

**Supplement I to the paper “Asymptotic cumulants of ability
 estimators using fallible item parameters”
 – Proofs, partial derivatives and tables
 (Corrected version)
 March, 2014**

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This supplement includes Subsections A.2 to A.5 of the appendix of Ogasawara (2013) and additional Tables A1 to A4.

A.2 Expansions of $\hat{\boldsymbol{\gamma}}_{\theta_0}^{(k)}$ and $\hat{\mathbf{l}}_{\theta_0}^{(k)}$ ($k = 1, 2, 3$)

(a) $\hat{\boldsymbol{\gamma}}_{\theta_0}^{(k)}$ ($k = 1, 2, 3$)

By the Taylor series expansion and (2.6), it follows that

$$\begin{aligned} \hat{\boldsymbol{\gamma}}_{\theta_0}^{(k)} &= \boldsymbol{\gamma}_{\theta_0}^{(k)} + \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(k)}}{\partial \boldsymbol{\alpha}_0'} (\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) + \frac{1}{2} \frac{\partial^2 \boldsymbol{\gamma}_{\theta_0}^{(k)}}{(\partial \boldsymbol{\alpha}_0')^{<2>}} (\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)^{<2>} + O_p(N^{-3/2}) \\ &= \boldsymbol{\gamma}_{\theta_0}^{(k)} + \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(k)}}{\partial \boldsymbol{\alpha}_0'} \left(\sum_{k=1}^2 \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(k)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(k)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0} \right) + \frac{1}{2} \frac{\partial^2 \boldsymbol{\gamma}_{\theta_0}^{(k)}}{(\partial \boldsymbol{\alpha}_0')^{<2>}} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>} \\ &\quad + O_p(N^{-3/2}) \\ &= (\boldsymbol{\gamma}_{\theta_0}^{(k)})_{O(1)} + \left(\frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(k)}}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)} \right)_{O_p(N^{-1/2})} + \left\{ \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(k)}}{\partial \boldsymbol{\alpha}_0'} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^2 \boldsymbol{\gamma}_{\theta_0}^{(k)}}{(\partial \boldsymbol{\alpha}_0')^{<2>}} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>} \right\}_{O_p(N^{-1})} + O_p(N^{-3/2}) \\ &\equiv (\boldsymbol{\gamma}_{\theta_0}^{(k)})_{O(1)} + (\boldsymbol{\gamma}_{\theta_0}^{(\Delta k)})_{O_p(N^{-1/2})} + (\boldsymbol{\gamma}_{\theta_0}^{(\Delta \Delta k)})_{O_p(N^{-1})} + O_p(N^{-3/2}) \quad (k = 1, 2, 3). \end{aligned}$$

(b) $\hat{\mathbf{l}}_{\theta_0}^{(1)} = \hat{l}_{\theta_0}^{(1)} = \frac{\partial \hat{l}_{\theta_0}}{\partial \theta_0}$

$$\begin{aligned}
&= \left(\frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0} \right)_{O_p(n^{-1/2})} + \left\{ \left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0'} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{l}_{\mathbf{a}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2})} \\
&+ \left[\left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0'} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{l}_{\mathbf{a}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0})_{O_p(N^{-1})} \right. \\
&\quad \left. + \left(\frac{1}{2} \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0')^{<2>}} \right)_{O_p(1)} \{ (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{l}_{\mathbf{a}_0}^{(1)})^{<2>} \}_{O_p(N^{-1})} \right]_{O_p(N^{-1})} \\
&+ \left[\left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0'} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(3)} \mathbf{l}_{\mathbf{a}_0}^{(3)})_{O_p(N^{-3/2})} + \left(\frac{1}{2} \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0')^{<2>}} \right)_{O_p(1)} \right. \\
&\quad \left. \times \sum_{\otimes}^2 \{ (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{l}_{\mathbf{a}_0}^{(1)}) \otimes (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{l}_{\mathbf{a}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0}) \}_{O_p(N^{-3/2})} \right. \\
&\quad \left. + \left(\frac{1}{6} \frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0')^{<3>}} \right)_{O_p(1)} \{ (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{l}_{\mathbf{a}_0}^{(1)})^{<3>} \}_{O_p(N^{-3/2})} \right]_{O_p(N^{-3/2})} + O_p(N^{-2}) \\
&= \left(\frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0} \right)_{O_p(n^{-1/2})} + \left\{ \left(\mathbf{E}_{\mathbb{T}\theta_0} \frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0'} \right)_{O(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{l}_{\mathbf{a}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2})} \\
&+ \left\{ \left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0'} - \mathbf{E}_{\mathbb{T}\theta_0}(\cdot) \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{l}_{\mathbf{a}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(n^{-1/2} N^{-1/2})} \\
&+ \left[\left(\mathbf{E}_{\mathbb{T}\theta_0} \frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0'} \right)_{O(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{l}_{\mathbf{a}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0})_{O_p(N^{-1})} \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[\left(\frac{1}{2} \mathbf{E}_{\mathbf{T}\theta_0} \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0')^{<2>}} \right)_{O(1)} \{(\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})^{<2>}\}_{O_p(N^{-1})} \right]_{O_p(N^{-1})} \\
& + \left[\left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0'} - \mathbf{E}_{\mathbf{T}\theta_0}(\cdot) \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{I}_{\mathbf{a}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0})_{O_p(N^{-1})} \right. \\
& \quad \left. + \frac{1}{2} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0')^{<2>}} - \mathbf{E}_{\mathbf{T}\theta_0}(\cdot) \right)_{O_p(n^{-1/2})} \{(\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})^{<2>}\}_{O_p(N^{-1})} \right]_{O_p(n^{-1/2} N^{-1})} \\
& + \left[\left(\mathbf{E}_{\mathbf{T}\theta_0} \frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0'} \right)_{O(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(3)} \mathbf{I}_{\mathbf{a}_0}^{(3)})_{O_p(N^{-3/2})} \right. \\
& \quad \left. + \frac{1}{2} \mathbf{E}_{\mathbf{T}\theta_0} \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0')^{<2>}} \sum_{\otimes}^2 \{(\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)}) \otimes (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{I}_{\mathbf{a}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0})\}_{O_p(N^{-3/2})} \right. \\
& \quad \left. + \left(\frac{1}{6} \mathbf{E}_{\mathbf{T}\theta_0} \frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0')^{<3>}} \right)_{O(1)} \{(\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})^{<3>}\}_{O_p(N^{-3/2})} \right]_{O_p(N^{-3/2})} \\
& + O_p(n^{-1/2} N^{-3/2}) + O_p(N^{-2}) \\
& \equiv (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} + (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (l_{\theta_0}^{(\Delta \Delta a 1)})_{O_p(n^{-1/2} N^{-1/2})} + (l_{\theta_0}^{(\Delta \Delta b 1)})_{O_p(N^{-1})} \\
& \quad + (l_{\theta_0}^{(\Delta \Delta \Delta a 1)})_{O_p(n^{-1/2} N^{-1})} + (l_{\theta_0}^{(\Delta \Delta \Delta b 1)})_{O_p(N^{-3/2})} + O_p(n^{-1/2} N^{-3/2}) + O_p(N^{-2}),
\end{aligned}$$

where $\sum_{\otimes}^2 \mathbf{x} \otimes \mathbf{y} = \mathbf{x} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{x}$ and $\mathbf{A} - \mathbf{E}_{\mathbf{T}\theta_0}(\cdot) = \mathbf{A} - \mathbf{E}_{\mathbf{T}\theta_0}(\mathbf{A})$.

$$\text{(c) } \hat{\mathbf{I}}_{\theta_0}^{(2)} = \left\{ \hat{m} \frac{\partial \hat{l}_{\theta_0}}{\partial \theta_0}, \left(\frac{\partial \hat{l}_{\theta_0}}{\partial \theta_0} \right)^2 \right\}'$$

In the above expression,

$$\begin{aligned}
\hat{m} &\equiv \frac{\partial^2 \hat{l}_{\theta_0}}{\partial \theta_0^2} - \hat{\lambda}_{\theta_0} \left(\text{recall } \lambda_{\theta_0} \equiv \mathbf{E}_{\text{T}\theta_0} \frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0^2} \right), \\
\frac{\partial^2 \hat{l}_{\theta_0}}{\partial \theta_0^2} &= \left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0^2} \right)_{O_p(1)} + \left\{ \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0'} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2})} \\
&+ \left[\left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0'} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{I}_{\mathbf{a}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0})_{O_p(N^{-1})} \right. \\
&\quad \left. + \left(\frac{1}{2} \frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^2 (\partial \mathbf{a}_0')^{<2>}} \right)_{O_p(1)} \{ (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})^{<2>} \}_{O_p(N^{-1})} \right]_{O_p(N^{-1})} + O_p(N^{-3/2}) \\
&= \lambda_{\theta_0} + \left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0^2} - \lambda_{\theta_0} \right)_{O_p(N^{-1/2})} \\
&+ \left\{ \left(\mathbf{E}_{\text{T}\theta_0} \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0'} \right)_{O(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2})} \\
&+ \left\{ \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0'} - \mathbf{E}_{\text{T}\theta_0}(\cdot) \right)_{O_p(N^{-1/2})} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2} N^{-1/2})} \\
&+ \left[\left(\mathbf{E}_{\text{T}\theta_0} \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0'} \right)_{O(1)} (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{I}_{\mathbf{a}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0})_{O_p(N^{-1})} \right. \\
&\quad \left. + \left(\frac{1}{2} \mathbf{E}_{\text{T}\theta_0} \frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^2 (\partial \mathbf{a}_0')^{<2>}} \right)_{O(1)} \{ (\boldsymbol{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})^{<2>} \}_{O_p(N^{-1})} \right]_{O_p(N^{-1})} \\
&+ O_p(N^{-1/2} N^{-1}) + O_p(N^{-3/2}),
\end{aligned}$$

$$\begin{aligned}
-\hat{\lambda}_{\theta_0} = & -\lambda_{\theta_0} - \left(\frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0}, \mathbf{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)} \right)_{O_p(N^{-1/2})} - \left\{ \frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0}, (\mathbf{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{I}_{\mathbf{a}_0}^{(2)} - N^{-1} \mathbf{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0}) \right. \\
& \left. + \left(\frac{1}{2} \mathbf{E}_{\mathbb{T}\theta_0} \frac{\partial^2 \lambda_{\theta_0}}{(\partial \mathbf{a}_0)'^{<2>}} \right) (\mathbf{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})^{<2>} \right\}_{O_p(N^{-1})} + O_p(N^{-3/2}),
\end{aligned}$$

where note that under m.m. $\mathbf{E}_{\mathbb{T}\theta_0} \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0} = \frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0}$, whereas under c.m.s. this

does not hold since $P_{\mathbb{T}k}(\cdot) = P_k(\cdot)$ ($k = 1, \dots, n$) are functions of \mathbf{a} . Similar

properties are also found for $\mathbf{E}_{\mathbb{T}\theta_0} \frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^2 (\partial \mathbf{a}_0)'^{<2>}}$ and $\frac{\partial^2 \lambda_{\theta_0}}{(\partial \mathbf{a}_0)'^{<2>}}$. Then, it

follows that

$$\begin{aligned}
\hat{m} &= \frac{\partial^2 \hat{l}_{\theta_0}}{\partial \theta_0^2} - \hat{\lambda}_{\theta_0} \\
&= \left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0^2} - \lambda_{\theta_0} \right)_{O_p(n^{-1/2})} \\
&+ \left[\left\{ \mathbf{E}_{\mathbb{T}\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} \right\}_{O(1)} (\mathbf{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})_{O_p(N^{-1/2})} \right]_{O_p(N^{-1/2})} \\
&+ \left\{ \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0'} - \mathbf{E}_{\mathbb{T}\theta_0}(\cdot) \right)_{O_p(n^{-1/2})} (\mathbf{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{I}_{\mathbf{a}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(n^{-1/2} N^{-1/2})} \\
&+ \left[\left\{ \mathbf{E}_{\mathbb{T}\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \mathbf{a}_0'} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} \right\}_{O(1)} (\mathbf{\Gamma}_{\mathbf{a}_0}^{(2)} \mathbf{I}_{\mathbf{a}_0}^{(2)} - N^{-1} \mathbf{\Lambda}_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0})_{O_p(N^{-1})} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left\{ \mathbf{E}_{\mathbf{T}\theta_0} \left(\frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^2 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right) - \frac{\partial^2 \lambda_{\theta_0}}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right\}_{O_p(1)} \left[\{(\mathbf{\Gamma}_{\mathbf{a}_0}^{(1)} \mathbf{l}_{\mathbf{a}_0}^{(1)})^{\langle 2 \rangle}\}_{O_p(N^{-1})} \right]_{O_p(N^{-1})} \\
& + O_p(n^{-1/2} N^{-1}) + O_p(N^{-3/2}) \\
& \equiv m_{O_p(n^{-1/2})} + (m^{(\Delta)})_{O_p(N^{-1/2})} + (m^{(\Delta\Delta a)})_{O_p(n^{-1/2} N^{-1/2})} + (m^{(\Delta\Delta b)})_{O_p(N^{-1})} \\
& + O_p(n^{-1/2} N^{-1}) + O_p(N^{-3/2}),
\end{aligned}$$

where the terms of orders $O_p(N^{-1/2})$ and $O_p(N^{-1})$ vanish under m.m.

Using the above definitions, we have

$$\begin{aligned}
\hat{\mathbf{l}}_{\theta_0}^{(2)} & = [m_{O_p(n^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}, \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})}]'_{O_p(n^{-1})} \\
& + \left[\begin{array}{l} m_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} \\ + (m^{(\Delta)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}, \end{array} \quad \begin{array}{l} 2(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} \\ \end{array} \right]'_{O_p(n^{-1/2} N^{-1/2})} \\
& + [(m^{(\Delta)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}, \{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}]'_{O_p(N^{-1})} \\
& + [m_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta\Delta a 1)})_{O_p(n^{-1/2} N^{-1/2})} + (m^{(\Delta\Delta a)})_{O_p(n^{-1/2} N^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}, \\
& \quad 2(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta\Delta a 1)})_{O_p(n^{-1/2} N^{-1/2})}]'_{O_p(n^{-1} N^{-1/2})} \\
& + [m_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta\Delta b 1)})_{O_p(N^{-1})} + (m^{(\Delta)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(\Delta\Delta a 1)})_{O_p(n^{-1/2} N^{-1/2})} \\
& \quad + (m^{(\Delta\Delta a)})_{O_p(n^{-1/2} N^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (m^{(\Delta\Delta b)})_{O_p(N^{-1})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}, \\
& \quad 2(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta\Delta b 1)})_{O_p(N^{-1})} + 2(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(\Delta\Delta a 1)})_{O_p(n^{-1/2} N^{-1/2})}]'_{O_p(n^{-1/2} N^{-1})} \\
& + [(m^{(\Delta)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(\Delta\Delta b 1)})_{O_p(N^{-1})} + (m^{(\Delta\Delta b)})_{O_p(N^{-1})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}, \\
& \quad 2(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(\Delta\Delta b 1)})_{O_p(N^{-1})}]'_{O_p(N^{-3/2})} \\
& \equiv (\mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})} + (\mathbf{l}_{\theta_0}^{(\Delta a 2)})_{O_p(n^{-1/2} N^{-1/2})} + (\mathbf{l}_{\theta_0}^{(\Delta b 2)})_{O_p(N^{-1})} \\
& + (\mathbf{l}_{\theta_0}^{(\Delta\Delta a 2)})_{O_p(n^{-1} N^{-1/2})} + (\mathbf{l}_{\theta_0}^{(\Delta\Delta b 2)})_{O_p(n^{-1/2} N^{-1})} + (\mathbf{l}_{\theta_0}^{(\Delta\Delta c 2)})_{O_p(N^{-3/2})} \\
& + O_p(n^{-1} N^{-1}) + O_p(n^{-1/2} N^{-3/2}) + O_p(N^{-2}).
\end{aligned}$$

$$(d) \hat{\mathbf{l}}_{\theta_0}^{(3)} = \left[\hat{m}^2 \frac{\partial \hat{l}_{\theta_0}}{\partial \theta_0}, \hat{m} \left(\frac{\partial \hat{l}_{\theta_0}}{\partial \theta_0} \right)^2, \left\{ \frac{\partial^3 \hat{l}_{\theta_0}}{\partial \theta_0^3} - \mathbf{E}_{T_{\theta_0}} \left(\frac{\partial^3 \hat{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\} \left(\frac{\partial \hat{l}_{\theta_0}}{\partial \theta_0} \right)^2, \right. \\ \left. \left(\frac{\partial \hat{l}_{\theta_0}}{\partial \theta_0} \right)^3, n^{-1} \left(\hat{m}, \frac{\partial \hat{l}_{\theta_0}}{\partial \theta_0} \right) \right]'$$

Define $\hat{m}^{(3)} = \frac{\partial^3 \hat{l}_{\theta_0}}{\partial \theta_0^3} - \mathbf{E}_{T_{\theta_0}} \left(\frac{\partial^3 \hat{l}_{\theta_0}}{\partial \theta_0^3} \right)$. Then, as in \hat{m} ,

$$\hat{m}^{(3)} = \left\{ \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} - \mathbf{E}_{T_{\theta_0}} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}_{O_p(n^{-1/2})} \\ + \left[\left\{ \mathbf{E}_{T_{\theta_0}} \left(\frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^3 \partial \alpha_0'} \right) - \frac{\partial}{\partial \alpha_0'} \mathbf{E}_{T_{\theta_0}} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}_{O(1)} (\boldsymbol{\Gamma}_{\alpha_0}^{(1)} \mathbf{I}_{\alpha_0}^{(1)})_{O_p(N^{-1/2})} \right]_{O_p(N^{-1/2})}$$

$$+ O_p(n^{-1/2} N^{-1/2}) + O_p(N^{-1})$$

$$\equiv (m^{(3)})_{O_p(n^{-1/2})} + (m^{(\Delta 3)})_{O_p(N^{-1/2})} + O_p(n^{-1/2} N^{-1/2}) + O_p(N^{-1}),$$

where the term of order $O_p(N^{-1/2})$, and the residual term $O_p(N^{-1})$ vanish under m.m. From the above definitions,

$$\hat{\mathbf{l}}_{\theta_0}^{(3)} = [(m^2)_{O_p(n^{-1})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}, m_{O_p(n^{-1/2})} \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})}, \\ (m^{(3)})_{O_p(n^{-1/2})} \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})}, \{(l_{\theta_0}^{(1)})^3\}_{O_p(n^{-3/2})}, n^{-1} (m, l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}]'_{O_p(n^{-3/2})} \\ + [2 m_{O_p(n^{-1/2})} (m^{(\Delta)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} + (m^2)_{O_p(n^{-1})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}, \\ 2 m_{O_p(n^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (m^{(\Delta)})_{O_p(N^{-1/2})} \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})}, \\ 2(m^{(3)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (m^{(\Delta 3)})_{O_p(N^{-1/2})} \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})}, \\ 3\{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}, n^{-1} (m^{(\Delta)}, l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}]'_{O_p(n^{-1} N^{-1/2})}$$

$$\begin{aligned}
& + [2 m_{O_p(n^{-1/2})} (m^{(\Delta)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + \{(m^{(\Delta)})^2\}_{O_p(N^{-1})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}, \\
& \quad 2(m^{(\Delta)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + m_{O_p(n^{-1/2})} \{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}, \\
& \quad 2(m^{(\Delta 3)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (m^{(3)})_{O_p(n^{-1/2})} \{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}, \\
& \quad 3\{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}, (0, 0)]'_{O_p(n^{-1/2}N^{-1})} \\
& + [\{(m^{(\Delta)})^2\}_{O_p(N^{-1})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}, (m^{(\Delta)})_{O_p(N^{-1/2})} \{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}, \\
& \quad (m^{(\Delta 3)})_{O_p(N^{-1/2})} \{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}, \{(l_{\theta_0}^{(\Delta 1)})^3\}_{O_p(N^{-3/2})}, (0, 0)]'_{O_p(N^{-3/2})} \\
& + O_p(n^{-1}N^{-1}) + O_p(n^{-1/2}N^{-3/2}) + O_p(N^{-2}) \\
& \equiv (\mathbf{I}_{\theta_0}^{(3)})_{O_p(n^{-3/2})} + (\mathbf{I}_{\theta_0}^{(\Delta a 3)})_{O_p(n^{-1}N^{-1/2})} + (\mathbf{I}_{\theta_0}^{(\Delta b 3)})_{O_p(n^{-1/2}N^{-1})} + (\mathbf{I}_{\theta_0}^{(\Delta c 3)})_{O_p(N^{-3/2})} \\
& \quad + O_p(n^{-1}N^{-1}) + O_p(n^{-1/2}N^{-3/2}) + O_p(N^{-2}).
\end{aligned}$$

$$\begin{aligned}
\text{(e)} \quad & -(n^{-1} \hat{\lambda}_{\theta_0}^{-1} \hat{\eta}_{\theta_0}) \\
& = -(n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0})_{O(n^{-1})} \\
& - \left\{ n^{-1} \left(-\lambda_{\theta_0}^{-2} \eta_{\theta_0} \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} + \lambda_{\theta_0}^{-1} \frac{\partial \eta_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right) (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{I}_{\boldsymbol{\alpha}_0}^{(1)}) \right\}_{O_p(n^{-1}N^{-1/2})} + O_p(n^{-1}N^{-1}) \\
& \equiv -(n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0})_{O(n^{-1})} - \{n^{-1} (\lambda_{\theta_0}^{-1} \eta_{\theta_0})^{(\Delta)}\}_{O_p(n^{-1}N^{-1/2})} + O_p(n^{-1}N^{-1}).
\end{aligned}$$

A.3 Asymptotic cumulants of $\hat{\theta}$: Proofs of Theorems 1, 2 and 3

Define $E_{T_{\boldsymbol{\alpha}_0}}(\cdot)$ similarly to $E_{T_{\theta_0}}(\cdot)$ under p.m.m., where the true marginal multinomial distribution for 2^n response patterns is employed for the expectation associated with item calibration. Denote the two-fold expectation $E_{T_{\theta_0}}\{E_{T_{\boldsymbol{\alpha}_0}}(\cdot)\}$ by $E_T(\cdot)$ for simplicity. The notation like $\left[\cdot \right]_{(A) (A)}$ is for ease of finding correspondence.

(a) Proof of Theorem 1 under Condition A:

$$N = O(n) \quad (\bar{c} = n / N = O(1))$$

Recall $w = n^{1/2}(\hat{\theta} - \theta_0)$. Then, from (A.1) the following results are obtained and give Theorem 1.

$$\begin{aligned}
\kappa_1(w) &= n^{1/2} \{ \mathbf{E}_T(q_{O_p(n^{-1})}^{(2)}) - (n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0})_{O(n^{-1})} \}_{O(n^{-1})} + O(n^{-3/2}) \\
&= n^{1/2} [\{ \mathbf{E}_{T\theta_0}(q_{O_p(n^{-1})}^{(20)}) - (n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0}) \} + \mathbf{E}_{T\alpha_0}(q_{O_p(N^{-1})}^{(22)})]_{O(n^{-1})} + O(n^{-3/2}) \\
&\equiv n^{-1/2} (\beta_1^{(0)} + \bar{\beta}_1^{(\Delta)}) + O(n^{-3/2}) \\
&\equiv n^{-1/2} \bar{\beta}_1 + O(n^{-3/2}) \\
&\equiv n^{-1/2} \beta_1^{(0)} + N^{-1/2} \bar{c}^{-1/2} \beta_1^{(\Delta)} + O(n^{-3/2}),
\end{aligned}$$

where $\bar{c} \beta_1^{(\Delta)} = \bar{\beta}_1^{(\Delta)}$.

$$\begin{aligned}
\kappa_2(w) &= n \mathbf{E}_T [\{ (q_{O_p(n^{-1/2})}^{(1)})^2 \}_{O_p(n^{-1})}] + n \mathbf{E}_T [\{ (q_{O_p(n^{-1})}^{(2)}) \}_{O_p(n^{-2})} \\
&\quad + 2(q_{O_p(n^{-1/2})}^{(1)} q_{O_p(n^{-1})}^{(2)})_{O_p(n^{-3/2})} + 2(q_{O_p(n^{-1/2})}^{(1)} q_{O_p(n^{-3/2})}^{(3)})_{O_p(n^{-2})}] \\
&\quad - n^{-1} (\beta_1^{(0)} + \bar{\beta}_1^{(\Delta)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0})^2 + O(n^{-2}) \\
&= \{ n \mathbf{E}_{T\theta_0} [\{ (q_{O_p(n^{-1/2})}^{(10)})^2 \}_{O_p(n^{-1})}] \}_{O(1)} + \{ n \mathbf{E}_{T\alpha_0} [\{ (q_{O_p(N^{-1/2})}^{(11)})^2 \}_{O_p(N^{-1})}] \}_{O(nN^{-1})} \\
&+ [n \mathbf{E}_{T\theta_0} [\{ (q_{O_p(n^{-1})}^{(20)})^2 \}_{O_p(n^{-2})} + 2q_{O_p(n^{-1/2})}^{(10)} q_{O_p(n^{-1})}^{(20)} + 2q_{O_p(n^{-1/2})}^{(10)} q_{O_p(n^{-3/2})}^{(30)}] \\
&\quad - n^{-1} (\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0})^2]_{O(n^{-1})} \\
&+ [\{ n \{ \mathbf{E}_T \{ (q_{O_p(n^{-1/2}N^{-1/2})}^{(21)})^2 + 2q_{O_p(n^{-1})}^{(20)} q_{O_p(N^{-1})}^{(22)} \} \}_{O(N^{-1})} \\
&\quad + \{ n \mathbf{E}_{T\alpha_0} \{ (q_{O_p(N^{-1})}^{(22)})^2 \} \}_{O(nN^{-2})} + \{ 2n \mathbf{E}_{T\alpha_0} (q_{O_p(N^{-1/2})}^{(11)} q_{O_p(N^{-1})}^{(22)}) \}_{O(nN^{-2})} \\
&+ \{ 2n \mathbf{E}_T [q_{O_p(n^{-1/2})}^{(10)} q_{O_p(n^{-1/2}N^{-1})}^{(32)} + q_{O_p(N^{-1/2})}^{(11)} \{ q_{O_p(n^{-1}N^{-1/2})}^{(31)} \\
&\quad - (n^{-1} (\lambda_{\theta_0}^{-1} \eta_{\theta_0})^{(\Delta)})_{O_p(n^{-1}N^{-1/2})} \}] \}_{O(N^{-1})} \\
&+ \{ 2n \mathbf{E}_{T\alpha_0} (q_{O_p(N^{-1/2})}^{(11)} q_{O_p(N^{-3/2})}^{(33)}) \}_{O(nN^{-2})} \\
&\quad - n^{-1} \{ 2\bar{\beta}_1^{(\Delta)} (\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) + (\bar{\beta}_1^{(\Delta)})^2 \}]_{(A)O(n^{-1})} + O(n^{-2})
\end{aligned}$$

$$\equiv \beta_2^{(0)} + \bar{\beta}_2^{(\Delta)} + n^{-1} \beta_{H2}^{(0)} + n^{-1} \bar{\beta}_{H2}^{(\Delta)} + O(n^{-2})$$

$$\equiv \bar{\beta}_2 + n^{-1} \bar{\beta}_{H2} + O(n^{-2})$$

$$\equiv \beta_2^{(0)} + \bar{c} \beta_2^{(\Delta)} + n^{-1} \beta_{H2}^{(0)} + N^{-1} \beta_{H2}^{(\Delta a)} + N^{-1} \bar{c} \beta_{H2}^{(\Delta b)} + O(n^{-2}),$$

where $\bar{\beta}_2 = \beta_2^{(0)} + \bar{\beta}_2^{(\Delta)}$, $\bar{\beta}_2^{(\Delta)} = \bar{c} \beta_2^{(\Delta)}$, $\bar{\beta}_{H2}^{(\Delta)} = \bar{c} \beta_{H2}^{(\Delta a)} + \bar{c}^2 \beta_{H2}^{(\Delta b)}$;

$N^{-1} \beta_{H2}^{(\Delta a)}$ and $N^{-1} \bar{c} \beta_{H2}^{(\Delta b)}$ denote the sums of orders $O(N^{-1})$ and

$O(nN^{-2})$ inside $[\cdot]_{(A)}^{(A)}$. Since the constant term with no expectations in

$$[\cdot]_{(A)}^{(A)} \text{ is } -n^{-1} \{ 2\bar{\beta}_1^{(\Delta)} (\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) + (\bar{\beta}_1^{(\Delta)})^2 \}$$

$$= -N^{-1} 2\beta_1^{(\Delta)} (\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) - N^{-1} \bar{c} (\beta_1^{(\Delta)})^2, \text{ the two terms on the}$$

right-hand side of the last equation are included in $N^{-1} \beta_{H2}^{(\Delta a)}$ and $N^{-1} \bar{c} \beta_{H2}^{(\Delta b)}$, respectively.

$$\begin{aligned} \kappa_3(w) &= n^{3/2} [\{ \mathbf{E}_T \{ (q_{O_p(n^{-1/2})}^{(1)})^3 + 3(q_{O_p(n^{-1/2})}^{(1)})^2 q_{O_p(n^{-1})}^{(2)} \} \}_{O(n^{-2})} \\ &\quad - 3n^{-2} (\beta_1^{(0)} + \bar{\beta}_1^{(\Delta)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) (\beta_2^{(0)} + \bar{\beta}_2^{(\Delta)}) \}_{O(n^{-2})} + O(n^{-3/2}) \end{aligned}$$

$$\begin{aligned} &= n^{3/2} [\{ \mathbf{E}_{T\theta_0} \{ (q_{O_p(n^{-1/2})}^{(10)})^3 + 3(q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(n^{-1})}^{(20)} \} \}_{O(n^{-2})} \\ &\quad - 3n^{-2} (\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) \beta_2^{(0)} \}_{O(n^{-2})} \end{aligned}$$

$$+ [[n^{3/2} \mathbf{E}_{T\alpha_0} \{ (q_{O_p(N^{-1/2})}^{(11)})^3 \}]_{O(n^{3/2} N^{-2})}^{(A)}$$

$$\begin{aligned} &+ [3n^{3/2} \mathbf{E}_T \{ (q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(N^{-1})}^{(22)} + 2q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} q_{O_p(n^{-1/2} N^{-1/2})}^{(21)} \\ &\quad + (q_{O_p(N^{-1/2})}^{(11)})^2 q_{O_p(n^{-1})}^{(20)} \}]_{O(n^{1/2} N^{-1})} \end{aligned}$$

$$+ [3n^{3/2} \mathbf{E}_{T\alpha_0} \{ (q_{O_p(N^{-1/2})}^{(11)})^2 q_{O_p(N^{-1})}^{(22)} \}]_{O(n^{3/2} N^{-2})}$$

$$\begin{aligned} &- 3n^{-1/2} \{ (\beta_1^{(0)} + \bar{\beta}_1^{(\Delta)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) \bar{\beta}_2^{(\Delta)} + \bar{\beta}_1^{(\Delta)} \beta_2^{(0)} \}]_{(A)O(n^{-1/2})} + O(n^{-3/2}) \end{aligned}$$

$$\equiv n^{-1/2} (\beta_3^{(0)} + \bar{\beta}_3^{(\Delta)}) + O(n^{-3/2})$$

$$\equiv n^{-1/2} \bar{\beta}_3 + O(n^{-3/2})$$

$$\equiv n^{-1/2} \beta_3^{(0)} + N^{-1/2} (\bar{c}^{-1/2} \beta_3^{(\Delta a)} + \bar{c}^{3/2} \beta_3^{(\Delta b)}) + O(n^{-3/2}),$$

where $\bar{\beta}_3^{(\Delta)} = \bar{c} \beta_3^{(\Delta a)} + \bar{c}^2 \beta_3^{(\Delta b)}$. The terms $N^{-1/2} \bar{c}^{-1/2} \beta_3^{(\Delta a)}$ and $N^{-1/2} \bar{c}^{-3/2} \beta_3^{(\Delta b)}$ shown above denote the sums of the terms of orders $O(n^{1/2} N^{-1})$ (given by $E_T(\cdot)$) and $O(n^{3/2} N^{-2})$ (given by $E_{T\alpha_0}(\cdot)$), respectively. Since the constant terms with no expectations in $\left[\begin{smallmatrix} \cdot \\ (A) \end{smallmatrix} \right]_{(A)}$ are $-3n^{-1/2} \bar{c} \{(\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) \beta_2^{(\Delta)} + \beta_1^{(\Delta)} \beta_2^{(0)}\} - 3n^{-1/2} \bar{c}^2 \beta_1^{(\Delta)} \beta_2^{(\Delta)}$
 $= -3N^{-1/2} \bar{c}^{-1/2} \{(\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) \beta_2^{(\Delta)} + \beta_1^{(\Delta)} \beta_2^{(0)}\} - 3N^{-1/2} \bar{c}^{-3/2} \beta_1^{(\Delta)} \beta_2^{(\Delta)}$,
the two terms on the right-hand side of the equation are included in $N^{-1/2} \bar{c}^{-1/2} \beta_3^{(\Delta a)}$ and $N^{-1/2} \bar{c}^{-3/2} \beta_3^{(\Delta b)}$, respectively.

Define $\bar{\beta}_1^* = \bar{\beta}_1 + \lambda_{\theta_0}^{-1} \eta_{\theta_0}$. Then,

$$\begin{aligned} \kappa_4(w) &= n^2 \left[\left\{ E_T \left\{ (q_{O_p(n^{-1/2})}^{(1)})^4 - 3n^{-2} \bar{\beta}_2^2 + 4(q_{O_p(n^{-1/2})}^{(1)})^3 q_{O_p(n^{-1})}^{(2)} \right. \right. \right. \\ &\quad \left. \left. \left. + 6(q_{O_p(n^{-1/2})}^{(1)})^2 (q_{O_p(n^{-1})}^{(2)})^2 + 4(q_{O_p(n^{-1/2})}^{(1)})^3 q_{O_p(n^{-3/2})}^{(3)} \right\} \right\}_{O(n^{-3})} \right. \\ &\quad \left. - 4n^{-3} \bar{\beta}_1^* (\bar{\beta}_3 + 3\bar{\beta}_1^* \bar{\beta}_2) - 6n^{-3} \bar{\beta}_2 \bar{\beta}_{H2} + 6n^{-3} \bar{\beta}_2 (\bar{\beta}_1^*)^2 \right]_{O(n^{-3})} + O(n^{-2}) \\ &= \left[\begin{smallmatrix} n^2 \left\{ E_{T\theta_0} \left\{ (q_{O_p(n^{-1/2})}^{(10)})^4 - 3n^{-2} (\beta_2^{(0)})^2 + 4(q_{O_p(n^{-1/2})}^{(10)})^3 q_{O_p(n^{-1})}^{(20)} \right. \right. \right. \\ &\quad \left. \left. \left. + 6(q_{O_p(n^{-1/2})}^{(10)})^2 (q_{O_p(n^{-1})}^{(20)})^2 + 4(q_{O_p(n^{-1/2})}^{(10)})^3 q_{O_p(n^{-3/2})}^{(30)} \right\} \right\}_{O(n^{-3})} \right. \\ &\quad \left. - 4n^{-1} \beta_1^{(0)} \beta_3^{(0)} - 6n^{-1} \beta_2^{(0)} \beta_{H2}^{(0)} - 6n^{-1} \beta_2^{(0)} (\beta_1^{(0)})^2 \right]_{(A)O(n^{-1})} \end{smallmatrix} \end{aligned}$$

(the following 9 terms are numbered as Terms (1) to (9))

$$\begin{aligned} &+ \left[\left\{ 6n^2 E_T \left\{ (q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)})^2 - n^{-2} \beta_2^{(0)} \bar{\beta}_2^{(\Delta)} \right\} \right\}_0 \right. \\ &\quad \left. + \left\{ n^2 E_{T\alpha_0} \left\{ (q_{O_p(N^{-1/2})}^{(11)})^4 - 3n^{-2} (\bar{\beta}_2^{(\Delta)})^2 \right\} \right\}_{O(n^2 N^{-3})} \right. \\ &\quad \left. + \left\{ 4n^2 E_T \left\{ (q_{O_p(n^{-1/2})}^{(10)})^3 q_{O_p(N^{-1})}^{(22)} + 3(q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(N^{-1/2})}^{(11)} \right. \right. \right. \\ &\quad \quad \left. \left. \left. \times (q_{O_p(n^{-1/2} N^{-1/2})}^{(21)} + q_{O_p(N^{-1})}^{(22)}) \right\} \right\}_{O(N^{-1}) + O(nN^{-2})} \right. \\ &\quad \left. + \left\{ 4n^2 E_T \left\{ 3q_{O_p(n^{-1/2})}^{(10)} (q_{O_p(N^{-1/2})}^{(11)})^2 (q_{O_p(n^{-1})}^{(20)} + q_{O_p(n^{-1/2} N^{-1/2})}^{(21)}) \right\} \right\}_{O(N^{-1}) + O(nN^{-2})} \right. \\ &\quad \left. + \left\{ 4n^2 E_T \left\{ (q_{O_p(N^{-1/2})}^{(11)})^3 (q_{O_p(n^{-1})}^{(20)} + q_{O_p(N^{-1})}^{(22)}) \right\} \right\}_{O(nN^{-2}) + O(n^2 N^{-3})} \right. \end{aligned}$$

$$\begin{aligned}
& + \{6n^2 \mathbf{E}_T \{ (q_{O_p(n^{-1/2})}^{(10)})^2 ((q_{O_p(n^{-1/2}N^{-1/2})}^{(21)})^2 + (q_{O_p(N^{-1})}^{(22)})^2 \\
& \quad + 2q_{O_p(n^{-1})}^{(20)} q_{O_p(N^{-1})}^{(22)}) \} \} \}_{O(N^{-1})+O(nN^{-2})} \\
& + \{6n^2 \mathbf{E}_T \{ 2q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} 2q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} (q_{O_p(n^{-1})}^{(20)} + q_{O_p(N^{-1})}^{(22)}) \} \} \}_{O(N^{-1})+O(nN^{-2})} \\
& + \{6n^2 \mathbf{E}_T \{ (q_{O_p(N^{-1/2})}^{(11)})^2 ((q_{O_p(n^{-1})}^{(20)})^2 + (q_{O_p(n^{-1/2}N^{-1/2})}^{(21)})^2 \\
& \quad + 2q_{O_p(n^{-1})}^{(20)} q_{O_p(N^{-1})}^{(22)}) \} \} \}_{O(N^{-1})+O(nN^{-2})} \\
& + \{6n^2 \mathbf{E}_{T\alpha_0} \{ (q_{O_p(N^{-1/2})}^{(11)})^2 (q_{O_p(N^{-1})}^{(22)})^2 \} \} \}_{O(n^2N^{-3})}
\end{aligned}$$

(the following 5 terms are numbered as Terms (10) to (14))

$$\begin{aligned}
& + \{4n^2 \mathbf{E}_T \{ (q_{O_p(n^{-1/2})}^{(10)})^3 q_{O_p(n^{-1/2}N^{-1})}^{(32)} \} \} \}_{O(N^{-1})} \\
& + \{4n^2 \mathbf{E}_T \{ 3(q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(N^{-1/2})}^{(11)} (q_{O_p(n^{-1}N^{-1/2})}^{(31)} + q_{O_p(N^{-3/2})}^{(33)} \\
& \quad - (n^{-1}(\lambda_{\theta_0}^{-1} \eta_{\theta_0})^{(\Delta)})_{O_p(n^{-1}N^{-1/2})} \} \} \}_{O(N^{-1})+O(nN^{-2})} \\
& + \{4n^2 \mathbf{E}_T \{ 3q_{O_p(n^{-1/2})}^{(10)} (q_{O_p(N^{-1/2})}^{(11)})^2 (q_{O_p(n^{-3/2})}^{(30)} + q_{O_p(n^{-1/2}N^{-1})}^{(32)}) \} \} \}_{O(N^{-1})+O(nN^{-2})} \\
& + \{4n^2 \mathbf{E}_T \{ (q_{O_p(N^{-1/2})}^{(11)})^3 (q_{O_p(n^{-1}N^{-1/2})}^{(31)} - (n^{-1}(\lambda_{\theta_0}^{-1} \eta_{\theta_0})^{(\Delta)})_{O_p(n^{-1}N^{-1/2})} \} \} \}_{O(nN^{-2})} \\
& + \{4n^2 \mathbf{E}_{T\alpha_0} \{ (q_{O_p(N^{-1/2})}^{(11)})^3 q_{O_p(N^{-3/2})}^{(33)} \} \} \}_{O(n^2N^{-3})} \\
& \quad - 4n^{-1} \{ (\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) \bar{\beta}_3^{(\Delta)} + \bar{\beta}_1^{(\Delta)} (\beta_3^{(0)} + \bar{\beta}_3^{(\Delta)}) \} \\
& \quad - 6n^{-1} \{ \beta_2^{(0)} \bar{\beta}_{H2}^{(\Delta)} + \bar{\beta}_2^{(\Delta)} (\beta_{H2}^{(0)} + \bar{\beta}_{H2}^{(\Delta)}) \} \\
& \quad - 6n^{-1} \{ \bar{\beta}_2^{(\Delta)} (\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0} + \bar{\beta}_1^{(\Delta)})^2 + \beta_2^{(0)} \{ 2(\beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) \bar{\beta}_1^{(\Delta)} \\
& \quad \quad \quad + (\bar{\beta}_1^{(\Delta)})^2 \} \} \} \}_{(B)O(n^{-1})} + O(n^{-2}) \\
& \equiv n^{-1} (\beta_4^{(0)} + \bar{\beta}_4^{(\Delta)}) + O(n^{-2}),
\end{aligned}$$

where Term (1) is 0 and Term (2) is

$$\begin{aligned}
& n^2 \mathbf{E}_{T\alpha_0} [\{ (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^4 \}_{O_p(N^{-2})}] - 3(\bar{\beta}_2^{(\Delta)})^2 \\
& = \bar{c}^2 [N^2 \mathbf{E}_{T\alpha_0} [\{ (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^4 \}_{O_p(N^{-2})}] - 3(\beta_2^{(\Delta)})^2] \\
& = N^{-1} \bar{c}^2 \{ N^3 \kappa_4 (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)}) \}_{O(1)}.
\end{aligned}$$

Define $\beta_1^{(0)*} = \beta_1^{(0)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0}$. Then, the constant term with no expectations in $\left[\begin{smallmatrix} \cdot \\ (B) \end{smallmatrix} \right]_{(B)}$ is

$$\begin{aligned}
& -4n^{-1} \{ (\beta_1^{(0)*} + \bar{c} \beta_1^{(\Delta)}) (\bar{c} \beta_3^{(\Delta a)} + \bar{c}^2 \beta_3^{(\Delta b)}) + \bar{c} \beta_1^{(\Delta)} \beta_3^{(0)} \} \\
& -6n^{-1} \{ (\beta_2^{(0)} + \bar{c} \beta_2^{(\Delta)}) (\bar{c} \beta_{H2}^{(\Delta a)} + \bar{c}^2 \beta_{H2}^{(\Delta b)}) + \bar{c} \beta_2^{(\Delta)} \beta_{H2}^{(0)} \} \\
& -6n^{-1} [\bar{c} \beta_2^{(\Delta)} (\beta_1^{(0)*} + \bar{c} \beta_1^{(\Delta)})^2 + \beta_2^{(0)} \{ 2\bar{c} \beta_1^{(0)*} \beta_1^{(\Delta)} + \bar{c}^2 (\beta_1^{(\Delta)})^2 \}] \\
& = -N^{-1} [\{ 4\beta_1^{(0)*} \beta_3^{(\Delta a)} + 4\beta_1^{(\Delta)} \beta_3^{(0)} + 6\beta_2^{(0)} \beta_{H2}^{(\Delta a)} + 6\beta_2^{(\Delta)} \beta_{H2}^{(0)} \\
& \quad + 6\beta_2^{(\Delta)} (\beta_1^{(0)*})^2 + 12\beta_2^{(0)} \beta_1^{(0)*} \beta_1^{(\Delta)} \} \\
& \quad + \bar{c} \{ 4\beta_1^{(0)*} \beta_3^{(\Delta b)} + 4\beta_1^{(\Delta)} \beta_3^{(\Delta a)} + 6\beta_2^{(\Delta)} \beta_{H2}^{(\Delta a)} + 6\beta_2^{(0)} \beta_{H2}^{(\Delta b)} \\
& \quad + 12\beta_2^{(\Delta)} \beta_1^{(0)*} \beta_1^{(\Delta)} + 6\beta_2^{(0)} (\beta_1^{(\Delta)})^2 \} \\
& \quad + \bar{c}^2 \{ 4\beta_1^{(\Delta)} \beta_3^{(\Delta b)} + 6\beta_2^{(\Delta)} \beta_{H2}^{(\Delta b)} + 6\beta_2^{(\Delta)} (\beta_1^{(\Delta)})^2 \}]
\end{aligned}$$

(note that $n^{-1} = N^{-1} \bar{c}^{-1}$ and $N^{-1} = n^{-1} \bar{c}$).

The above results are summarized as

$$\begin{aligned}
\kappa_4(w) &= n^{-1} (\beta_4^{(0)} + \bar{\beta}_4^{(\Delta)}) + O(n^{-2}) \\
&\equiv n^{-1} \beta_4^{(0)} + N^{-1} (\beta_4^{(\Delta a)} + \bar{c} \beta_4^{(\Delta b)} + \bar{c}^2 \beta_4^{(\Delta c)}) + O(n^{-2}).
\end{aligned}$$

The terms, except Term (2), other than the constant terms are included in $N^{-1} \beta_4^{(\Delta a)}$ for the terms of order $O(N^{-1})$, $N^{-1} \bar{c} \beta_4^{(\Delta b)}$ for the terms of order $O(nN^{-2})$, and $N^{-1} \bar{c}^2 \beta_4^{(\Delta c)}$ for the terms of order $O(n^2 N^{-3})$. The cumulant term or Term (2) is included in $N^{-1} \bar{c}^{-2} \beta_4^{(\Delta c)}$. The constants have terms included in $N^{-1} \beta_4^{(\Delta a)}$, $N^{-1} \bar{c} \beta_4^{(\Delta b)}$ and $N^{-1} \bar{c}^2 \beta_4^{(\Delta c)}$.

(b) Proof of Theorem 2 under Condition B: $N = O(n^{3/2})$

$$(\bar{c}^* = n^{3/2} / N = O(1))$$

From (A.2), the following results giving Theorem 2 are obtained.

$$\begin{aligned}
\kappa_1(w) &= n^{1/2} \{ \mathbf{E}_T (q_{O_p(n^{-1})}^{(2)}) - (n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0})_{O(n^{-1})} \} + O(n^{-1}) \\
&= n^{1/2} [\mathbf{E}_{T\theta_0} \{ (\boldsymbol{\gamma}_{\theta_0}^{(2)} \mathbf{I}_{\theta_0}^{(2)})_{O_p(n^{-1})} \}]_{O(n^{-1})} - n^{-1/2} \lambda_{\theta_0}^{-1} \eta_{\theta_0} + O(n^{-1}) \\
&= n^{-1/2} \beta_1^{(0)} + O(n^{-1}),
\end{aligned}$$

where $\bar{\beta}_1 = \beta_1^{(0)} = nE_{T\theta_0} \{(\gamma_{\theta_0}^{(2)} \mathbf{1}_{\theta_0}^{(2)})_{O_p(n^{-1})}\} - \lambda_{\theta_0}^{-1} \eta_{\theta_0}$ ($\bar{\beta}_1^{(\Delta)} = 0$)

(the order of the residual is not $O(n^{-3/2})$ but $O(n^{-1})$, which is due to the term of order $O_p(N^{-1}) = O_p(n^{-3/2})$ in (A.2)).

$$\begin{aligned} \kappa_2(w) &= nE_{T\theta_0} \{(q_{O_p(n^{-1/2})}^{(10)})^2\} + nE_T \{(q_{O_p(N^{-1/2})}^{(1a)})^2\} + nE_{T\theta_0} \{(q_{O_p(n^{-1})}^{(20)})^2\} \\ &\quad + 2q_{O_p(n^{-1/2})}^{(10)} (q_{O_p(n^{-1})}^{(20)} + q_{O_p(n^{-3/2})}^{(30)}) - n^{-1} (\beta_1^{(0)*})^2 + O(n^{-3/2}) \\ &= \beta_2^{(0)} + nE_{T\alpha_0} [\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}] + n^{-1} \beta_{H2}^{(0)} + O(n^{-3/2}) \\ &\equiv \beta_2^{(0)} + n^{-1/2} \bar{\beta}_{h2}^{(\Delta)} + n^{-1} \beta_{H2}^{(0)} + O(n^{-3/2}) \\ &\equiv \beta_2^{(0)} + n^{-1/2} \bar{c}^* \beta_{h2}^{(\Delta)} + n^{-1} \beta_{H2}^{(0)} + O(n^{-3/2}), \end{aligned}$$

where $\bar{\beta}_{h2}^{(\Delta)} = n^{3/2} E_{T\alpha_0} [\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}] = \bar{c}^* \beta_{h2}^{(\Delta)} = O(1) > 0$ and

$\bar{\beta}_2 = \beta_2^{(0)}$ (the order $O(n^{-3/2})$ rather than $O(n^{-2})$ of the residual is due to $n\{(q_{O_p(n^{-1/2}N^{-1/2})}^{(2a)})_{O_p(n^{-5/4})}\}^2$).

$$\begin{aligned} \kappa_3(w) &= [n^{3/2} [\{E_{T\theta_0} \{(q_{O_p(n^{-1/2})}^{(10)})^3\}\}_{O(n^{-2})} \\ &\quad + 3E_{T\theta_0} \{(q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(n^{-1})}^{(20)}\}\}_{O(n^{-2})} - 3n^{-2} \beta_1^{(0)*} \beta_2^{(0)}]]_{O(n^{-1/2})} + O(n^{-1}) \\ &= n^{-1/2} \beta_3^{(0)} + O(n^{-1}), \end{aligned}$$

where $\bar{\beta}_3 = \beta_3^{(0)}$ ($\bar{\beta}_3^{(\Delta)} = 0$) (the order $O(n^{-1})$ rather than $O(n^{-3/2})$ of

the residual is due to $n^{3/2} (-3n^{-5/2} \beta_1^{(0)} \bar{\beta}_{h2}^{(\Delta)})$ and

$$3n^{3/2} E_T \{(q_{O_p(N^{-1/2})}^{(1a)})^2 q_{O_p(n^{-1})}^{(20)}\}.$$

Denote temporarily the expressions $q_{O_p(\cdot)}^{(\cdot)}$ by $q^{(\cdot)}$. Then,

$$\begin{aligned} \kappa_4(w) &= n^2 [E_T \{(q^{(1)})^4 + 6(q^{(1)})^2 (q^{(1a)})^2 + (q^{(1a)})^4 + 4(q^{(1)})^3 q^{(2)} \\ &\quad + 6(q^{(1)})^2 (q^{(2)})^2 + 4(q^{(1)})^3 q^{(3)}\} \\ &\quad - 3n^{-2} (\beta_2^{(0)})^2 - 6n^{-5/2} \beta_2^{(0)} \bar{\beta}_{h2}^{(\Delta)} - 6n^{-3} \beta_2^{(0)} \beta_{H2}^{(0)} - 3n^{-3} (\bar{\beta}_{h2}^{(\Delta)})^2 \\ &\quad - 4n^{-3} \beta_1^{(0)*} \beta_3^{(0)} - 6n^{-3} \beta_2^{(0)} (\beta_1^{(0)*})^2]_{O(n^{-3})} + O(n^{-3/2}) \end{aligned}$$

$$\begin{aligned}
&= [\underset{(A)}{n^2} [\underset{(B)}{[E_{T\theta_0} \{(q^{(10)})^4\}]_{O(n^{-2})}} + 4[E_{T\theta_0} \{(q^{(10)})^3 q^{(20)}\}]_{O(n^{-3})} \\
&\quad + 6[E_{T\theta_0} \{(q^{(10)})^2 (q^{(20)})^2\}]_{O(n^{-3})} + \underline{6n^{-5/2} \beta_2^{(0)} \bar{\beta}_{h2}^{(\Delta)} + 3n^{-3} (\bar{\beta}_{h2}^{(\Delta)})^2} \\
&\quad + 4[E_{T\theta_0} \{(q^{(10)})^3 q^{(30)}\}]_{O(n^{-3})} - 3n^{-2} (\beta_2^{(0)})^2 - \underline{6n^{-5/2} \beta_2^{(0)} \bar{\beta}_{h2}^{(\Delta)}} \\
&\quad \underline{-3n^{-3} (\bar{\beta}_{h2}^{(\Delta)})^2} - 6n^{-3} \beta_2^{(0)} \beta_{H2}^{(0)} - 4n^{-3} \beta_1^{(0)*} \beta_3^{(0)} \\
&\quad \underline{-6n^{-3} \beta_2^{(0)} (\beta_1^{(0)*})^2}]]_{(B) (A) O(n^{-1})} + O(n^{-3/2}) \\
&= n^{-1} \beta_4^{(0)} + O(n^{-3/2}),
\end{aligned}$$

where the sum of the underscored terms is zero, and $\bar{\beta}_4 = \beta_4^{(0)}$ ($\bar{\beta}_4^{(\Delta)} = 0$).

(c) Proof of Theorem 3 under Condition C: $N = O(n^2)$

$$(\bar{c}^{**} = n^2 / N = O(1))$$

From (A.3), the following results giving Theorem 3 are obtained.

$$\begin{aligned}
\kappa_1(w) &= n^{1/2} \{E_T(q_{O_p(n^{-1})}^{(2)}) - (n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0})_{O(n^{-1})}\} + O(n^{-3/2}) \\
&= n^{1/2} [E_{T\theta_0} \{(\gamma_{\theta_0}^{(2)} \mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})}\}]_{O(n^{-1})} - n^{-1/2} \lambda_{\theta_0}^{-1} \eta_{\theta_0} + O(n^{-3/2}) \\
&= n^{-1/2} \beta_1^{(0)} + O(n^{-3/2}),
\end{aligned}$$

where $\bar{\beta}_1 = \beta_1^{(0)} = n E_{T\theta_0} \{(\gamma_{\theta_0}^{(2)} \mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})}\} - \lambda_{\theta_0}^{-1} \eta_{\theta_0}$ ($\bar{\beta}_1^{(\Delta)} = 0$).

$$\begin{aligned}
\kappa_2(w) &= n E_T \{(q_{O_p(n^{-1/2})}^{(1)})^2\} + n E_T \{(q_{O_p(n^{-1})}^{(2)})^2 \\
&\quad + \underline{2q_{O_p(n^{-1/2})}^{(1)} (q_{O_p(n^{-1})}^{(2)} + q_{O_p(n^{-3/2})}^{(3)})\}} - n^{-1} (\beta_1^{(0)*})^2 + O(n^{-2}) \\
&= \beta_2^{(0)} + n^{-1} \beta_{H2}^{(0)} + n E_{T\alpha_0} [\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}] + O(n^{-2}) \\
&= \beta_2^{(0)} + n^{-1} [\beta_{H2}^{(0)} + n^2 E_{T\alpha_0} [\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}]]_{O(1)+O(n^2 N^{-1})} + O(n^{-2}) \\
&\equiv \beta_2^{(0)} + n^{-1} (\beta_{H2}^{(0)} + \bar{\beta}_{H2}^{(\Delta)}) + O(n^{-2}) \\
&\equiv \beta_2^{(0)} + n^{-1} (\beta_{H2}^{(0)} + \bar{c}^{**} \beta_{H2}^{(\Delta)}) + O(n^{-2}),
\end{aligned}$$

where $\bar{\beta}_{H2}^{(\Delta)} = n^2 E_{T\alpha_0} [\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}] = \bar{c}^{**} \beta_{H2}^{(\Delta)} > 0$, $\bar{\beta}_2 = \beta_2^{(0)}$,

$\bar{\beta}_{H2} = \beta_{H2}^{(0)} + \bar{\beta}_{H2}^{(\Delta)} > \beta_{H2}^{(0)}$, and the underscored term after taking the expectation leaves only the quantity associated with $\beta_{H2}^{(0)}$.

$$\begin{aligned} \kappa_3(w) &= [n^{3/2} [\{E_T\{(q_{O_p(n^{-1/2})}^{(1)})^3\}\}_{O(n^{-2})} \\ &+ 3E_T\{(q_{O_p(n^{-1/2})}^{(1)})^2 q_{O_p(n^{-1})}^{(2)}\}\}_{O(n^{-2})} - 3n^{-2} \beta_1^{(0)*} \beta_2^{(0)}]]_{O(n^{-1/2})} + O(n^{-3/2}) \\ &= n^{-1/2} \beta_3^{(0)} + O(n^{-3/2}), \end{aligned}$$

where $E_T\{\cdot\} = E_{T\theta_0}\{\cdot\}$ and $\bar{\beta}_3 = \beta_3^{(0)}$ ($\bar{\beta}_3^{(\Delta)} = 0$).

Using simple notations like $q^{(1)}$,

$$\begin{aligned} \kappa_4(w) &= n^2 [E_T\{(q^{(1)})^4 + 4(q^{(1)})^3 q^{(2)} + 6(q^{(1)})^2 (q^{(2)})^2 + 4(q^{(1)})^3 q^{(3)}\} \\ &\quad - 3n^{-2} (\beta_2^{(0)})^2 - 4n^{-3} \beta_1^{(0)*} \beta_3^{(0)} - 6n^{-3} \beta_2^{(0)} (\beta_{H2}^{(0)} + \bar{\beta}_{H2}^{(\Delta)}) \\ &\quad - 6n^{-3} \beta_2^{(0)} (\beta_1^{(0)*})^2]_{O(n^{-3})} + O(n^{-2}) \\ &= [n^2 [[E_{T\theta_0}\{(q^{(10)})^4\}]_{O(n^{-2})} + 4[E_{T\theta_0}\{(q^{(10)})^3 q^{(20)}\}]_{O(n^{-3})} \\ &\quad + 6[E_{T\theta_0}\{(q^{(10)})^2 (q^{(20)})^2\}]_{O(n^{-3})} + 6n^{-3} \beta_2^{(0)} \bar{\beta}_{H2}^{(\Delta)} \\ &\quad + 4[E_{T\theta_0}\{(q^{(10)})^3 q^{(30)}\}]_{O(n^{-3})} - 3n^{-2} (\beta_2^{(0)})^2 - 4n^{-3} \beta_1^{(0)*} \beta_3^{(0)} \\ &\quad - 6n^{-3} \beta_2^{(0)} (\beta_{H2}^{(0)} + \bar{\beta}_{H2}^{(\Delta)}) - 6n^{-3} \beta_2^{(0)} (\beta_1^{(0)*})^2]]_{O(n^{-1})} + O(n^{-2}) \\ &= n^{-1} \beta_4^{(0)} + O(n^{-2}), \end{aligned}$$

where the sum of the underscored terms is zero, and $\bar{\beta}_4 = \beta_4^{(0)}$ ($\bar{\beta}_4^{(\Delta)} = 0$).

A.4 Asymptotic cumulants of the studentized $\hat{\theta}$: Proof of Theorem 4 under Condition A: $N = O(n)$ ($\bar{c} = n / N = O(1)$)

Define the asymptotic cumulants $\bar{\beta}_{tk}$ ($k = 1, \dots, 4$) and the higher-order asymptotic variance $\bar{\beta}_{tH2}$ with the associated quantities for t , which are independent of n , in the following equations.

$$\begin{aligned}
\kappa_1(t) &= n^{-1/2} \{ \mathbf{E}_T (nt_{O_p(n^{-1})}^{(2)}) - \lambda_{\theta_0}^{-1} \eta_{\theta_0} \bar{\beta}_{2I}^{-1/2} \} + O(n^{-3/2}) \\
&= n^{-1/2} \{ \bar{\beta}_1 \bar{\beta}_{2I}^{-1/2} + \mathbf{E}_T (nq_{O_p(n^{-1/2})}^{(1)} b_{O_p(n^{-1/2})}^{(1)}) \} + O(n^{-3/2}) \\
&= n^{-1/2} \{ \bar{\beta}_1 \bar{\beta}_{2I}^{-1/2} + \mathbf{E}_{T\theta_0} (nq_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)}) \\
&\quad + \bar{c} \mathbf{E}_{T\alpha_0} (Nq_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)}) \} + O(n^{-3/2}) \\
&\equiv n^{-1/2} (\bar{\beta}_1 \bar{\beta}_{2I}^{-1/2} + \beta_1^{(t0)} + \bar{\beta}_1^{(t\Delta)}) + O(n^{-3/2}) \\
&\equiv n^{-1/2} (\bar{\beta}_1 \bar{\beta}_{2I}^{-1/2} + \beta_1^{(t0)} + \bar{c} \beta_1^{(t\Delta)}) + O(n^{-3/2}) \\
&\equiv n^{-1/2} \bar{\beta}_{t1} + O(n^{-3/2}) \\
&= n^{-1/2} (\bar{\beta}_1 \bar{\beta}_{2I}^{-1/2} + \beta_1^{(t0)}) + N^{-1/2} \bar{c}^{-1/2} \beta_1^{(t\Delta)} + O(n^{-3/2}),
\end{aligned}$$

where the term $-n^{-1/2} \lambda_{\theta_0}^{-1} \eta_{\theta_0} \bar{\beta}_{2I}^{-1/2}$ is included in $n^{-1/2} \bar{\beta}_1 \bar{\beta}_{2I}^{-1/2}$.

Let $\bar{\beta}_{t1}^* = \bar{\beta}_{t1} + \lambda_{\theta_0}^{-1} \eta_{\theta_0} \bar{\beta}_{2I}^{-1/2}$. Then,

$$\begin{aligned}
\kappa_2(t) &= \mathbf{E}_T \{ n(q_{O_p(n^{-1/2})}^{(1)})^2 \} \bar{\beta}_{2I}^{-1} + n \mathbf{E}_T [\{ (t_{O_p(n^{-1})}^{(2)})^2 \\
&\quad + 2t_{O_p(n^{-1/2})}^{(1)} (t_{O_p(n^{-1})}^{(2)} + t_{O_p(n^{-3/2})}^{(3)}) \}]_{O(n^{-2})} - n^{-1} (\bar{\beta}_{t1}^*)^2 + O(n^{-2}) \\
&= \bar{\beta}_2 \bar{\beta}_{2I}^{-1} + n [\mathbf{E}_T \{ (q_{O_p(n^{-1})}^{(2)})^2 + 2q_{O_p(n^{-1/2})}^{(1)} (q_{O_p(n^{-1})}^{(2)} + q_{O_p(n^{-3/2})}^{(3)}) \} \bar{\beta}_{2I}^{-1} \\
&\quad - n^{-2} (\bar{\beta}_1^* \bar{\beta}_{2I}^{-1/2})^2]_{O(n^{-2})} + O(n^{-2}) \quad (\text{recall that } \bar{\beta}_1^* = \bar{\beta}_1 + \lambda_{\theta_0}^{-1} \eta_{\theta_0}) \\
&+ n \underset{(A)}{[\mathbf{E}_{T\theta_0} \{ (q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)})^2 + 2q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)} q_{O_p(n^{-1})}^{(20)} \bar{\beta}_{2I}^{-1/2} \} \\
&\quad + \mathbf{E}_T \{ (q_{O_p(n^{-1/2})}^{(10)} b_{O_p(N^{-1/2})}^{(11)})^2 + (q_{O_p(N^{-1/2})}^{(11)} b_{O_p(n^{-1/2})}^{(10)})^2 \\
&\quad + 4q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)} \\
&\quad + 2((q_{O_p(n^{-1/2})}^{(10)} b_{O_p(N^{-1/2})}^{(11)} + q_{O_p(N^{-1/2})}^{(11)} b_{O_p(n^{-1/2})}^{(10)}) q_{O_p(n^{-1/2} N^{-1/2})}^{(21)} \\
&\quad + q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1})}^{(22)} + q_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)} q_{O_p(n^{-1})}^{(20)} \} \bar{\beta}_{2I}^{-1/2} \} \\
&+ \mathbf{E}_{T\alpha_0} \{ (q_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)})^2 + 2q_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)} q_{O_p(N^{-1})}^{(22)} \bar{\beta}_{2I}^{-1/2} \}
\end{aligned}$$

$$\begin{aligned}
& +2\mathbf{E}_{T\theta_0} \{ (q_{O_p(n^{-1/2})}^{(10)})^2 b_{O_p(n^{-1/2})}^{(10)} \} \bar{\beta}_{2I}^{-1/2} + 2\mathbf{E}_{T\alpha_0} \{ (q_{O_p(N^{-1/2})}^{(11)})^2 b_{O_p(N^{-1/2})}^{(11)} \} \bar{\beta}_{2I}^{-1/2} \\
& + 2\mathbf{E}_{T\theta_0} \{ (q_{O_p(n^{-1/2})}^{(10)})^2 b_{O_p(n^{-1})}^{(20)} \} \bar{\beta}_{2I}^{-1/2} + 2\mathbf{E}_T \{ 2q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} b_{O_p(n^{-1/2}N^{-1/2})}^{(21)} \\
& \quad + (q_{O_p(n^{-1/2})}^{(10)})^2 b_{O_p(N^{-1})}^{(22)} + (q_{O_p(N^{-1/2})}^{(11)})^2 b_{O_p(n^{-1})}^{(20)} \} \bar{\beta}_{2I}^{-1/2} \\
& + 2n^{-1}(\beta_2^{(0)} + \bar{\beta}_2^{(\Delta)}) \bar{\beta}_{2I}^{-1/2} \\
& \quad \times \left(n^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \theta_0} \lambda_{\theta_0}^{-1} \eta_{\theta_0} + N^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \alpha_0} \Lambda_{\alpha_0}^{-1} \boldsymbol{\eta}_{\alpha_0} \right) \\
& + 2\mathbf{E}_{T\theta_0} (q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1})}^{(20)} b_{O_p(n^{-1/2})}^{(10)}) \bar{\beta}_{2I}^{-1/2} \\
& + 2\mathbf{E}_T \{ q_{O_p(n^{-1/2})}^{(10)} (q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} b_{O_p(N^{-1/2})}^{(11)} + q_{O_p(N^{-1})}^{(22)} b_{O_p(n^{-1/2})}^{(10)}) \\
& \quad + q_{O_p(N^{-1/2})}^{(11)} (q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} b_{O_p(n^{-1/2})}^{(10)} + q_{O_p(n^{-1})}^{(20)} b_{O_p(N^{-1/2})}^{(11)}) \} \bar{\beta}_{2I}^{-1/2} \\
& + 2\mathbf{E}_{T\alpha_0} (q_{O_p(N^{-1/2})}^{(11)} q_{O_p(N^{-1})}^{(22)} b_{O_p(N^{-1/2})}^{(11)}) \bar{\beta}_{2I}^{-1/2} - 2n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0} \\
& \quad \times \{ \mathbf{E}_{T\theta_0} (q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)}) + \mathbf{E}_{T\alpha_0} (q_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)}) \} \bar{\beta}_{2I}^{-1/2} \Big]_{(A)O(n^{-2})} \\
& - n^{-1} (\beta_{t1}^*)^2 + n^{-1} (\bar{\beta}_1^* \bar{\beta}_{2I}^{-1/2})^2 + O(n^{-2})
\end{aligned}$$

$$\begin{aligned}
& \equiv \bar{\beta}_2 \bar{\beta}_{2I}^{-1} + n^{-1} \bar{\beta}_{H2} \bar{\beta}_{2I}^{-1} + n^{-1} \bar{\beta}_{H2}^{(t0\Delta)} + O(n^{-2}) \\
& \equiv \bar{\beta}_2 \bar{\beta}_{2I}^{-1} + n^{-1} \bar{\beta}_{tH2} + O(n^{-2}) \\
& \equiv \bar{\beta}_{t2} + n^{-1} \bar{\beta}_{tH2} + O(n^{-2}),
\end{aligned}$$

$$\text{where } n^{-1} \bar{\beta}_{H2}^{(t0\Delta)} = n \Big[\cdot \Big]_{(A)} - n^{-1} (\beta_{t1}^*)^2 + n^{-1} (\beta_1^* \bar{\beta}_{2I}^{-1/2})^2.$$

$$\begin{aligned}
\kappa_3(t) & = n^{3/2} \left[\{ \mathbf{E}_T \{ (t_{O_p(n^{-1/2})}^{(1)})^3 + 3(t_{O_p(n^{-1/2})}^{(1)})^2 t_{O_p(n^{-1})}^{(2)} \} \} \right]_{O(n^{-2})} \\
& \quad - 3n^{-2} \bar{\beta}_{t1}^* \bar{\beta}_{t2} \Big]_{O(n^{-1/2})} + O(n^{-3/2}) \\
& = n^{3/2} \left[\mathbf{E}_T \{ (q_{O_p(n^{-1/2})}^{(1)})^3 + 3(q_{O_p(n^{-1/2})}^{(1)})^2 q_{O_p(n^{-1})}^{(2)} \} - 3n^{-2} \bar{\beta}_1^* \bar{\beta}_2 \right]_{O(n^{-2})} \bar{\beta}_{2I}^{-3/2}
\end{aligned}$$

$$\left(\begin{array}{l} \text{note that } \bar{\beta}_1^* = \bar{\beta}_1 + \lambda_{\theta_0}^{-1} \eta_{\theta_0} \text{ and} \\ \bar{\beta}_{t1}^* = \bar{\beta}_{t1} + \lambda_{\theta_0}^{-1} \eta_{\theta_0} \bar{\beta}_{2I}^{-1/2} = \bar{\beta}_1 \bar{\beta}_{2I}^{-1/2} + \beta_1^{(t0)} + \bar{\beta}_1^{(t\Delta)} + \lambda_{\theta_0}^{-1} \eta_{\theta_0} \bar{\beta}_{2I}^{-1/2} \\ = \bar{\beta}_1^* \bar{\beta}_{2I}^{-1/2} + \beta_1^{(t0)} + \bar{\beta}_1^{(t\Delta)} \end{array} \right)$$

$$\begin{aligned}
& + n^{3/2} [3\mathbb{E}_{T\theta_0} \{ (q_{O_p(n^{-1/2})}^{(10)})^3 b_{O_p(n^{-1/2})}^{(10)} \bar{\beta}_{2I}^{-1} - 3n^{-2} \beta_1^{(t0)} \bar{\beta}_{t2} \}]_{O(n^{-2})} \\
& + n^{3/2} [9\mathbb{E}_T \{ (q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)} \\
& \quad + (q_{O_p(N^{-1/2})}^{(11)})^2 q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)} \} \bar{\beta}_{2I}^{-1} \\
& \quad + 3\mathbb{E}_{T\alpha_0} \{ (q_{O_p(N^{-1/2})}^{(11)})^3 b_{O_p(N^{-1/2})}^{(11)} \} \bar{\beta}_{2I}^{-1} - 3n^{-2} \bar{\beta}_1^{(t\Delta)} \bar{\beta}_{t2} \}]_{O(n^{-2})} + O(n^{-3/2}) \\
& \equiv n^{-1/2} (\bar{\beta}_3 \bar{\beta}_{2I}^{-3/2} + \beta_3^{(t0)} + \bar{\beta}_3^{(t\Delta)}) + O(n^{-3/2}) \\
& \equiv n^{-1/2} \bar{\beta}_{t3} + O(n^{-3/2}).
\end{aligned}$$

$$\begin{aligned}
\kappa_4(t) & = n^2 [\{ \mathbb{E}_T \{ (t_{O_p(n^{-1/2})}^{(1)})^4 - 3n^{-2} \bar{\beta}_2^2 \bar{\beta}_{2I}^{-2} + 4(t_{O_p(n^{-1/2})}^{(1)})^3 t_{O_p(n^{-1})}^{(2)} \\
& \quad + 6(t_{O_p(n^{-1/2})}^{(1)})^2 (t_{O_p(n^{-1})}^{(2)})^2 + 4(t_{O_p(n^{-1/2})}^{(1)})^3 t_{O_p(n^{-3/2})}^{(3)} \} \} \}_{O(n^{-3})} \\
& \quad - 4n^{-3} \bar{\beta}_{t1}^* \bar{\beta}_{t3} - 6n^{-3} \bar{\beta}_{t2} \bar{\beta}_{tH2} - 6n^{-3} \bar{\beta}_{t2} (\bar{\beta}_{t1}^*)^2 \}]_{O(n^{-3})} + O(n^{-2}) \\
& \text{(note that } \bar{\beta}_{t3} = \bar{\beta}_3 \bar{\beta}_{2I}^{-3/2} + \beta_3^{(t0)} + \bar{\beta}_3^{(t\Delta)} \text{ and } \bar{\beta}_{tH2} = \bar{\beta}_{H2} \bar{\beta}_{2I}^{-1} + \bar{\beta}_{H2}^{(t0\Delta)} \text{)}
\end{aligned}$$

$$\begin{aligned}
& = n^2 [\{ \mathbb{E}_T \{ (q_{O_p(n^{-1/2})}^{(1)})^4 - 3n^{-2} \bar{\beta}_2^2 + 4(q_{O_p(n^{-1/2})}^{(1)})^3 q_{O_p(n^{-1})}^{(2)} \\
& \quad + 6(q_{O_p(n^{-1/2})}^{(1)})^2 (q_{O_p(n^{-1})}^{(2)})^2 + 4(q_{O_p(n^{-1/2})}^{(1)})^3 q_{O_p(n^{-3/2})}^{(3)} \} \} \}_{O(n^{-3})} \\
& \quad - 4n^{-3} \bar{\beta}_1^* \bar{\beta}_3 - 6n^{-3} \bar{\beta}_2 \bar{\beta}_{H2} - 6n^{-3} \bar{\beta}_2 (\bar{\beta}_1^*)^2 \}]_{O(n^{-3})} \bar{\beta}_{2I}^{-2} + O(n^{-2}) \\
& + n^2 [\mathbb{E}_{T\theta_0} \{ 4(q_{O_p(n^{-1/2})}^{(10)})^4 b_{O_p(n^{-1/2})}^{(10)} \bar{\beta}_{2I}^{-3/2} + 6(q_{O_p(n^{-1/2})}^{(10)})^4 (b_{O_p(n^{-1/2})}^{(10)})^2 \bar{\beta}_{2I}^{-1} \\
& \quad + 12(q_{O_p(n^{-1/2})}^{(10)})^3 b_{O_p(n^{-1/2})}^{(10)} q_{O_p(n^{-1})}^{(20)} \bar{\beta}_{2I}^{-3/2}
\end{aligned}$$

$$\begin{aligned}
& +4(q_{O_p(n^{-1/2})}^{(10)})^4 \\
& \times \left(b_{O_p(n^{-1})}^{(20)} + n^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \theta_0} \lambda_{\theta_0}^{-1} \eta_{\theta_0} + N^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \mathbf{a}_0} \Lambda_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0} \right) \bar{\beta}_{2I}^{-3/2} \\
& + 4(q_{O_p(n^{-1/2})}^{(10)})^3 (q_{O_p(n^{-1})}^{(20)} - n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0}) b_{O_p(n^{-1/2})}^{(10)} \bar{\beta}_{2I}^{-3/2} \} \\
& + \mathbf{E}_T \{ \underset{\text{(B)}}{16(q_{O_p(n^{-1/2})}^{(10)})^3 q_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)} \bar{\beta}_{2I}^{-3/2}} \\
& \quad + 24(q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)})^2 (b_{O_p(n^{-1/2})}^{(10)} + b_{O_p(N^{-1/2})}^{(11)}) \bar{\beta}_{2I}^{-3/2}} \\
& + 16(q_{O_p(N^{-1/2})}^{(11)})^3 q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)} \bar{\beta}_{2I}^{-3/2} \\
& + 6(q_{O_p(n^{-1/2})}^{(10)})^4 (b_{O_p(N^{-1/2})}^{(11)})^2 \bar{\beta}_{2I}^{-1} + 6(q_{O_p(N^{-1/2})}^{(11)})^4 (b_{O_p(n^{-1/2})}^{(10)})^2 \bar{\beta}_{2I}^{-1} \\
& + 36(q_{O_p(n^{-1/2})}^{(10)})^2 (q_{O_p(N^{-1/2})}^{(11)})^2 \{ (b_{O_p(n^{-1/2})}^{(10)})^2 + (b_{O_p(N^{-1/2})}^{(11)})^2 \} \bar{\beta}_{2I}^{-1} \\
& + 24q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} \{ (q_{O_p(n^{-1/2})}^{(10)})^2 + (q_{O_p(N^{-1/2})}^{(11)})^2 \} 2b_{O_p(n^{-1/2})}^{(10)} b_{O_p(N^{-1/2})}^{(11)} \bar{\beta}_{2I}^{-1} \\
& + 12(q_{O_p(n^{-1/2})}^{(10)})^3 (b_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1})}^{(22)} + b_{O_p(N^{-1/2})}^{(11)} q_{O_p(n^{-1/2}N^{-1/2})}^{(21)}) \bar{\beta}_{2I}^{-3/2} \\
& + 12(q_{O_p(N^{-1/2})}^{(11)})^3 (b_{O_p(N^{-1/2})}^{(11)} q_{O_p(n^{-1})}^{(20)} + b_{O_p(n^{-1/2})}^{(10)} q_{O_p(n^{-1/2}N^{-1/2})}^{(21)}) \bar{\beta}_{2I}^{-3/2} \\
& + 36q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} (q_{O_p(n^{-1/2})}^{(10)} b_{O_p(n^{-1/2})}^{(10)} q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} \\
& \quad + q_{O_p(N^{-1/2})}^{(11)} b_{O_p(N^{-1/2})}^{(11)} q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} \\
& \quad + q_{O_p(n^{-1/2})}^{(10)} b_{O_p(N^{-1/2})}^{(11)} q_{O_p(N^{-1})}^{(22)} + q_{O_p(N^{-1/2})}^{(11)} b_{O_p(n^{-1/2})}^{(10)} q_{O_p(n^{-1})}^{(20)}) \bar{\beta}_{2I}^{-3/2} \\
& + 4(q_{O_p(n^{-1/2})}^{(10)})^4 b_{O_p(N^{-1})}^{(22)} \bar{\beta}_{2I}^{-3/2} + 4(q_{O_p(N^{-1/2})}^{(11)})^4 b_{O_p(n^{-1})}^{(20)} \bar{\beta}_{2I}^{-3/2} \\
& + 16q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)} \{ (q_{O_p(n^{-1/2})}^{(10)})^2 + (q_{O_p(N^{-1/2})}^{(11)})^2 \} b_{O_p(n^{-1/2}N^{-1/2})}^{(21)} \bar{\beta}_{2I}^{-3/2} \\
& + 24(q_{O_p(n^{-1/2})}^{(10)} q_{O_p(N^{-1/2})}^{(11)})^2 \left(b_{O_p(n^{-1})}^{(20)} + b_{O_p(N^{-1})}^{(22)} \right. \\
& \quad \left. + n^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \theta_0} \lambda_{\theta_0}^{-1} \eta_{\theta_0} + N^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \mathbf{a}_0} \Lambda_{\mathbf{a}_0}^{-1} \boldsymbol{\eta}_{\mathbf{a}_0} \right) \bar{\beta}_{2I}^{-3/2}
\end{aligned}$$

$$\begin{aligned}
& +4(q_{O_p(n^{-1/2})}^{(10)})^3 (q_{O_p(N^{-1})}^{(22)} b_{O_p(n^{-1/2})}^{(10)} + q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} b_{O_p(N^{-1/2})}^{(11)}) \bar{\beta}_{2I}^{-3/2} \\
& +4(q_{O_p(N^{-1/2})}^{(11)})^3 (q_{O_p(n^{-1})}^{(20)} b_{O_p(N^{-1/2})}^{(11)} + q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} b_{O_p(n^{-1/2})}^{(10)}) \bar{\beta}_{2I}^{-3/2} \\
& +12(q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(N^{-1/2})}^{(11)} \{ (q_{O_p(n^{-1})}^{(20)} + q_{O_p(N^{-1})}^{(22)} - n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0}) b_{O_p(N^{-1/2})}^{(11)} \\
& \quad + q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} b_{O_p(n^{-1/2})}^{(10)} \} \bar{\beta}_{2I}^{-3/2} \\
& +12(q_{O_p(N^{-1/2})}^{(11)})^2 q_{O_p(n^{-1/2})}^{(10)} \{ (q_{O_p(n^{-1})}^{(20)} + q_{O_p(N^{-1})}^{(22)} - n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0}) b_{O_p(n^{-1/2})}^{(10)} \\
& \quad + q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} b_{O_p(N^{-1/2})}^{(11)} \} \bar{\beta}_{2I}^{-3/2} \quad \} \quad \text{(B)}
\end{aligned}$$

$$\begin{aligned}
& +E_{T\alpha_0} \{ 4(q_{O_p(N^{-1/2})}^{(11)})^4 b_{O_p(N^{-1/2})}^{(11)} \bar{\beta}_{2I}^{-3/2} + 6(q_{O_p(N^{-1/2})}^{(11)})^4 (b_{O_p(N^{-1/2})}^{(11)})^2 \bar{\beta}_{2I}^{-1} \\
& \quad + 12(q_{O_p(N^{-1/2})}^{(11)})^3 b_{O_p(N^{-1/2})}^{(11)} q_{O_p(N^{-1})}^{(22)} \bar{\beta}_{2I}^{-3/2} \\
& +4(q_{O_p(N^{-1/2})}^{(11)})^4 \\
& \quad \times \left(b_{O_p(N^{-1})}^{(22)} + n^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \theta_0} \lambda_{\theta_0}^{-1} \eta_{\theta_0} + N^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \alpha_0} \Lambda_{\alpha_0}^{-1} \mathbf{n}_{\alpha_0} \right) \bar{\beta}_{2I}^{-3/2} \\
& \quad + 4(q_{O_p(N^{-1/2})}^{(11)})^3 (q_{O_p(N^{-1})}^{(22)} - n^{-1} \lambda_{\theta_0}^{-1} \eta_{\theta_0}) b_{O_p(N^{-1/2})}^{(11)} \bar{\beta}_{2I}^{-3/2} \} \\
& -4n^{-3} \{ (\beta_1^{(t0)} + \bar{\beta}_1^{(t\Delta)}) \bar{\beta}_{t3} + \bar{\beta}_1^* \bar{\beta}_{2I}^{-1/2} (\beta_3^{(t0)} + \bar{\beta}_3^{(t\Delta)}) \} \\
& -6n^{-3} \bar{\beta}_{H2}^{(t0\Delta)} \bar{\beta}_{t2} \\
& -6n^{-3} \{ (\beta_1^{(t0)} + \bar{\beta}_1^{(t\Delta)}) \bar{\beta}_{t1}^* \\
& \quad + \bar{\beta}_1^* \bar{\beta}_{2I}^{-1/2} (\beta_1^{(t0)} + \bar{\beta}_1^{(t\Delta)}) \} \bar{\beta}_{t2} \quad \text{] } \quad \text{(A)}_{O(n^{-3})} \quad + O(n^{-2}) \\
& \equiv n^{-1} (\bar{\beta}_4 \bar{\beta}_{2I}^{-2} + \bar{\beta}_4^{(t0\Delta)}) + O(n^{-2}) \\
& \equiv n^{-1} \bar{\beta}_{t4} + O(n^{-2}).
\end{aligned}$$

A.5 Partial derivatives

A.5.1 Partial derivatives associated with the non-studentized $\hat{\theta}$ under m.m.

Note that under m.m. $P_{Tk} (k = 1, \dots, n)$ are assumed to be not functions of \mathbf{a} . Define

$$\bar{l}_{\theta_0}(\mathbf{a}_0, \theta_0) \equiv n^{-1} \sum_{k=1}^n \{U_k \log P_k + (1 - U_k) Q_k\}.$$

$$(a.1) \quad \boldsymbol{\gamma}_{\theta_0}^{(1)} \cdot \mathbf{l}_{\theta_0}^{(1)} = \boldsymbol{\gamma}_{\theta_0}^{(1)} l_{\theta_0}^{(1)} = -\lambda_{\theta_0}^{-1} \frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0}$$

$$\begin{aligned} \boldsymbol{\gamma}_{\theta_0}^{(\Delta 1)} : \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(1)}}{\partial \mathbf{a}_0} &= -\frac{\partial \lambda_{\theta_0}^{-1}}{\partial \mathbf{a}_0} = \lambda_{\theta_0}^{-2} \frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} = \lambda_{\theta_0}^{-2} \frac{\partial}{\partial \mathbf{a}_0} \mathbf{E}_{T\theta_0} \left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0^2} \right) \\ &= \lambda_{\theta_0}^{-2} \frac{\partial}{\partial \mathbf{a}_0} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 P_{Tk} \left\{ -\frac{1}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 + \frac{1}{P_k} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \\ &= \lambda_{\theta_0}^{-2} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 P_{Tk} \left[\left\{ \frac{2}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 - \frac{1}{P_k^2} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \frac{\partial P_k}{\partial \mathbf{a}_0} \right. \\ &\quad \left. - \frac{2}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} + \frac{1}{P_k} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_0} \right], \end{aligned}$$

where $\sum_{P(Q)}^2$ indicates the sum of two terms replacing P by Q with other summations, shown later, defined similarly. Note that under c.m.s., the above result becomes

$$= \lambda_{\theta_0}^{-2} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 \left[\left\{ \frac{2}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 - \frac{1}{P_k} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \frac{\partial P_k}{\partial \mathbf{a}_0} - \frac{2}{P_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right].$$

On the other hand, since $P_{Tk} (k = 1, \dots, n)$ under c.m.s. are functions of \mathbf{a} ,

$$\begin{aligned}\lambda_{\theta_0}^{-2} \frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} &= \lambda_{\theta_0}^{-2} \frac{\partial}{\partial \mathbf{a}_0} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 \left\{ -\frac{1}{P_k} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \right\} \\ &= \lambda_{\theta_0}^{-2} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 \left\{ \frac{1}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial P_k}{\partial \mathbf{a}_0} - \frac{2}{P_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right\},\end{aligned}$$

which is different from the former result under m.m.

$$\begin{aligned}\gamma_{\theta_0}^{(\Delta\Delta 1)} : \frac{\partial^2 \gamma_{\theta_0}^{(1)}}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} &= -2\lambda_{\theta_0}^{-3} \left(\frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} + \lambda_{\theta_0}^{-2} \frac{\partial^2 \lambda_{\theta_0}}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \\ &= -2\lambda_{\theta_0}^{-3} \left(\frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} \\ &+ \lambda_{\theta_0}^{-2} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 P_{Tk} \left[\left\{ -\frac{6}{P_k^4} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 + \frac{2}{P_k^3} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \left(\frac{\partial P_k}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} \right. \\ &+ \frac{4}{P_k^3} \frac{\partial P_k}{\partial \theta_0} \sum_{\otimes}^2 \frac{\partial P_k}{\partial \mathbf{a}_0} \otimes \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} - \frac{1}{P_k^2} \sum_{\otimes}^2 \frac{\partial P_k}{\partial \mathbf{a}_0} \otimes \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_0} \\ &+ \left. \left\{ \frac{2}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 - \frac{1}{P_k^2} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \frac{\partial^2 P_k}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right. \\ &\left. - \frac{2}{P_k^2} \left\{ \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right)^{\langle 2 \rangle} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right\} + \frac{1}{P_k} \frac{\partial^4 P_k}{\partial \theta_0^2 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right].\end{aligned}$$

$$\begin{aligned}l_{\theta_0}^{(\Delta 1)} : \frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0} &= \frac{\partial}{\partial \mathbf{a}_0} n^{-1} \sum_{k=1}^n \sum_{\substack{P,U \\ (Q, 1-U)}}^2 \frac{U_k}{P_k} \frac{\partial P_k}{\partial \theta_0} \\ &= n^{-1} \sum_{k=1}^n \sum_{\substack{P,U \\ (Q, 1-U)}}^2 U_k \left(-\frac{1}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} + \frac{1}{P_k} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right),\end{aligned}$$

$$l_{\theta_0}^{(\Delta\Delta b1)} : \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} = n^{-1} \sum_{k=1}^n \sum_{\substack{P,U \\ (Q, 1-U)}}^2 U_k \left\{ \frac{2}{P_k^3} \frac{\partial P_k}{\partial \theta_0} \left(\frac{\partial P_k}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} \right. \\ \left. - \frac{1}{P_k^2} \left(\sum_{\otimes}^2 \frac{\partial P_k}{\partial \mathbf{a}_0} \otimes \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right) + \frac{1}{P_k} \frac{\partial^3 P_k}{\partial \theta_0 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right\},$$

$$l_{\theta_0}^{(\Delta\Delta\Delta b1)} : \frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0 (\partial \mathbf{a}_0)^{\langle 3 \rangle}} = n^{-1} \sum_{k=1}^n \sum_{\substack{P,U \\ (Q, 1-U)}}^2 U_k \left[-\frac{6}{P_k^4} \frac{\partial P_k}{\partial \theta_0} \left(\frac{\partial P_k}{\partial \mathbf{a}_0} \right)^{\langle 3 \rangle} \right. \\ \left. + \frac{2}{P_k^3} \sum_{\otimes}^3 \left\{ \left(\frac{\partial P_k}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} \otimes \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} \otimes \frac{\partial^2 P_k}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right\} \right. \\ \left. - \frac{1}{P_k^2} \left\{ \sum_{\otimes}^3 \left(\frac{\partial P_k}{\partial \mathbf{a}_0} \otimes \frac{\partial^3 P_k}{\partial \theta_0 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} + \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \otimes \frac{\partial^2 P_k}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right) \right. \right. \\ \left. \left. + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{(\partial \mathbf{a}_0)^{\langle 3 \rangle}} \right\} + \frac{1}{P_k} \frac{\partial^4 P_k}{\partial \theta_0 (\partial \mathbf{a}_0)^{\langle 3 \rangle}} \right],$$

where the corresponding expectations are given by replacing U_k with P_{Tk} ($k = 1, \dots, n$), which hold in the following similar results.

$$(a.2) \quad \boldsymbol{\gamma}_{\theta_0}^{(2)} \cdot \mathbf{I}_{\theta_0}^{(2)} = \left\{ \lambda_{\theta_0}^{-2}, -\frac{\lambda_{\theta_0}^{-3}}{2} \mathbf{E}_{T\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\} \left\{ m \frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0}, \left(\frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0} \right)^2 \right\}' \\ \left(\text{recall } m = \frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0^2} - \lambda_{\theta_0} \right),$$

$$\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)} : \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(2)}}{\partial \boldsymbol{\alpha}_0} = \left\{ -2\lambda_{\theta_0}^{-3}, \frac{3\lambda_{\theta_0}^{-4}}{2} \mathbf{E}_{T\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\} \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} + \left\{ \mathbf{0}, -\frac{\lambda_{\theta_0}^{-3}}{2} \frac{\partial}{\partial \boldsymbol{\alpha}_0} \mathbf{E}_{T\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\},$$

where

$$\begin{aligned} & \frac{\partial}{\partial \boldsymbol{\alpha}_0} \mathbf{E}_{T\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \\ &= \frac{\partial}{\partial \boldsymbol{\alpha}_0} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 P_{Tk} \left\{ \frac{2}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 - \frac{3}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} + \frac{1}{P_k} \frac{\partial^3 P_k}{\partial \theta_0^3} \right\} \\ &= n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 P_{Tk} \left[\left\{ -\frac{6}{P_k^4} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 + \frac{6}{P_k^3} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} - \frac{1}{P_k^2} \frac{\partial^3 P_k}{\partial \theta_0^3} \right\} \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} \right. \\ & \quad \left. + \left\{ \frac{6}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 - \frac{3}{P_k^2} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0} \right. \\ & \quad \left. - \frac{3}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} + \frac{1}{P_k} \frac{\partial^4 P_k}{\partial \theta_0^3 \partial \boldsymbol{\alpha}_0} \right]. \end{aligned}$$

$$\mathbf{I}_{\theta_0}^{(\Delta a 2)}, \mathbf{I}_{\theta_0}^{(\Delta b 2)}, \mathbf{I}_{\theta_0}^{(\Delta \Delta a 2)}, \mathbf{I}_{\theta_0}^{(\Delta \Delta b 2)}, \mathbf{I}_{\theta_0}^{(\Delta \Delta c 2)} :$$

$$\text{In } \frac{\partial m}{\partial \boldsymbol{\alpha}_0} = \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0}, \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} \text{ is given by } \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \text{ shown}$$

earlier with P_{Tk} replaced by U_k :

$$\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} = \frac{\partial}{\partial \boldsymbol{\alpha}_0} n^{-1} \sum_{k=1}^n \sum_{\substack{P,U \\ (Q,1-U)}}^2 U_k \left\{ -\frac{1}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 + \frac{1}{P_k} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\}$$

$$\begin{aligned}
&= n^{-1} \sum_{k=1}^n \sum_{\substack{P,U \\ (Q,1-U)}}^2 U_k \left[\left\{ \frac{2}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 - \frac{1}{P_k^2} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \frac{\partial P_k}{\partial \mathbf{a}_0} \right. \\
&\quad \left. - \frac{2}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} + \frac{1}{P_k} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_0} \right], \\
\text{and in } \frac{\partial^2 m}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} &= \mathbf{E}_{T\theta_0} \left(\frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^2 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right) - \frac{\partial^2 \lambda_{\theta_0}}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}}, \\
\mathbf{E}_{T\theta_0} \left(\frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^2 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right) &= n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 P_{Tk} \left[\left\{ -\frac{6}{P_k^4} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 + \frac{2}{P_k^3} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \left(\frac{\partial P_k}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} \right. \\
&+ \sum_{\otimes}^2 \left(\frac{4}{P_k^3} \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} \otimes \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} - \frac{1}{P_k^2} \frac{\partial P_k}{\partial \mathbf{a}_0} \otimes \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_0} \right) \\
&+ \left\{ \frac{2}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 - \frac{1}{P_k^2} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \frac{\partial^2 P_k}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \\
&\left. - \frac{2}{P_k^2} \left\{ \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right)^{\langle 2 \rangle} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right\} + \frac{1}{P_k} \frac{\partial^4 P_k}{\partial \theta_0^2 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right].
\end{aligned}$$

(a.3)

$$\begin{aligned}
& \boldsymbol{\gamma}_{\theta_0}^{(3)} \mathbf{I}_{\theta_0}^{(3)} \\
&= \left[-\lambda_{\theta_0}^{-3}, \frac{3}{2} \lambda_{\theta_0}^{-4} \mathbf{E}_{T\theta_0}(j_0^{(3)}), -\frac{\lambda_{\theta_0}^{-3}}{2}, \right. \\
&\quad \left. -\frac{\lambda_{\theta_0}^{-5}}{2} \{\mathbf{E}_{T\theta_0}(j_0^{(3)})\}^2 + \frac{\lambda_{\theta_0}^{-4}}{6} \mathbf{E}_{T\theta_0}(j_0^{(4)}), \right. \\
&\quad \left. \left\{ \lambda_{\theta_0}^{-2} \eta_{\theta_0}, \lambda_{\theta_0}^{-2} \frac{\partial \eta_{\theta_0}}{\partial \theta_0} - \lambda_{\theta_0}^{-3} \mathbf{E}_{T\theta_0}(j_0^{(3)}) \eta_{\theta_0} \right\} \right] \\
&\times \left[m^2 \frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0}, m \left(\frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0} \right)^2, \{j_0^{(3)} - \mathbf{E}_{T\theta_0}(j_0^{(3)})\} \left(\frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0} \right)^2, \right. \\
&\quad \left. \left(\frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0} \right)^3, n^{-1} \left(m, \frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0} \right) \right]
\end{aligned}$$

For $\boldsymbol{\gamma}_{\theta_0}^{(3)}$, no partial derivatives are required since $\boldsymbol{\gamma}_{\theta_0}^{(3)}$ is not expended.

$\mathbf{I}_{\theta_0}^{(\Delta a3)}, \mathbf{I}_{\theta_0}^{(\Delta b3)}, \mathbf{I}_{\theta_0}^{(\Delta c3)}$:

In $\frac{\partial}{\partial \boldsymbol{\alpha}_0} \left\{ \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} - \mathbf{E}_{T\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}$, the following is required and is given

by $\frac{\partial}{\partial \boldsymbol{\alpha}_0} \mathbf{E}_{T\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right)$ shown earlier with P_{Tk} replaced by U_k :

$$\begin{aligned}
\frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^3 \partial \mathbf{a}} &= \frac{\partial}{\partial \mathbf{a}_0} n^{-1} \sum_{k=1}^n \sum_{\substack{P,U \\ (Q,1-U)}}^2 U_k \left\{ \frac{2}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 - \frac{3}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} + \frac{1}{P_k} \frac{\partial^3 P_k}{\partial \theta_0^3} \right\} \\
&= n^{-1} \sum_{k=1}^n \sum_{\substack{P,U \\ (Q,1-U)}}^2 U_k \left[\left\{ -\frac{6}{P_k^4} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 + \frac{6}{P_k^3} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} - \frac{1}{P_k^2} \frac{\partial^3 P_k}{\partial \theta_0^3} \right\} \frac{\partial P_k}{\partial \mathbf{a}_0} \right. \\
&\quad \left. + \left\{ \frac{6}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 - \frac{3}{P_k^2} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}} - \frac{3}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}} + \frac{1}{P_k} \frac{\partial^4 P_k}{\partial \theta_0^3 \partial \mathbf{a}} \right].
\end{aligned}$$

$$(a.4) \quad -n^{-1} (\lambda_{\theta_0}^{-1} \eta_{\theta_0})^{(\Delta)}$$

The required $\frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0}$ was given earlier while $\frac{\partial \eta_{\theta_0}}{\partial \mathbf{a}_0}$ depends on the functional form of η_{θ_0} .

A.5.2 Partial derivatives associated with the non-studentized $\hat{\theta}$ under c.m.s.

Note that under c.m.s. P_{Tk} ($k = 1, \dots, n$) are functions of \mathbf{a} . The results different from those in Subsection A.5.1 are only for $\gamma_{\theta_0}^{(k)}$ ($k = 1, 2$).

$$(a.1) \quad \gamma_{\theta_0}^{(1)} \mathbf{l}_{\theta_0}^{(1)} = \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(1)} = -\lambda_{\theta_0}^{-1} \frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0}$$

$$\gamma_{\theta_0}^{(\Delta 1)} : \frac{\partial \gamma_{\theta_0}^{(1)}}{\partial \mathbf{a}_0} = -\frac{\partial \lambda_{\theta_0}^{-1}}{\partial \mathbf{a}_0} = \lambda_{\theta_0}^{-2} \frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} = \lambda_{\theta_0}^{-2} \frac{\partial}{\partial \mathbf{a}_0} \mathbf{E}_{\theta_0} \left(\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0^2} \right)$$

$$= \lambda_{\theta_0}^{-2} \frac{\partial}{\partial \mathbf{a}_0} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 \left\{ -\frac{1}{P_k} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \right\}$$

$$= \lambda_{\theta_0}^{-2} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 \left\{ \frac{1}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial P_k}{\partial \mathbf{a}_0} - \frac{2}{P_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right\},$$

$$\begin{aligned} \gamma_{\theta_0}^{(\Delta\Delta 1)} : \frac{\partial^2 \gamma_{\theta_0}^{(1)}}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} &= -2\lambda_{\theta_0}^{-3} \left(\frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} + \lambda_{\theta_0}^{-2} \frac{\partial^2 \lambda_{\theta_0}}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \\ &= -2\lambda_{\theta_0}^{-3} \left(\frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} + \lambda_{\theta_0}^{-2} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 \left[-\frac{2}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \left(\frac{\partial P_k}{\partial \mathbf{a}_0} \right)^{\langle 2 \rangle} \right. \\ &\quad \left. + \frac{2}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \sum_{\otimes}^2 \frac{\partial P_k}{\partial \mathbf{a}_0} \otimes \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} + \frac{1}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{(\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right. \\ &\quad \left. - \frac{2}{P_k} \left\{ \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right)^{\langle 2 \rangle} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0 (\partial \mathbf{a}_0)^{\langle 2 \rangle}} \right\} \right]. \end{aligned}$$

$$(a.2) \quad \boldsymbol{\gamma}_{\theta_0}^{(2)} \mathbf{l}_{\theta_0}^{(2)} = \left\{ \lambda_{\theta_0}^{-2}, -\frac{\lambda_{\theta_0}^{-3}}{2} \mathbf{E}_{\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\} \left\{ m \frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0}, \left(\frac{\partial \bar{l}_{\theta_0}}{\partial \theta_0} \right)^2 \right\},$$

$$\begin{aligned} \boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)} : \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(2)}}{\partial \mathbf{a}_0} &= \left\{ -2\lambda_{\theta_0}^{-3}, \frac{3\lambda_{\theta_0}^{-4}}{2} \mathbf{E}_{\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\} \frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0} \\ &\quad + \left\{ \mathbf{0}, -\frac{\lambda_{\theta_0}^{-3}}{2} \frac{\partial}{\partial \mathbf{a}_0} \mathbf{E}_{\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}, \end{aligned}$$

where

$$\frac{\partial}{\partial \mathbf{a}_0} \mathbf{E}_{\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right)$$

$$\begin{aligned}
&= \frac{\partial}{\partial \mathbf{a}_0} n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 \left\{ \frac{2}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 - \frac{3}{P_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\} \\
&= n^{-1} \sum_{k=1}^n \sum_{P(Q)}^2 \left\{ -\frac{4}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial P_k}{\partial \mathbf{a}_0} + \frac{6}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right. \\
&\quad \left. + \frac{3}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial P_k}{\partial \mathbf{a}_0} - \frac{3}{P_k} \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \frac{\partial^2 P_k}{\partial \theta_0^2} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_0} \right) \right\}
\end{aligned}$$

(a.3) $\boldsymbol{\gamma}_{\theta_0}^{(3)} \mathbf{1}_{\theta_0}^{(3)}$

For $\boldsymbol{\gamma}_{\theta_0}^{(3)}$, no partial derivatives are required since $\boldsymbol{\gamma}_{\theta_0}^{(3)}$ is not expended.

(a.4) $-n^{-1} (\lambda_{\theta_0}^{-1} \eta_{\theta_0})^{(\Delta)}$

$\frac{\partial \lambda_{\theta_0}}{\partial \mathbf{a}_0}$ was given earlier while $\frac{\partial \eta_{\theta_0}}{\partial \mathbf{a}_0}$ depends on the functional form of

η_{θ_0} .

A.5.3 Partial derivatives when the 3PLM is employed in Subsections A.5.1 and A.5.2

Define $\mathbf{a}_0 = \{\mathbf{a}_{0(1)}, \dots, \mathbf{a}_{0(n)}\}'$, $\mathbf{a}_{0(k)} = (a_k, b_k, c_k)'$,

$$P_k = c_k + \frac{1 - c_k}{1 + \exp\{-Da_k(\theta_0 - b_k)\}} = c_k + (1 - c_k)B_k \quad (k = 1, \dots, n) \quad \text{and}$$

$D = 1.7$. Then, assuming $k = 1, \dots, n$ when unspecified from now on,

$$\frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} = \{(\theta_0 - b_k, -a_k)D(1 - c_k)B_k(1 - B_k), 1 - B_k\}',$$

$$\begin{aligned}
\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} &= [\{D + (\theta_0 - b_k)D^2 a_k(1 - 2B_k), -D^2 a_k^2(1 - 2B_k)\}(1 - c_k), \\
&\quad -Da_k] B_k(1 - B_k),
\end{aligned}$$

$$\frac{\partial^2 P_k}{\partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)}} = \left[\begin{array}{c} \left(\begin{array}{cc} (\theta_0 - b_k)^2 & -(\theta_0 - b_k)a_k \\ -(\theta_0 - b_k)a_k & a_k^2 \end{array} \right) D^2 (1 - c_k) (1 - 2B_k) \\ + \left(\begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right) D (1 - c_k) \\ \{ -(\theta_0 - b_k), a_k \} D \end{array} \right] \begin{array}{c} \text{sym.} \\ B_k (1 - B_k), \\ 0 \end{array}$$

$$\frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_{0(k)}} = \left[\begin{array}{c} \left\{ \begin{array}{l} D^2 a_k (1 - 2B_k) \\ + (\theta_0 - b_k) D^3 a_k^2 (1 - 6B_k + 6B_k^2), \\ - D^2 a_k^2 (1 - 2B_k) \end{array} \right\} (1 - c_k), \\ - D^3 a_k^3 (1 - 6B_k + 6B_k^2) \end{array} \right] B_k (1 - B_k).$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)}} = \left\{ \begin{array}{l} 2(\theta_0 - b_k) D^2 (1 - c_k) (1 - 2B_k) + (\theta_0 - b_k)^2 a_k D^3 (1 - c_k) \\ \times (1 - 6B_k + 6B_k^2) \end{array} \right\} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial b_k \partial a_k} = \left\{ \begin{array}{l} -2a_k D^2 (1 - c_k) (1 - 2B_k) - (\theta_0 - b_k) a_k^2 D^3 (1 - c_k) \\ \times (1 - 6B_k + 6B_k^2) \end{array} \right\} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial b_k^2} = a_k^3 D^3 (1 - c_k) (1 - 6B_k + 6B_k^2) B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial c_k \partial a_k} = -\left\{ D + (\theta_0 - b_k) a_k D^2 (1 - 2B_k) \right\} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial c_k \partial b_k} = a_k^2 D^2 (1 - 2B_k) B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial c_k^2} = 0.$$

$$\frac{\partial^3 P_k}{\partial a_k \partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)}} :$$

$$\frac{\partial^3 P_k}{\partial a_k^3} = (\theta_0 - b_k)^3 D^3 (1 - c_k) (1 - 6B_k + 6B_k^2) B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial b_k \partial a_k} = \{ -2(\theta_0 - b_k) D^2 (1 - c_k) (1 - 2B_k) \\ - (\theta_0 - b_k)^2 a_k D^3 (1 - c_k) (1 - 6B_k + 6B_k^2) \} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial b_k^2} = \{ 2a_k D^2 (1 - c_k) (1 - 2B_k) \\ + a_k^2 D^3 (\theta_0 - b_k) (1 - c_k) (1 - 6B_k + 6B_k^2) \} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial c_k \partial a_k} = -(\theta_k - b_k)^2 D^2 (1 - 2B_k) B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial c_k \partial b_k} = \{ D + (\theta_k - b_k) a_k D^2 (1 - 2B_k) \} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial c_k^2} = 0.$$

$$\frac{\partial^3 P_k}{\partial b_k \partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)}} :$$

$$\frac{\partial^3 P_k}{\partial b_k \partial a_k^2} = \{ -2(\theta_0 - b_k) D^2 (1 - c_k) (1 - 2B_k) \\ - (\theta_0 - b_k)^2 a_k D^3 (1 - c_k) (1 - 6B_k + 6B_k^2) \} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k^2 \partial a_k} = \{ 2a_k D^2 (1 - c_k)(1 - 2B_k) + (\theta_0 - b_k) a_k^2 D^3 (1 - c_k)(1 - 6B_k + 6B_k^2) \} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k^3} = -a_k^3 D^3 (1 - c_k)(1 - 6B_k + 6B_k^2) B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k \partial c_k \partial a_k} = \{ D + (\theta_0 - b_k) a_k D^2 (1 - 2B_k) \} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k \partial c_k \partial b_k} = -a_k^2 D^2 (1 - 2B_k) B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k \partial c_k^2} = 0.$$

$$\frac{\partial^3 P_k}{\partial c_k \partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)'}} = \begin{bmatrix} \left(\begin{array}{cc} -(\theta_0 - b_k)^2 & (\theta_0 - b_k) a_k \\ (\theta_0 - b_k) a_k & -a_k^2 \end{array} \right) D^2 (1 - 2B_k) & 0 \\ & 0 \\ + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D & \\ 0 & 0 & 0 \end{bmatrix} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0^3 \partial \mathbf{a}_{0(k)}} = \left[\begin{array}{l} \left\{ \begin{array}{l} 2D^3 a_k^2 (1 - 6B_k + 6B_k^2) \\ + (\theta_0 - b_k) D^4 a_k^3 \\ \times (1 - 14B_k + 36B_k^2 - 24B_k^3), \end{array} \right. \\ \left. \begin{array}{l} -D^4 a_k^4 (1 - 14B_k \\ + 36B_k^2 - 24B_k^3) \end{array} \right\} (1 - c_k), \\ \\ -D^3 a_k^3 (1 - 6B_k + 6B_k^2) \end{array} \right] B_k (1 - B_k).$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)}} :$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial a_k^2} = \{ 2D^2 (1 - c_k) (1 - 2B_k) \\ + 4(\theta_0 - b_k) a_k D^3 (1 - c_k) (1 - 6B_k + 6B_k^2) \\ + (\theta_0 - b_k)^2 a_k^2 D^4 (1 - c_k) (1 - 14B_k + 36B_k^2 - 24B_k^3) \} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial b_k \partial a_k} = \{ -3a_k^2 D^3 (1 - c_k) (1 - 6B_k + 6B_k^2) \\ - (\theta_0 - b_k) a_k^3 D^4 (1 - c_k) (1 - 14B_k + 36B_k^2 - 24B_k^3) \} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial b_k^2} = a_k^4 D^4 (1 - c_k) (1 - 14B_k + 36B_k^2 - 24B_k^3) B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial c_k \partial a_k} = \{ -2a_k D^2 (1 - 2B_k) \\ - (\theta_0 - b_k) a_k^2 D^3 (1 - 6B_k + 6B_k^2) \} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial c_k \partial b_k} = a_k^3 D^3 (1 - 6B_k + 6B_k^2) B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial c_k^2} = 0.$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)}} :$$

$$\begin{aligned} \frac{\partial^4 P_k}{\partial \theta_0 \partial a_k^3} = & \{ 3(\theta_0 - b_k)^2 D^3 (1 - c_k)(1 - 6B_k + 6B_k^2) \\ & + (\theta_0 - b_k)^3 a_k D^4 (1 - c_k)(1 - 14B_k + 36B_k^2 - 24B_k^3) \} B_k (1 - B_k), \end{aligned}$$

$$\begin{aligned} \frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial b_k \partial a_k} = & \{ -2D^2 (1 - c_k)(1 - 2B_k) \\ & - 4(\theta_0 - b_k) a_k D^3 (1 - c_k)(1 - 6B_k + 6B_k^2) \\ & - (\theta_0 - b_k)^2 a_k^2 D^4 (1 - c_k)(1 - 14B_k + 36B_k^2 - 24B_k^3) \} B_k (1 - B_k), \end{aligned}$$

$$\begin{aligned} \frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial b_k^2} = & \{ 3a_k^2 D^3 (1 - c_k)(1 - 6B_k + 6B_k^2) \\ & + (\theta_0 - b_k) a_k^3 D^4 (1 - c_k)(1 - 14B_k + 36B_k^2 - 24B_k^3) \} B_k (1 - B_k), \end{aligned}$$

$$\begin{aligned} \frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial c_k \partial a_k} = & \{ -2(\theta_0 - b_k) D^2 (1 - 2B_k) \\ & - (\theta_0 - b_k)^2 a_k D^3 (1 - 6B_k + 6B_k^2) \} B_k (1 - B_k), \end{aligned}$$

$$\begin{aligned} \frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial c_k \partial b_k} = & \{ 2a_k D^2 (1 - 2B_k) \\ & + (\theta_0 - b_k) a_k^2 D^3 (1 - 6B_k + 6B_k^2) \} B_k (1 - B_k), \end{aligned}$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial c_k^2} = 0.$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial b_k \partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)}} :$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial b_k \partial a_k^2} = \{ -2D^2(1-c_k)(1-2B_k) \\ -4(\theta_0 - b_k)a_k D^3(1-c_k)(1-6B_k + 6B_k^2) \\ -(\theta_0 - b_k)^2 a_k^2 D^4(1-c_k)(1-14B_k + 36B_k^2 - 24B_k^3) \} B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial b_k^2 \partial a_k} = \{ 3a_k^2 D^3(1-c_k)(1-6B_k + 6B_k^2) \\ +(\theta_0 - b_k)a_k^3 D^4(1-c_k)(1-14B_k + 36B_k^2 - 24B_k^3) \} B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial b_k^3} = -a_k^4 D^4(1-c_k)(1-14B_k + 36B_k^2 - 24B_k^3) B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial b_k \partial c_k \partial a_k} = \{ 2a_k D^2(1-2B_k) \\ +(\theta_0 - b_k)a_k^2 D^3(1-6B_k + 6B_k^2) \} B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial b_k \partial c_k \partial b_k} = -a_k^3 D^3(1-6B_k + 6B_k^2) B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial b_k \partial c_k^2} = 0.$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial c_k \partial \mathbf{a}_{0(k)} \partial \mathbf{a}_{0(k)}} :$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial c_k \partial a_k^2} = \{ -2(\theta_0 - b_k)D^2(1-2B_k) \\ -(\theta_0 - b_k)^2 a_k D^3(1-6B_k + 6B_k^2) \} B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial c_k \partial b_k \partial a_k} = \{ 2a_k D^2(1-2B_k) \\ +(\theta_0 - b_k)a_k^2 D^3(1-6B_k + 6B_k^2) \} B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial c_k \partial b_k^2} = -a_k^3 D^3 (1 - 6B_k + 6B_k^2) B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial c_k^2 \partial a_k} = \frac{\partial^4 P_k}{\partial \theta_0 \partial c_k^2 \partial b_k} = \frac{\partial^4 P_k}{\partial \theta_0 \partial c_k^3} = 0.$$

A.5.4 Partial derivatives associated with the studentized $\hat{\theta}$

$$(a.1) \quad \bar{i}_{\theta_0}^{(1)} = \frac{\bar{i}_{\theta_0}^{-1/2}}{2} \bar{i}_{\theta_0}^{(D1)}$$

$$\frac{\partial \bar{i}_{\theta_0}^{(1)}}{\partial \theta_0} = -\frac{\bar{i}_{\theta_0}^{-3/2}}{4} (\bar{i}_{\theta_0}^{(D1)})^2 + \frac{\bar{i}_{\theta_0}^{-1/2}}{2} \bar{i}_{\theta_0}^{(D2)},$$

$$\text{where } \bar{i}_{\theta_0}^{(D1)} = n^{-1} \sum_{k=1}^n \left\{ \frac{2}{P_k Q_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} - \frac{1 - 2P_k}{(P_k Q_k)^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \right\} \text{ and}$$

$$\begin{aligned} \bar{i}_{\theta_0}^{(D2)} = n^{-1} \sum_{k=1}^n & \left[\frac{2}{P_k Q_k} \left\{ \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^3} + \left(\frac{\partial^2 P_k}{\partial \theta_0^2} \right)^2 \right\} + \frac{1}{(P_k Q_k)^2} \right. \\ & \left. \times \left\{ -5(1 - 2P_k) \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \theta_0^2} + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^4 \right\} + \frac{2(1 - 2P_k)^2}{(P_k Q_k)^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^4 \right]. \end{aligned}$$

$$\frac{\partial \bar{i}_{\theta_0}^{(1)}}{\partial \alpha_{0(k)}} = -\frac{\bar{i}_{\theta_0}^{-3/2}}{4} \bar{i}_{\theta_0}^{(D1)} \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} + \frac{\bar{i}_{\theta_0}^{-1/2}}{2} \frac{\partial \bar{i}_{\theta_0}^{(D1)}}{\partial \alpha_{0(k)}},$$

$$\begin{aligned} \frac{\partial^2 \bar{i}_{\theta_0}^{(1)}}{(\partial \alpha_{0(k)})^{<2>}} &= \frac{3}{8} \bar{i}_{\theta_0}^{-5/2} \bar{i}_{\theta_0}^{(D1)} \left(\frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} \right)^{<2>} - \frac{1}{4} \sum_{\otimes}^2 \bar{i}_{\theta_0}^{-3/2} \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} \otimes \frac{\partial \bar{i}_{\theta_0}^{(D1)}}{\partial \alpha_{0(k)}} \\ &\quad - \frac{1}{4} \bar{i}_{\theta_0}^{-3/2} \bar{i}_{\theta_0}^{(D1)} \frac{\partial^2 \bar{i}_{\theta_0}}{(\partial \alpha_{0(k)})^{<2>}} + \frac{\bar{i}_{\theta_0}^{-1/2}}{2} \frac{\partial^2 \bar{i}_{\theta_0}^{(D1)}}{(\partial \alpha_{0(k)})^{<2>}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \dot{i}_{\theta_0}^{(1)}}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} &= \frac{3}{8} \bar{i}_{\theta_0}^{-5/2} (\bar{i}_{\theta_0}^{(D1)})^2 \frac{\partial \bar{i}_{\theta_0}}{\partial \mathbf{a}_{0(k)}} - \frac{\bar{i}_{\theta_0}^{-3/2}}{2} \bar{i}_{\theta_0}^{(D1)} \frac{\partial \bar{i}_{\theta_0}^{(D1)}}{\partial \mathbf{a}_{0(k)}} \\ &\quad - \frac{\bar{i}_{\theta_0}^{-3/2}}{4} \bar{i}_{\theta_0}^{(D2)} \frac{\partial \bar{i}_{\theta_0}}{\partial \mathbf{a}_{0(k)}} + \frac{\bar{i}_{\theta_0}^{-1/2}}{2} \frac{\partial \bar{i}_{\theta_0}^{(D2)}}{\partial \mathbf{a}_{0(k)}}, \end{aligned}$$

$$\frac{\partial^2 \dot{i}_{\theta_0}^{(1)}}{\partial \theta_0^2} = \frac{3}{8} \bar{i}_{\theta_0}^{-5/2} (\bar{i}_{\theta_0}^{(D1)})^3 - \frac{3}{4} \bar{i}_{\theta_0}^{-3/2} \bar{i}_{\theta_0}^{(D1)} \bar{i}_{\theta_0}^{(D2)} + \frac{\bar{i}_{\theta_0}^{-1/2}}{2} \bar{i}_{\theta_0}^{(D3)},$$

where

$$\begin{aligned} \frac{\partial \bar{i}_{\theta_0}}{\partial \mathbf{a}_{0(k)}} &= n^{-1} \frac{\partial}{\partial \mathbf{a}_{0(k)}} \sum_{k=1}^n \frac{1}{P_k Q_k} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \\ &= n^{-1} \left\{ \frac{2}{P_k Q_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} - \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{1 - 2P_k}{(P_k Q_k)^2} \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right\}, \\ \frac{\partial^2 \bar{i}_{\theta_0}}{(\partial \mathbf{a}_{0(k)})^{<2>}} &= n^{-1} \left[\frac{2}{P_k Q_k} \left\{ \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \right)^{<2>} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0 (\partial \mathbf{a}_{0(k)})^{<2>}} \right\} \right. \\ &\quad + \frac{1}{(P_k Q_k)^2} \left\{ - \sum_{\otimes}^2 2 \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \otimes \left((1 - 2P_k) \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right) \right. \\ &\quad \left. \left. + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \left(\frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right)^{<2>} - \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 (1 - 2P_k) \frac{\partial^2 P_k}{(\partial \mathbf{a}_{0(k)})^{<2>}} \right\} \right. \\ &\quad \left. + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{(1 - 2P_k)^2}{(P_k Q_k)^3} \left(\frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right)^{<2>} \right], \end{aligned}$$

$$\bar{i}_{\theta_0}^{(D3)} = n^{-1} \sum_{k=1}^n \left[\frac{2}{P_k Q_k} \left(3 \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial^3 P_k}{\partial \theta_0^3} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^4 P_k}{\partial \theta_0^4} \right) \right]$$

$$\begin{aligned}
& + \frac{1}{(P_k Q_k)^2} \left\{ (1 - 2P_k) \left\{ -2 \left(\frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^3} + \left(\frac{\partial^2 P_k}{\partial \theta_0^2} \right)^2 \right) \frac{\partial P_k}{\partial \theta_0} \right. \right. \\
& - 5 \left. \left(2 \frac{\partial P_k}{\partial \theta_0} \left(\frac{\partial^2 P_k}{\partial \theta_0^2} \right)^2 + \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^3 P_k}{\partial \theta_0^3} \right) \right\} + 18 \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial^2 P_k}{\partial \theta_0^2} \left. \right\} \\
& + \frac{1}{(P_k Q_k)^3} \left\{ 18(1 - 2P_k)^2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial^2 P_k}{\partial \theta_0^2} - 12(1 - 2P_k) \left(\frac{\partial P_k}{\partial \theta_0} \right)^5 \right\} \\
& \quad \left. - \frac{6(1 - 2P_k)^3}{(P_k Q_k)^4} \left(\frac{\partial P_k}{\partial \theta_0} \right)^5 \right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \bar{i}_{\theta_0}^{(D1)}}{\partial \mathbf{a}_{0(k)}} & = n^{-1} \frac{\partial}{\partial \mathbf{a}_{0(k)}} \sum_{k=1}^n \left\{ \frac{2}{P_k Q_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} - \frac{1 - 2P_k}{(P_k Q_k)^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \right\} \\
& = n^{-1} \left[\frac{2}{P_k Q_k} \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \frac{\partial^2 P_k}{\partial \theta_0^2} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_{0(k)}} \right) \right. \\
& + \frac{1}{(P_k Q_k)^2} \left\{ -(1 - 2P_k) \left(2 \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} + 3 \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \right) \right. \\
& \quad \left. + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right\} + \frac{2(1 - 2P_k)^2}{(P_k Q_k)^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \left. \right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \bar{i}_{\theta_0}^{(D1)}}{(\partial \mathbf{a}_{0(k)})^{<2>}} & = n^{-1} \left[\frac{2}{P_k Q_k} \left(\frac{\partial^3 P_k}{\partial \theta_0 (\partial \mathbf{a}_{0(k)})^{<2>}} \frac{\partial^2 P_k}{\partial \theta_0^2} \right) \right. \\
& \quad \left. + \sum_{\otimes}^2 \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_{0(k)}} \otimes \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^4 P_k}{\partial \theta_0^2 (\partial \mathbf{a}_{0(k)})^{<2>}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(P_k Q_k)^2} \left\{ -(1-2P_k) \left\{ 2 \sum_{\otimes}^2 \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \frac{\partial^2 P_k}{\partial \theta_0^2} + \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_{0(k)}} \frac{\partial P_k}{\partial \theta_0} \right) \right. \right. \\
& \quad \left. \left. \otimes \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} + 2 \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial^2 P_k}{(\partial \mathbf{a}_{0(k)})^{<2>}} \right. \right. \\
& \quad \left. \left. + 6 \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \right)^{<2>} \frac{\partial P_k}{\partial \theta_0} + 3 \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^3 P_k}{\partial \theta_0 (\partial \mathbf{a}_{0(k)})^{<2>}} \right\} \\
& \quad + 4 \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \left(\frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right)^{<2>} \\
& \quad + 6 \sum_{\otimes}^2 \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \otimes \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial^2 P_k}{(\partial \mathbf{a}_{0(k)})^{<2>}} \left. \right\} \\
& \quad + \frac{1}{(P_k Q_k)^3} \left\{ 2(1-2P_k)^2 \left\{ 2 \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \left(\frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right)^{<2>} \right. \right. \\
& \quad \left. \left. + 3 \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \sum_{\otimes}^2 \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \otimes \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial^2 P_k}{(\partial \mathbf{a}_{0(k)})^{<2>}} \right\} \right. \\
& \quad \left. - 12(1-2P_k) \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \left(\frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right)^{<2>} \right\} - \frac{6(1-2P_k)^3}{(P_k Q_k)^4} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \left(\frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right)^{<2>} \left. \right].
\end{aligned}$$

$$\text{(a.2)} \quad i_{\theta_0}^{(2)} = \frac{\bar{i}_{\theta_0}^{-1/2}}{4} \bar{i}_{\theta_0}^{(D2)} - \frac{\bar{i}_{\theta_0}^{-3/2}}{8} (\bar{i}_{\theta_0}^{(D1)})^2$$

$$\begin{aligned}\frac{\partial \bar{i}_{\theta_0}^{(2)}}{\partial \theta_0} &= -\frac{3}{8} \bar{i}_{\theta_0}^{-3/2} \bar{i}_{\theta_0}^{(D1)} \bar{i}_{\theta_0}^{(D2)} + \frac{\bar{i}_{\theta_0}^{-1/2}}{4} \bar{i}_{\theta_0}^{(D3)} + \frac{3}{16} \bar{i}_{\theta_0}^{-5/2} (\bar{i}_{\theta_0}^{(D1)})^3, \\ \frac{\partial \bar{i}_{\theta_0}^{(2)}}{\partial \mathbf{a}_0} &= -\frac{\bar{i}_{\theta_0}^{-3/2}}{8} \frac{\partial \bar{i}_{\theta_0}}{\partial \mathbf{a}_0} \bar{i}_{\theta_0}^{(D2)} + \frac{\bar{i}_{\theta_0}^{-1/2}}{4} \frac{\partial \bar{i}_{\theta_0}^{(D2)}}{\partial \mathbf{a}_0} + \frac{3}{16} \bar{i}_{\theta_0}^{-5/2} \frac{\partial \bar{i}_{\theta_0}}{\partial \mathbf{a}_0} (\bar{i}_{\theta_0}^{(D1)})^2 \\ &\quad - \frac{\bar{i}_{\theta_0}^{-3/2}}{4} \bar{i}_{\theta_0}^{(D1)} \frac{\partial \bar{i}_{\theta_0}^{(D1)}}{\partial \mathbf{a}_0},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \bar{i}_{\theta_0}^{(D2)}}{\partial \mathbf{a}_{0(k)}} &= n^{-1} \frac{\partial}{\partial \mathbf{a}_{0(k)}} \left[\frac{2}{P_k Q_k} \left\{ \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^3} + \left(\frac{\partial^2 P_k}{\partial \theta_0^2} \right)^2 \right\} \right. \\ &\quad \left. + \frac{1}{(P_k Q_k)^2} \left\{ -5(1-2P_k) \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \theta_0^2} + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^4 \right\} \right. \\ &\quad \left. + \frac{1}{(P_k Q_k)^3} 2(1-2P_k)^2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^4 \right] \\ &= n^{-1} \left[\frac{2}{P_k Q_k} \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \frac{\partial^3 P_k}{\partial \theta_0^3} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^4 P_k}{\partial \theta_0^3 \partial \mathbf{a}_{0(k)}} + 2 \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_{0(k)}} \right) \right. \\ &\quad \left. + \frac{1}{(P_k Q_k)^2} \left\{ (1-2P_k) \left\{ -2 \left(\frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^3} + \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \right) \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right. \right. \right. \\ &\quad \left. \left. - 5 \left(2 \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} + \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_{0(k)}} \right) \right\} \right. \\ &\quad \left. \left. \right\} \right]\end{aligned}$$

$$\begin{aligned}
& +10 \left\{ \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} + 8 \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \right\} \\
& + \frac{1}{(P_k Q_k)^3} \left\{ (1-2P_k)^2 \left\{ 10 \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} + 8 \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_{0(k)}} \right\} \right. \\
& \quad \left. - 12(1-2P_k) \left(\frac{\partial P_k}{\partial \theta_0} \right)^4 \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \right\} - \frac{6(1-2P_k)^3}{(P_k Q_k)^4} \left(\frac{\partial P_k}{\partial \theta_0} \right)^4 \frac{\partial P_k}{\partial \mathbf{a}_{0(k)}} \Big].
\end{aligned}$$

$$\text{(a.3)} \quad \bar{\beta}_{2I} = \bar{i}_{\theta_0}^{-1} + \bar{c} \bar{i}_{\theta_0}^{-2} \mathbf{E}_{\theta_0} \left(-\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0} \right) \{ \mathbf{E}_{\mathbf{a}_0}(\mathbf{G}_0) \}^{-1} \mathbf{E}_{\theta_0} \left(-\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0} \right)$$

Note that \bar{i}_{θ_0} and $\mathbf{E}_{\theta_0}(\cdot)$ in $\bar{\beta}_{2I}$ include θ_0 and \mathbf{a}_0 while

$\mathbf{E}_{\mathbf{a}_0}(\mathbf{G}_0) \equiv \mathbf{\Gamma}_{\mathbf{G}_0} \equiv \mathbf{I}_{\mathbf{a}_0}$ includes only \mathbf{a}_0 . Define

$$\mathbf{E}_{\theta_0} \left(-\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \mathbf{a}_0} \right) = n^{-1} \sum_{k=1}^n \frac{1}{P_k Q_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} \equiv \mathbf{d}. \text{ Let } \alpha_{0(k)} \text{ and } \alpha_{0(k^*)} \text{ be}$$

possibly different elements in $\mathbf{a}_{0(k)}$ (recall that $\mathbf{a}_0 = (\mathbf{a}_{0(1)}, \dots, \mathbf{a}_{0(n)})'$);

and let $\gamma_{jk}^{(\mathbf{G}_0)} = (\mathbf{\Gamma}_{\mathbf{G}_0})_{jk}$ ($q \geq j \geq k \geq 1$) or the (j, k) th element of $\mathbf{\Gamma}_{\mathbf{G}_0}$ with q being the number of item parameters. Then,

$$\bar{\beta}_{2I} = \bar{i}_{\theta_0}^{-1} + \bar{c} \bar{i}_{\theta_0}^{-2} \mathbf{d}' \mathbf{\Gamma}_{\mathbf{G}_0}^{-1} \mathbf{d},$$

$$\frac{\partial \bar{\beta}_{2I}}{\partial \theta_0} = -(\bar{i}_{\theta_0}^{-2} + 2\bar{c} \bar{i}_{\theta_0}^{-3} \mathbf{d}' \mathbf{\Gamma}_{\mathbf{G}_0}^{-1} \mathbf{d}) \bar{i}_{\theta_0}^{(D1)} + 2\bar{c} \bar{i}_{\theta_0}^{-2} \mathbf{d}' \mathbf{\Gamma}_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \theta_0},$$

$$\frac{\partial \bar{\beta}_{2I}}{\partial \alpha_{0(k)}} = -(\bar{i}_{\theta_0}^{-2} + 2\bar{c} \bar{i}_{\theta_0}^{-3} \mathbf{d}' \mathbf{\Gamma}_{\mathbf{G}_0}^{-1} \mathbf{d}) \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} + 2\bar{c} \bar{i}_{\theta_0}^{-2} \mathbf{d}' \mathbf{\Gamma}_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \alpha_{0(k)}}$$

$$- \bar{c} \bar{i}_{\theta_0}^{-2} \mathbf{d}' \mathbf{\Gamma}_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{\Gamma}_{\mathbf{G}_0}}{\partial \alpha_{0(k)}} \mathbf{\Gamma}_{\mathbf{G}_0}^{-1} \mathbf{d} \quad (k = 1, \dots, n),$$

$$\begin{aligned}\frac{\partial \bar{\beta}_{2I}}{\partial \gamma_{jk}^{(\mathbf{G}_0)}} &= -\bar{c} \bar{i}_{\theta_0}^{-2} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{2 - \delta_{jk}}{2} (\mathbf{E}_{jk} + \mathbf{E}_{jk}') \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d} \\ &= -\bar{c} \bar{i}_{\theta_0}^{-2} (2 - \delta_{jk}) (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_j (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_k \quad (q \geq j \geq k \geq 1),\end{aligned}$$

where δ_{jk} is the Kronecker delta, \mathbf{E}_{jk} is the matrix of an appropriate size whose (j, k) th element is 1 with the remaining ones being 0, and $(\cdot)_j$ denote the j -th element of a vector.

$$\text{The matrix } \frac{\partial \mathbf{G}_0}{\partial \alpha_{0(k)}} = N^{-1} \sum_{j=1}^N \left(\frac{\partial^2 l_{\mathbf{a}(j)}}{\partial \mathbf{a}_0 \partial \alpha_{0(k)}} \frac{\partial l_{\mathbf{a}(j)}}{\partial \mathbf{a}_0'} + \frac{\partial l_{\mathbf{a}(j)}}{\partial \mathbf{a}_0} \frac{\partial^2 l_{\mathbf{a}(j)}}{\partial \mathbf{a}_0' \partial \alpha_{0(k)}} \right) \text{ is}$$

also used when evaluating the sampling behavior of stochastic $\mathbf{G}_0 = O_p(1)$ including \mathbf{a}_0 and $\hat{\mathbf{G}} = O_p(1)$ using $\hat{\mathbf{a}}$ (recall (4.3); see also Ogasawara, 2010).

$$\begin{aligned}\frac{\partial \mathbf{d}}{\partial \theta_0} &= n^{-1} \sum_{k=1}^n \left\{ \frac{1}{P_k Q_k} \left(\frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial P_k}{\partial \mathbf{a}_0} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right) - \frac{1 - 2P_k}{(P_k Q_k)^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial P_k}{\partial \mathbf{a}_0} \right\}, \\ \frac{\partial \mathbf{d}}{\partial \alpha_{0(k)}} &= n^{-1} \left\{ \frac{1}{P_k Q_k} \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \alpha_{0(k)}} \frac{\partial P_k}{\partial \mathbf{a}_0} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \mathbf{a}_0 \partial \alpha_{0(k)}} \right) \right. \\ &\quad \left. - \frac{1 - 2P_k}{(P_k Q_k)^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} \frac{\partial P_k}{\partial \alpha_{0(k)}} \right\} \quad (k = 1, \dots, n),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \bar{\beta}_{2I}}{\partial \theta_0^2} &= (2\bar{i}_{\theta_0}^{-3} + 6\bar{c} \bar{i}_{\theta_0}^{-4} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d}) (\bar{i}_{\theta_0}^{(D1)})^2 - 8\bar{c} \bar{i}_{\theta_0}^{-3} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \theta_0} \bar{i}_{\theta_0}^{(D1)} \\ &\quad - (\bar{i}_{\theta_0}^{-2} + 2\bar{c} \bar{i}_{\theta_0}^{-3} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d}) \bar{i}_{\theta_0}^{(D2)} + 2\bar{c} \bar{i}_{\theta_0}^{-2} \left(\frac{\partial \mathbf{d}'}{\partial \theta_0} \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \theta_0} + \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial^2 \mathbf{d}}{\partial \theta_0^2} \right),\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \bar{\beta}_{2I}}{\partial \theta_0 \partial \alpha_{0(k)}} &= (2\bar{i}_{\theta_0}^{-3} + 6\bar{c}\bar{i}_{\theta_0}^{-4} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d}) \bar{i}_{\theta_0}^{(D1)} \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} \\
&\quad - 4\bar{c}\bar{i}_{\theta_0}^{-3} \left(\mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \alpha_{0(k)}} \bar{i}_{\theta_0}^{(D1)} + \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \theta_0} \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} \right) \\
&\quad - (\bar{i}_{\theta_0}^{-2} + 2\bar{c}\bar{i}_{\theta_0}^{-3} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d}) \frac{\partial \bar{i}_{\theta_0}^{(D1)}}{\partial \alpha_{0(k)}} + 2\bar{c}\bar{i}_{\theta_0}^{-3} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \Gamma_{\mathbf{G}_0}}{\partial \alpha_{0(k)}} \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d} \bar{i}_{\theta_0}^{(D1)} \\
&\quad + 2\bar{c}\bar{i}_{\theta_0}^{-2} \left(\frac{\partial \mathbf{d}'}{\partial \alpha_{0(k)}} \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \theta_0} + \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial^2 \mathbf{d}}{\partial \theta_0 \partial \alpha_{0(k)}} - \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \Gamma_{\mathbf{G}_0}}{\partial \alpha_{0(k)}} \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \theta_0} \right)
\end{aligned}$$

$(k = 1, \dots, n),$

$$\begin{aligned}
\frac{\partial^2 \bar{\beta}_{2I}}{\partial \theta_0 \partial \gamma_{jk}^{(\mathbf{G}_0)}} &= 2\bar{c}\bar{i}_{\theta_0}^{-3} (2 - \delta_{jk}) (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_j (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_k \bar{i}_{\theta_0}^{(D1)} \\
&\quad - \bar{c}\bar{i}_{\theta_0}^{-2} (2 - \delta_{jk}) \left\{ \left(\Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \theta_0} \right)_j (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_k + (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_j \left(\Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \theta_0} \right)_k \right\} \\
&\quad (q \geq j \geq k \geq 1),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \bar{\beta}_{2I}}{\partial \alpha_{0(j)} \partial \alpha_{0(k)}} &= -(\bar{i}_{\theta_0}^{-2} + 2\bar{c}\bar{i}_{\theta_0}^{-3} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d}) \frac{\partial^2 \bar{i}_{\theta_0}}{\partial \alpha_{0(j)} \partial \alpha_{0(k)}} \\
&\quad + (2\bar{i}_{\theta_0}^{-3} + 6\bar{c}\bar{i}_{\theta_0}^{-4} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d}) \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(j)}} \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} \\
&\quad - 4\bar{c}\bar{i}_{\theta_0}^{-3} \left(\mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \alpha_{0(j)}} \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} + \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \alpha_{0(k)}} \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(j)}} \right) \\
&\quad + \sum_{(j,k)}^2 2\bar{c}\bar{i}_{\theta_0}^{-3} \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \Gamma_{\mathbf{G}_0}}{\partial \alpha_{0(j)}} \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d} \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}}
\end{aligned}$$

$$\begin{aligned}
& + \bar{c} \bar{i}_{\theta_0}^{-2} \left(- \sum_{(j,k)}^2 2 \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \Gamma_{\mathbf{G}_0}}{\partial \alpha_{0(j)}} \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \alpha_{0(k)}} + 2 \frac{\partial^2 \mathbf{d}}{\partial \alpha_{0(j)} \partial \alpha_{0(k)}} \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d} \right. \\
& \quad + 2 \frac{\partial \mathbf{d}'}{\partial \alpha_{0(j)}} \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \alpha_{0(k)}} + 2 \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \Gamma_{\mathbf{G}_0}}{\partial \alpha_{0(j)}} \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \Gamma_{\mathbf{G}_0}}{\partial \alpha_{0(k)}} \mathbf{d} \\
& \quad \left. - \mathbf{d}' \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial^2 \Gamma_{\mathbf{G}_0}}{\partial \alpha_{0(j)} \partial \alpha_{0(k)}} \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d} \right) \quad (j, k = 1, \dots, n),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \bar{\beta}_{2l}}{\partial \gamma_{jk}^{(\mathbf{G}_0)} \partial \alpha_{0(l^*)}} & = 2 \bar{c} \bar{i}_{\theta_0}^{-3} (2 - \delta_{jk}) (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_j (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_k \frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(l^*)}} \\
& + \bar{c} \bar{i}_{\theta_0}^{-2} (2 - \delta_{jk}) \sum_{(j,k)}^2 \left(\Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \Gamma_{\mathbf{G}_0}}{\partial \alpha_{0(l^*)}} \Gamma_{\mathbf{G}_0}^{-1} \mathbf{d} - \Gamma_{\mathbf{G}_0}^{-1} \frac{\partial \mathbf{d}}{\partial \alpha_{0(l^*)}} \right)_j (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_k \\
& \quad (q \geq j \geq k \geq 1; l^* = 1, \dots, n),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \bar{\beta}_{2l}}{\partial \gamma_{jk}^{(\mathbf{G}_0)} \partial \gamma_{l^* m^*}^{(\mathbf{G}_0)}} & = \frac{\bar{c}}{2} \bar{i}_{\theta_0}^{-2} (2 - \delta_{jk}) (2 - \delta_{l^* m^*}) \\
& \quad \times \sum_{(j,k)}^2 \sum_{(l^*, m^*)}^2 (\Gamma_{\mathbf{G}_0}^{-1})_{jl^*} (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_{m^*} (\Gamma_{\mathbf{G}_0}^{-1} \mathbf{d})_k
\end{aligned}$$

$$(q \geq j \geq k \geq 1; q \geq l^* \geq m^* \geq 1),$$

where

$$\begin{aligned}
\frac{\partial^2 \mathbf{d}}{\partial \theta_0^2} & = n^{-1} \sum_{k=1}^n \left[\frac{1}{P_k Q_k} \left(\frac{\partial^3 P_k}{\partial \theta_0^3} \frac{\partial P_k}{\partial \mathbf{a}_0} + 2 \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \mathbf{a}_0} \right) \right. \\
& + \frac{1}{(P_k Q_k)^2} \left\{ -(1 - 2P_k) \left(3 \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right) \right. \\
& \quad \left. \left. + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial P_k}{\partial \mathbf{a}_0} \right\} + \frac{2(1 - 2P_k)^2}{(P_k Q_k)^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial P_k}{\partial \mathbf{a}_0} \right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \mathbf{d}}{\partial \theta_0 \partial \alpha_{0(k)}} &= n^{-1} \left[\frac{1}{P_k Q_k} \left(\frac{\partial^3 P_k}{\partial \theta_0^2 \partial \alpha_{0(k)}} \frac{\partial P_k}{\partial \mathbf{a}_0} + \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial^2 P_k}{\partial \mathbf{a}_0 \partial \alpha_{0(k)}} \right. \right. \\
&\quad \left. \left. + \frac{\partial^2 P_k}{\partial \theta_0 \partial \alpha_{0(k)}} \frac{\partial^2 P_k}{\partial \theta_0 \partial \mathbf{a}_0} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0 \partial \mathbf{a}_0 \partial \alpha_{0(k)}} \right) \right. \\
&\quad + \frac{1}{(P_k Q_k)^2} \left\{ -(1-2P_k) \left\{ \left(\frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial P_k}{\partial \mathbf{a}_0} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0 \partial \mathbf{a}_0} \right) \frac{\partial P_k}{\partial \alpha_{0(k)}} \right. \right. \\
&\quad \left. \left. + 2 \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \alpha_{0(k)}} \frac{\partial P_k}{\partial \mathbf{a}_0} + \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \mathbf{a}_0 \partial \alpha_{0(k)}} \right\} \right. \\
&\quad \left. + 2 \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial P_k}{\partial \mathbf{a}_0} \frac{\partial P_k}{\partial \alpha_{0(k)}} \right\} + \frac{2(1-2P_k)^2}{(P_k Q_k)^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial P_k}{\partial \mathbf{a}_0} \frac{\partial P_k}{\partial \alpha_{0(k)}} \left. \right]
\end{aligned}$$

$(k = 1, \dots, n),$

$$\begin{aligned}
\frac{\partial^2 \mathbf{d}}{\partial \alpha_{0(k)} \partial \alpha_{0(k^*)}} &= n^{-1} \left[\frac{1}{P_k Q_k} \left(\frac{\partial^3 P_k}{\partial \theta_0 \partial \alpha_{0(k)} \partial \alpha_{0(k^*)}} \frac{\partial P_k}{\partial \mathbf{a}_0} \right. \right. \\
&\quad \left. \left. + \sum_{(k, k^*)}^2 \frac{\partial^2 P_k}{\partial \theta_0 \partial \alpha_{0(k)}} \frac{\partial^2 P_k}{\partial \mathbf{a}_0 \partial \alpha_{0(k^*)}} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \mathbf{a}_0 \partial \alpha_{0(k)} \partial \alpha_{0(k^*)}} \right) \right. \\
&\quad + \frac{1}{(P_k Q_k)^2} \left\{ -(1-2P_k) \left\{ \sum_{(k, k^*)}^2 \left(\frac{\partial P_k}{\partial \alpha_{0(k)}} \frac{\partial^2 P_k}{\partial \theta_0 \partial \alpha_{0(k^*)}} \frac{\partial P_k}{\partial \mathbf{a}_0} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \alpha_{0(k)}} \frac{\partial^2 P_k}{\partial \mathbf{a}_0 \partial \alpha_{0(k^*)}} \right) \right\} \right.
\end{aligned}$$

$$\left. + \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} \frac{\partial^2 P_k}{\partial \alpha_{0(k)} \partial \alpha_{0(k^*)}} \right\} + 2 \left. \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} \frac{\partial P_k}{\partial \alpha_{0(k)}} \frac{\partial P_k}{\partial \alpha_{0(k^*)}} \right\} \\ + \left. \frac{2(1-2P_k)^3}{(P_k Q_k)^3} \frac{\partial P_k}{\partial \theta_0} \frac{\partial P_k}{\partial \mathbf{a}_0} \frac{\partial P_k}{\partial \alpha_{0(k)}} \frac{\partial P_k}{\partial \alpha_{0(k^*)}} \right] (k = 1, \dots, n),$$

$$\frac{\partial^2 \mathbf{G}_0}{\partial \alpha_{0(j)} \partial \alpha_{0(k)}} = N^{-1} \sum_{m^*=1}^N \left(\frac{\partial^3 l_{\mathbf{a}(m^*)}}{\partial \mathbf{a}_0 \partial \alpha_{0(j)} \partial \alpha_{0(k)}} \frac{\partial l_{\mathbf{a}(m^*)}}{\partial \mathbf{a}_0} \right. \\ \left. + \sum_{(j,k)}^2 \frac{\partial^2 l_{\mathbf{a}(m^*)}}{\partial \mathbf{a}_0 \partial \alpha_{0(j)}} \frac{\partial^2 l_{\mathbf{a}(m^*)}}{\partial \mathbf{a}_0 \partial \alpha_{0(k)}} + \frac{\partial l_{\mathbf{a}(m^*)}}{\partial \mathbf{a}_0} \frac{\partial^3 l_{\mathbf{a}(m^*)}}{\partial \mathbf{a}_0 \partial \alpha_{0(j)} \partial \alpha_{0(k)}} \right) \\ (j, k = 1, \dots, n).$$

When the 3PLM is used, recall that

$$P_k = c_k + \frac{1 - c_k}{1 + \exp\{-Da_k(\theta_0 - b_k)\}} = c_k + (1 - c_k)B_k \quad (k = 1, \dots, n) \quad \text{and}$$

$D = 1.7$. Then,

$$\frac{\partial P_k}{\partial \theta_0} = (1 - c_k)Da_k B_k (1 - B_k),$$

$$\frac{\partial^2 P_k}{\partial \theta_0^2} = (1 - c_k)(Da_k)^2 (1 - 2B_k)B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0^3} = (1 - c_k)(Da_k)^3 (1 - 6B_k + 6B_k^2)B_k (1 - B_k).$$

Reference

Ogasawara, H. (2013). Asymptotic cumulants of ability estimators using fallible item parameters. *Journal of Multivariate Analysis*, 119, 144-162.

Table A1. Simulated and asymptotic standard errors of the studentized $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 50$)

Standard error		$t = n^{1/2}(\hat{\theta} - \theta_0)\hat{\beta}_{2I}^{-1/2}$						
		$t_{\alpha_0}^* = n^{1/2}(\hat{\theta} - \theta_0)\hat{i}_{\alpha_0}^{1/2}$			Estimated item parameters			
		Known item parameters			N = 500		N = 1,000	
		SD ^(t0)	ASE ^(t0)	HASE ^(t0)	SD ^(t1)	ASE ^(t1)	SD ^(t1)	ASE ^(t1)
$\theta = -1$	ML	.989	1	.999	1.044	1	.984	1
	BM	.897	1	.883	.946	1	.889	1
	WL	.987	1	1.000	1.043	1	.983	1
$\theta = 0$	ML	1.006	1	.992	1.055	1	.996	1
	BM	.938	1	.916	.979	1	.924	1
	WL	.993	1	.978	1.043	1	.984	1
$\theta = 1$	ML	.996	1	.992	1.016	1	.964	1
	BM	.932	1	.919	.949	1	.899	1
	WL	.980	1	.976	1.002	1	.950	1
$\theta = 2$	ML	.976	1	.981	.995	1	1.014	1
	BM	.893	1	.866	.914	1	.929	1
	WL	.956	1	.953	.979	1	.997	1

Note. SD^(t0) (SD^(t1)) = the standard deviation from simulations, ASE^(t0) = $(\beta_2^{(0)}\bar{i}_{\theta_0})^{1/2}$, ASE^(t1) = $\bar{\beta}_{I2}^{1/2} = (\bar{\beta}_2\bar{\beta}_{2I}^{-1})^{1/2}$, HASE^(t0) = $(\beta_2^{(0)}\bar{i}_{\theta_0} + n^{-1}\beta_{tH2}^{(0)})^{1/2}$
= $\{(ASE^{(t0)})^2 + n^{-1}\beta_{tH2}^{(0)}\}^{1/2}$, HASE^(t1) = $(\bar{\beta}_{I2}\bar{i}_{\theta_0} + n^{-1}\beta_{tH2}^{(0)})^{1/2} = \{(ASE^{(t1)})^2 + n^{-1}\beta_{tH2}^{(0)}\}^{1/2}$.

Under c.m.s., ASE^(t0) = ASE^(t1) = 1 and HASE^(t0) = HASE^(t1) = $(1 + n^{-1}\beta_{tH2}^{(0)})^{1/2}$. The \hat{i}_{α_0} in $t_{\alpha_0}^*$ is given by $\hat{\theta}$ and known α_0 ($t_{\alpha_0}^*$ and \hat{i}_{α_0} should not be confused with t^* and \hat{i} using $\hat{\theta}$ and estimated $\hat{\alpha}$ in the text pages). The numbers of deleted cases are the same as those of Table 2. See also the footnote of Table 1.

Table A2. Simulated and asymptotic biases of the studentized $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 50$)

Bias	$t_{\alpha_0}^* = n^{1/2}(\hat{\theta} - \theta_0)\hat{i}_{\alpha_0}^{1/2}$		$t = n^{1/2}(\hat{\theta} - \theta_0)\hat{\beta}_{2I}^{-1/2}$				
	Known		Estimated item parameters				
	item parameters		$N = 500$		$N = 1,000$		
		Sim.	Th.	Sim.	Th.	Sim.	Th.
$\theta = -1$	ML	-.09	0	.10	.02	-.26	.01
	BM	2.05	2.33	2.23	2.36	1.92	2.34
	WL	.40	.47	.56	.50	.20	.49
$\theta = 0$	ML	.04	0	-.12	.00	.04	.00
	BM	.06	0	-.10	.00	.05	.00
	WL	.32	.30	.15	.31	.31	.30
$\theta = 1$	ML	.03	0	.68	-.02	1.18	-.01
	BM	-1.71	-1.86	-1.12	-1.88	-.68	-1.87
	WL	-.16	-.20	.49	-.22	.99	-.21
$\theta = 2$	ML	.00	0	.72	-.06	1.74	-.03
	BM	-4.16	-4.61	-3.44	-4.66	-2.58	-4.63
	WL	-.73	-.76	.03	-.82	1.04	-.79

Note. Sim. = $n^{1/2}$ times the simulated bias, Th. = $\bar{\beta}_{t1}$ ($n^{1/2}$ times the theoretical or asymptotic bias). See also the footnote of Table 1.

Table A3. Simulated and asymptotic third cumulants of the studentized $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 50$)

Third cumulant		$t_{a_0}^* = n^{1/2}(\hat{\theta} - \theta_0)\hat{i}_{a_0}^{1/2}$		$t = n^{1/2}(\hat{\theta} - \theta_0)\hat{\beta}_{2I}^{-1/2}$			
		Known		Estimated item parameters			
		item parameters		$N = 500$		$N = 1,000$	
		Sim.	Th.	Sim.	Th.	Sim.	Th.
$\theta = -1$	ML	1.14	.95	1.74	1.18	.62	1.06
	BM	.92	*	1.36	*	.50	*
	WL	1.18	*	1.80	*	.67	*
$\theta = 0$	ML	.23	.60	.28	.67	1.14	.64
	BM	.29	*	.31	*	1.00	*
	WL	.11	*	.14	*	1.00	*
$\theta = 1$	ML	-.29	-.41	.10	-.58	-.69	-.49
	BM	-.20	*	.16	*	-.52	*
	WL	-.24	*	.13	*	-.63	*
$\theta = 2$	ML	-1.49	-1.52	-1.78	-2.17	-3.20	-1.85
	BM	-1.04	*	-1.19	*	-2.28	*
	WL	-1.32	*	-1.57	*	-2.93	*

Note. Sim. = $n^{1/2}$ times the simulated third cumulant, Th. = $\overline{\beta}_{t3}$ ($n^{1/2}$ times the theoretical or asymptotic third cumulant).

See also the footnote of Table 1.

Table A4. Simulated and asymptotic fourth cumulants of the studentized $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 50$)

Fourth cumulant		$t_{a_0}^* = n^{1/2}(\hat{\theta} - \theta_0)\hat{t}_{a_0}^{1/2}$		$t = n^{1/2}(\hat{\theta} - \theta_0)\hat{\beta}_{2I}^{-1/2}$	
		Known item parameters		Estimated item parameters	
		Sim.	Th.	$N = 500$	$N = 1,000$
$\theta = -1$	ML	-1.18	.49	-13.31	-6.73
	BM	.51	*	-7.76	-3.26
	WL	-1.94	*	-14.07	-7.54
$\theta = 0$	ML	-3.33	-4.23	-12.84	4.33
	BM	-2.57	*	-9.51	3.49
	WL	-2.83	*	-11.75	4.22
$\theta = 1$	ML	-4.23	-4.65	-.39	2.03
	BM	-3.02	*	.58	1.72
	WL	-4.06	*	-.42	1.71
$\theta = 2$	ML	-8.31	-8.00	-3.04	10.10
	BM	-4.71	*	-.58	8.12
	WL	-6.67	*	-2.04	10.42

Note. Sim.= n times the simulated fourth cumulant, Th. = $\overline{\beta}_{t4}$ (n times the theoretical or asymptotic fourth cumulant). See also the footnote of Table 1.