\Omega}_\mathrm{T})_{kl^*}$$

$$+(\boldsymbol{\Omega}_\mathrm{T})_{i^*k}(\boldsymbol{\Omega}_\mathrm{T})_{jl^*}+(\boldsymbol{\Omega}_\mathrm{T})_{i^*l^*}(\boldsymbol{\Omega}_\mathrm{T})_{jk}\}$$

$$\times\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}k}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\right)$$

$$+\frac{1}{4}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\sum_{i^*,j,k,l^*=1}^{2^n}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}i^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}j}}\right)$$

$$\times\{(\boldsymbol{\Omega}_\mathrm{T})_{i^*j}(\boldsymbol{\Omega}_\mathrm{T})_{kl^*}+(\boldsymbol{\Omega}_\mathrm{T})_{i^*k}(\boldsymbol{\Omega}_\mathrm{T})_{jl^*}+(\boldsymbol{\Omega}_\mathrm{T})_{i^*l^*}(\boldsymbol{\Omega}_\mathrm{T})_{jk}\}$$

$$\times\left(\frac{\partial\boldsymbol{\alpha}_0'}{\partial\pi_{\mathrm{T}k}}\otimes\frac{\partial\boldsymbol{\alpha}_0'}{\partial\pi_{\mathrm{T}l^*}}\right)\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<2>}}\right)$$

$$-\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_\mathrm{T}'}\right)^{<2>}\mathrm{vec}(\boldsymbol{\Omega}_\mathrm{T})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}+O(N^{-1}).$$

Term (3): $N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta 1)})^2\}$

$$=N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\left\{\left(\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}\right)^2\right\}$$

$$= \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} + 2 \left(\frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right)^2 + O(N^{-1}).$$

Term (4): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} (\mathbf{l}_{\theta_0}^{(\Delta b2)} l_{\theta_0}^{(\Delta\Delta b1)})$

$$= N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} \underset{(\mathrm{A})}{[} \{ m^{(\Delta)} l_{\theta_0}^{(\Delta 1)},\ (l_{\theta_0}^{(\Delta 1)})^2 \}' \left[\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} '(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) \right.$$
$$\left. + \frac{1}{2} \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>} \right] \underset{(\mathrm{A})}{]},$$

where the first element of the above vector is

$$\left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \left[\sum_{i^*, j, k, l^* = 1}^{2^n} \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}j}} \right.$$
$$\times \{ (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* j} (\boldsymbol{\Omega}_{\mathrm{T}})_{k l^*} + (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* k} (\boldsymbol{\Omega}_{\mathrm{T}})_{j l^*} + (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* l^*} (\boldsymbol{\Omega}_{\mathrm{T}})_{jk} \}$$
$$\times \frac{1}{2} \left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial^2 \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k} \partial \pi_{\mathrm{T}l^*}} + \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) \left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k}} \otimes \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}l^*}} \right) \right\}$$
$$\left. - \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0} \right] + O(N^{-1}),$$

and the second element of the vector is

$$\sum_{i^*, j, k, l^* = 1}^{2^n} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}j}}$$
$$\times \{ (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* j} (\boldsymbol{\Omega}_{\mathrm{T}})_{k l^*} + (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* k} (\boldsymbol{\Omega}_{\mathrm{T}})_{j l^*} + (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* l^*} (\boldsymbol{\Omega}_{\mathrm{T}})_{jk} \}$$
$$\times \frac{1}{2} \left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial^2 \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k} \partial \pi_{\mathrm{T}l^*}} + \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) \left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k}} \otimes \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}l^*}} \right) \right\}$$
$$- \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0} + O(N^{-1}).$$

Term (5): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}_{\theta_0}^{(\Delta b2)} \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta 1)})$

$$= N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\left[\{m^{(\Delta)} l_{\theta_0}^{(\Delta 1)},\ (l_{\theta_0}^{(\Delta 1)})^2\}' l_{\theta_0}^{(\Delta 1)} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)} \right]$$

$$= \underset{(\mathrm{A})}{[} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0'} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left(\frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right.$$

$$\left. + 2\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right),$$

$$3\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \quad \underset{(\mathrm{A})}{]}' + O(N^{-1}).$$

Term (6): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta\Delta b1)} \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta 1)})$

$$= N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} \left\{ \left[\quad \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) \right. \right.$$

$$\left. \left. + \frac{1}{2} \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0')^{<2>}} \right) (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>} \right] \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)} \right\}$$

$$= \sum_{i^*, j, k, l^* = 1}^{2^n} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0'} \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}j}}$$

$$\times \{ (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* j} (\boldsymbol{\Omega}_{\mathrm{T}})_{k l^*} + (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* k} (\boldsymbol{\Omega}_{\mathrm{T}})_{j l^*} + (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* l^*} (\boldsymbol{\Omega}_{\mathrm{T}})_{jk} \}$$

$$\times \frac{1}{2} \left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \frac{\partial^2 \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k} \partial \pi_{\mathrm{T}l^*}} + \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0')^{<2>}} \right) \left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k}} \otimes \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}l^*}} \right) \right\}$$

$$- \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0} + O(N^{-1}).$$

Term (7): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)} \mathbf{l}_{\theta_0}^{(\Delta b2)}{}')$

$$= N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[l_{\theta_0}^{(\Delta 1)}\{m^{(\Delta)} l_{\theta_0}^{(\Delta 1)},\ (l_{\theta_0}^{(\Delta 1)})^2\}]$$

$$= \underset{(\mathrm{A})}{[} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \left\{ \frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}} '} \otimes \left(\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}} '} \right)^{<2>} \right\} N^2 \kappa_3(\mathbf{p}),$$

$$\left(\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}} '} \right)^{<3>} N^2 \kappa_3(\mathbf{p}) \ \underset{(\mathrm{A})}{]} .$$

Term (8): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta\Delta b1)})$

$$= N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\left[l_{\theta_0}^{(\Delta 1)} \left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} '(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) \right.\right.$$

$$\left.\left. + \frac{1}{2} \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>} \right\} \right]$$

$$= \frac{1}{2}\left[\left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}} ')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) \left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}} '} \right)^{<2>} \right\} \right.$$

$$\left. \otimes \left(\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}} '} \right) \right] N^2 \kappa_3(\mathbf{p}),$$

where $\kappa_3(\mathbf{p}) = \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\mathbf{p} - \boldsymbol{\pi}_{\mathrm{T}})^{<3>}\}$.

Term (9): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)} \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta 1)})$

$$= \left\{ \left(\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}} '} \right)^{<2>} \otimes \left(\frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}} '} \right) \right\} N^2 \kappa_3(\mathbf{p}) .$$

Term (10): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)} \mathbf{l}_{\theta_0}^{(\Delta c3)} ')$ ($m^{(\Delta)} = m^{(\Delta 3)} = 0$ under m.m.)

$$= N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{ \ l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} [\ (m_{O_p(N^{-1/2})}^{(\Delta)})^2 l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)},\ m_{O_p(N^{-1/2})}^{(\Delta)} (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2$$

$$m_{O_p(N^{-1/2})}^{(\Delta 3)} (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2,\ (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^3,\ (0,0)\ ]\ \}$$

$$= \underset{(A)}{[} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\}$$

$$\times \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} + 2\left(\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right)^2,$$

$$3\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$3\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0^3 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial}{\partial \boldsymbol{\alpha}_0 '} \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^3} \right) \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$3(\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0})^2,\ (0,0) \quad \underset{(A)}{]} + O(N^{-1}).$$

Term (11): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)} \mathbf{l}_{\theta_0}^{(\Delta\Delta c2)} ')$ ($m^{(\Delta)} = m^{(\Delta\Delta b)} = 0$ under m.m.)

$$= N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{\ l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} [\ m^{(\Delta)}_{O_p(N^{-1/2})} l^{(\Delta\Delta b1)}_{\theta_0 O_p(N^{-1})} + m^{(\Delta\Delta b)}_{O_p(N^{-1})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})},$$

$$2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} l^{(\Delta\Delta b1)}_{\theta_0 O_p(N^{-1})}\ ]\ \},$$

where the first element of the above vector is

$$\sum_{i^*, j, k, l^*=1}^{2^n} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}j}}$$

$$\times \frac{1}{2}\left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \frac{\partial^2 \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k} \partial \pi_{\mathrm{T}l^*}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) \left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k}} \otimes \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}l^*}} \right) \right\}$$

$$\times \{ (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* j} (\boldsymbol{\Omega}_{\mathrm{T}})_{k l^*} + (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* k} (\boldsymbol{\Omega}_{\mathrm{T}})_{j l^*} + (\boldsymbol{\Omega}_{\mathrm{T}})_{i^* l^*} (\boldsymbol{\Omega}_{\mathrm{T}})_{jk} \}$$

$$-\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0}$$

$$+ \underset{(A)}{[} \ \frac{1}{2}\ \underset{(B)}{\{} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}} ')^{<2>}}$$

$$+\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4\overline{l}_{\theta_0}}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)-\frac{\partial^2\lambda_{\theta_0}}{(\partial\boldsymbol{\alpha}_0')^{<2>}}\right\}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\underset{(\mathrm{B})}{\}}\ \mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})$$

$$-\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\ \underset{(\mathrm{A})}{]}\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+\underset{(\mathrm{C})}{[}\ \left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}')^{<2>}}$$

$$+\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4\overline{l}_{\theta_0}}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)-\frac{\partial^2\lambda_{\theta_0}}{(\partial\boldsymbol{\alpha}_0')^{<2>}}\right\}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\underset{(\mathrm{C})}{]}\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)^{<2>}$$

$$+O(N^{-1}),$$

and the second element of the vector is

$$\underset{(\mathrm{A})}{[}\ \left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}')^{<2>}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\right\}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})$$

$$-2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\ \underset{(\mathrm{A})}{]}\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+2\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}')^{<2>}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\right\}$$

$$\times\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)^{<2>}+O(N^{-1}).$$

Term (12): $N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta1)}\boldsymbol{\gamma}_{\theta_0}^{(\Delta2)}{}'\mathbf{l}_{\theta_0}^{(\Delta b2)})$ ($m^{(\Delta)}=0$ under m.m.)

$$=N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\underset{(\mathrm{A})}{[}\ l_{\theta_0 O_p(N^{-1/2})}^{(\Delta1)}\left\{\frac{\partial\boldsymbol{\gamma}_{\theta_0}^{(2)}}{\partial\boldsymbol{\alpha}_0'}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})}\right\}'$$

$$\times\{m_{O_p(N^{-1/2})}^{(\Delta)}l_{\theta_0 O_p(N^{-1/2})}^{(\Delta1)},\ (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta1)})^2\}'\ \underset{(\mathrm{A})}{]}$$

$$= \frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+2 \frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0'} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+3 \frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_2}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} + O(N^{-1}) .$$

Term (13): $N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} (l_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta\Delta\Delta b1)})$

$$= N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} \underset{(\mathrm{A})}{\{} \; l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \; \underset{(\mathrm{B})}{[} \; \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(3)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(3)})_{O_p(N^{-3/2})}$$

$$+ \frac{1}{2} \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0')^{<2>}} \right) \sum_{\otimes}^{2} \{ (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}) \otimes (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) \}_{O_p(N^{-3/2})}$$

$$+ \frac{1}{6} \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0')^{<3>}} \right) \{ (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<3>} \}_{O_p(N^{-3/2})} \; \underset{(\mathrm{B})}{]} \; \underset{(\mathrm{A})}{\}}$$

$$= \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}' \left[\frac{1}{2} \frac{\partial^3 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}}')^{<3>}} \left\{ \left(\boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial \boldsymbol{\alpha}_0'}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right) \otimes \mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}}) \right\} \right.$$

$$\left. + \frac{\partial \boldsymbol{\alpha}_{\Delta \mathrm{W}}}{\partial \boldsymbol{\pi}_{\mathrm{T}}'} \boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial \boldsymbol{\alpha}_0'}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right]$$

$$+ \frac{1}{2} \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0')^{<2>}} \right) \left\{ \left(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right) \otimes \left(\frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}}')^{<2>}} \mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}}) \right) \right\}$$

$$+ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0')^{<2>}} \right) \left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}'} \otimes \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}}')^{<2>}} \right)$$

$$\times \left\{ \mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}}) \otimes \left(\boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial \boldsymbol{\alpha}_0'}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right) \right\}$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<3>}}\right)\{(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})\otimes\mathrm{vec}(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0})\}$$

$$-\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\{(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})\otimes(\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})\}+O(N^{-1}),$$

where

$$\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(3)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(3)}=\frac{1}{6}\frac{\partial^3\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<3>}}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})^{<3>}+N^{-1}\frac{\partial\boldsymbol{\alpha}_{\Delta\mathrm{W}}}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})$$

$$\frac{\partial\boldsymbol{\alpha}_{\Delta\mathrm{W}}}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}=\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\left\{-\sum_{k=1}^{q}\frac{\partial^3\overline{l}_{\boldsymbol{\alpha}_0\mathrm{ML}}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\alpha}_0{}'\partial(\boldsymbol{\alpha}_0)_k}(\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_k+\frac{\partial\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\frac{\partial^2\overline{l}_{\boldsymbol{\alpha}_0\mathrm{ML}}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}$$

$$+\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\sum_{k=1}^{q}\frac{\partial^3\overline{l}_{\boldsymbol{\alpha}_0\mathrm{ML}}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\pi}_{\mathrm{T}}{}'\partial(\boldsymbol{\alpha}_0)_k}(\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_k$$

and $\overline{l}_{\boldsymbol{\alpha}_0\mathrm{ML}}$ is $\overline{l}_{\boldsymbol{\alpha}_0}$ for ML estimation (Ogasawara, 2012a, Equation (3.4)).

Term (14): $N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta 1)}\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta\Delta b1)})$

$$=N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\underset{(\mathrm{A})}{\{}\; l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})}$$

$$\times\underset{(\mathrm{B})}{[}\;\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)}-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\{(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}\}_{O_p(N^{-1})}\underset{(\mathrm{B})}{]}\quad\underset{(\mathrm{A})}{\}}$$

$$=\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}\left[\frac{1}{2}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}\right.\right.$$

$$\left.\left.+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right\}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right]$$

$$+\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right\}$$

$$\times\left\{\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)\otimes\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}\right)\right\}+O(N^{-1}).$$

Term (15): $N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(l_{\theta_0}^{(\Delta1)}\gamma_{\theta_0}^{(\Delta\Delta1)}l_{\theta_0}^{(\Delta1)})$

$$=N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\left[(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta1)})^2\left\{\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)}-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})\right.\right.$$

$$\left.\left.+\frac{1}{2}\frac{\partial^2\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0')^{<2>}}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}\right\}\right]$$

$$=\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\left[\frac{1}{2}\left\{\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}')^{<2>}}+\frac{\partial^2\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0')^{<2>}}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\right\}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})\right.$$

$$\left.-\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right]$$

$$+\left\{\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}')^{<2>}}+\frac{\partial^2\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0')^{<2>}}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\right\}\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)^{<2>}$$

$$+O(N^{-1}).$$

(a.3) The third asymptotic cumulant

(a.3.1) $\beta_3^{(\Delta a)}$ **(the term with** $\bar{c}$ **in** $\overline{\beta}_3^{(\Delta)}$ **)**

$$=3nN\mathrm{E}_{\mathrm{T}}\{(q_{O_p(n^{-1/2})}^{(10)})^2q_{O_p(N^{-1})}^{(22)}+2q_{O_p(n^{-1/2})}^{(10)}q_{O_p(N^{-1/2})}^{(11)}q_{O_p(n^{-1/2}N^{-1/2})}^{(21)}$$

$$+(q_{O_p(N^{-1/2})}^{(11)})^2q_{O_p(n^{-1})}^{(20)}\}-3\{(\beta_1^{(0)}+\lambda_{\theta_0}^{-1}\eta_{\theta_0})\beta_2^{(\Delta)}+\beta_1^{(\Delta)}\beta_2^{(0)}\}$$

$$=6nN\mathrm{E}_{\mathrm{T}}(q_{O_p(n^{-1/2})}^{(10)}q_{O_p(N^{-1/2})}^{(11)}q_{O_p(n^{-1/2}N^{-1/2})}^{(21)}),$$

where

$$nN\mathrm{E}_{\mathrm{T}}(q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})}q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})\quad (m^{(\Delta)}=0 \text{ under m.m.})$$

$$= nN\mathrm{E}_{\mathrm{T}}\{(\gamma^{(1)}_{\theta_0}l^{(1)}_{\theta_0})_{O_p(n^{-1/2})}(\gamma^{(1)}_{\theta_0}l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1/2})}$$

$$\times(\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta a2)}_{\theta_0}+\gamma^{(1)}_{\theta_0}l^{(\Delta\Delta a1)}_{\theta_0}+\gamma^{(\Delta 1)}_{\theta_0}l^{(1)}_{\theta_0})_{O_p(n^{-1/2}N^{-1/2})}\}$$

$$=(\gamma^{(1)}_{\theta_0})^2\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'nN\mathrm{E}_{\mathrm{T}}\{l^{(1)}_{\theta_0 O_p(n^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}$$

$$\times[m_{O_p(n^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}+m^{(\Delta)}_{O_p(N^{-1/2})}l^{(1)}_{\theta_0 O_p(n^{-1/2})},\ 2l^{(1)}_{\theta_0 O_p(n^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}]'$$

$$+(\gamma^{(1)}_{\theta_0})^3 nN\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}\ l^{(1)}_{\theta_0 O_p(n^{-1/2})}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})\boldsymbol{\alpha}_0}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'}-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\right)_{O_p(n^{-1/2})}$$

$$\times\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}\ \underset{(\mathrm{A})}{\}}$$

$$+(\gamma^{(1)}_{\theta_0})^2 nN\mathrm{E}_{\mathrm{T}}\left\{(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}\frac{\partial\gamma^{(1)}_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}\right\}$$

$$=\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\underset{(\mathrm{A})}{[}\ (\gamma^{(1)}_{\theta_0})^2 n\,\mathrm{cov}(l^{(1)}_{\theta_0},m)\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+\beta^{(0)}_2\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

$$2\beta^{(0)}_2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\ \underset{(\mathrm{A})}{]}{}'$$

$$+(\gamma^{(1)}_{\theta_0})^3 n\,\mathrm{cov}\left(l^{(1)}_{\theta_0},\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}+\beta^{(0)}_2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma^{(1)}_{\theta_0}}{\partial\boldsymbol{\alpha}_0}$$

$$+O(N^{-1}).$$

(a.3.2) $\beta^{(\Delta b)}_3$ **(the term with** $\overline{c}^2$ **in** $\overline{\beta}^{(\Delta)}_3$**)** ($m^{(\Delta)}=0$ under m.m.)

$$= N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^3+3(q^{(11)}_{O_p(N^{-1/2})})^2 q^{(22)}_{O_p(N^{-1})}\}-3\beta^{(\Delta)}_1\beta^{(\Delta)}_2$$

$$= N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\gamma^{(1)}_{\theta_0}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^3+3(\gamma^{(1)}_{\theta_0}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2$$

$$\times(\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta b2)}_{\theta_0}+\gamma^{(1)}_{\theta_0}l^{(\Delta\Delta b1)}_{\theta_0}+\gamma^{(\Delta 1)}_{\theta_0}l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1})}\}-3\beta^{(\Delta)}_1\beta^{(\Delta)}_2$$

$$= (\gamma_{\theta_0}^{(1)})^3 N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^3\} + 3(\gamma_{\theta_0}^{(1)})^2 N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{\ (\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^2$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m^{(\Delta)}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)},\ (\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^2]'\ \}$$

$$+3(\gamma_{\theta_0}^{(1)})^3 N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\ \underset{(\mathrm{A})}{[}\ (\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^2\ \underset{(\mathrm{B})}{\{}\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}\ \underset{(\mathrm{B})}{\}}\ \ \underset{(\mathrm{A})}{]}$$

$$+3(\gamma_{\theta_0}^{(1)})^2 N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\left\{(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^3\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}\right\} - 3\beta_1^{(\Delta)}\beta_2^{(\Delta)}$$

$$= (\gamma_{\theta_0}^{(1)})^3\left(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<3>} N^2\kappa_3(\mathbf{p})$$

$$+6(\gamma_{\theta_0}^{(1)})^2\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\ \underset{(\mathrm{A})}{[}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

$$(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2\quad \underset{(\mathrm{A})}{]}'$$

$$+3(\gamma_{\theta_0}^{(1)})^3\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\right\}$$

$$\times\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)^{<2>}$$

$$+6(\gamma_{\theta_0}^{(1)})^2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} + O(N^{-1})$$

(a.4) The fourth asymptotic cumulants

$n^{-1}\overline{\beta}_4^{(\Delta)} = N^{-1}(\beta_4^{(\Delta a)} + \overline{c}\beta_4^{(\Delta b)} + \overline{c}^2\beta_4^{(\Delta c)})$.

In the following, the definitions of Terms (1) to (14) (see Subsection A.3) are used. The notation $\to x$ below indicates that the associated term is a member of the summarized term *x*.

Term (1): 0.

Term (2):

$$[n^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^4 - 3n^{-2}(\overline{\beta}^{(\Delta)}_2)^2\}]_{O(n^2 N^{-3})}$$

$$= n^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\{(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^4\}_{O_p(N^{-2})}] - 3(\overline{\beta}^{(\Delta)}_2)^2$$

$$= \overline{c}^2[\ N^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\{(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^4\}_{O_p(N^{-2})}] - 3(\beta^{(\Delta)}_2)^2\ ]\ (\because \overline{\beta}^{(\Delta)}_2 = \overline{c}\beta^{(\Delta)}_2)$$

$$= N^{-1}\overline{c}^2\{N^3 \kappa_4(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})\}_{O(1)}$$

$$= N^{-1}\overline{c}^2(\gamma^{(1)}_{\theta_0})^4 \left(\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<4>} \{N^3 \kappa_4(\mathbf{p})\}_{O(1)}\ \ (\to N^{-1}\overline{c}^2 \beta^{(\Delta c)}_4),$$

where $\kappa_4(\mathbf{p})$ is the $2^{4n} \times 1$ vector of the fourth multivariate cumulants of **p**.

Term (3):

$$[\ \ 4n^2 \mathrm{E}_{\mathrm{T}}\{(q^{(10)}_{O_p(n^{-1/2})})^3 q^{(22)}_{O_p(N^{-1})} + 3(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})}$$

$$\times (q^{(21)}_{O_p(n^{-1/2}N^{-1/2})} + q^{(22)}_{O_p(N^{-1})})\}\ \ ]_{O(N^{-1})+O(nN^{-2})}$$

$$= 4N^{-1} \underset{(\mathrm{A})}{[}\ \mathrm{E}_{\mathrm{T}\theta_0}\{n^2(q^{(10)}_{O_p(n^{-1/2})})^3\}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(Nq^{(22)}_{O_p(N^{-1})})\ (\text{known; given earlier})$$

$$+ 3\mathrm{E}_{\mathrm{T}}\{n^2 N(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})} q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}\}$$

$$+ 3\overline{c}\beta^{(0)}_2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(N^2 q^{(11)}_{O_p(N^{-1/2})} q^{(22)}_{O_p(N^{-1})})\ \ \underset{(\mathrm{A})}{]}\ (\text{given earlier})$$

(the first and second terms in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]} \to \beta^{(\Delta a)}_4$ and the third term $\to \overline{c}\beta^{(\Delta b)}_4$),

where $\mathrm{E}_{\mathrm{T}}\{n^2 N(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})} q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}\}$

$$= \mathrm{E}_{\mathrm{T}}\{n^2 N(\gamma^{(1)}_{\theta_0} l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 \gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}$$

$$\times (\boldsymbol{\gamma}^{(2)}_{\theta_0}{}' \mathbf{l}^{(\Delta a 2)}_{\theta_0} + \gamma^{(1)}_{\theta_0} l^{(\Delta\Delta a 1)}_{\theta_0} + \gamma^{(\Delta 1)}_{\theta_0} l^{(1)}_{\theta_0})_{O_p(n^{-1/2}N^{-1/2})}\}$$

($m^{(\Delta)} = 0$ under m.m.)

$$= \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{}\ n^2 N(\gamma^{(1)}_{\theta_0} l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 \gamma^{(1)}_{\theta_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}$$

$$\times \underset{(\mathrm{B})}{[}\ \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\ \underset{(\mathrm{C})}{\{}\ m_{O_p(n^{-1/2})}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$+\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}_{O(1)}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}l_{\theta_0 O_p(n^{-1/2})}^{(1)},$$

$$2l_{\theta_0 O_p(n^{-1/2})}^{(1)}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}\ \underset{(\mathrm{C})}{\}}{}'$$

$$+\gamma_{\theta_0}^{(1)}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'}-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\right)_{O_p(n^{-1/2})}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$+\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}l_{\theta_0 O_p(n^{-1/2})}^{(1)}\ \underset{(\mathrm{B})}{]}\ \underset{(\mathrm{A})}{\}}$$

$$=(\gamma_{\theta_0}^{(1)})^3(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1\ \underset{(\mathrm{A})}{[}\ n^2\kappa_3(l_{\theta_0}^{(1)},l_{\theta_0}^{(1)},m)\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+n^2\kappa_3(l_{\theta_0}^{(1)})\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\ \underset{(\mathrm{A})}{]}$$

$$+2(\gamma_{\theta_0}^{(1)})^3(\boldsymbol{\gamma}_{\theta_0}^{(2)})_2 n^2\kappa_3(l_{\theta_0}^{(1)})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+(\gamma_{\theta_0}^{(1)})^4 n^2\kappa_3\left(l_{\theta_0}^{(1)},l_{\theta_0}^{(1)}\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+(\gamma_{\theta_0}^{(1)})^3 n^2\kappa_3(l_{\theta_0}^{(1)})\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

where

$$n^2\kappa_3(l_{\theta_0}^{(1)},l_{\theta_0}^{(1)},m)=n^{-1}\sum_{k=1}^{n}n^2\kappa_3(U_k)\left(\frac{1}{P_k}\frac{\partial P_k}{\partial\theta_0}-\frac{1}{Q_k}\frac{\partial Q_k}{\partial\theta_0}\right)^2$$

$$\times\left\{-\frac{1}{P_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2+\frac{1}{Q_k^2}\left(\frac{\partial Q_k}{\partial\theta_0}\right)^2+\frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0^2}-\frac{1}{Q_k}\frac{\partial^2 Q_k}{\partial\theta_0^2}\right\}$$

$$=n^{-1}\sum_{k=1}^{n}P_{\mathrm{T}k}Q_{\mathrm{T}k}(1-2P_{\mathrm{T}k})\left(\frac{1}{P_kQ_k}\frac{\partial P_k}{\partial\theta_0}\right)^2\left\{\frac{P_k-Q_k}{P_k^2Q_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2+\frac{1}{P_kQ_k}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\},$$

$$n^2\kappa_3(l_{\theta_0}^{(1)}) = n^{-1}\sum_{k=1}^{n} P_{\mathrm{T}k}Q_{\mathrm{T}k}(1-2P_{\mathrm{T}k})\left(\frac{1}{P_kQ_k}\frac{\partial P_k}{\partial\theta_0}\right)^3,$$

$$n^2\kappa_3\left(l_{\theta_0}^{(1)}, l_{\theta_0}^{(1)}, \frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right) = n^{-1}\sum_{k=1}^{n} P_{\mathrm{T}k}Q_{\mathrm{T}k}(1-2P_{\mathrm{T}k})$$

$$\times\left(\frac{1}{P_kQ_k}\frac{\partial P_k}{\partial\theta_0}\right)^2\left(\frac{P_k-Q_k}{P_k^2Q_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}+\frac{1}{P_kQ_k}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right).$$

Term (4):

$$[\ 4n^2\mathrm{E}_{\mathrm{T}}\{\ 3q_{O_p(n^{-1/2})}^{(10)}(q_{O_p(N^{-1/2})}^{(11)})^2(q_{O_p(n^{-1})}^{(20)}+q_{O_p(n^{-1/2}N^{-1/2})}^{(21)})\ \}]_{O(N^{-1})+O(nN^{-2})}$$

$$= 4N^{-1}[\ 3\mathrm{E}_{\mathrm{T}\theta_0}(n^2q_{O_p(n^{-1/2})}^{(10)}q_{O_p(n^{-1})}^{(20)})\beta_2^{(\Delta)}$$

$$+3\overline{c}\mathrm{E}_{\mathrm{T}}\{nN^2q_{O_p(n^{-1/2})}^{(10)}(q_{O_p(N^{-1/2})}^{(11)})^2q_{O_p(n^{-1/2}N^{-1/2})}^{(21)}\}\ ]$$

(the known first term in $[\cdot]\to\beta_4^{(\Delta a)}$; and the second term $\to\overline{c}\beta_4^{(\Delta b)}$).

The second term of Term (4): ($m^{(\Delta)}=0$ under m.m.)

$$\mathrm{E}_{\mathrm{T}}\{nN^2q_{O_p(n^{-1/2})}^{(10)}(q_{O_p(N^{-1/2})}^{(11)})^2q_{O_p(n^{-1/2}N^{-1/2})}^{(21)}\}$$

$$= \mathrm{E}_{\mathrm{T}}\{\ nN^2\gamma_{\theta_0}^{(1)}l_{\theta_0O_p(n^{-1/2})}^{(1)}(\gamma_{\theta_0}^{(1)}l_{\theta_0O_p(N^{-1/2})}^{(\Delta 1)})^2$$

$$\times(\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta a2)}+\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta a1)}+\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1/2})}\ \}$$

$$=\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}\ nN^2\gamma_{\theta_0}^{(1)}l_{\theta_0O_p(n^{-1/2})}^{(1)}(\gamma_{\theta_0}^{(1)}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}^{(1)})^2$$

$$\times\underset{(\mathrm{B})}{[}\ \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\underset{(\mathrm{C})}{\{}\ m_{O_p(n^{-1/2})}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}^{(1)}$$

$$+\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}_{O(1)}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}^{(1)}l_{\theta_0O_p(n^{-1/2})}^{(1)},$$

$$2l_{\theta_0O_p(n^{-1/2})}^{(1)}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}^{(1)}\ \underset{(\mathrm{C})}{\}}{}'$$

$$+\gamma_{\theta_0}^{(1)}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\right)_{O_p(n^{-1/2})}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$+\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}l_{\theta_0 O_p(n^{-1/2})}^{(1)}\ \underset{(\mathrm{B})}{]}\ \underset{(\mathrm{A})}{\}}$$

$$=\underset{(\mathrm{A})}{[}\ (\boldsymbol{\gamma}_{\theta_0}^{(2)})_1\ \underset{(\mathrm{B})}{\{}\ (\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}(l_{\theta_0}^{(1)},m)\left(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<3>}$$

$$+\gamma_{\theta_0}^{(1)}\beta_2^{(0)}\left\{\left(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\otimes\left(\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)\right\}\ \underset{(\mathrm{B})}{\}}$$

$$+2\gamma_{\theta_0}^{(1)}(\boldsymbol{\gamma}_{\theta_0}^{(2)})_2\beta_2^{(0)}\left(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<3>}$$

$$+(\gamma_{\theta_0}^{(1)})^4 n\,\mathrm{cov}\left(l_{\theta_0}^{(1)},\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\left\{\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\otimes\left(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\right\}$$

$$+\gamma_{\theta_0}^{(1)}\beta_2^{(0)}\left\{\left(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\otimes\left(\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)\right\}\ \underset{(\mathrm{A})}{]}\ N^2\kappa_3(\mathbf{p})\,.$$

Term (5): ($m^{(\Delta)}=0$ under m.m.)

$$[\ 4n^2\mathrm{E}_{\mathrm{T}}\{\ (q_{O_p(N^{-1/2})}^{(11)})^3(q_{O_p(n^{-1})}^{(20)}+q_{O_p(N^{-1})}^{(22)})\ \}]_{O(nN^{-2})+O(n^2N^{-3})}$$

$$=4N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{N^2(q_{O_p(N^{-1/2})}^{(11)})^3\}\mathrm{E}_{\mathrm{T}\theta_0}(nq_{O_p(n^{-1})}^{(20)})$$

$$+4N^{-1}\overline{c}^2\ \underset{(\mathrm{A})}{[}\ \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{N^2(q_{O_p(N^{-1/2})}^{(11)})^3\}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(Nq_{O_p(N^{-1})}^{(22)})$$ (the term associated

with $-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}$ is included only in this term)

$$+3\beta_2^{(\Delta)}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(N^2q_{O_p(N^{-1/2})}^{(11)}q_{O_p(N^{-1})}^{(22)})$$

$$+\sum_{i^*,j}^{2^n}(\gamma_{\theta_0}^{(1)})^3 \underset{(B)}{\{} \ \boldsymbol{\gamma}_{\theta_0}^{(2)}{}' \ \underset{(C)}{[} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0{}'}\right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'}\right\}$$

$$\times \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}j}}, \quad \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}j}} \ \underset{(C)}{]}{}'$$

$$+(\gamma_{\theta_0}^{(1)})\frac{1}{2}\left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial^2 \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}\partial \pi_{\mathrm{T}j}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}} \otimes \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}j}}\right)\right\}$$

$$+\frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'} \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}i^*}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}j}} \ \underset{(B)}{\}}$$

$$\times \sum_{k,l^*,m^*}^{2^n} \{ \ \sum_{(i^*,j)}^{2} \sum_{(k,l^*,m^*)}^{3} (\boldsymbol{\Omega}_{\mathrm{T}})_{i^*k} N^2 \kappa_3(p_j, p_{l^*}, p_{m^*})$$

$$\times \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}k}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}l^*}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial \boldsymbol{\alpha}_0}{\partial \pi_{\mathrm{T}m^*}} \ \} \quad \underset{(A)}{]} + O(N^{-2})$$

$(\to N^{-1}\overline{c}\beta_4^{(\Delta b)}$ and $\to N^{-1}\overline{c}^2\beta_4^{(\Delta c)})$,

where $\sum_{(i^*,j)}^{2}$ indicates the sum of two terms exchanging i^* and j, with $\sum_{(k,l^*,m^*)}^{3}$ defined similarly; and

$$q_{O_p(N^{-1})}^{(22)} = \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta b2)} + \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta b1)} + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta 1)}$$

$$= \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m^{(\Delta)} l_{\theta_0}^{(\Delta 1)}, (l_{\theta_0}^{(\Delta 1)})^2]' + \gamma_{\theta_0}^{(1)} \underset{(A)}{\{} \ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0{}')^{<2>}}\right)(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^2 \ \underset{(A)}{\}}$$

$$+\frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)} l_{\theta_0}^{(\Delta 1)} \quad \text{with} \quad l_{\theta_0}^{(\Delta 1)} = \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}.$$

Term (6):

$$\underset{(A)}{\{}\ 6n^2\mathrm{E_T}[\ (q^{(10)}_{O_p(n^{-1/2})})^2\{(q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})^2$$

$$+(q^{(22)}_{O_p(N^{-1})})^2+2q^{(20)}_{O_p(n^{-1})}q^{(22)}_{O_p(N^{-1})}\}]\ \}_{(A)O(N^{-1})+O(nN^{-2})}$$

The first term of Term (6):

$$6n^2\mathrm{E_T}\{(q^{(10)}_{O_p(n^{-1/2})})^2(q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})^2\}\quad(\to N^{-1}\beta_4^{(\Delta a)})$$

$$=6n^2\mathrm{E_T}[(\gamma^{(1)}_{\theta_0}l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2\{(\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta a2)}_{\theta_0}+\gamma^{(1)}_{\theta_0}l^{(\Delta\Delta a1)}_{\theta_0}+\gamma^{(\Delta 1)}_{\theta_0}l^{(1)}_{\theta_0})_{O_p(n^{-1/2}N^{-1/2})}\ \}^2]$$

$$=6n^2\mathrm{E_T}[(\gamma^{(1)}_{\theta_0}l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2\{(\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta a2)}_{\theta_0})^2+(\gamma^{(1)}_{\theta_0}l^{(\Delta\Delta a1)}_{\theta_0})^2+(\gamma^{(\Delta 1)}_{\theta_0}l^{(1)}_{\theta_0})^2$$

$$+2\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta a2)}_{\theta_0}(\gamma^{(1)}_{\theta_0}l^{(\Delta\Delta a1)}_{\theta_0}+\gamma^{(\Delta 1)}_{\theta_0}l^{(1)}_{\theta_0})+2\gamma^{(1)}_{\theta_0}l^{(\Delta\Delta a1)}_{\theta_0}\gamma^{(\Delta 1)}_{\theta_0}l^{(1)}_{\theta_0}\}_{O_p(n^{-1}N^{-1})}],\quad(*)$$

where the first term of (*) is ($m^{(\Delta)}=0$ under m.m.)

$$=6N^{-1}(\gamma^{(1)}_{\theta_0})^2\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathrm{E_T}\underset{(A)}{\{}\ n^2N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2\begin{bmatrix}e_{11} & \text{sym.}\\ e_{21} & e_{22}\end{bmatrix}\}_{(A)}\boldsymbol{\gamma}^{(2)}_{\theta_0}$$

with $(\gamma^{(1)}_{\theta_0})^2\mathrm{E_T}\underset{(A)}{\{}\ n^2N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2e_{11}\ \underset{(A)}{\}}$

$$=(\gamma^{(1)}_{\theta_0})^2\mathrm{E_T}\underset{(A)}{\{}\ n^2N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2\ \underset{(B)}{[}\ m_{O_p(n^{-1/2})}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}$$

$$+\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}_{O(1)}\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}l^{(1)}_{\theta_0 O_p(n^{-1/2})}\ ]^2_{(B)}\ \ \}_{(A)}$$

$$=\{n\,\mathrm{var}(m)\beta_2^{(0)}+2(\gamma^{(1)}_{\theta_0})^2(n\,\mathrm{cov}(m,l^{(1)}_{\theta_0}))^2\}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+6n\,\mathrm{cov}(m,l^{(1)}_{\theta_0})\beta_2^{(0)}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}_{O(1)}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+3(\beta_2^{(0)})^2(\gamma^{(1)}_{\theta_0})^{-2}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}$$

$$\times\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\}+O(N^{-1}),$$

$$(\gamma_{\theta_0}^{(1)})^2 \mathrm{E}_{\mathrm{T}}\{n^2 N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 e_{21}\}$$

$$=(\gamma_{\theta_0}^{(1)})^2 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \; n^2 N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 2 l^{(1)}_{\theta_0 O_p(n^{-1/2})} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}$$

$$\times \underset{(\mathrm{B})}{[} \; m_{O_p(n^{-1/2})} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}$$

$$+ \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right\}_{O(1)} \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})} \; \underset{(\mathrm{B})}{]} \; \underset{(\mathrm{A})}{\}}$$

$$= 6 \beta_2^{(0)} n \operatorname{cov}(l^{(1)}_{\theta_0}, m) \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ 6(\beta_2^{(0)})^2 (\gamma_{\theta_0}^{(1)})^{-2} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right\}_{O(1)} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} + O(N^{-1}),$$

and $(\gamma_{\theta_0}^{(1)})^2 \mathrm{E}_{\mathrm{T}}\{n^2 N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 e_{22}\}$

$$=(\gamma_{\theta_0}^{(1)})^2 \mathrm{E}_{\mathrm{T}}\{n^2 N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 4(l^{(1)}_{\theta_0 O_p(n^{-1/2})} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})})^2\}$$

$$= 12(\beta_2^{(0)})^2 (\gamma_{\theta_0}^{(1)})^{-2} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} + O(N^{-1}),$$

the second term of (*) is

$$= 6N^{-1} (\gamma_{\theta_0}^{(1)})^4 \mathrm{E}_{\mathrm{T}}\{n^2 N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2 (l^{(\Delta\Delta a1)}_{\theta_0 O_p(n^{-1/2} N^{-1/2})})^2\}$$

$$= 6N^{-1} (\gamma_{\theta_0}^{(1)})^4 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{[} \; n^2 N(l^{(1)}_{\theta_0 O_p(n^{-1/2})})^2$$

$$\times \left\{ \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \right)_{O_p(n^{-1/2})} \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})} \right\}^2 \; \underset{(\mathrm{A})}{]}$$

$$= 6N^{-1} (\gamma_{\theta_0}^{(1)})^2 \beta_2^{(0)} \operatorname{tr} \left\{ n \operatorname{cov} \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \right\}$$

$$+12N^{-1} (\gamma_{\theta_0}^{(1)})^4 n \operatorname{cov} \left(l^{(1)}_{\theta_0}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} n \operatorname{cov} \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0}, l^{(1)}_{\theta_0} \right) + O(N^{-2}),$$

where

$$n\,\mathrm{cov}\left(\frac{\partial^2\overline{l_{\theta_0}}}{\partial\theta_0\partial\boldsymbol{\alpha}_0},\frac{\partial^2\overline{l_{\theta_0}}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)=n^{-1}\sum_{k=1}^{n}P_{\mathrm{T}k}Q_{\mathrm{T}k}$$

$$\times\left(\frac{P_k-Q_k}{P_k^2Q_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}+\frac{1}{P_kQ_k}\frac{\partial^2P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right)(\cdot)',$$

the third term of (*) is

$$=6N^{-1}(\gamma_{\theta_0}^{(1)})^2\mathrm{E_T}\{n^2N(l^{(1)}_{\theta_0O_p(n^{-1/2})})^4(\gamma^{(\Delta1)}_{\theta_0O_p(N^{-1/2})})^2\}$$

$$=18N^{-1}(\gamma_{\theta_0}^{(1)})^{-2}(\beta_2^{(0)})^2\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}+O(N^{-2}),$$

the fourth term of (*) is ($m^{(\Delta)}=0$ under m.m.)

$$=12N^{-1}\mathrm{E_T}\{n^2N(\gamma_{\theta_0}^{(1)}l^{(1)}_{\theta_0O_p(n^{-1/2})})^2$$

$$\times\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}^{(\Delta a2)}_{\theta_0O_p(n^{-1/2}N^{-1/2})}(\gamma_{\theta_0}^{(1)}l^{(\Delta\Delta a1)}_{\theta_0O_p(n^{-1/2}N^{-1/2})}+\gamma^{(\Delta1)}_{\theta_0O_p(N^{-1/2})}l^{(1)}_{\theta_0O_p(n^{-1/2})})\}$$

$$=12N^{-1}(\gamma_{\theta_0}^{(1)})^2\mathrm{E_T}\underset{(\mathrm{A})}{\{}\;n^2N(l^{(1)}_{\theta_0O_p(n^{-1/2})})^2$$

$$\times\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\underset{(\mathrm{B})}{[}\;m_{O_p(n^{-1/2})}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}$$

$$+\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l_{\theta_0}}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}_{O(1)}\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}l^{(1)}_{\theta_0O_p(n^{-1/2})},$$

$$2l^{(1)}_{\theta_0O_p(n^{-1/2})}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}\;\underset{(\mathrm{B})}{]}'$$

$$\times\underset{(\mathrm{C})}{[}\;\gamma_{\theta_0}^{(1)}\left(\frac{\partial^2\overline{l_{\theta_0}}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\right)_{O_p(n^{-1/2})}\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}$$

$$+\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0O_p(N^{-1/2})}l^{(1)}_{\theta_0O_p(n^{-1/2})}\;\underset{(\mathrm{C})}{]}\;\;\underset{(\mathrm{A})}{\}}$$

$$=12N^{-1}\underset{(\mathrm{A})}{[}\;(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1\underset{(\mathrm{B})}{\{}\;\gamma_{\theta_0}^{(1)}\beta_2^{(0)}n\,\mathrm{cov}\left(m,\frac{\partial^2\overline{l_{\theta_0}}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+2(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}(m, l_{\theta_0}^{(1)}) n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+3\gamma_{\theta_0}^{(1)} \beta_2^{(0)} n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\}$$

$$+3\beta_2^{(0)} n\,\mathrm{cov}(m, l_{\theta_0}^{(1)}) \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+3(\beta_2^{(0)})^2 (\gamma_{\theta_0}^{(1)})^{-2} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\} \quad \underset{(\mathrm{B})}{\}}$$

$$+(\boldsymbol{\gamma}_{\theta_0}^{(2)})_2 \underset{(\mathrm{C})}{\{}\ 6\gamma_{\theta_0}^{(1)} \beta_2^{(0)} n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ 6(\beta_2^{(0)})^2 (\gamma_{\theta_0}^{(1)})^{-2} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \underset{(\mathrm{C})}{\}} \quad \underset{(\mathrm{A})}{]} + O(N^{-2}),$$

where

$$n\,\mathrm{cov}\left(m, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0} \right) = n^{-1} \sum_{k=1}^{n} P_{\mathrm{T}k} Q_{\mathrm{T}k} \left\{ \frac{P_k - Q_k}{P_k^2 Q_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 + \frac{1}{P_k Q_k} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\}$$

$$\times \left(\frac{P_k - Q_k}{P_k^2 Q_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} + \frac{1}{P_k Q_k} \frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0} \right),$$

$$n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0} \right) = n^{-1} \sum_{k=1}^{n} P_{\mathrm{T}k} Q_{\mathrm{T}k} \frac{1}{P_k Q_k} \frac{\partial P_k}{\partial \theta_0}$$

$$\times \left(\frac{P_k - Q_k}{P_k^2 Q_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} + \frac{1}{P_k Q_k} \frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0} \right),$$

$$n\,\mathrm{cov}(m, l_{\theta_0}^{(1)}) = n^{-1} \sum_{k=1}^{n} P_{\mathrm{T}k} Q_{\mathrm{T}k} \left\{ \frac{P_k - Q_k}{P_k^2 Q_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 + \frac{1}{P_k Q_k} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \frac{1}{P_k Q_k} \frac{\partial P_k}{\partial \theta_0},$$

the fifth term of (*) is

$$= 12N^{-1}(\gamma_{\theta_0}^{(1)})^3 \mathrm{E}_{\mathrm{T}}\{n^2 N (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2}N^{-1/2})}^{(\Delta\Delta a1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}\}$$

$$= 12N^{-1}(\gamma_{\theta_0}^{(1)})^3 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} n^2 N (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^3 \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0 '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \right)_{O_p(n^{-1/2})}$$

$$\times \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)} \underset{(\mathrm{A})}{\}}$$

$$= 36N^{-1}\gamma_{\theta_0}^{(1)} \beta_2^{(0)} n \operatorname{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} + O(N^{-2}) \; .$$

The second term of Term (6):

$$6n^2 \mathrm{E}_{\mathrm{T}}\{(q_{O_p(n^{-1/2})}^{(10)})^2 (q_{O_p(N^{-1})}^{(22)})^2\} \quad (\to N^{-1}\overline{c}\beta_4^{(\Delta b)})$$

$$= 6N^{-1}\overline{c}\beta_2^{(0)} \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{N^2 (q_{O_p(N^{-1})}^{(22)})^2\} \text{ (given in } \beta_{\mathrm{H}2}^{(\Delta b)}),$$

the third term of Term (6):

$$12n^2 \mathrm{E}_{\mathrm{T}}\{(q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(n^{-1})}^{(20)} q_{O_p(N^{-1})}^{(22)}\} \quad (\to N^{-1}\beta_4^{(\Delta a)})$$

$$= 12N^{-1} \mathrm{E}_{\mathrm{T}\theta_0}\{n^2 (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(1)})^2 [m l_{\theta_0}^{(1)}, (l_{\theta_0}^{(1)})^2] \boldsymbol{\gamma}_{\theta_0}^{(2)}\} \beta_1^{(\Delta)}$$

$$= 36N^{-1}[\beta_2^{(0)} n \operatorname{cov}(m, l_{\theta_0}^{(1)}), (\gamma_{\theta_0}^{(1)})^{-2} (\beta_2^{(0)})^2] \boldsymbol{\gamma}_{\theta_0}^{(2)} \beta_1^{(\Delta)} + O(N^{-2}).$$

Term (7):

$$\underset{(\mathrm{A})}{[} \; 6n^2 \mathrm{E}_{\mathrm{T}}\{ \, 2q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} 2q_{O_p(n^{-1/2}N^{-1/2})}^{(21)}$$

$$\times (q_{O_p(n^{-1})}^{(20)} + q_{O_p(N^{-1})}^{(22)}) \} \; \underset{(\mathrm{A})}{]}_{O(N^{-1}) + O(nN^{-2})}$$

The first term of Term (7):

$$24n^2 \mathrm{E}_{\mathrm{T}}(q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} q_{O_p(n^{-1})}^{(20)}) \; (\to N^{-1}\beta_4^{(\Delta a)})$$

$$= 24n^2 \mathrm{E}_{\mathrm{T}}\{ \; \gamma_{\theta_0}^{(1)} l_{\theta_0 O_p(n^{-1/2})}^{(1)} \gamma_{\theta_0}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times (\boldsymbol{\gamma}_{\theta_0}^{(2)} ' \mathbf{l}_{\theta_0}^{(\Delta a 2)} + \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta a1)} + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1/2})} \boldsymbol{\gamma}_{\theta_0}^{(2)} ' \mathbf{l}_{\theta_0 O_p(n^{-1})}^{(2)} \; \}, \quad (*)$$

the first term of (*) is ($m^{(\Delta)} = 0$ under m.m.)

$$24n^2 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \ (\gamma_{\theta_0}^{(1)})^2 l_{\theta_0 O_p(n^{-1/2})}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m_{O_p(n^{-1/2})} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} + m_{\theta_0 O_p(N^{-1/2})}^{(\Delta)} l_{\theta_0 O_p(n^{-1/2})}^{(1)},\ 2 l_{\theta_0 O_p(n^{-1/2})}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}]'$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m_{O_p(n^{-1/2})} l_{\theta_0 O_p(n^{-1/2})}^{(1)},\ (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2]' \ \underset{(\mathrm{A})}{\}}$$

$$= 24 N^{-1} \mathrm{E}_{\mathrm{T}\theta_0} \underset{(\mathrm{A})}{\{} \ n^2 (\gamma_{\theta_0}^{(1)})^2 l_{\theta_0 O_p(n^{-1/2})}^{(1)}$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)}{}' \underset{(\mathrm{B})}{[} \ m_{O_p(n^{-1/2})} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ l_{\theta_0 O_p(n^{-1/2})}^{(1)} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$2 l_{\theta_0 O_p(n^{-1/2})}^{(1)} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \ \underset{(\mathrm{B})}{]}{}'$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m_{O_p(n^{-1/2})} l_{\theta_0 O_p(n^{-1/2})}^{(1)},\ (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2]' \ \underset{(\mathrm{A})}{\}}$$

$$= 24 N^{-1} \boldsymbol{\gamma}_{\theta_0}^{(2)}{}' \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \boldsymbol{\gamma}_{\theta_0}^{(2)} + O(N^{-2})$$

with

$$e_{11} = [\ \beta_2^{(0)} n \operatorname{var}(m) + 2(\gamma_{\theta_0}^{(1)})^2 \{ n \operatorname{cov}(m, l_{\theta_0}^{(1)}) \}^2 \ ] \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ 3\beta_2^{(0)} n \operatorname{cov}(m, l_{\theta_0}^{(1)}) \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$e_{21} = 6\beta_2^{(0)} n \operatorname{cov}(m, l_{\theta_0}^{(1)}) \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$e_{12} = 3\beta_2^{(0)} n \operatorname{cov}(m, l_{\theta_0}^{(1)}) \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ 3(\beta_2^{(0)})^2 (\gamma_{\theta_0}^{(1)})^{-2} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$e_{22} = 6(\beta_2^{(0)})^2 (\gamma_{\theta_0}^{(1)})^{-2} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

the second term of (*) is

$$24n^2 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \ (\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_0 O_p(n^{-1/2}N^{-1/2})}^{(\Delta\Delta a1)}$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)} '[m_{O_p(n^{-1/2})} l_{\theta_0 O_p(n^{-1/2})}^{(1)}, \ (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2]' \ \underset{(\mathrm{A})}{\}}$$

$$= 24N^{-1} \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \ n^2 N (\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$\times \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \right)_{O_p(n^{-1/2})} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)} '[m_{O_p(n^{-1/2})} l_{\theta_0 O_p(n^{-1/2})}^{(1)}, \ (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2]' \ \underset{(\mathrm{A})}{\}}$$

$$= 24N^{-1} \boldsymbol{\gamma}_{\theta_0}^{(2)} ' \underset{(\mathrm{A})}{[} \ \beta_2^{(0)} \gamma_{\theta_0}^{(1)} n \operatorname{cov}\left(m, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ 2(\gamma_{\theta_0}^{(1)})^3 n \operatorname{cov}(m, l_{\theta_0}^{(1)}) n \operatorname{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$3\beta_2^{(0)} \gamma_{\theta_0}^{(1)} n \operatorname{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \ \underset{(\mathrm{A})}{]} ' + O(N^{-2}),$$

the third term of (*) is

$$24n^2 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} \ (\gamma_{\theta_0}^{(1)})^2 (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2 l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)} '[m_{O_p(n^{-1/2})} l_{\theta_0 O_p(n^{-1/2})}^{(1)}, \ (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2]' \ \underset{(\mathrm{A})}{\}}$$

$$= 72N^{-1} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0}$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)} '[\beta_2^{(0)} n \operatorname{cov}(m, l_{\theta_0 O_p(n^{-1/2})}^{(1)}), \ (\beta_2^{(0)})^2 (\gamma_{\theta_0}^{(1)})^{-2}]' + O(N^{-2}).$$

The second term of Term (7): ($m^{(\Delta)} = 0$ under m.m.)

$$24n^2 \mathrm{E}_{\mathrm{T}} (q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} q_{O_p(N^{-1})}^{(22)}) (\to N^{-1} \overline{c} \beta_4^{(\Delta b)})$$

$$= 24n^2 \mathrm{E}_{\mathrm{T}} \{ \ \gamma_{\theta_0}^{(1)} l_{\theta_0 O_p(n^{-1/2})}^{(1)} \gamma_{\theta_0}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times(\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta a2)}+\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta a1)}+\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1/2})}$$
$$\times(\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta b2)}+\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta b1)}+\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1})}\ \}, \qquad (*)$$

the first term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}\ nN^2(\gamma_{\theta_0}^{(1)})^2 l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}$$
$$\times\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m_{O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}+m^{(\Delta)}_{O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})}, 2l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}]'$$
$$[m^{(\Delta)}_{O_p(N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}, (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2]\boldsymbol{\gamma}_{\theta_0}^{(2)}\ \underset{(\mathrm{A})}{\}}$$
$$=24N^{-1}\overline{c}\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}\boldsymbol{\gamma}_{\theta_0}^{(2)}+O(N^{-2})$$

with

$$e_{11}=3(\gamma_{\theta_0}^{(1)})^2 n\,\mathrm{cov}(m, l_{\theta_0}^{(1)})\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$
$$\times\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}+\beta_2^{(0)}\underset{(\mathrm{A})}{[}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$
$$\times\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$
$$+2\left[\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right]^2\underset{(\mathrm{A})}{]},$$
$$e_{21}=6\beta_2^{(0)}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$
$$e_{12}=3(\gamma_{\theta_0}^{(1)})^2 n\,\mathrm{cov}(m, l_{\theta_0}^{(1)})(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2$$
$$+3\beta_2^{(0)}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

$$e_{22} = 6\beta_2^{(0)}(\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0})^2 ,$$

the second term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} nN^2(\gamma_{\theta_0}^{(1)})^3 l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m_{O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta)}_{O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})},\ 2l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}]'$$

$$\times \underset{(\mathrm{B})}{\{} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'(\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0} - N^{-1}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})$$

$$+ \frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})^2 \quad \underset{(\mathrm{B})O_p(N^{-1})}{\}} \quad \underset{(\mathrm{A})}{\}}$$

$$= 24N^{-1}\overline{c}\,\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'[e_1, e_2]' + O(N^{-2})$$

with

$$e_1 = (\gamma^{(1)}_{\theta_0})^3 n \operatorname{cov}(m, l^{(1)}_{\theta_0}) \underset{(\mathrm{A})}{\{} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$\times \underset{(\mathrm{B})}{[} \ \frac{1}{2} \underset{(\mathrm{C})}{\{} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}$$

$$+ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>} \underset{(\mathrm{C})}{\}} \operatorname{vec}(\boldsymbol{\Omega}_{\mathrm{T}}) - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0} \ \underset{(\mathrm{B})}{]}$$

$$+ \left\{\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right\}$$

$$\times \left(\boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial \boldsymbol{\alpha}_0{}'}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}\right)^{<2>} \quad \underset{(\mathrm{A})}{\}}$$

$$+ \gamma^{(1)}_{\theta_0}\beta^{(0)}_2 \underset{(\mathrm{D})}{\{} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'}\right\}$$

$$\times \underset{(E)}{[} \frac{1}{2}\left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_T{}')^{<2>}} + \mathrm{E}_{T\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0{}')^{<2>}} \right)\left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_T{}'} \right)^{<2>} \right\} \mathrm{vec}(\boldsymbol{\Omega}_T)$$

$$- \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0} \underset{(E)}{]}$$

$$+\left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_T{}')^{<2>}} + \mathrm{E}_{T\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0{}')^{<2>}} \right)\left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_T{}'} \right)^{<2>} \right\}$$

$$\times\left[\left\{ \boldsymbol{\Omega}_T \frac{\partial \boldsymbol{\alpha}_0{}'}{\partial \boldsymbol{\pi}_T} \left\{ \mathrm{E}_{T\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\} \right\} \otimes \left(\boldsymbol{\Omega}_T \frac{\partial \boldsymbol{\alpha}_0{}'}{\partial \boldsymbol{\pi}_T} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right) \right] \underset{(D)}{\}},$$

$$e_2 = \gamma_{\theta_0}^{(1)} \beta_2^{(0)} \underset{(A)}{\{} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \underset{(B)}{[} \underset{(C)}{\{} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_T{}')^{<2>}}$$

$$+\mathrm{E}_{T\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0{}')^{<2>}} \right)\left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_T{}'} \right)^{<2>} \underset{(C)}{\}} \mathrm{vec}(\boldsymbol{\Omega}_T) - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0} \underset{(B)}{]}$$

$$+2\left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_T{}')^{<2>}} + \mathrm{E}_{T\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0{}')^{<2>}} \right)\left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_T{}'} \right)^{<2>} \right\}$$

$$\times\left(\boldsymbol{\Omega}_T \frac{\partial \boldsymbol{\alpha}_0{}'}{\partial \boldsymbol{\pi}_T} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right)^{<2>} \underset{(A)}{\}},$$

the third term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_T \underset{(A)}{\{} nN^2 (\gamma_{\theta_0}^{(1)})^2 l_{\theta_0 O_p(n^{-1/2})}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m_{O_p(n^{-1/2})} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} + m_{O_p(N^{-1/2})}^{(\Delta)} l_{\theta_0 O_p(n^{-1/2})}^{(1)},\ 2 l_{\theta_0 O_p(n^{-1/2})}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}]'$$

$$\times \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \underset{(A)}{\}}$$

$$= 24N^{-1}\overline{c}\, \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[e_1, e_2]' + O(N^{-2})$$

with

$$e_1 = (3\gamma_{\theta_0}^{(1)})^2 n \operatorname{cov}(m, l_{\theta_0}^{(1)}) \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ \beta_2^{(0)} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+2\beta_2^{(0)} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$e_2 = 6\beta_2^{(0)} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

the fourth term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{nN^2(\gamma_{\theta_0}^{(1)})^3 l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} l^{(\Delta\Delta a1)}_{\theta_0 O_p(n^{-1/2}N^{-1/2})} \boldsymbol{\gamma}_{\theta_0}^{(2)} ' \mathbf{l}^{(\Delta b2)}_{\theta_0 O_p(N^{-1})}\}$$

$$= 24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} nN^2(\gamma_{\theta_0}^{(1)})^3 l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}$$

$$\times \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \right)_{O_p(n^{-1/2})} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}$$

$$\times [m^{(\Delta)}_{O_p(N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}, (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2] \boldsymbol{\gamma}_{\theta_0}^{(2)} \underset{(\mathrm{A})}{\}}$$

$$= 24N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^3 n \operatorname{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times \underset{(\mathrm{A})}{[} \underset{(\mathrm{B})}{\{} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+2\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \underset{(\mathrm{B})}{\}} (\boldsymbol{\gamma}_{\theta_0}^{(2)})_1$$

$$+ 3\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} (\boldsymbol{\gamma}_{\theta_0}^{(2)})_2 \underset{(\mathrm{A})}{]} + O(N^{-2}),$$

the fifth term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{nN^2(\gamma_{\theta_0}^{(1)})^4 l_{\theta_0 O_p(n^{-1/2})}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_0 O_p(n^{-1/2}N^{-1/2})}^{(\Delta\Delta a1)} l_{\theta_0 O_p(N^{-1})}^{(\Delta\Delta b1)}\}$$

$$= 24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{[} nN^2(\gamma_{\theta_0}^{(1)})^4 l_{\theta_0 O_p(n^{-1/2})}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0 '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}'\right)_{O_p(n^{-1/2})} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$\times \underset{(\mathrm{B})}{\{} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}'(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}}\right)(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)})^{<2>} \underset{(\mathrm{B})}{\}} \ \underset{(\mathrm{A})}{]}$$

$$= 24N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^4 \underset{(\mathrm{A})}{[} n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0 '}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$\times \underset{(\mathrm{B})}{\{} \frac{1}{2}\left\{\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}'\frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}}')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}}\right)\left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\right\}$$

$$\times \mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}}) - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0} \underset{(\mathrm{B})}{\}}$$

$$+\left\{\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}'\frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}}')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}}\right)\left(\frac{\partial \boldsymbol{\alpha}_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}\right\}$$

$$\times\left\{\left\{\frac{\partial \boldsymbol{\alpha}_0 '}{\partial \boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0}\right)\right\} \otimes \left(\frac{\partial \boldsymbol{\alpha}_0 '}{\partial \boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}\right)\right\} \underset{(\mathrm{A})}{]} + O(N^{-2}),$$

the sixth term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{nN^2(\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)} (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2 l_{\theta_0 O_p(n^{-1/2}N^{-1/2})}^{(\Delta\Delta a1)} \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}\}$$

$$= 24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} nN^2(\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)} (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2$$

$$\times\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0 '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}'\right)_{O_p(n^{-1/2})} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)} \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \underset{(\mathrm{A})}{\}}$$

$$= 24N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^3 n \operatorname{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times \left(\frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} + 2\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \right) + O(N^{-2}),$$

the seventh term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{nN^2(\gamma_{\theta_0}^{(1)})^2 (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2 l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \boldsymbol{\gamma}_{\theta_0}^{(2)} ' \mathbf{l}_{\theta_0 O_p(N^{-1})}^{(\Delta b 2)}\}$$

$$= 24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} nN^2(\gamma_{\theta_0}^{(1)})^2 (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2 l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times [m_{O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}, (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2] \boldsymbol{\gamma}_{\theta_0}^{(2)} \underset{(\mathrm{A})}{\}}$$

$$= 24N^{-1}\overline{c}\beta_2^{(0)}$$

$$\times \underset{(\mathrm{A})}{[} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+2 \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0},$$

$$3\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \underset{(\mathrm{A})}{]} \boldsymbol{\gamma}_{\theta_0}^{(2)} + O(N^{-2}),$$

the eighth term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{nN^2(\gamma_{\theta_0}^{(1)})^3 (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2 l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_0 O_p(N^{-1})}^{(\Delta\Delta b 1)}\}$$

$$= 24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{[} nN^2(\gamma_{\theta_0}^{(1)})^3 (l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2 l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times \underset{(\mathrm{B})}{\{} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} '(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+ \frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)})^{<2>} \underset{(\mathrm{B})}{\}} \underset{(\mathrm{A})}{]}$$

$$= 24N^{-1}\overline{c}\beta_2^{(0)}\gamma_{\theta_0}^{(1)}$$

$$\times \underset{(A)}{[}\ \underset{(B)}{\{}\ \frac{1}{2}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right\}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})$$

$$-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\ \underset{(B)}{\}}\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}$$

$$+\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right\}$$

$$\times\left\{\left(\frac{\partial\boldsymbol{\alpha}_0{}'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)\otimes\left(\frac{\partial\boldsymbol{\alpha}_0{}'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}\right)\right\}\ \underset{(A)}{]} + O(N^{-2}),$$

the ninth term of (*) is

$$24N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{nN^2(\gamma_{\theta_0}^{(1)})^2(l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2(\gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2\}$$

$$= 24N^{-1}\overline{c}\beta_2^{(0)}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0} + 2\left(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}\right)^2\right\}$$

$$+ O(N^{-2}).$$

Term (8):

$$\underset{(A)}{[}\ 6n^2\mathrm{E}_{\mathrm{T}}[\ (q_{O_p(N^{-1/2})}^{(11)})^2\{(q_{O_p(n^{-1})}^{(20)})^2 + (q_{O_p(n^{-1/2}N^{-1/2})}^{(21)})^2$$

$$+ 2q_{O_p(n^{-1})}^{(20)}q_{O_p(N^{-1})}^{(22)}\}]\ \underset{(A)}{]}_{O(N^{-1})+O(nN^{-2})}$$

The first term of Term (8):

$$6n^2\mathrm{E}_{\mathrm{T}}\{(q_{O_p(N^{-1/2})}^{(11)})^2(q_{O_p(n^{-1})}^{(20)})^2\}\ (\to N^{-1}\beta_4^{(\Delta a)})$$

$$= 6N^{-1}\beta_2^{(\Delta)}\mathrm{E}_{\mathrm{T}\theta_0}\{n^2(q_{O_p(n^{-1})}^{(20)})^2\}\ \text{(known)}$$

$$= 6N^{-1}\beta_2^{(\Delta)}\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathrm{E}_{\mathrm{T}\theta_0}[n^2(ml_{\theta_0}^{(1)},\ (l_{\theta_0}^{(1)})^2)'\ (ml_{\theta_0}^{(1)},\ (l_{\theta_0}^{(1)})^2)]\boldsymbol{\gamma}_{\theta_0}^{(2)}$$

$$= 6N^{-1}\beta_2^{(\Delta)}\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\begin{bmatrix} n\,\mathrm{var}(m)\lambda_{\theta_0}^{(11)} + 2\{n\,\mathrm{cov}(m,l_{\theta_0}^{(1)})\}^2 & \mathrm{sym.} \\ 3n\,\mathrm{cov}(m,l_{\theta_0}^{(1)})\lambda_{\theta_0}^{(11)} & 3(\lambda_{\theta_0}^{(11)})^2 \end{bmatrix}\boldsymbol{\gamma}_{\theta_0}^{(2)} + O(N^2)\cdot$$

The second term of Term (8):

$$6n^2\mathrm{E}_{\mathrm{T}}\{(q_{O_p(N^{-1/2})}^{(11)})^2(q_{O_p(n^{-1/2}N^{-1/2})}^{(21)})^2\}\ \ (\to N^{-1}\overline{c}\,\beta_4^{(\Delta b)})$$

$$= 6N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}[nN^2(\gamma_{\theta_0}^{(1)}l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2$$

$$\times\{(\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta a2)} + \gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta a1)} + \gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1/2})}\}^2]$$

$$= 6N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}[nN^2(\gamma_{\theta_0}^{(1)}l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2\{(\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta a2)})^2 + (\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta a1)})^2 + (\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(1)})^2$$

$$+ 2\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta a2)}(\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta a1)} + \gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(1)}) + 2\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta a1)}\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(1)}\}_{O_p(n^{-1}N^{-1})}],\quad (*)$$

the first term of (*) is ($m^{(\Delta)} = 0$ under m.m.)

$$= 6N^{-1}\overline{c}\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}\boldsymbol{\gamma}_{\theta_0}^{(2)} + O(N^{-2})$$

with

$$e_{11} = (\gamma_{\theta_0}^{(1)})^2\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}\ nN^2(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2\ \underset{(\mathrm{B})}{[}\ m_{O_p(n^{-1/2})}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$+\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}_{O(1)}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}l_{\theta_0 O_p(n^{-1/2})}^{(1)}\ \underset{(\mathrm{B})}{]}^2\ \underset{(\mathrm{A})}{\}}$$

$$= 3(\gamma_{\theta_0}^{(1)})^2 n\,\mathrm{var}(m)(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2 + 6(\gamma_{\theta_0}^{(1)})^2 n\,\mathrm{cov}(m,l_{\theta_0}^{(1)})$$

$$\times\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+\beta_2^{(0)}\underset{(\mathrm{A})}{[}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+2\left[\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right]^2 \underset{(\mathrm{A})}{]}$$

$$e_{21} = (\gamma_{\theta_0}^{(1)})^2 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} nN^2 (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2 2 l^{(1)}_{\theta_0 O_p(n^{-1/2})} \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}$$

$$\times \underset{(\mathrm{B})}{[} m_{O_p(n^{-1/2})}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})} + \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}_{O(1)}$$

$$\times\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})} l^{(1)}_{\theta_0 O_p(n^{-1/2})} \underset{(\mathrm{B})}{]} \quad \underset{(\mathrm{A})}{\}}$$

$$= 6(\gamma_{\theta_0}^{(1)})^2 n\,\mathrm{cov}(m, l^{(1)}_{\theta_0})(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2$$

$$+6\beta_2^{(0)}\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$e_{22} = (\gamma_{\theta_0}^{(1)})^2 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} nN^2 (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2 4 (l^{(1)}_{\theta_0 O_p(n^{-1/2})} \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})})^2 \underset{(\mathrm{A})}{\}}$$

$$= 12\beta_2^{(0)}(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2,$$

the second term of (*) is

$$= 6N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^4 \mathrm{E}_{\mathrm{T}}\{nN^2 (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2 (l^{(\Delta\Delta a 1)}_{\theta_0 O_p(n^{-1/2}N^{-1/2})})^2\}$$

$$= 6N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^4 \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} nN^2 (l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2$$

$$\times \underset{(\mathrm{B})}{[} \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\right)_{O_p(n^{-1/2})} \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})} \underset{(\mathrm{B})}{]}^2 \underset{(\mathrm{A})}{\}}$$

$$= 6N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^4 \underset{(A)}{[} \mathrm{tr}\left\{ n\,\mathrm{cov}\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \right\} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ 2\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} n\,\mathrm{cov}\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \underset{(A)}{]} + O(N^{-2}),$$

the third term of (*) is

$$= 6N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^2 \mathrm{E_T}\{ nN^2 (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2 (\gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_0 O_p(n^{-1/2})}^{(1)})^2 \}$$

$$= 6N^{-1}\overline{c}\beta_2^{(0)} \left\{ \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} + 2\left(\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \right)^2 \right\}$$

$$+ O(N^{-2}),$$

the fourth term of (*) is ($m^{(\Delta)} = 0$ under m.m.)

$$= 12N^{-1}\overline{c}\mathrm{E_T}\{ nN^2 (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2 \boldsymbol{\gamma}_{\theta_0}^{(2)} ' \mathbf{l}_{\theta_0}^{(\Delta a 2)} (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta a 1)} + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(1)}) \}$$

$$= 12N^{-1}\overline{c}\mathrm{E_T} \underset{(A)}{\{} nN^2 (\gamma_{\theta_0}^{(1)})^2 (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2$$

$$\times \boldsymbol{\gamma}_{\theta_0}^{(2)} ' \underset{(B)}{[} m_{O_p(n^{-1/2})} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} + \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\}_{O(1)}$$

$$\times \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)} l_{\theta_0 O_p(n^{-1/2})}^{(1)}, \quad 2 l_{\theta_0 O_p(n^{-1/2})}^{(1)} l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} \underset{(B)}{]} '$$

$$\times \underset{(C)}{[} \gamma_{\theta_0}^{(1)} \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \right)_{O_p(n^{-1/2})} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$+ \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)} l_{\theta_0 O_p(n^{-1/2})}^{(1)} \underset{(C)}{]} \underset{(A)}{\}}$$

$$= 12N^{-1}\overline{c} \underset{(A)}{[} (\boldsymbol{\gamma}_{\theta_0}^{(2)})_1 \underset{(B)}{\{} 3(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}\left(m, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+3(\gamma_{\theta_0}^{(1)})^2 n\,\mathrm{cov}(m, l_{\theta_0}^{(1)}) \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \underset{(\mathrm{C})}{[} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\}$$

$$\times \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} + 2 \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \underset{(\mathrm{C})}{]}$$

$$+\beta_2^{(0)} \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right\} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times \left(\frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} + 2 \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0} \right) \underset{(\mathrm{B})}{\}}$$

$$+6(\boldsymbol{\gamma}_{\theta_0}^{(2)})_2 \underset{(\mathrm{D})}{[} (\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ \beta_2^{(0)} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'} \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \underset{(\mathrm{D})}{]} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \underset{(\mathrm{A})}{]} + O(N^{-2}),$$

the fifth term of (*) is

$$= 12 N^{-1} \overline{c} \mathrm{E}_{\mathrm{T}} \{ n N^2 (\gamma_{\theta_0}^{(1)})^3 (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2 l_{\theta_0 O_p(n^{-1/2} N^{-1/2})}^{(\Delta\Delta a 1)} \gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_0 O_p(n^{-1/2})}^{(1)} \}$$

$$= 12 N^{-1} \overline{c} \mathrm{E}_{\mathrm{T}} \underset{(\mathrm{A})}{\{} n N^2 (\gamma_{\theta_0}^{(1)})^3 (l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2 \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \right)_{O_p(n^{-1/2})}$$

$$\times \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)} \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_0{}'} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)} l_{\theta_0 O_p(n^{-1/2})}^{(1)} \underset{(\mathrm{A})}{\}}$$

$$= 12 N^{-1} \overline{c} (\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0{}'} \right) \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times\left(\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}+2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}\right)+O(N^{-2}).$$

The third term of Term (8):

$$12n^2\mathrm{E}_{\mathrm{T}}\{(q_{O_p(N^{-1/2})}^{(11)})^2 q_{O_p(n^{-1})}^{(20)} q_{O_p(N^{-1})}^{(22)}\}\ (\to N^{-1}\overline{c}\beta_4^{(\Delta b)})$$

$$=12N^{-1}\overline{c}(\beta_1^{(0)}+\lambda_{\theta_0}^{-1}\eta_{\theta_0})\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{N^2(q_{O_p(N^{-1/2})}^{(11)})^2 q_{O_p(N^{-1})}^{(22)}\},$$

where $\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{\cdot\}$ was given earlier in $\beta_3^{(\Delta b)}$.

Term (9): ($m^{(\Delta)}=0$ under m.m.)

$$[6n^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{\,(q_{O_p(N^{-1/2})}^{(11)})^2(q_{O_p(N^{-1})}^{(22)})^2\,\}]_{O(n^2N^{-3})}\ (\to N^{-1}\overline{c}^2\beta_4^{(\Delta c)})$$

$$=6N^{-1}\overline{c}^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\,N^3(\gamma_{\theta_0}^{(1)})^2(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2$$

$$\times\{(\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta b2)}+\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta b1)}+\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1})}\}^2\ ]$$

$$=6N^{-1}\overline{c}^2(\gamma_{\theta_0}^{(1)})^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\underset{(\mathrm{A})}{[}\ N^3(l_{\theta_0}^{(\Delta 1)})^2\underset{(\mathrm{B})}{\{}\ \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'[m^{(\Delta)}l_{\theta_0}^{(\Delta 1)},(l_{\theta_0}^{(\Delta 1)})^2]'$$

$$+\gamma_{\theta_0}^{(1)}\underset{(\mathrm{C})}{[}\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)}-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)$$

$$\times(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)})^{<2>}\ \underset{(\mathrm{C})}{]}+\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}l_{\theta_0}^{(\Delta 1)}\ \underset{(\mathrm{B})}{\}}^2\ \underset{(\mathrm{A})}{]}$$

$$=6N^{-1}\overline{c}^2(\gamma_{\theta_0}^{(1)})^2\ \underset{(\mathrm{A})}{[}\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}(\gamma_{\theta_0}^{(1)}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})^2$$

$$-2\gamma_{\theta_0}^{(1)}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\ \underset{(\mathrm{B})}{\{}\ \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\ \underset{(\mathrm{C})}{[}\ 3\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$\times\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},\ 3(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2\ \underset{(\mathrm{C})}{]}'$$

$$+\gamma_{\theta_0}^{(1)}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right\}$$

$$\times\left\{\frac{1}{2}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}+\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0{}'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)^{<2>}\right\}$$

$$+3\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\underset{(\mathrm{B})}{\}}$$

$$+\sum_{i^*,j,k,l^*,m^*,n^*=1}^{2^n}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}i^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}j}}$$

$$\times\underset{(\mathrm{D})}{\{}\ \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\left[\ \left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}k}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}},\right.$$

$$\left.\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}k}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\right]'$$

$$+\gamma_{\theta_0}^{(1)}\frac{1}{2}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}k}\partial\pi_{\mathrm{T}l^*}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}k}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\right)\right\}$$

$$+\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}k}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\underset{(\mathrm{D})}{\}}$$

$$\times\underset{(\mathrm{E})}{\{}\ \boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\left[\ \left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}},\right.$$

$$\left.\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\right]'$$

$$+\gamma_{\theta_0}^{(1)}\frac{1}{2}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}\partial\pi_{\mathrm{T}n^*}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\right)\right\}$$

$$+\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\underset{(\mathrm{E})}{\}}$$

$$\times\sum_{(i^*,j,k,l^*,m^*,n^*)}^{15}(\boldsymbol{\Omega}_{\mathrm{T}})_{i^*j}(\boldsymbol{\Omega}_{\mathrm{T}})_{kl^*}(\boldsymbol{\Omega}_{\mathrm{T}})_{m^*n^*}\underset{(\mathrm{A})}{]}+O(N^{-2}).$$

Term (10):

$$[4n^2\mathrm{E_T}\{(q^{(10)}_{O_p(n^{-1/2})})^3 q^{(32)}_{O_p(n^{-1/2}N^{-1})}\}]_{O(N^{-1})} (\to N^{-1}\beta_4^{(\Delta a)})$$

$$= 12N^{-1}\beta_2^{(0)}\mathrm{E_T}(Nq^{(10)}_{O_p(n^{-1/2})} q^{(32)}_{O_p(n^{-1/2}N^{-1})}),$$

where $\mathrm{E_T}(\cdot)$ was given earlier in Terms (7) to (12) of $\beta^{(\Delta a)}_{\mathrm{H}2}$ in (a.2.2).

Term (11):

$$[4n^2\mathrm{E_T} \underset{(\mathrm{A})}{\{} 3(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})}(q^{(31)}_{O_p(n^{-1}N^{-1/2})} + q^{(33)}_{O_p(N^{-3/2})}$$

$$-\{(n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0})^{(\Delta)}\}_{O_p(n^{-1}N^{-1/2})}) \underset{(\mathrm{A})}{\}}]_{O(N^{-1})+O(nN^{-2})}$$

The first term of Term (11):

$$12n^2\mathrm{E_T}\{(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})} q^{(31)}_{O_p(n^{-1}N^{-1/2})}\} \; (\to N^{-1}\beta_4^{(\Delta a)})$$

$$= 12N^{-1}(\gamma^{(1)}_{\theta_0})^3 \underset{(\mathrm{A})}{[} \; \mathrm{E_T}\{Nn^2(l^{(1)}_{\theta_0})^2 l^{(\Delta 1)}_{\theta_0} \mathbf{l}^{(\Delta a3)}_{\theta_0}{}'\}\boldsymbol{\gamma}^{(3)}_{\theta_0}$$

$$+ \mathrm{E_T}\{Nn^2(l^{(1)}_{\theta_0})^2 l^{(\Delta 1)}_{\theta_0} \mathbf{l}^{(\Delta\Delta a2)}_{\theta_0}{}'\}\boldsymbol{\gamma}^{(2)}_{\theta_0} + \mathrm{E_T}\{Nn^2(l^{(1)}_{\theta_0})^2 l^{(\Delta 1)}_{\theta_0} \boldsymbol{\gamma}^{(\Delta 2)}_{\theta_0}{}' \mathbf{l}^{(2)}_{\theta_0}\}$$

$$- \frac{\partial \lambda^{-1}_{\theta_0}\eta_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathrm{E_T}\{Nn\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0} l^{(\Delta 1)}_{\theta_0}(l^{(1)}_{\theta_0})^2\} \underset{(\mathrm{A})}{]}, \quad (*)$$

where Term (13) of $\beta^{(\Delta a)}_{\mathrm{H}2}$ in (a.2.2) can be used here, but it is not used since the use does not yield much simplification,

the first term of (*) is ($m^{(\Delta)} = m^{(\Delta 3)} = 0$ under m.m.)

$$= 12N^{-1}(\gamma^{(1)}_{\theta_0})^3 \mathrm{E_T}\{Nn^2(l^{(1)}_{\theta_0})^2 l^{(\Delta 1)}_{\theta_0} \mathbf{l}^{(\Delta a3)}_{\theta_0}{}'\}\boldsymbol{\gamma}^{(3)}_{\theta_0}$$

$$= 12N^{-1}\mathrm{E_T} \underset{(\mathrm{A})}{\{} \; Nn^2(\gamma^{(1)}_{\theta_0})^3 (l^{(1)}_{\theta_0})^2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}$$

$$\times \underset{(\mathrm{B})}{[} \; 2m\, m^{(\Delta)}_{O_p(N^{-1/2})} l^{(1)}_{\theta_0} + m^2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}, 2m l^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta)}_{O_p(N^{-1/2})}(l^{(1)}_{\theta_0})^2,$$

$$2m^{(3)} l^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta 3)}_{O_p(N^{-1/2})}(l^{(1)}_{\theta_0})^2, 3(l^{(1)}_{\theta_0})^2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})},$$

$$n^{-1}(m^{(\Delta)}, l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1/2})} \underset{(\mathrm{B})}{]} \underset{(\mathrm{A})}{\}} \; \boldsymbol{\gamma}^{(3)}_{\theta_0}$$

$$= 12N^{-1} \underset{(A)}{[} 6\gamma_{\theta_0}^{(1)}\beta_2^{(0)} n\,\mathrm{cov}(m, l_{\theta_0}^{(1)}) \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\}$$

$$+ \{\gamma_{\theta_0}^{(1)}\beta_2^{(0)} n\,\mathrm{var}(m) + 2(\gamma_{\theta_0}^{(1)})^3 (n\,\mathrm{cov}(m, l_{\theta_0}^{(1)}))^2\} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$6\gamma_{\theta_0}^{(1)}\beta_2^{(0)} n\,\mathrm{cov}(m, l_{\theta_0}^{(1)}) \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+3(\gamma_{\theta_0}^{(1)})^{-1}(\beta_2^{(0)})^2 \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\},$$

$$6\gamma_{\theta_0}^{(1)}\beta_2^{(0)} n\,\mathrm{cov}(m^{(3)}, l_{\theta_0}^{(1)}) \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}$$

$$+ 3(\gamma_{\theta_0}^{(1)})^{-1}(\beta_2^{(0)})^2 \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0^3 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial}{\partial \boldsymbol{\alpha}_0} \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^3} \right) \right\},$$

$$9(\gamma_{\theta_0}^{(1)})^{-1}(\beta_2^{(0)})^2 \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0},$$

$$\gamma_{\theta_0}^{(1)}\beta_2^{(0)} \left[\boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_0} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\}, \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} \right] \underset{(A)}{]} \boldsymbol{\gamma}_{\theta_0}^{(3)}$$

$$+O(N^{-2}),$$

the second term of (*) is

$$= 12N^{-1}(\gamma_{\theta_0}^{(1)})^3 \mathrm{E}_{\mathrm{T}}\{Nn^2 (l_{\theta_0}^{(1)})^2 l_{\theta_0}^{(\Delta 1)} \mathbf{l}_{\theta_0}^{(\Delta\Delta a2)}{}'\} \boldsymbol{\gamma}_{\theta_0}^{(2)}$$

$$= 12N^{-1}\mathrm{E}_{\mathrm{T}} \underset{(A)}{\{} Nn^2 (\gamma_{\theta_0}^{(1)})^3 (l_{\theta_0}^{(1)})^2 l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times [m\, l_{\theta_0 O_p(n^{-1/2}N^{-1/2})}^{(\Delta\Delta a1)} + m_{O_p(n^{-1/2}N^{-1/2})}^{(\Delta\Delta a)} l_{\theta_0}^{(1)},\ 2 l_{\theta_0}^{(1)} l_{\theta_0 O_p(n^{-1/2}N^{-1/2})}^{(\Delta\Delta a1)}] \boldsymbol{\gamma}_{\theta_0}^{(2)} \underset{(A)}{\}}$$

$$= 12N^{-1}\mathrm{E}_{\mathrm{T}} \underset{(A)}{\{} Nn^2 (\gamma_{\theta_0}^{(1)})^3 (l_{\theta_0}^{(1)})^2 l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times \underset{(B)}{[}\ m \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \right) \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}$$

$$+ \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial \boldsymbol{\alpha}_0{}'} - \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}{}' \right) \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)} l_{\theta_0}^{(1)},$$

$$2l^{(1)}_{\theta_0}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\right)\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}\underset{(\mathrm{B})}{]}\underset{(\mathrm{A})}{\}}\boldsymbol{\gamma}^{(2)}_{\theta_0}$$

(note that $l^{(\Delta\Delta a1)}_{\theta_0}=m^{(\Delta\Delta a)}$)

$$=12N^{-1}\underset{(\mathrm{A})}{[}\left\{\gamma^{(1)}_{\theta_0}\beta^{(0)}_2 n\,\mathrm{cov}\left(m,\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\right.$$

$$\left.+2(\gamma^{(1)}_{\theta_0})^3 n\,\mathrm{cov}(m,l^{(1)}_{\theta_0})\,n\,\mathrm{cov}\left(l^{(1)}_{\theta_0},\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+3\gamma^{(1)}_{\theta_0}\beta^{(0)}_2 n\,\mathrm{cov}\left(l^{(1)}_{\theta_0},\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

$$6\gamma^{(1)}_{\theta_0}\beta^{(0)}_2 n\,\mathrm{cov}\left(l^{(1)}_{\theta_0},\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\underset{(\mathrm{A})}{]}\boldsymbol{\gamma}^{(2)}_{\theta_0}+O(N^{-2}),$$

the third term of (*) is

$$=12N^{-1}(\gamma^{(1)}_{\theta_0})^3\mathrm{E}_{\mathrm{T}}\{Nn^2(l^{(1)}_{\theta_0})^2 l^{(\Delta 1)}_{\theta_0}\boldsymbol{\gamma}^{(\Delta 2)}_{\theta_0}{}'\mathbf{l}^{(2)}_{\theta_0}\}$$

$$=12N^{-1}\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}Nn^2(\gamma^{(1)}_{\theta_0})^3(l^{(1)}_{\theta_0})^2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}$$

$$\times\left(\frac{\partial\boldsymbol{\gamma}^{(2)}_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}\right)'[m\,l^{(1)}_{\theta_0},\ (l^{(1)}_{\theta_0})^2]'\underset{(\mathrm{A})}{\}}$$

$$=36N^{-1}\{\gamma^{(1)}_{\theta_0}\beta^{(0)}_2 n\,\mathrm{cov}(m,l^{(1)}_{\theta_0}),\ (\gamma^{(1)}_{\theta_0})^{-1}(\beta^{(0)}_2)^2\}\frac{\partial\boldsymbol{\gamma}^{(2)}_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}+(N^{-2}),$$

the fourth term of (*) is

$$=-12N^{-1}(\gamma^{(1)}_{\theta_0})^3\frac{\partial\lambda^{-1}_{\theta_0}\eta_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathrm{E}_{\mathrm{T}}\{Nn\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0}l^{(\Delta 1)}_{\theta_0}(l^{(1)}_{\theta_0})^2\}$$

$$=-12N^{-1}\gamma^{(1)}_{\theta_0}\beta^{(0)}_2\frac{\partial\lambda^{-1}_{\theta_0}\eta_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}.$$

The second term of Term (11):

$$12n^2\mathrm{E_T}\{(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})} q^{(33)}_{O_p(N^{-3/2})}\}\ (\to N^{-1}\overline{c}\beta_4^{(\Delta b)})$$

$$=12N^{-1}\overline{c}\beta_2^{(0)}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(N^2 q^{(11)}_{O_p(N^{-1/2})} q^{(33)}_{O_p(N^{-3/2})}),$$

where $\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\cdot)$ was given earlier in Terms (10) to (15) of $\beta_{\mathrm{H2}}^{(\Delta b)}$ in (a.2.3),

the third term of Term (11):

$$-12n^2\mathrm{E_T}[(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})}\{(n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})^{(\Delta)}\}_{O_p(n^{-1}N^{-1/2})}]\ (\to N^{-1}\beta_4^{(\Delta a)})$$

$$=-12N^{-1}\mathrm{E_T}\left\{n(\gamma_{\theta_0}^{(1)})^3(l_{\theta_0}^{(1)})^2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}\frac{\partial\lambda_{\theta_0}^{-1}\eta_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}\right\}$$

$$=-12N^{-1}\gamma_{\theta_0}^{(1)}\beta_2^{(0)}\frac{\partial\lambda_{\theta_0}^{-1}\eta_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}.$$

Term (12):

$$[4n^2\mathrm{E_T}\underset{(\mathrm{A})}{\{}3q^{(10)}_{O_p(n^{-1/2})}(q^{(11)}_{O_p(N^{-1/2})})^2(q^{(30)}_{O_p(n^{-3/2})}+q^{(32)}_{O_p(n^{-1/2}N^{-1})})\underset{(\mathrm{A})}{\}}]_{O(N^{-1})+O(nN^{-2})}$$

The first term of Term (12):

$$12n^2\mathrm{E_T}\{q^{(10)}_{O_p(n^{-1/2})}(q^{(11)}_{O_p(N^{-1/2})})^2 q^{(30)}_{O_p(n^{-3/2})}\}\ (\to N^{-1}\beta_4^{(\Delta a)})$$

$$=12N^{-1}\mathrm{E_T}\{Nn^2(\gamma_{\theta_0}^{(1)} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2 q^{(10)}_{O_p(n^{-1/2})} q^{(30)}_{O_p(n^{-3/2})}\}$$

$$=12N^{-1}(\gamma_{\theta_0}^{(1)})^2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(n^2 q^{(10)}_{O_p(n^{-1/2})} q^{(30)}_{O_p(n^{-3/2})}),$$

where $\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\cdot)$ is known in $\beta_{\mathrm{H2}}^{(0)}$,

the second term of Term (12):

$$12n^2\mathrm{E_T}\{q^{(10)}_{O_p(n^{-1/2})}(q^{(11)}_{O_p(N^{-1/2})})^2 q^{(32)}_{O_p(n^{-1/2}N^{-1})}\}\ (\to N^{-1}\overline{c}\beta_4^{(\Delta b)})$$

$$=12N^{-1}\overline{c}\mathrm{E_T}\underset{(\mathrm{A})}{\{}N^2 n(\gamma_{\theta_0}^{(1)})^3 l^{(1)}_{\theta_0 O_p(n^{-1/2})}(l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2$$

$$\times(\boldsymbol{\gamma}_{\theta_0}^{(3)}{}'\mathbf{l}_{\theta_0}^{(\Delta b3)}+\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(\Delta\Delta b2)}+\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)}{}'\mathbf{l}_{\theta_0}^{(\Delta a2)}$$

$$+\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(\Delta\Delta\Delta a1)}+\gamma_{\theta_0}^{(\Delta 1)}l_{\theta_0}^{(\Delta\Delta a1)}+\gamma_{\theta_0}^{(\Delta\Delta 1)}l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1})}\underset{(\mathrm{A})}{\}}, \ (*)$$

the first term of (*) is ($m^{(\Delta)}=0$ under m.m.)

$$=12N^{-1}\overline{c}\mathrm{E_T}\{N^2 n(\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2\boldsymbol{\gamma}_{\theta_0}^{(3)}{}'\mathbf{l}_{\theta_0 O_p(n^{-1/2}N^{-1})}^{(\Delta b3)}\}$$

$$=12N^{-1}\overline{c}\mathrm{E_T}\underset{(\mathrm{A})}{\{}N^2 n(\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2$$

$$\times\underset{(\mathrm{B})}{[}\ 2m_{O_p(n^{-1/2})}m^{(\Delta)}l_{\theta_0}^{(\Delta 1)}+(m^{(\Delta)})^2 l_{\theta_0 O_p(n^{-1/2})}^{(1)},$$

$$2m^{(\Delta)}l_{\theta_0 O_p(n^{-1/2})}^{(1)}l_{\theta_0}^{(\Delta 1)}+m_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta 1)})^2,$$

$$2m^{(\Delta 3)}l_{\theta_0 O_p(n^{-1/2})}^{(1)}l_{\theta_0}^{(\Delta 1)}+m_{O_p(n^{-1/2})}^{(3)}(l_{\theta_0}^{(\Delta 1)})^2,$$

$$3(l_{\theta_0}^{(\Delta 1)})^2 l_{\theta_0 O_p(n^{-1/2})}^{(1)},(0,0)\ ]_{(\mathrm{B})O_p(n^{-1/2}N^{-1})}\ \underset{(\mathrm{A})}{\}}\ \boldsymbol{\gamma}_{\theta_0}^{(3)}$$

$$=12N^{-1}\overline{c}$$

$$\times\underset{(\mathrm{A})}{[}\ 6(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}(m,l_{\theta_0}^{(1)})N\,\mathrm{cov}(m^{(\Delta)},l_{\theta_0}^{(\Delta 1)})N\,\mathrm{var}(l_{\theta_0}^{(\Delta 1)})$$

$$+\gamma_{\theta_0}^{(1)}\beta_2^{(0)}\{N\,\mathrm{var}(m^{(\Delta)})N\,\mathrm{var}(l_{\theta_0}^{(\Delta 1)})+2(N\,\mathrm{cov}(m^{(\Delta)},l_{\theta_0}^{(\Delta 1)}))^2\},$$

$$6\gamma_{\theta_0}^{(1)}\beta_2^{(0)}N\,\mathrm{cov}(m^{(\Delta)},l_{\theta_0}^{(\Delta 1)})N\,\mathrm{var}(l_{\theta_0}^{(\Delta 1)})$$

$$+3(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}(m,l_{\theta_0}^{(1)})\{N\,\mathrm{var}(l_{\theta_0}^{(\Delta 1)})\}^2,$$

$$6\gamma_{\theta_0}^{(1)}\beta_2^{(0)}N\,\mathrm{cov}(m^{(\Delta 3)},l_{\theta_0}^{(\Delta 1)})N\,\mathrm{var}(l_{\theta_0}^{(\Delta 1)})$$

$$+3(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}(m^{(3)},l_{\theta_0}^{(1)})\{N\,\mathrm{var}(l_{\theta_0}^{(\Delta 1)})\}^2,$$

$$9\gamma_{\theta_0}^{(1)}\beta_2^{(0)}\{N\,\mathrm{var}(l_{\theta_0}^{(\Delta 1)})\}^2,(0,0)\ \underset{(\mathrm{A})}{]}\ \boldsymbol{\gamma}_{\theta_0}^{(3)}+O(N^{-2}),$$

where

$$N\,\mathrm{cov}(m^{(\Delta)},l_{\theta_0}^{(\Delta 1)})=\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

$$N\,\mathrm{var}(l_{\theta_0}^{(\Delta 1)}) = \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

$$N\,\mathrm{var}(m^{(\Delta)}) = \left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\{\cdot\}',$$

$$N\,\mathrm{cov}(m^{(\Delta 3)}, l_{\theta_0}^{(\Delta 1)}) = \left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4\overline{l}_{\theta_0}}{\partial\theta_0^3\partial\boldsymbol{\alpha}_0{}'}\right) - \frac{\partial}{\partial\boldsymbol{\alpha}_0{}'}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^3}\right)\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

the second term of (*) is ($m^{(\Delta)} = 0$ under m.m.)

$$= 12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{N^2 n(\gamma_{\theta_0}^{(1)})^3 l^{(1)}_{\theta_0 O_p(n^{-1/2})}(l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}^{(\Delta\Delta b2)}_{\theta_0 O_p(n^{-1/2}N^{-1})}\}$$

$$= 12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}\ N^2 n(\gamma_{\theta_0}^{(1)})^3 l^{(1)}_{\theta_0 O_p(n^{-1/2})}(l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2$$

$$\times \underset{(\mathrm{B})}{[}\ m_{O_p(n^{-1/2})} l^{(\Delta\Delta b1)}_{\theta_0 O_p(N^{-1})} + m^{(\Delta)}_{O_p(N^{-1/2})} l^{(\Delta\Delta a1)}_{\theta_0 O_p(n^{-1/2}N^{-1/2})}$$

$$+ m^{(\Delta\Delta a)}_{O_p(n^{-1/2}N^{-1/2})} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} + m^{(\Delta\Delta b)}_{O_p(N^{-1})} l^{(1)}_{\theta_0 O_p(n^{-1/2})},$$

$$2 l^{(1)}_{\theta_0 O_p(n^{-1/2})} l^{(\Delta\Delta b1)}_{\theta_0 O_p(N^{-1})} + 2 l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} l^{(\Delta\Delta a1)}_{\theta_0 O_p(n^{-1/2}N^{-1/2})}\ \underset{(\mathrm{B})}{]}\ \underset{(\mathrm{A})}{\}}\ \boldsymbol{\gamma}_{\theta_0}^{(2)}$$

$$= 12N^{-1}\overline{c}[e_1,\, e_2]\boldsymbol{\gamma}_{\theta_0}^{(2)} + O(N^{-2})$$

with

$$e_1 = (\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}(m, l_{\theta_0}^{(1)})$$

$$\times \underset{(\mathrm{A})}{[}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right\}$$

$$\times\left\{\frac{1}{2}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} + \left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0{}'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)^{<2>}\right\}$$

$$-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\ \underset{(\mathrm{A})}{]}$$

$$+(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\boldsymbol{\alpha}_0{}'} \right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \underset{(\mathrm{B})}{[} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial\boldsymbol{\alpha}_0} \right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0} \right\}$$

$$\times \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} + 2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial\boldsymbol{\alpha}_0{}'} \right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'} \right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} \underset{(\mathrm{B})}{]}$$

$$+3(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\boldsymbol{\alpha}_0{}'} \right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+\gamma_{\theta_0}^{(1)}\beta_2^{(0)} \underset{(\mathrm{C})}{[} \underset{(\mathrm{D})}{\{} \left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial\boldsymbol{\alpha}_0} \right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0} \right\} \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}$$

$$+\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0^2 (\partial\boldsymbol{\alpha}_0{}')^{<2>}} \right) - \frac{\partial^2\lambda_{\theta_0}}{(\partial\boldsymbol{\alpha}_0{}')^{<2>}} \right\}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'} \right)^{<2>} \underset{(\mathrm{D})}{\}}$$

$$\times\left\{ \frac{1}{2}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} + \left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0{}'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} \right)^{<2>} \right\}$$

$$-\left\{ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0^2 \partial\boldsymbol{\alpha}_0{}'} \right) - \frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'} \right\}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} \underset{(\mathrm{C})}{]},$$

$$e_2 = 2\gamma_{\theta_0}^{(1)}\beta_2^{(0)}$$

$$\times \underset{(\mathrm{A})}{[} \left\{ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}} + \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial\boldsymbol{\alpha}_0{}')^{<2>}} \right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'} \right)^{<2>} \right\}$$

$$\times\left\{ \frac{1}{2}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} + \left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0{}'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} \right)^{<2>} \right\}$$

$$-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0} \underset{(\mathrm{A})}{]}$$

$$+6(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\boldsymbol{\alpha}_0{}'} \right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0},$$

the third term of (*) is ($m^{(\Delta)}=0$ under m.m.)

$$=12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{N^2 n(\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2 \boldsymbol{\gamma}_{\theta_0 O_p(N^{-1/2})}^{(\Delta 2)}{}'\mathbf{l}_{\theta_0 O_p(n^{-1/2}N^{-1/2})}^{(\Delta a 2)}\}$$

$$=12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}N^2 n(\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2\left(\frac{\partial\boldsymbol{\gamma}_{\theta_0}^{(2)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}\right)'$$

$$\times[m_{O_p(n^{-1/2})}l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}+m_{O_p(N^{-1/2})}^{(\Delta)}l_{\theta_0 O_p(n^{-1/2})}^{(1)},\ 2l_{\theta_0 O_p(n^{-1/2})}^{(1)}l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}]'\underset{(\mathrm{A})}{\}}$$

$$=12N^{-1}\overline{c}\underset{(\mathrm{A})}{[}3(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}(m,l_{\theta_0}^{(1)})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1}{\partial\boldsymbol{\alpha}_0}$$

$$+\gamma_{\theta_0}^{(1)}\beta_2^{(0)}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times\left\{\frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1}{\partial\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}+2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right\}$$

$$+6\gamma_{\theta_0}^{(1)}\beta_2^{(0)}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\partial(\boldsymbol{\gamma}_{\theta_0}^{(2)})_2}{\partial\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\underset{(\mathrm{A})}{]}+O(N^{-2}),$$

the fourth term of (*) is

$$12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{N^2 n(\gamma_{\theta_0}^{(1)})^4 l_{\theta_0 O_p(n^{-1/2})}^{(1)}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2 l_{\theta_0 O_p(n^{-1/2}N^{-1})}^{(\Delta\Delta\Delta a 1)}\}$$

$$=12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{[}N^2 n(\gamma_{\theta_0}^{(1)})^4 l_{\theta_0 O_p(n^{-1/2})}^{(1)}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2$$

$$\times\underset{(\mathrm{B})}{\{}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0{}'}-\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\right)_{O_p(n^{-1/2})}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)}-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+\frac{1}{2}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}-\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)\right)_{O_p(n^{-1/2})}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)})^{<2>}\underset{(\mathrm{B})}{\}}\underset{(\mathrm{A})}{]}$$

$$=12N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^4$$

$$\times \underset{(A)}{[}\ n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0 \partial\,\boldsymbol{\alpha}_0\,'} \right) \underset{(B)}{\{}\ \left(\frac{1}{2}\frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}\,')^{<2>}}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}}) - \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0} \right)$$

$$\times \boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}\,'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0} + \frac{\partial^2 \boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}\,')^{<2>}}\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial \boldsymbol{\alpha}_0\,'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0} \right)^{<2>} \underset{(B)}{\}}$$

$$+ n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial\,\boldsymbol{\alpha}_0\,')^{<2>}} \right) \underset{(C)}{\{}\ \left(\frac{\partial \boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}\,'} \right)^{<2>}$$

$$\times \left\{ \frac{1}{2}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}\,'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0} + \left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial \boldsymbol{\alpha}_0\,'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0} \right)^{<2>} \right\} \underset{(C)}{\}}\ \underset{(A)}{]} + O(N^{-2}),$$

where

$$n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0 (\partial\,\boldsymbol{\alpha}_0)^{<2>}} \right) = n^{-1}\sum_{k=1}^{n}\frac{P_{\mathrm{T}k}Q_{\mathrm{T}k}}{P_k Q_k}\sum_{P(Q)}^{2}\left\{ \frac{2}{P_k^3}\frac{\partial P_k}{\partial\theta_0}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} \right)^{<2>} \right.$$

$$\left. -\frac{1}{P_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_0)^{<2>}} + \sum_{\otimes}^{2}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^2 P_k}{\partial\theta_0\partial\,\boldsymbol{\alpha}_0} \right) + \frac{1}{P_k}\frac{\partial^3 P_k}{\partial\theta_0(\partial\,\boldsymbol{\alpha}_0)^{<2>}} \right\}\frac{\partial P_k}{\partial\theta_0},$$

the fifth term of (*) is

$$=12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{N^2 n(\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2\gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)} l_{\theta_0 O_p(n^{-1/2}N^{-1/2})}^{(\Delta\Delta a1)}\}$$

$$=12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\ \underset{(A)}{\{}\ N^2 n(\gamma_{\theta_0}^{(1)})^3 l_{\theta_0 O_p(n^{-1/2})}^{(1)}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^2\gamma_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)}$$

$$\times\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0\partial\,\boldsymbol{\alpha}_0\,'} - \boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}\,' \right)_{O_p(n^{-1/2})} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}\ \underset{(A)}{\}}$$

$$=12N^{-1}\overline{c}(\gamma_{\theta_0}^{(1)})^3 n\,\mathrm{cov}\left(l_{\theta_0}^{(1)}, \frac{\partial^2 \overline{l}_{\theta_0}}{\partial\theta_0\partial\,\boldsymbol{\alpha}_0\,'} \right)\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}$$

$$\times\left(\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}\,'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0} + 2\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0\,'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\,\boldsymbol{\alpha}_0} \right) + O(N^{-2}),$$

the sixth term of (*) is

$$=12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\{N^{2}n(\gamma_{\theta_0}^{(1)})^{3}(l_{\theta_0 O_p(n^{-1/2})}^{(1)})^{2}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^{2}\gamma_{\theta_0 O_p(N^{-1})}^{(\Delta\Delta 1)}\}$$

$$=12N^{-1}\overline{c}\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}N^{2}n(\gamma_{\theta_0}^{(1)})^{3}(l_{\theta_0 O_p(n^{-1/2})}^{(1)})^{2}(l_{\theta_0 O_p(N^{-1/2})}^{(\Delta 1)})^{2}$$

$$\times\underset{(\mathrm{B})}{[}\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)}-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+\frac{1}{2}\frac{\partial^{2}\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0{}')^{<2>}}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)})^{<2>}\underset{(\mathrm{B})}{]}\underset{(\mathrm{A})}{\}}$$

$$=12N^{-1}\overline{c}\gamma_{\theta_0}^{(1)}\beta_{2}^{(0)}\underset{(\mathrm{A})}{[}\left(\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\frac{\partial^{2}\boldsymbol{\alpha}_0}{(\partial\boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}}+\frac{\partial^{2}\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}{}'}\right)^{<2>}\right)$$

$$\times\left\{\frac{1}{2}\mathrm{vec}(\boldsymbol{\Omega}_{\mathrm{T}})\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}+\left(\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0{}'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\right)^{<2>}\right\}$$

$$-\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\underset{(\mathrm{A})}{]}+O(N^{-2}).$$

Term (13):

$$[4n^{2}\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{\{}(q_{O_p(N^{-1/2})}^{(11)})^{3}(q_{O_p(n^{-1}N^{-1/2})}^{(31)}-\{(n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})^{(\Delta)}\}_{O_p(n^{-1}N^{-1/2})})\underset{(\mathrm{A})}{\}}]_{O(nN^{-2})}$$

$$(\to N^{-1}\overline{c}\beta_{4}^{(\Delta b)})$$

$$=12N^{-1}\overline{c}\beta_{2}^{(\Delta)}\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{A})}{(}N^{2}q_{O_p(N^{-1/2})}^{(11)}q_{O_p(n^{-1}N^{-1/2})}^{(31)}$$

$$-\gamma_{\theta_0}^{(1)}l_{O_p(N^{-1/2})}^{(\Delta 1)}\frac{\partial\lambda_{\theta_0}^{-1}\eta_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}^{(1)}\underset{(\mathrm{A})}{)}+O(N^{-2}),$$

where the first term of Term (13) was given earlier in Terms (13) to (15) of $\beta_{\mathrm{H2}}^{(\Delta a)}$ in (a.2.2) and the second term of Term (13) is

$$-12N^{-1}\overline{c}\beta_{2}^{(\Delta)}\gamma_{\theta_0}^{(1)}\frac{\partial\lambda_{\theta_0}^{-1}\eta_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}.$$

Term (14): ($m^{(\Delta)} = m^{(\Delta 3)} = m^{(\Delta\Delta b)} = 0$ under m.m.)

$$[4n^2 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^3 q^{(33)}_{O_p(N^{-3/2})}\}]_{O(n^2 N^{-3})} \quad (\to N^{-1}\bar{c}^2 \beta_4^{(\Delta c)})$$

$$= 4N^{-1}\bar{c}^2(\gamma^{(1)}_{\theta_0})^3 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} \underset{(\mathrm{A})}{\{} N^3 (l^{(\Delta 1)}_{\theta_0})^3 (\boldsymbol{\gamma}^{(3)}_{\theta_0}{}'\mathbf{l}^{(\Delta c3)}_{\theta_0} + \boldsymbol{\gamma}^{(2)}_{\theta_0}{}'\mathbf{l}^{(\Delta\Delta c2)}_{\theta_0} + \boldsymbol{\gamma}^{(\Delta 2)}_{\theta_0}{}'\mathbf{l}^{(\Delta b2)}_{\theta_0}$$

$$+\gamma^{(1)}_{\theta_0} l^{(\Delta\Delta\Delta b1)}_{\theta_0} + \gamma^{(\Delta 1)}_{\theta_0} l^{(\Delta\Delta b1)}_{\theta_0} + \gamma^{(\Delta\Delta 1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-3/2})} \underset{(\mathrm{A})}{\}}$$

$$= 4N^{-1}\bar{c}^2(\gamma^{(1)}_{\theta_0})^3 \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} \underset{(\mathrm{A})}{[} N^3 (l^{(\Delta 1)}_{\theta_0})^3$$

$$\times \underset{(\mathrm{B})}{\{} \boldsymbol{\gamma}^{(3)}_{\theta_0}{}'[(m^{(\Delta)})^2 l^{(\Delta 1)}_{\theta_0}, m^{(\Delta)}(l^{(\Delta 1)}_{\theta_0})^2, m^{(\Delta 3)}(l^{(\Delta 1)}_{\theta_0})^2, (l^{(\Delta 1)}_{\theta_0})^3, (0,0)]'$$

$$+\boldsymbol{\gamma}^{(2)}_{\theta_0}{}'[m^{(\Delta)} l^{(\Delta\Delta b1)}_{\theta_0} + m^{(\Delta\Delta b)} l^{(\Delta 1)}_{\theta_0},\ 2l^{(\Delta 1)}_{\theta_0} l^{(\Delta\Delta b1)}_{\theta_0}]'$$

$$+\boldsymbol{\gamma}^{(\Delta 2)}_{\theta_0}{}'[m^{(\Delta)} l^{(\Delta 1)}_{\theta_0}, (l^{(\Delta 1)}_{\theta_0})^2]',$$

$$+\gamma^{(1)}_{\theta_0} \underset{(\mathrm{C})}{[} \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial\theta_0 \partial\boldsymbol{\alpha}_0{}'}\right)\boldsymbol{\Gamma}^{(3)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(3)}_{\boldsymbol{\alpha}_0}$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)\sum_{\otimes}^{2}\{(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0}) \otimes (\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0} - N^{-1}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})\}$$

$$+\frac{1}{6}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4 \bar{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<3>}}\right)(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})^{<3>} \underset{(\mathrm{C})}{]}$$

$$+\gamma^{(\Delta 1)}_{\theta_0} \underset{(\mathrm{D})}{\{} \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}{}'(\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0} - N^{-1}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0{}')^{<2>}}\right)(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})^{<2>} \underset{(\mathrm{D})}{\}}$$

$$+\left\{\frac{\partial\gamma^{(1)}_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}(\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0} - N^{-1}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) + \frac{1}{2}\frac{\partial^2\gamma^{(1)}_{\theta_0}}{(\partial\boldsymbol{\alpha}_0{}')^{<2>}}(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})^{<2>}\right\} l^{(\Delta 1)}_{\theta_0} \underset{(\mathrm{B})}{\}} \underset{(\mathrm{A})}{]}$$

$$= 4N^{-1}\overline{c}^2(\gamma_{\theta_0}^{(1)})^3 \underset{(\mathrm{A})}{[}$$

$$-3(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$\times\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}$$

$$-3(\boldsymbol{\gamma}_{\theta_0}^{(2)})_1(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}$$

$$-6(\boldsymbol{\gamma}_{\theta_0}^{(2)})_2(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}$$

$$-3\gamma_{\theta_0}^{(1)}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\{(\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})\otimes(\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})\}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$-3\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}$$

$$-3(\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0})^2\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}$$

$$+3\gamma_{\theta_0}^{(1)}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\frac{\partial\boldsymbol{\alpha}_{\Delta\mathrm{W}}}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\boldsymbol{\Omega}_{\mathrm{T}}\frac{\partial\boldsymbol{\alpha}_0'}{\partial\boldsymbol{\pi}_{\mathrm{T}}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}$$

$$+\sum_{i^*,j,k,l^*,m^*,n^*=1}^{2^n}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}i^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}j}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}k}}$$

$$\times\underset{(\mathrm{B})}{\{}\boldsymbol{\gamma}_{\theta_0}^{(3)}{}'\underset{(\mathrm{C})}{[}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}$$

$$\times\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}},$$

$$\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}},$$

$$\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4\overline{l}_{\theta_0}}{\partial\theta_0^3\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial}{\partial\boldsymbol{\alpha}_0'}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^3}\right)\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}},$$

$$\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}},\ (0,0)\ ]_{(\mathrm{C})}'$$

$$+\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\ \underset{(\mathrm{D})}{[}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}$$

$$\times\frac{1}{2}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}\partial\pi_{\mathrm{T}n^*}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\right)\right\}$$

$$+\frac{1}{2}\underset{(\mathrm{E})}{[}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}\partial\pi_{\mathrm{T}m^*}}$$

$$+\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4\overline{l}_{\theta_0}}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)-\frac{\partial^2\lambda_{\theta_0}}{(\partial\boldsymbol{\alpha}_0')^{<2>}}\right\}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\right)\ ]_{(\mathrm{E})}$$

$$\times\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}},$$

$$\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\underset{(\mathrm{F})}{\{}\ \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}\partial\pi_{\mathrm{T}n^*}}$$

$$+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\right)\}_{(\mathrm{F})}\ ]_{(\mathrm{D})}'$$

$$+\left(\frac{\partial\boldsymbol{\gamma}_{\theta_0}^{(2)}}{\partial\boldsymbol{\alpha}_0'}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\right)'\underset{(\mathrm{G})}{[}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}},$$

$$\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\underset{(\mathrm{G})}{]}'$$

$$+\gamma_{\theta_0}^{(1)}\underset{(\mathrm{H})}{\{}\ \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^2\bar{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\frac{1}{6}\frac{\partial^3\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}\partial\pi_{\mathrm{T}m^*}\partial\pi_{\mathrm{T}n^*}}$$

$$+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\bar{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\otimes\frac{1}{2}\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}\partial\pi_{\mathrm{T}n^*}}\right)$$

$$+\frac{1}{6}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4\bar{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<3>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\right)\underset{(\mathrm{H})}{\}}$$

$$+\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}$$

$$\times\frac{1}{2}\left\{\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}\partial\pi_{\mathrm{T}n^*}}+\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\bar{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\right)\right\}$$

$$+\frac{1}{2}\left\{\frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0'}\frac{\partial^2\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}\partial\pi_{\mathrm{T}m^*}}+\frac{\partial^2\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0')^{<2>}}\left(\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}l^*}}\otimes\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}m^*}}\right)\right\}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\frac{\partial\boldsymbol{\alpha}_0}{\partial\pi_{\mathrm{T}n^*}}\underset{(\mathrm{B})}{\}}$$

$$\times\sum_{(i^*,j,k,l^*,m^*,n^*)}^{15}(\boldsymbol{\Omega}_{\mathrm{T}})_{i^*j}(\boldsymbol{\Omega}_{\mathrm{T}})_{kl^*}(\boldsymbol{\Omega}_{\mathrm{T}})_{m^*n^*}\underset{(\mathrm{A})}{]}+O(N^{-2}),$$

where recall that

$$m^{(\Delta)}=\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\bar{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}\right\}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)},$$

$$m^{(\Delta 3)}=\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4\bar{l}_{\theta_0}}{\partial\theta_0^3\partial\boldsymbol{\alpha}_0'}\right)-\frac{\partial}{\partial\boldsymbol{\alpha}_0'}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\bar{l}_{\theta_0}}{\partial\theta_0^3}\right)\right\}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)},$$

$$l_{\theta_0}^{(\Delta\Delta b1)}=\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)}-N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\bar{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>},$$

$$m^{(\Delta\Delta b)} = \left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0})$$

$$+ \frac{1}{2} \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial \theta_0^2 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} - \frac{\partial^2 \lambda_{\theta_0}}{(\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>},$$

$$\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(3)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(3)} = \frac{1}{6} \frac{\partial^3 \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}} ')^{<3>}} (\mathbf{p} - \boldsymbol{\pi}_{\mathrm{T}})^{<3>} + N^{-1} \frac{\partial \boldsymbol{\alpha}_{\Delta \mathrm{W}}}{\partial \boldsymbol{\pi}_{\mathrm{T}} '} (\mathbf{p} - \boldsymbol{\pi}_{\mathrm{T}}).$$

(b) Non-studentized estimator $\hat{\theta}$ under Condition B and m.m.:
$N = O(n^{3/2})$ $(\overline{c}^* = n^{3/2} / N = O(1))$

The asymptotic cumulants up to the fourth order are the same as those with known item parameters except that the following higher-order asymptotic variance of order $O(n^{-1/2})$ for *w* is added.

$$n^{-1/2} \overline{\beta}_{\mathrm{h}2}^{(\Delta)} = n^{-1/2} \overline{c}^* \beta_{\mathrm{h}2}^{(\Delta)} = n \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} [\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}]$$

$$= n^{-1/2} \overline{c}^* \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} [N \{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}]$$

$$= n^{-1/2} \overline{c}^* (\gamma_{\theta_0}^{(1)})^2 \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}.$$

(c) Non-studentized estimator $\hat{\theta}$ under Condition C and m.m.:
$N = O(n^2)$ $(\overline{c}^{**} = n^2 / N = O(1))$

The asymptotic cumulants up to the fourth order are the same as those with known item parameters except that the following higher-order asymptotic variance of order $O(n^{-1})$ for *w* is added.

$$n^{-1} \overline{\beta}_{\mathrm{H}2}^{(\Delta)} = n^{-1} \overline{c}^{**} \beta_{\mathrm{H}2}^{(\Delta)} = n \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} [\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}]$$

$$= n^{-1} \overline{c}^{**} \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0} [N \{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}]$$

$$= n^{-1} \overline{c}^{**} (\gamma_{\theta_0}^{(1)})^2 \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0} ' \boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} \boldsymbol{\lambda}_{\theta_0 \boldsymbol{\alpha}_0}.$$

A.6.2 Studentized estimator of $\hat{\theta}$

(a) Studentized estimator $t = n^{1/2}(\hat{\theta} - \theta_0)\hat{\overline{\beta}}_{2I}^{-1/2}$ **under Condition A and m.m.:** $N = O(n)$ $(\overline{c} = n / N = O(1))$

Only the expectations for the first and third asymptotic cumulants are shown.

(a.1) The first asymptotic cumulant

$$n^{-1/2}\overline{\beta}_1^{(t\Delta)} = n^{-1/2}\overline{c}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(Nq^{(11)}_{O_p(N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})})$$

$$= -n^{-1/2}\overline{c}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\underset{(\mathrm{A})}{\{}N\gamma^{(1)}_{\theta_0}l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})}\frac{\overline{\beta}_{2I}^{-3/2}}{2}\frac{\partial\overline{\beta}_{2I}}{\partial(\boldsymbol{\gamma}_{\mathbf{G}_0}', \theta_0, \boldsymbol{\alpha}_0')}$$

$$\times[\mathbf{m}_{\mathbf{G}_0}', q^{(11)}_{O_p(N^{-1/2})}, (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})})']'\underset{(\mathrm{A})}{\}}.$$

Noting $l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})} = \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0 O_p(N^{-1/2})}$, the above result becomes

$$= -n^{-1/2}\overline{c}\gamma^{(1)}_{\theta_0}\frac{\overline{\beta}_{2I}^{-3/2}}{2}\frac{\partial\overline{\beta}_{2I}}{\partial(\boldsymbol{\gamma}_{\mathbf{G}_0}', \theta_0, \boldsymbol{\alpha}_0')}$$

$$\times[\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{m}_{\mathbf{G}_0}'), \gamma^{(1)}_{\theta_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}, \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}]',$$

where recall that $\mathbf{m}_{\mathbf{G}_0} = \mathrm{v}(\mathbf{G}_0 - \boldsymbol{\Gamma}_{\mathbf{G}_0})$, $\boldsymbol{\Gamma}_{\mathbf{G}_0} = \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{G}_0)$ and $\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0} = \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}N\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0}{}')\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}{}'$. Incidentally, under c.m.s. from Ogasawara (2010, Theorem 2), we have $N\,\mathrm{acov}\{\mathrm{v}(\hat{\mathbf{G}}^{-1}), \hat{\boldsymbol{\alpha}}'\} = N\,\mathrm{acov}\{\mathrm{v}(\hat{\mathbf{I}}_{\boldsymbol{\alpha}}^{-1}), \hat{\boldsymbol{\alpha}}'\}$ and consequently $N\,\mathrm{acov}\{\mathrm{v}(\hat{\mathbf{G}}), \hat{\boldsymbol{\alpha}}'\} = N\,\mathrm{acov}\{\mathrm{v}(\hat{\mathbf{I}}_{\boldsymbol{\alpha}}), \hat{\boldsymbol{\alpha}}'\}$ ($\hat{\mathbf{I}}_{\boldsymbol{\alpha}}$ is the estimator of the information matrix $\mathbf{I}_{\boldsymbol{\alpha}_0}$ per observation). That is, when the IRT model holds,

$$\boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}N\mathrm{E}_{\boldsymbol{\alpha}_0}(\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{m}_{\mathbf{G}_0}') = \boldsymbol{\lambda}_{\theta_0\boldsymbol{\alpha}_0}'\boldsymbol{\Omega}_{\boldsymbol{\alpha}_0}\frac{\{\partial\mathrm{v}(\mathbf{I}_{\boldsymbol{\alpha}_0})\}'}{\partial\boldsymbol{\alpha}_0}$$

with $\boldsymbol{\Gamma}_{\mathbf{G}_0} = \mathbf{I}_{\boldsymbol{\alpha}_0} = \boldsymbol{\Omega}^{-1}_{\boldsymbol{\alpha}_0}$.

(a.2) The third asymptotic cumulant

$$n^{-1/2}\overline{\beta}_3^{(t\Delta)} = n^{3/2} \underset{(\mathrm{A})}{[} 9\mathrm{E_T}\{(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})} b^{(11)}_{O_p(N^{-1/2})}$$

$$+ (q^{(11)}_{O_p(N^{-1/2})})^2 q^{(10)}_{O_p(n^{-1/2})} b^{(10)}_{O_p(n^{-1/2})}\}\overline{\beta}_{2I}^{-1}$$

$$+ 3\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^3 b^{(11)}_{O_p(N^{-1/2})}\}\overline{\beta}_{2I}^{-1} - 3n^{-2}\overline{\beta}_1^{(t\Delta)}\overline{\beta}_{t2} \ \underset{(\mathrm{A})O(n^{-2})}{]}$$

$$= 9n^{-1/2}\overline{c}\{\beta_2^{(0)}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(Nq^{(11)}_{O_p(N^{-1/2})} b^{(11)}_{O_p(N^{-1/2})})$$

$$+ \beta_2^{(\Delta)}\mathrm{E}_{\mathrm{T}\theta_0}(nq^{(10)}_{O_p(n^{-1/2})} b^{(10)}_{O_p(n^{-1/2})})\}\overline{\beta}_{2I}^{-1}$$

$$+9n^{-1/2}\overline{c}^{\,2}\beta_2^{(\Delta)}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(Nq^{(11)}_{O_p(N^{-1/2})} b^{(11)}_{O_p(N^{-1/2})})\overline{\beta}_{2I}^{-1} - 3n^{-1/2}\overline{\beta}_1^{(t\Delta)}\overline{\beta}_{t2} + O(n^{-3/2})$$

$$= 9n^{-1/2}\overline{\beta}_1^{(t\Delta)}\beta_2^{(0)}\overline{\beta}_{2I}^{-1} + 9n^{-1/2}\overline{c}\beta_1^{(t0)}\beta_2^{(\Delta)}\overline{\beta}_{2I}^{-1} + 9n^{-1/2}\overline{c}\,\overline{\beta}_1^{(t\Delta)}\beta_2^{(\Delta)}\overline{\beta}_{2I}^{-1}$$

$$- 3n^{-1/2}\overline{\beta}_1^{(t\Delta)}\overline{\beta}_{t2} + O(n^{-3/2}),$$

where
$\overline{\beta}_1^{(t\Delta)} = \overline{c}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(Nq^{(11)}_{O_p(N^{-1/2})} b^{(11)}_{O_p(N^{-1/2})})$ and $\beta_1^{(t0)} = \mathrm{E}_{\mathrm{T}\theta_0}(nq^{(10)}_{O_p(n^{-1/2})} b^{(10)}_{O_p(n^{-1/2})})$
are used.

(b) Studentized estimator $t^* = n^{1/2}(\hat{\theta} - \theta_0)\hat{i}^{1/2}$ **under Condition B and m.m.:** $N = O(n^{3/2})$ $(\overline{c}^* = n^{3/2}/N = O(1))$

The expectation $\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(1a)}_{O_p(N^{-1/2})})^2\} = \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\}$ associated with the only added term $n^{-1/2}\overline{c}^*\beta^{(\Delta)}_{t\mathrm{h}2}$ was given in $\beta_2^{(\Delta)}$ of (a.2.1).

(c) Studentized estimator $t^* = n^{1/2}(\hat{\theta} - \theta_0)\hat{i}^{1/2}$ **under Condition C and m.m.:** $N = O(n^2)$ $(\overline{c}^{**} = n^2/N = O(1))$

The expectation $\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(21)}_{O_p(N^{-1/2})})^2\} = \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(\gamma^{(1)}_{\theta_0} l^{(\Delta 1)}_{\theta_0 O_p(N^{-1/2})})^2\}$ associated with the only added term $n^{-1/2}\overline{c}^{**}\beta^{(t*\Delta)}_{\mathrm{H}2}$ was given in $\beta_2^{(\Delta)}$ of (a.2.1). Note that the added term is algebraically equal to the that of (b) i.e., $n^{-1/2}\overline{c}^*\beta^{(\Delta)}_{t\mathrm{h}2} = n^{-1/2}\overline{c}^{**}\beta^{(t*\Delta)}_{\mathrm{H}2}$.

Reference

Ogasawara, H. (2013). Asymptotic cumulants of ability estimators using fallible item parameters. *Journal of Multivariate Analysis, 119,* 144-162.