Euler Product Expression of Triple Zeta Functions

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Abstract

We construct multiple zeta functions considered as absolute tensor products of usual zeta functions. We establish Euler product expressions for triple zeta functions $\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)$ with p,q,r distinct primes, via multiple sine functions by using the signatured Poisson summation formula. We also establish Euler product expressions for triple zeta functions $\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)$ with a prime p, via the theory of multiple sine functions.

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1 Introduction and Main Theorem

Let

$$Z_j(s)=\prod_{\rho\in\mathbb{C}}(s-\rho)^{m_j(\rho)}$$

be "zeta functions" expressed as regularized products in the notation of Deninger [2], [3], where $m_j:\mathbb{C}\to\mathbb{Z}$ denotes the multiplicity function for $j=1,\ldots,r$. (Later we will specify "zeta functions" to be treated.) Then, as in [4] we define the absolute tensor product $Z_1(s)\otimes\cdots\otimes Z_r(s)$ as

$$Z_1(s)\otimes\cdots\otimes Z_r(s)=\prod_{\rho_1,\ldots,\rho_r\in\mathbb{C}}(s-(\rho_1+\cdots+\rho_r))^{m(\rho_1,\ldots,\rho_r)},$$

where

$$m(\rho_1,\dots,\rho_r) := m_1(\rho_1)\cdots m_r(\rho_r) \times \begin{cases} 1 & \text{if } \mathrm{Im}(\rho_1),\dots,\mathrm{Im}(\rho_r) \geq 0, \\ (-1)^{r-1} & \text{if } \mathrm{Im}(\rho_1),\dots,\mathrm{Im}(\rho_r) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We refer to [8] for an excellent survey on the absolute tensor product written by Manin.

We define Hasse zeta functions $\zeta(s,A)$ for commutative rings A as

$$\zeta(s,A) := \prod_{\mathfrak{m}} (1 - N(\mathfrak{m})^{-s})^{-1},$$

where $\mathfrak{m}$ runs over maximal ideals of A, and $N(\mathfrak{m}) := \#(A/\mathfrak{m})$.

To simplify the description, for two functions $F(s), G(s)$ we write $F(s) \cong G(s)$ if there exists a polynomial $Q(s)$ satisfying $F(s) = G(s)e^{Q(s)}$. The following result about the absolute tensor product of Hasse zeta functions is known:

Theorem KK . *([5, Theorem 1, Theorem 4]) The following expressions hold in* $\mathrm{Re}(s) > 0$*:*
(1) When $p \neq q$*, it holds that*

$$\begin{aligned} \zeta(s,\mathbb{F}_p) \otimes \zeta(s,\mathbb{F}_q) \;&\cong\; (1-p^{-s})^{\frac{1}{2}}(1-q^{-s})^{\frac{1}{2}} \\ &\times\; \exp\left(\frac{1}{2i}\sum_{k=1}^{\infty} \frac{\cot\left(\pi k \frac{\log p}{\log q}\right)}{k} p^{-ks} + \frac{1}{2i}\sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{\log q}{\log p}\right)}{n} q^{-ns} \right). \end{aligned}$$

(2) When $p = q$*, it holds that*

$$\zeta(s,\mathbb{F}_p) \otimes \zeta(s,\mathbb{F}_p) \cong (1-p^{-s})^{1-\frac{is\log p}{2\pi}} \exp\left(-\frac{\mathrm{Li}_2(p^{-s})}{2\pi i}\right),$$

where

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$

In this paper we treat $\zeta(s,\mathbb{F}_p) \otimes \zeta(s,\mathbb{F}_q) \otimes \zeta(s,\mathbb{F}_r)$ when p, q, r are distinct primes and when $p = q = r$. In Section 2 - Section 4 we prove the following theorem, which is similar to Theorem KK (1):

Theorem 1. *Suppose that p, q, r are distinct primes and* $\mathrm{Re}(s) > 0$*. Then, we have*

$$
\begin{aligned}
&\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)\\
\cong\ &(1-p^{-s})^{-\frac{1}{4}}(1-q^{-s})^{-\frac{1}{4}}(1-r^{-s})^{-\frac{1}{4}}\\
&\exp\Bigg(-\frac{1}{4}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi n_1\frac{\log p}{\log q}\right)\cot\left(\pi n_1\frac{\log p}{\log r}\right)p^{-n_1 s}\\
&\quad-\frac{1}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi n_2\frac{\log q}{\log p}\right)\cot\left(\pi n_2\frac{\log q}{\log r}\right)q^{-n_2 s}\\
&\quad-\frac{1}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi n_3\frac{\log r}{\log p}\right)\cot\left(\pi n_3\frac{\log r}{\log q}\right)r^{-n_3 s}\\
&\quad+\frac{i}{4}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\left(\cot\left(\pi n_1\frac{\log p}{\log q}\right)+\cot\left(\pi n_1\frac{\log p}{\log r}\right)\right)p^{-n_1 s}\\
&\quad+\frac{i}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\left(\cot\left(\pi n_2\frac{\log q}{\log p}\right)+\cot\left(\pi n_2\frac{\log q}{\log r}\right)\right)q^{-n_2 s}\\
&\quad+\frac{i}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\left(\cot\left(\pi n_3\frac{\log r}{\log p}\right)+\cot\left(\pi n_3\frac{\log r}{\log q}\right)\right)r^{-n_3 s}\Bigg).
\end{aligned}
$$

To prove Theorem 1, we show a corresponding expression for the triple sine function. We recall the multiple sine function in [6]. We define the multiple Hurwitz zeta function due to Barnes as

$$\zeta_r(s,z,\underline{\omega}) := \sum_{n_1,\ldots,n_r\geq 0}(n_1\omega_1+\cdots+n_r\omega_r+z)^{-s}$$

for $\underline{\omega}=(\omega_1,\ldots,\omega_r)\in(\mathbb{R}_{>0})^r$, the multiple gamma and the multiple sine as

$$
\begin{aligned}
\Gamma_r(z,\underline{\omega}) &:= \exp\left(\frac{\partial}{\partial s}\zeta_r(s,z,\underline{\omega})\bigg|_{s=0}\right),\\
S_r(z,\underline{\omega}) &:= \Gamma_r(z,\underline{\omega})^{-1}\Gamma_r(\omega_1+\cdots+\omega_r-z,\underline{\omega})^{(-1)^r}.
\end{aligned}
$$

We say that $\alpha\in\mathbb{R}$ is *generic* if

$$\lim_{m\to\infty}\|m\alpha\|^{\frac{1}{m}} = 1,$$

where we put $\|x\| := \min\{|x-n| : n\in\mathbb{Z}\}$ for $x\in\mathbb{R}$. For example:

(1) Let α and β be in $\overline{\mathbb{Q}}\cap\mathbb{R}_{>0}$. If $\frac{\log\alpha}{\log\beta}\notin\mathbb{Q}$, then $\frac{\log\alpha}{\log\beta}$ is generic (Baker, [1, Theorem 3.1]).

(2) If $\alpha\in\mathbb{Q}$, then α is *not* generic (Lemma 2.1 below).

Theorem 2. *Let $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ be in $(\mathbb{R}_{>0})^3$ such that ω_j/ω_k are generic $(j, k = 1, 2, 3; j \neq k)$. Then the triple sine function has the following expression in $\mathrm{Im}(z) > 0$:*

$$
\begin{aligned}
S_3(z, \underline{\omega}) = \exp\Bigg(& \frac{1}{4}\sum_{n_1=1}^{\infty} \frac{1}{n_1} \cot\left(\pi \frac{n_1\omega_2}{\omega_1}\right) \cot\left(\pi \frac{n_1\omega_3}{\omega_1}\right) e^{2\pi i n_1 \frac{z}{\omega_1}} \\
& + \frac{1}{4}\sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2\omega_1}{\omega_2}\right) \cot\left(\pi \frac{n_2\omega_3}{\omega_2}\right) e^{2\pi i n_2 \frac{z}{\omega_2}} \\
& + \frac{1}{4}\sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3\omega_1}{\omega_3}\right) \cot\left(\pi \frac{n_3\omega_2}{\omega_3}\right) e^{2\pi i n_3 \frac{z}{\omega_3}} \\
& + \frac{1}{4i}\sum_{n_1=1}^{\infty} \frac{1}{n_1} \left(\cot\left(\pi \frac{n_1\omega_2}{\omega_1}\right) + \cot\left(\pi \frac{n_1\omega_3}{\omega_1}\right)\right) e^{2\pi i n_1 \frac{z}{\omega_1}} \\
& + \frac{1}{4i}\sum_{n_2=1}^{\infty} \frac{1}{n_2} \left(\cot\left(\pi \frac{n_2\omega_1}{\omega_2}\right) + \cot\left(\pi \frac{n_2\omega_3}{\omega_2}\right)\right) e^{2\pi i n_2 \frac{z}{\omega_2}} \\
& + \frac{1}{4i}\sum_{n_3=1}^{\infty} \frac{1}{n_3} \left(\cot\left(\pi \frac{n_3\omega_1}{\omega_3}\right) + \cot\left(\pi \frac{n_3\omega_2}{\omega_3}\right)\right) e^{2\pi i n_3 \frac{z}{\omega_3}} \\
& + \frac{1}{4}\log(1 - e^{2\pi i \frac{z}{\omega_1}}) + \frac{1}{4}\log(1 - e^{2\pi i \frac{z}{\omega_2}}) \\
& + \frac{1}{4}\log(1 - e^{2\pi i \frac{z}{\omega_3}}) - \frac{\pi i z^3}{6\omega_1\omega_2\omega_3} + \frac{\pi i}{4}\left(\frac{1}{\omega_1\omega_2} + \frac{1}{\omega_2\omega_3} + \frac{1}{\omega_3\omega_1}\right) z^2 \\
& - \frac{\pi i}{12}\left(\frac{3}{\omega_1} + \frac{3}{\omega_2} + \frac{3}{\omega_3} + \frac{\omega_1}{\omega_2\omega_3} + \frac{\omega_2}{\omega_3\omega_1} + \frac{\omega_3}{\omega_1\omega_2}\right) z \\
& + \frac{\pi i}{24}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} + \frac{\omega_3}{\omega_1} + \frac{\omega_1}{\omega_3} + 3\right)\Bigg).
\end{aligned}
$$

Since

$$
\zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) \otimes \zeta(s, \mathbb{F}_r) \cong S_3\left(is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}, \frac{2\pi}{\log r}\right)\right)^{-1}
$$

by Lemma 4.1 below, Theorem 2 implies Theorem 1 directly. The key point to prove Theorem 2 is to establish the signatured triple Poisson summation formula as follows:

Theorem 3. *For some $R > 0$ let $H(t)$ be an even regular function on $\{z = x + iy : x \in \mathbb{R}, y \in (-R, R)\}$, which satisfies (i) and (ii):*
(i) There exists $\delta > 0$ such that $H(t) = O(t^{-3-\delta})$ $(|t| \to \infty)$.
(ii) There exists $\mu \in (0, 1)$ such that $\widetilde{H}(x) = O(\mu^x)$ $(x \to \infty)$, where we denote by $\widetilde{H}$ the

Fourier transform of H:

$$\widetilde{H}(x) := \int_{-\infty}^{\infty} H(t)e^{itx}dt.$$

Let $a,b,c \in \mathbb{R}$ be positive real numbers such that $a/b, b/c, c/a$ and their inverses are generic. Then we have

$$\begin{aligned}
&\sum_{n_1,n_2,n_3\geq 0} \varepsilon_{n_1,n_2,n_3} H\left(2\pi\left(\frac{n_1}{a}+\frac{n_2}{b}+\frac{n_3}{c}\right)\right)\\
&= -\frac{a}{8\pi}\sum_{n_1>0}\cot\left(\pi\frac{n_1 a}{b}\right)\cot\left(\pi\frac{n_1 a}{c}\right)\widetilde{H}(n_1 a) - \frac{b}{8\pi}\sum_{n_2>0}\cot\left(\pi\frac{n_2 b}{c}\right)\cot\left(\pi\frac{n_2 b}{a}\right)\widetilde{H}(n_2 b)\\
&\quad -\frac{c}{8\pi}\sum_{n_3>0}\cot\left(\pi\frac{n_3 c}{a}\right)\cot\left(\pi\frac{n_3 c}{b}\right)\widetilde{H}(n_3 c) + \frac{abc}{48\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\widetilde{H}(0) - \frac{abc}{32\pi^3}\widetilde{H}''(0),
\end{aligned}$$

where $\varepsilon_{n_1,\dots,n_r} := \varepsilon_{n_1}\times\cdots\times\varepsilon_{n_r}$ and

$$\varepsilon_n := \begin{cases} 1 & \text{if } n \neq 0,\\ \frac{1}{2} & \text{if } n = 0.\end{cases}$$

.

First we prove Theorem 3 in Section 2. Then we prove Theorem 2 in Section 3 by applying the signatured double Poisson summation formula, the expression of double sine functions in [5], and Theorem 3. We show Theorem 1 from Theorem 2 in Section 4.

We remark that Kurokawa and Wakayama calculated the Euler product of $\zeta(s,\mathbb{F}_{p_1})\otimes\cdots\otimes\zeta(s,\mathbb{F}_{p_r})$ for distinct primes $p_1,\dots,p_r$ by an entirely different method in [7] after this paper was written.

In Section 5 we treat the case $p=q=r$. We obtain the following results.

Theorem 4. *For $\mathrm{Re}(s)>0$ and a prime p we have*

$$\begin{aligned}
&\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)\\
\cong\ &\exp\left(-\frac{1}{4\pi^2}\,\mathrm{Li}_3(p^{-s}) - \frac{s\log p+3\pi i}{4\pi^2}\,\mathrm{Li}_2(p^{-s}) - \frac{(s\log p+2\pi i)(s\log p+4\pi i)}{8\pi^2}\,\mathrm{Li}_1(p^{-s})\right).
\end{aligned}$$

Theorem 5. *For $\mathrm{Im}(z)>0$ $S_3(z) := S_3(z,(1,1,1))$ has a following expression:*

$$\begin{aligned}
S_3(z) = \exp\Bigg(&\frac{1}{4\pi^2}\,\mathrm{Li}_3(e^{2\pi iz}) + \frac{i}{4\pi}(-2z+3)\,\mathrm{Li}_2(e^{2\pi iz}) - \frac{(z-1)(z-2)}{2}\,\mathrm{Li}_1(e^{2\pi iz})\\
&-\frac{\pi i}{6}z^3 + \frac{3\pi i}{4}z^2 - \pi iz + \frac{3\pi i}{8}\Bigg). \qquad (1.1)
\end{aligned}$$

Remark 1.1. *It turns out that the polynomial part*

$$-\frac{\pi i}{6}z^3 + \frac{3\pi i}{4}z^2 - \pi i z + \frac{3\pi i}{8}$$

is identified with $\pi i \zeta_3(0, z, (1,1,1))$.

In the proof we use the theory of multiple sine functions [6].

Remark 1.2. *We can treat the case* $\zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q)$ *for* $p \neq q$ *also. Then we obtain the following result:*

$$\begin{aligned} &\zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) \\ \cong\ & \exp\left(-\frac{\log p}{\log q} \sum_{n=1}^{\infty} \frac{e^{2\pi i n \frac{\log p}{\log q}}}{n(e^{2\pi i n \frac{\log p}{\log q}} - 1)^2} p^{-ns} + \frac{is \log p}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi i n \frac{\log p}{\log q}} - 1)} p^{-ns} \right. \\ & - \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n^2(e^{2\pi i n \frac{\log p}{\log q}} - 1)} p^{-ns} - \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi i n \frac{\log p}{\log q}} - 1)} p^{-ns} \\ & \left. + \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi i n \frac{\log q}{\log p}} - 1)^2} q^{-ns} \right). \end{aligned}$$

Since the needed calculation is rather long, we will publish the detailed proof in another opportunity.

2 Signatured Triple Poisson Summation Formula

In this section we prove Theorem 3.

Lemma 2.1. *If* $\alpha \in \mathbb{Q}$*, then* α *is not generic.*

Proof. Let $\alpha = a/b$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{>0}$. Then $\|(bm)\alpha\| = 0$ for all $m \in \mathbb{Z}$. Hence α is not generic. □

Lemma 2.2. *Let* α *and* β *be generic. Then*

$$\sum_{n=1}^{\infty} \cot(\pi n \alpha) \cot(\pi n \beta) x^n \tag{2.1}$$

converges absolutely in $|x| < 1$.

Proof. Since $\alpha,\ \beta$ are generic, $\|n\alpha\|^{-1},\ \|n\beta\|^{-1} = O(e^{\varepsilon n})$ as $n \to \infty$ for any $\varepsilon > 0$. Since $\cot(\pi x) \sim 1/(\pi x)$ $(x \to 0)$, $\cot(\pi n\alpha)\cot(\pi n\beta) = O(e^{2\varepsilon n})$ $(n \to \infty)$. Hence the radius of convergence of (2.1), say R, satisfies

$$R = \limsup_{n\to\infty} \left| \frac{1}{\cot(\pi n\alpha)\cot(\pi n\beta)} \right|^{\frac{1}{n}} \geq e^{-2\varepsilon}.$$

Since $\varepsilon > 0$ is arbitary, $R \geq 1$. □

Until the end of this section, suppose that $H(t), a, b, c$ satisfy the assumptions of Theorem 3. $h(t)$ and $H_\alpha(t)$ denote $h(t) := H(t/i)$, $H_\alpha(t) := h(\alpha + it)$, respectively.

Lemma 2.3. *Let $\alpha \in (-R, R)$ and $x \in \mathbb{R}$. Then,*
(1)$\widetilde{H_\alpha}(x) = e^{-\alpha x}\widetilde{H}(x)$.
(2)$\widetilde{tH_\alpha(t)}(0) = i\alpha\widetilde{H}(0)$.

Proof. (1) By Cauchy's theorem we have

$$\int_{C_T} H(t)e^{itx}dt = 0,$$

where

$$C_T := \partial\{z \in \mathbb{C} : -T < \mathrm{Re}(z) < T,\ \ \min\{-\alpha, 0\} < \mathrm{Im}(z) < \max\{-\alpha, 0\}\}.$$

Considering the limit $T \to \infty$, (1) follows.

(2) Considering the same as (1), we have

$$\int_{-\infty}^{\infty} tH(t)dt = \int_{-\infty}^{\infty} (t - i\alpha)H_\alpha(t)dt. \tag{2.2}$$

Since $tH(t)$ is an odd function, the left hand side of (2.2) is equal to 0. Therefore we have

$$\widetilde{tH_\alpha(t)}(0) = i\alpha\widetilde{H_\alpha}(0).$$

Applying (1) with $x = 0$, (2) follows. □

We prepare some lemmas (Lemma 2.4 -Lemma 2.6) for interchanging the limit and the sum. We will apply Lemma 2.6 to (2.16) below.

Lemma 2.4. *Let n, m be positive integers and a, b be positive real numbers such that a/b is generic. Then, for any $\varepsilon > 0$ we have*

$$|na - mb|^{-1} \ll_{a,b,\varepsilon} m^{-1} e^{\varepsilon n}.$$

Here $A \ll_{c_1,\ldots,c_r} B$ means that there exists a constant C depending only on $c_1, \ldots, c_r$ such that $A \leq CB$.

Proof. Considering that n, a, b are fixed and that m is a positive integer variable, $m = [\frac{na}{b}]$ or $m = [\frac{na}{b}] + 1$ is a minimum for $|\frac{na}{mb} - 1|$. Here $[x] := \max\{l \in \mathbb{Z} : l \leq x\}$. Therefore we have

$$\begin{aligned}
\left|\frac{na}{mb} - 1\right| &\geq \min\left\{\frac{1}{[\frac{na}{b}]}\left|\frac{na}{b} - \left[\frac{na}{b}\right]\right|, \frac{1}{[\frac{na}{b}]+1}\left|\frac{na}{b} - \left[\frac{na}{b}\right] - 1\right|\right\} \\
&\geq \min\left\{\frac{1}{[\frac{na}{b}]}\left\|\frac{na}{b}\right\|, \frac{1}{[\frac{na}{b}]+1}\left\|\frac{na}{b}\right\|\right\} \\
&= \frac{1}{[\frac{na}{b}]+1}\left\|\frac{na}{b}\right\|.
\end{aligned}$$

Hence we have

$$|na - mb|^{-1} \leq \frac{[\frac{na}{b}]+1}{mb}\left\|\frac{na}{b}\right\|^{-1}.$$

Since a/b is generic, we have $\|\frac{na}{b}\|^{-1} \ll_{a,b,\varepsilon} e^{\varepsilon n}$ for any $\varepsilon > 0$. This completes the proof. □

Lemma 2.5. *(1) If $x_1 > x_2$, then we have*

$$e^{x_1} - e^{x_2} \leq (x_1 - x_2)e^{x_1}.$$

(2) If $y > 0$, then we have

$$e^{-y} \leq y^{-1/2}.$$

Proof. (1) follows from the mean value theorem. (2) follows from the following estimate:

$$(e^y)^2 = e^{2y} = 1 + 2y + \frac{(2y)^2}{2} + \cdots \geq 2y \geq y.$$

□

Lemma 2.6. *Suppose that $a, b, c > 0$ and the function H satisfy the assumptions of Theorem 3. Then we have*

$$
\begin{aligned}
(1)\qquad & \lim_{\alpha\downarrow 0}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_2b+n_3c)\alpha}\frac{(e^{n_1a\alpha}+e^{-n_1a\alpha})^2n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)\\
&= \sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}\frac{4n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a),\\
(2)\qquad & \lim_{\alpha\downarrow 0}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_2b+n_3c)\alpha}\frac{(e^{2n_1a\alpha}-e^{-2n_1a\alpha})n_1n_2ab}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)=0,\\
(3)\qquad & \lim_{\alpha\downarrow 0}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_2b+n_3c)\alpha}\frac{(e^{n_1a\alpha}-e^{-n_1a\alpha})^2n_2n_3bc}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)=0,
\end{aligned}
$$

where

$$\Gamma := \{(n_1,n_2,n_3)\in(\mathbb{Z}_{\geq 0})^3 : \#\{j=1,2,3 : n_j=0\}\leq 1\}. \tag{2.3}$$

Proof. Let $\beta := -\log\mu$ and $0<\alpha<\beta/6$, where μ appears in the assumption (ii) of Theorem 3.

(1) We have

$$
\begin{aligned}
& \sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_2b+n_3c)\alpha}\frac{(e^{n_1a\alpha}+e^{-n_1a\alpha})^2n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)\\
- & \sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}\frac{4n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)\\
= & \sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_2b+n_3c)\alpha}\frac{(e^{n_1a\alpha}-e^{-n_1a\alpha})^2n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)\\
+ & \sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}\frac{4e^{-n_2b\alpha}(e^{-n_3c\alpha}-1)n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)\\
+ & \sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}\frac{4(e^{-n_2b\alpha}-1)n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)\\
=: & \ A_1+A_2+A_3.
\end{aligned}
$$

It is sufficient to prove that A_1, A_2, A_3 tend to 0 as $\alpha\downarrow 0$. First we deal with A_1. By Lemma 2.5 (1), the assumption (ii) of Theorem 3 and $0<\alpha<\beta/6$ we have

$$|A_1| \ll \sum_{(n_1,n_2,n_3)\in\Gamma}\frac{(n_1^2\alpha e^{2n_1a\alpha})^2}{|n_1^2a^2-n_2b^2||n_1^2a^2-n_3^2c^2|}\mu^{n_1a}$$

$$
\begin{aligned}
&\ll \sum_{(n_1,n_2,n_3)\in\Gamma} \frac{n_1^4\alpha^2}{|n_1^2a^2 - n_2b^2||n_1^2a^2 - n_3^2c^2|}\mu^{\frac{2}{3}n_1a} \\
&= \alpha^2 \left(\sum_{n_1,n_2,n_3\geq 1} \frac{n_1^4\mu^{\frac{2}{3}n_1a}}{|n_1^2a^2 - n_2^2b^2||n_1^2a^2 - n_3^2c^2|} + \sum_{n_1,n_3\geq 1} \frac{n_1^2\mu^{\frac{2}{3}n_1a}}{a^2|n_1^2a^2 - n_3^2c^2|} \right. \\
&\quad \left. + \sum_{n_1,n_2\geq 1} \frac{n_1^2\mu^{\frac{2}{3}n_1a}}{a^2|n_1^2a^2 - n_2^2b^2|} \right) \\
&=: \alpha^2(A_{11} + A_{12} + A_{13}).
\end{aligned}
$$

We prove $A_{1j} < \infty$. By Lemma 2.4 with $\varepsilon = \frac{1}{6}a\beta$ we get

$$
\frac{1}{|n_1^2a^2 - n_2^2b^2|} \leq \frac{1}{n_2b|n_1a - n_2b|} \ll \frac{\mu^{-\frac{1}{6}n_1a}}{n_2^2}, \tag{2.4}
$$

$$
\frac{1}{|n_1^2a^2 - n_3^2c^2|} \ll \frac{\mu^{-\frac{1}{6}n_1a}}{n_3^2} \tag{2.5}
$$

for $n_1, n_2, n_3 \geq 1$. Hence it holds that

$$
\begin{aligned}
A_{11} &\ll \sum_{n_1,n_2,n_3\geq 1} \frac{n_1^4\mu^{\frac{1}{3}n_1a}}{n_2^2n_3^2} < \infty, \\
A_{12} &\ll \sum_{n_1,n_3\geq 1} \frac{n_1^2\mu^{\frac{1}{2}n_1a}}{n_3^2} < \infty, \\
A_{13} &\ll \sum_{n_1,n_2\geq 1} \frac{n_1^2\mu^{\frac{1}{2}n_1a}}{n_2^2} < \infty,
\end{aligned}
$$

Consequently A_1 tends to 0 as $\alpha \downarrow 0$.

Next we deal with A_2. Estimating A_2 similarly, we have

$$
\begin{aligned}
|A_2| &\ll \sum_{(n_1,n_2,n_3)\in\Gamma} \frac{(1 - e^{-n_3c\alpha})n_1^2\mu^{n_1a}}{|n_1^2a^2 - n_2^2b^2||n_1^2a^2 - n_3^2c^2|} \\
&= \sum_{n_1,n_2,n_3\geq 1} \frac{(1 - e^{-n_3c\alpha})n_1^2\mu^{n_1a}}{|n_1^2a^2 - n_2^2b^2||n_1^2a^2 - n_3^2c^2|} + \sum_{n_1,n_3\geq 1} \frac{(1 - e^{-n_3c\alpha})\mu^{n_1a}}{a^2|n_1^2a^2 - n_3^2c^2|} \\
&=: A_{21} + A_{22}.
\end{aligned}
$$

We prove A_{2j} tends to 0 as $\alpha \downarrow 0$. By (2.4) and (2.5) we get

$$
A_{21} \ll \sum_{n_1,n_2,n_3\geq 1} \frac{(1 - e^{-n_3c\alpha})n_1^2\mu^{\frac{2}{3}n_1a}}{n_2^2n_3^2}
$$

$$
\begin{aligned}
&= \left(\sum_{n_1=1}^{\infty} n_1^2 \mu^{\frac{2}{3}n_1 a}\right)\left(\sum_{n_2=1}^{\infty} \frac{1}{n_2^2}\right)\left(\sum_{n_3=1}^{\infty} \frac{1}{n_3^2} - \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3^2}\right) \\
&\to 0 \qquad \text{as } \alpha \downarrow 0, \\
A_{22} &\ll \sum_{n_1, n_2 \geq 1} \frac{(1-e^{-n_3 c\alpha})\mu^{\frac{5}{6}n_1 \alpha}}{n_3^2} \\
&= \left(\sum_{n_1=1}^{\infty} \mu^{\frac{5}{6}n_1 a}\right)\left(\sum_{n_3=1}^{\infty} \frac{1}{n_3^2} - \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3^2}\right) \\
&\to 0 \qquad \text{as } \alpha \downarrow 0.
\end{aligned}
$$

Hence we obtain $A_2 \to 0$ as $\alpha \downarrow 0$. Dealing with A_3 in the same manner as A_2, A_3 tends to 0 as $\alpha \downarrow 0$. Hence we obtain (1).

(2) By Lemma 2.5 (1) and $0 < \alpha < \beta/6$ we have

$$
\begin{aligned}
&\left| \sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_2 b + n_3 c)\alpha} \frac{(e^{2n_1 a\alpha} - e^{-2n_1 a\alpha}) n_1 n_2 ab}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} \widetilde{H}(n_1 a) \right| \\
\ll& \sum_{(n_1,n_2,n_3)\in\Gamma} e^{-n_2 b\alpha} \frac{\alpha n_1^2 e^{2n_1 a\alpha} n_2}{|n_1^2 a^2 - n_2^2 b^2||n_1^2 a^2 - n_3^2 c^2|} \mu^{n_1 a} \\
\ll& \sum_{(n_1,n_2,n_3)\in\Gamma} \frac{\alpha n_1^2 n_2 e^{-n_2 b\alpha}}{|n_1^2 a^2 - n_2^2 b^2||n_1^2 a^2 - n_3^2 c^2|} \mu^{\frac{2}{3}n_1 a} \\
=& \sum_{n_1,n_2,n_3 \geq 1} \frac{\alpha n_1^2 n_2 e^{-n_2 b\alpha}}{|n_1^2 a^2 - n_2^2 b^2||n_1^2 a^2 - n_3^2 c^2|} \mu^{\frac{2}{3}n_1 a} + \sum_{n_1,n_2 \geq 1} \frac{\alpha n_2 e^{-n_2 b\alpha}}{a^2 |n_1^2 a^2 - n_2^2 b^2|} \mu^{\frac{2}{3}n_1 a} \\
=:& \ A_4 + A_5.
\end{aligned}
$$

First we deal with A_4. By (2.4) and (2.5) we get

$$
\begin{aligned}
A_4 &\ll \alpha \sum_{n_1,n_2,n_3 \geq 1} \frac{n_1^2 n_2 e^{-n_2 b\alpha} \mu^{\frac{1}{3}n_1 a}}{n_2^2 n_3^2} \\
&= \alpha \left(\sum_{n_1=1}^{\infty} n_1^2 \mu^{\frac{1}{3}n_1 a}\right)\left(\sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2}\right)\left(\sum_{n_3=1}^{\infty} \frac{1}{n_3^2}\right) \\
&\ll \alpha \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2}.
\end{aligned}
$$

By Lemma 2.5 (2) we have

$$
A_4 \ll \alpha \sum_{n_2=1}^{\infty} \frac{1}{n_2 (n_2 b\alpha)^{1/2}} \ll \alpha^{1/2} \to 0 \quad \text{as } \alpha \downarrow 0.
$$

Dealing with A_5 in the same manner as A_4, A_5 tends to 0 as $\alpha \downarrow 0$. Hence (2) holds.

(3) Estimating the left hand side in the same manner as (2), we obtain the desired result. □

Proof of Theorem 3. Put $Z_k(s) = \sinh\left(\frac{ks}{2}\right)$ $(k = a, b, c)$. Let D_T be the region defined by

$$D_T := \{s \in \mathbb{C} : |s| > \alpha,\ |\operatorname{Re}(s)| < \alpha,\ 0 < \operatorname{Im}(s) < T\},$$

where $0 < \alpha < \min\left\{\frac{2\pi}{a}, \frac{2\pi}{b}, \frac{2\pi}{c}\right\}$.

By Cauchy's theorem we have

$$\sum_{0<\operatorname{Im}(\rho_1),\operatorname{Im}(\rho_2),\operatorname{Im}(\rho_3)<T} h(\rho_a + \rho_b + \rho_c)$$
$$= \frac{1}{(2\pi i)^3}\int_{\partial D_T}\int_{\partial D_T}\int_{\partial D_T} h(s_1+s_2+s_3)\frac{Z_a'}{Z_a}(s_1)\frac{Z_b'}{Z_b}(s_2)\frac{Z_c'}{Z_c}(s_3)ds_1ds_2ds_3, \tag{2.6}$$

where ρ_k denotes the zeros of $Z_k(s)(k = a, b, c)$, and the contour ∂D_T is taken counterclockwise. Considering $T \to \infty$ in (2.6), we have

$$\sum_{0<\operatorname{Im}(\rho_a),\operatorname{Im}(\rho_b),\operatorname{Im}(\rho_c)} h(\rho_a + \rho_b + \rho_c)$$
$$= \frac{1}{(2\pi i)^3}\int_{\partial D}\int_{\partial D}\int_{\partial D} h(s_1+s_2+s_3)\frac{Z_a'}{Z_a}(s_1)\frac{Z_b'}{Z_b}(s_2)\frac{Z_c'}{Z_c}(s_3)ds_1ds_2ds_3, \tag{2.7}$$

where

$$D := \{s \in \mathbb{C} : |\operatorname{Re}(s)| < \alpha,\ |s| > \alpha,\ \operatorname{Im}(s) > 0\}.$$

We decompose $\partial D = C_1 \cup C_2 \cup C_3$ with

$$C_1 := \{s \in \partial D : \operatorname{Re}(s) = -\alpha\},$$
$$C_2 := \{s \in \partial D : |s| = \alpha\},$$
$$C_3 := \{s \in \partial D : \operatorname{Re}(s) = \alpha\}.$$

We compute each triple integral $I_{i_1i_2i_3} = \frac{1}{(2\pi i)^3}\int_{C_{i_1}}\int_{C_{i_2}}\int_{C_{i_3}}$ in (2.7).

First we calculate $I_{i_1i_2i_3}$ with $(i_1, i_2, i_3) \in \{1, 3\}^3$.

$$I_{333} = \frac{1}{(2\pi)^3}\int_0^\infty\int_0^\infty\int_0^\infty h(3\alpha + i(t_1+t_2+t_3))\frac{Z_a'}{Z_a}(\alpha+it_1)\frac{Z_b'}{Z_b}(\alpha+it_2)\frac{Z_c'}{Z_c}(\alpha+it_3)dt_1dt_2dt_3.$$

$$(2.8)$$

Since

$$\frac{Z_k'}{Z_k}(s) = \frac{k}{2} + k\sum_{n=1}^{\infty} e^{-kns} \tag{2.9}$$

for $k = a, b, c$ and $\mathrm{Re}(s) > 0$, (2.8) turns to

$$I_{333} = \frac{1}{8\pi^3} \sum_{n_1,n_2,n_3 \geq 0} \varepsilon_{n_1,n_2,n_3} \int_0^\infty \int_0^\infty \int_0^\infty H_{3\alpha}(t_1+t_2+t_3) e^{-n_1 a(\alpha+it_1)} e^{-n_2 b(\alpha+it_2)} e^{-n_3 c(\alpha+it_3)} dt_1 dt_2 dt_3.$$

We replace t_3 with $t = t_1 + t_2 + t_3$ to get

$$\begin{aligned} I_{333} &= \frac{1}{8\pi^3} \sum_{n_1,n_2,n_3 \geq 0} \varepsilon_{n_1,n_2,n_3} \int_0^\infty H_{3\alpha}(t) \times \\ &\quad \left(\iint_{\substack{t_1,t_2 \geq 0 \\ t_1+t_2 \leq t}} e^{-n_1 a(\alpha+it_1)} e^{-n_2 b(\alpha+it_2)} e^{-n_3 c(\alpha+i(t-t_1-t_2))} dt_1 dt_2 \right) dt. \end{aligned} \tag{2.10}$$

By

$$\iint_{\substack{t_1,t_2 \geq 0 \\ t_1+t_2 \leq t}} \cdots dt_1 dt_2 = \int_0^t \int_0^{t-t_1} \cdots dt_2 dt_1 \qquad (t > 0)$$

and

$$\begin{aligned} &n_1 a = n_2 b \Leftrightarrow (n_1, n_2) = (0,0),\ n_2 b = n_3 c \Leftrightarrow (n_2, n_3) = (0,0), \\ &n_1 a = n_3 c \Leftrightarrow (n_1, n_3) = (0,0), \end{aligned} \tag{2.11}$$

which follows from Lemma 2.1, we calculate that

$$\begin{aligned} I_{333} = &-\frac{abc}{8\pi^3} \sum_{(n_1,n_2,n_3) \in \Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1 a + n_2 b + n_3 c)\alpha} \times \\ &\left(\int_0^\infty \frac{H_{3\alpha}(t) e^{-n_1 ait}}{(n_1 a - n_2 b)(n_1 a - n_3 c)} dt + \int_0^\infty \frac{H_{3\alpha}(t) e^{-n_2 bit}}{(n_2 b - n_1 a)(n_2 b - n_3 c)} dt \right. \\ &\left. + \int_0^\infty \frac{H_{3\alpha}(t) e^{-n_3 cit}}{(n_3 c - n_1 a)(n_3 c - n_2 b)} dt \right) \\ &+ \frac{abc}{32\pi^3} \sum_{n_1=1}^\infty e^{-n_1 a\alpha} \left(\int_0^\infty \frac{tH_{3\alpha}(t)}{n_1 ai} dt - \int_0^\infty \frac{H_{3\alpha}(t) e^{-n_1 ait}}{n_1^2 a^2} dt + \int_0^\infty \frac{H_{3\alpha}(t)}{n_1^2 a^2} dt \right) \\ &+ \frac{abc}{32\pi^3} \sum_{n_2=1}^\infty e^{-n_2 b\alpha} \left(\int_0^\infty \frac{tH_{3\alpha}(t)}{n_2 bi} dt - \int_0^\infty \frac{H_{3\alpha}(t) e^{-n_2 bit}}{n_2^2 b^2} dt + \int_0^\infty \frac{H_{3\alpha}(t)}{n_2^2 b^2} dt \right) \end{aligned}$$

$$+\frac{abc}{32\pi^3}\sum_{n_3=1}^{\infty}e^{-n_3c\alpha}\left(\int_0^{\infty}\frac{tH_{3\alpha}(t)}{n_3ci}dt-\int_0^{\infty}\frac{H_{3\alpha}(t)e^{-n_3cit}}{n_3^2c^2}dt+\int_0^{\infty}\frac{H_{3\alpha}(t)}{n_3^2c^2}dt\right)$$
$$+\frac{abc}{128\pi^3}\int_0^{\infty}t^2H_{3\alpha}(t)dt, \tag{2.12}$$

where Γ is defined as (2.3).

Similarly

$$I_{111}=-\frac{abc}{8\pi^3}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_1a+n_2b+n_3c)\alpha}\times$$
$$\left(\int_{-\infty}^0\frac{H_{3\alpha}(t)e^{-n_1ait}}{(n_1a-n_2b)(n_1a-n_3c)}dt+\int_{-\infty}^0\frac{H_{3\alpha}(t)e^{-n_2bit}}{(n_2b-n_1a)(n_2b-n_3c)}dt\right.$$
$$\left.+\int_{-\infty}^0\frac{H_{3\alpha}(t)e^{-n_3cit}}{(n_3c-n_1a)(n_3c-n_2b)}dt\right)$$
$$+\frac{abc}{32\pi^3}\sum_{n_1=1}^{\infty}e^{-n_1a\alpha}\left(\int_{-\infty}^0\frac{tH_{3\alpha}(t)}{n_1ai}dt-\int_{-\infty}^0\frac{H_{3\alpha}(t)e^{-n_1ait}}{n_1^2a^2}dt+\int_{-\infty}^0\frac{H_{3\alpha}(t)}{n_1^2a^2}dt\right)$$
$$+\frac{abc}{32\pi^3}\sum_{n_2=1}^{\infty}e^{-n_2b\alpha}\left(\int_{-\infty}^0\frac{tH_{3\alpha}(t)}{n_2bi}dt-\int_{-\infty}^0\frac{H_{3\alpha}(t)e^{-n_2bit}}{n_2^2b^2}dt+\int_{-\infty}^0\frac{H_{3\alpha}(t)}{n_2^2b^2}dt\right)$$
$$+\frac{abc}{32\pi^3}\sum_{n_3=1}^{\infty}e^{-n_3c\alpha}\left(\int_{-\infty}^0\frac{tH_{3\alpha}(t)}{n_3ci}dt-\int_{-\infty}^0\frac{H_{3\alpha}(t)e^{-n_3cit}}{n_3^2c^2}dt+\int_{-\infty}^0\frac{H_{3\alpha}(t)}{n_3^2c^2}dt\right)$$
$$+\frac{abc}{128\pi^3}\int_{-\infty}^0t^2H_{3\alpha}(t)dt. \tag{2.13}$$

By (2.12) and (2.13) we have

$$I_{111}+I_{333}$$
$$=-\frac{abc}{8\pi^3}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_1a+n_2b+n_3c)\alpha}\times$$
$$\left(\frac{\widetilde{H_{3\alpha}}(-n_1a)}{(n_1a-n_2b)(n_1a-n_3c)}+\frac{\widetilde{H_{3\alpha}}(-n_2b)}{(n_2b-n_1a)(n_2b-n_3c)}+\frac{\widetilde{H_{3\alpha}}(-n_3c)}{(n_3c-n_1a)(n_3c-n_2b)}\right)$$
$$+\frac{abc}{32\pi^3}\sum_{n_1=1}^{\infty}e^{-n_1a\alpha}\left(\frac{\widetilde{tH_{3\alpha}(t)}(0)}{n_1ai}-\frac{\widetilde{H_{3\alpha}}(-n_1a)}{n_1^2a^2}+\frac{\widetilde{H_{3\alpha}}(0)}{n_1^2a^2}\right)$$
$$+\frac{abc}{32\pi^3}\sum_{n_2=1}^{\infty}e^{-n_2b\alpha}\left(\frac{\widetilde{tH_{3\alpha}(t)}(0)}{n_2bi}-\frac{\widetilde{H_{3\alpha}}(-n_2b)}{n_2^2b^2}+\frac{\widetilde{H_{3\alpha}}(0)}{n_2^2b^2}\right)$$
$$+\frac{abc}{32\pi^3}\sum_{n_3=1}^{\infty}e^{-n_3c\alpha}\left(\frac{\widetilde{tH_{3\alpha}(t)}(0)}{n_3ci}-\frac{\widetilde{H_{3\alpha}}(-n_3c)}{n_3^2c^2}+\frac{\widetilde{H_{3\alpha}}(0)}{n_3^2c^2}\right)$$

$$+ \quad \frac{abc}{128\pi^3} \widetilde{t^2 H_{3\alpha}(t)}(0).$$

By Lemma 2.3 and $\widetilde{H}(-x) = \widetilde{H}(x)$, we have

$$\begin{aligned}
& I_{111} + I_{333} \\
= \quad & -\frac{abc}{8\pi^3} \sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1a+n_2b+n_3c)\alpha} \times \\
& \left(\frac{e^{3n_1a\alpha}\widetilde{H}(n_1a)}{(n_1a-n_2b)(n_1a-n_3c)} + \frac{e^{3n_2b\alpha}\widetilde{H}(n_2b)}{(n_2b-n_1a)(n_2b-n_3c)} + \frac{e^{3n_3c\alpha}\widetilde{H}(n_3c)}{(n_3c-n_1a)(n_3c-n_2b)} \right) \\
+ \quad & \frac{abc}{32\pi^3} \left(3\alpha \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1a} - \sum_{n_1=1}^{\infty} \frac{e^{2n_1a\alpha}\widetilde{H}(n_1a)}{n_1^2a^2} + \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1^2a^2} \right) \\
+ \quad & \frac{abc}{32\pi^3} \left(3\alpha \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2b} - \sum_{n_2=1}^{\infty} \frac{e^{2n_2b\alpha}\widetilde{H}(n_2b)}{n_2^2b^2} + \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2^2b^2} \right) \\
+ \quad & \frac{abc}{32\pi^3} \left(3\alpha \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3c} - \sum_{n_3=1}^{\infty} \frac{e^{2n_3c\alpha}\widetilde{H}(n_3c)}{n_3^2c^2} + \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3^2c^2} \right) \\
+ \quad & \frac{abc}{128\pi^3} \widetilde{t^2 H_{3\alpha}(t)}(0). \qquad (2.14)
\end{aligned}$$

Similarly we compute $I_{113} + I_{331}$, $I_{131} + I_{313}$ and $I_{311} + I_{133}$:

$$I_{113} + I_{331} = A(\alpha; a, b, c), \qquad I_{131} + I_{313} = A(\alpha; b, c, a), \qquad I_{311} + I_{133} = A(\alpha; c, a, b),$$

where

$$\begin{aligned}
& A(\alpha; a, b, c) \\
= \quad & -\frac{abc}{8\pi^3} \sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1a+n_2b+n_3c)\alpha} \times \\
& \left(\frac{e^{-n_1a\alpha}\widetilde{H}(n_1a)}{(n_1a+n_2b)(n_1a+n_3c)} + \frac{e^{n_2b\alpha}\widetilde{H}(n_2b)}{(n_2b+n_1a)(n_2b-n_3c)} + \frac{e^{n_3c\alpha}\widetilde{H}(n_3c)}{(n_3c+n_1a)(n_3c-n_2b)} \right) \\
+ \quad & \frac{abc}{32\pi^3} \left(-\alpha \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1a} - \sum_{n_1=1}^{\infty} \frac{e^{-2n_1a\alpha}\widetilde{H}(n_1a)}{n_1^2a^2} + \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1^2a^2} \right) \\
+ \quad & \frac{abc}{32\pi^3} \left(\alpha \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2b} - \sum_{n_2=1}^{\infty} \frac{\widetilde{H}(n_2b)}{n_2^2b^2} + \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2^2b^2} \right) \\
+ \quad & \frac{abc}{32\pi^3} \left(\alpha \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3c} - \sum_{n_3=1}^{\infty} \frac{\widetilde{H}(n_3c)}{n_3^2c^2} + \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3^2c^2} \right)
\end{aligned}$$

$$+ \quad \frac{abc}{128\pi^3}\widetilde{t^2 H_\alpha(t)}(0),$$

Considering the limit $\alpha \downarrow 0$ except for the sum through Γ, we have

$$\begin{aligned}
&\lim_{\alpha\downarrow 0}\left(\sum_{(i_1,i_2,i_3)\in\{1,3\}^3} I_{i_1 i_2 i_3} - B(\alpha;a,b,c) - B(\alpha;b,c,a) - B(\alpha;c,a,b)\right) \\
= &-\frac{abc}{8\pi^3}\left(\sum_{n_1=1}^{\infty}\frac{\widetilde{H}(n_1 a)}{n_1^2 a^2} + \sum_{n_2=1}^{\infty}\frac{\widetilde{H}(n_2 b)}{n_2^2 b^2} + \sum_{n_3=1}^{\infty}\frac{\widetilde{H}(n_3 c)}{n_3^2 c^2}\right) \\
+ &\frac{abc}{8\pi^3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\zeta(2)\widetilde{H}(0) - \frac{abc}{32\pi^3}\widetilde{t^2 H(t)}(0), \qquad (2.15)
\end{aligned}$$

where

$$\begin{aligned}
B(\alpha;a,b,c) \;=\; &\sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_2 b + n_3 c)\alpha} \times \\
&\left(\frac{(e^{n_1 a\alpha}+e^{-n_1 a\alpha})^2 n_1^2 a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)} + \frac{(e^{2n_1 a\alpha}-e^{-2n_1 a\alpha}) n_1 n_2 ab}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\right. \\
&\left.+\frac{(e^{2n_1 a\alpha}-e^{-2n_1 a\alpha}) n_1 n_3 ac}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)} + \frac{(e^{n_1 a\alpha}-e^{-n_1 a\alpha})^2 n_2 n_3 bc}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\right)\widetilde{H}(n_1 a) \qquad (2.16)
\end{aligned}$$

and $\zeta(s)$ is the Riemann zeta function. By Lemma 2.6 we have

$$\lim_{\alpha\downarrow 0}(B(\alpha;a,b,c)+B(\alpha;b,c,a)+B(\alpha;c,a,b)) = B(0;a,b,c)+B(0;b,c,a)+B(0;c,a,b)$$

Hence we have

$$\begin{aligned}
&\lim_{\alpha\downarrow 0}\sum_{(i_1,i_2,i_3)\in\{1,3\}^3} I_{i_1 i_2 i_3} \\
= &-\frac{abc}{8\pi^3}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3} \times \\
&\left(\frac{4n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1 a) + \frac{4n_2^2b^2}{(n_2^2b^2-n_1^2a^2)(n_2^2b^2-n_3^2c^2)}\widetilde{H}(n_2 b)\right. \\
&\left.+\frac{4n_3^2c^2}{(n_3^2c^2-n_1^2a^2)(n_3^2c^2-n_2^2b^2)}\widetilde{H}(n_3 c)\right) \\
- &\frac{abc}{8\pi^3}\left(\sum_{n_1=1}^{\infty}\frac{\widetilde{H}(n_1 a)}{n_1^2 a^2} + \sum_{n_2=1}^{\infty}\frac{\widetilde{H}(n_2 b)}{n_2^2 b^2} + \sum_{n_3=1}^{\infty}\frac{\widetilde{H}(n_3 c)}{n_3^2 c^2}\right)
\end{aligned}$$

$$+ \quad \frac{abc}{48\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\widetilde{H}(0)-\frac{abc}{32\pi^3}\widetilde{t^2H(t)}(0). \tag{2.17}$$

By

$$\sum_{n>0}\frac{2ku}{k^2u^2-n^2v^2}+\frac{1}{ku}=\frac{\pi}{v}\cot\left(\pi\frac{ku}{v}\right),$$

(2.17) turns to

$$\begin{aligned}
&\lim_{\alpha\downarrow 0}\sum_{(i_1,i_2,i_3)\in\{1,3\}^3} I_{i_1i_2i_3}\\
&=-\frac{a}{8\pi}\sum_{n_1>0}\cot\left(\pi\frac{n_1a}{b}\right)\cot\left(\pi\frac{n_1a}{c}\right)\widetilde{H}(n_1a)-\frac{b}{8\pi}\sum_{n_2>0}\cot\left(\pi\frac{n_2b}{c}\right)\cot\left(\pi\frac{n_2b}{a}\right)\widetilde{H}(n_2b)\\
&\quad-\frac{c}{8\pi}\sum_{n_3>0}\cot\left(\pi\frac{n_3c}{a}\right)\cot\left(\pi\frac{n_3c}{b}\right)\widetilde{H}(n_3c)+\frac{abc}{48\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\widetilde{H}(0)-\frac{abc}{32\pi^3}\widetilde{H}''(0).
\end{aligned} \tag{2.18}$$

Next we calculate $I_{i_1i_2i_3}$ with $i_1=2,\ i_2=2$ or $i_3=2$. Put

$$I^{(1)}:=\sum_{j,k=1}^{3}I_{2jk},\quad I^{(2)}:=\sum_{j,k=1}^{3}I_{j2k},\quad I^{(3)}:=\sum_{j,k=1}^{3}I_{jk2},$$
$$I^{(4)}:=\sum_{j=1}^{3}I_{22j},\quad I^{(5)}:=\sum_{j=1}^{3}I_{2j2},\quad I^{(6)}:=\sum_{j=1}^{3}I_{j22}.$$

Then we have

$$\begin{aligned}
I^{(1)}&=\frac{1}{(2\pi i)^3}\int_{C_2}\int_{\partial D}\int_{\partial D}h(s_1+s_2+s_3)\frac{Z_a'}{Z_a}(s_1)\frac{Z_b'}{Z_b}(s_2)\frac{Z_c'}{Z_c}(s_3)ds_1ds_2ds_3\\
&=\frac{1}{2\pi i}\int_{C_2}\sum_{\rho_a,\rho_b}h(\rho_a+\rho_b+s_3)\frac{Z_c'}{Z_c}(s_3)ds_3\\
&=\frac{1}{2\pi i}\int_{\pi}^{0}\sum_{\rho_a,\rho_b}h(\rho_a+\rho_b+\alpha e^{i\theta})\frac{Z_c'}{Z_c}(\alpha e^{i\theta})\alpha ie^{i\theta}d\theta,
\end{aligned}$$

where ρ_a,ρ_b run through the zeros of $Z_a(s),Z_b(s)$ with $\mathrm{Im}(\rho_a)>0,\mathrm{Im}(\rho_b)>0$, respectively. As $\alpha\downarrow 0$, we have

$$\lim_{\alpha\downarrow 0}I^{(1)}=\frac{1}{2\pi i}\int_{\pi}^{0}\sum_{\rho_a,\rho_b}h(\rho_a+\rho_b)id\theta=-\frac{1}{2}\sum_{\rho_a,\rho_b}h(\rho_a+\rho_b).$$

We similarly deal with $I^{(k)}(k=2,3,\ldots,6)$ and I_{222} to get

$$\lim_{\alpha\downarrow 0}I^{(2)}=-\frac{1}{2}\sum_{\rho_a,\rho_c}h(\rho_a+\rho_c),\qquad \lim_{\alpha\downarrow 0}I^{(3)}=-\frac{1}{2}\sum_{\rho_b,\rho_c}h(\rho_b+\rho_c),$$

$$\lim_{\alpha\downarrow 0} I^{(4)} = \frac{1}{4}\sum_{\rho_a} h(\rho_a), \qquad \lim_{\alpha\downarrow 0} I^{(5)} = \frac{1}{4}\sum_{\rho_b} h(\rho_b),$$
$$\lim_{\alpha\downarrow 0} I^{(6)} = \frac{1}{4}\sum_{\rho_c} h(\rho_c), \qquad \lim_{\alpha\downarrow 0} I_{222} = -\frac{1}{8}h(0),$$

where ρ_k runs over the zeros of $Z_k(s)$ with $\mathrm{Im}(\rho_k) > 0$. Therefore we obtain

$$\begin{aligned}
&\lim_{\alpha\downarrow 0} \sum_{(i_1,i_2,i_3)\in\{1,2,3\}^3\setminus\{1,3\}^3} I_{i_1i_2i_3} \\
&= \lim_{\alpha\downarrow 0}(I^{(1)} + I^{(2)} + I^{(3)} - (I^{(4)} + I^{(5)} + I^{(6)}) + I_{222}) \\
&= -\frac{1}{2}\left(\sum_{\rho_a,\rho_b} h(\rho_a+\rho_b) + \sum_{\rho_a,\rho_c} h(\rho_a+\rho_c) + \sum_{\rho_b,\rho_c} h(\rho_b+\rho_c)\right) \\
&\quad - \frac{1}{4}\left(\sum_{\rho_a} h(\rho_a) + \sum_{\rho_b} h(\rho_b) + \sum_{\rho_c} h(\rho_c)\right) - \frac{1}{8}h(0). \qquad (2.19)
\end{aligned}$$

We apply (2.18) and (2.19) to the limit $\alpha \downarrow 0$ in (2.7). This completes the proof. □

3 Expression of Triple Sine functions

In this section we prove Theorem 2 from Theorem 3.

Lemma 3.1.
$$\frac{d^3}{dz^3}\log(1-e^{iaz}) = 2\sum_{n=-\infty}^{\infty} \frac{1}{\left(z-\frac{2\pi n}{a}\right)^3}.$$

Proof. By
$$\log(1-e^{iaz}) = -\frac{\pi i}{2} + \frac{az}{2}i + \log\left(2\sin\frac{az}{2}\right)$$

and
$$2\sin\left(\frac{az}{2}\right) = az\prod_{n=1}^{\infty}\left(1-\left(\frac{az}{2\pi n}\right)^2\right)$$

we have
$$\begin{aligned}
\frac{d^3}{dz^3}\log(1-e^{iaz}) &= \frac{2}{z^3} + 2\sum_{n=1}^{\infty}\left(\frac{1}{(z-\frac{2\pi n}{a})^3} + \frac{1}{(z+\frac{2\pi n}{a})^3}\right) \\
&= 2\sum_{n=-\infty}^{\infty}\frac{1}{(z-\frac{2\pi n}{a})^3}.
\end{aligned}$$

□

Lemma 3.2. *([5, Theorem 3]) Let $I(t)$ be an odd function in $L^1(\mathbb{R})$ with satisfying following (i), (ii):*
(i)$I(t) = O(t^{-2})$ $(|t| \to \infty)$
(ii)There exists $\mu \in (0,1)$ such that $\widetilde{I}(x) = O(\mu^x)$ $(x \to \infty)$.
Let a, b be positive real numbers such that a/b and b/a are generic. Then we have

$$\begin{aligned}
&\sum_{k,n\geq 0} \varepsilon_{k,n} I\left(2\pi\left(\frac{k}{a}+\frac{n}{b}\right)\right) \\
&= -\frac{ia}{4\pi}\sum_{k>0}\cot\left(\pi\frac{ka}{b}\right)\widetilde{I}(ka) - \frac{ib}{4\pi}\sum_{n>0}\cot\left(\pi\frac{nb}{a}\right)\widetilde{I}(nb) - \frac{iab}{8\pi^2}\widetilde{I}'(0). \qquad (3.1)
\end{aligned}$$

Lemma 3.3. *([5, Theorem 2]) Assume ω_1/ω_2 and ω_2/ω_1 are generic, then the double sine function has the following expression in $\mathrm{Im}(z) > 0$:*

$$\begin{aligned}
S_2(z,(\omega_1,\omega_2)) = \exp\Bigg(&\frac{1}{2i}\sum_{k=1}^{\infty}\frac{1}{k}\cot\left(\pi k\frac{\omega_2}{\omega_1}\right)e^{2\pi i k\frac{z}{\omega_1}} \\
&+ \frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{n}\cot\left(\pi n\frac{\omega_1}{\omega_2}\right)e^{2\pi i n\frac{z}{\omega_2}} \\
&+ \frac{1}{2}\log(1-e^{2\pi i\frac{z}{\omega_1}}) + \frac{1}{2}\log(1-e^{2\pi i\frac{z}{\omega_2}}) \\
&+ \frac{\pi i z^2}{2\omega_1\omega_2} - \frac{\pi i}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)z + \frac{\pi i}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}+3\right)\Bigg).
\end{aligned}$$

Lemma 3.4. *([6, Proposition 2.4]) For $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in (\mathbb{R}_{>0})^r$ we have an expression:*

$$\begin{aligned}
S_r(z,\underline{\omega}) \;=\; & e^{Q_{\underline{\omega}}(z)} z(z-\omega_1-\cdots-\omega_r)^{(-1)^{r-1}} \times \\
& \prod_{n_1,\ldots,n_r\geq 0}{}' \left\{ P_r\left(-\frac{z}{n_1\omega_1+\cdots+n_r\omega_r}\right) P_r\left(\frac{z}{(n_1+1)\omega_1+\cdots+(n_r+1)\omega_r}\right)^{(-1)^{r-1}} \right\},
\end{aligned}$$

where $Q_{\underline{\omega}}(z)$ is a polynomial with $\deg Q_{\underline{\omega}}(z) \leq r$, the product runs through all $(n_1, \ldots, n_r) \in (\mathbb{Z}_{\geq 0})^r \setminus \{(0,\ldots,0)\}$ and $P_r(u) := (1-u)\exp(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r})$.

Lemma 3.5. *([6, Theorem 2.1 (a), (b)])*
(1) For $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in (\mathbb{R}_{>0})^r$ we put $\underline{\omega}(j) = (\omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_r)$. Then we have

$$S_r(z+\omega_j,\underline{\omega}) = S_r(z,\underline{\omega})S_{r-1}(z,\underline{\omega}(j))^{-1}.$$

(2) For a positive integer N and $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in (\mathbb{R}_{>0})^r$ we have

$$S_r(Nz, \underline{\omega}) = \prod_{0 \le k_1, \ldots, k_r \le N-1} S_r\left(z + \frac{k_1\omega_1 + \cdots + k_r\omega_r}{N}, \underline{\omega}\right).$$

Proof of Theorem 2. By Lemma 3.4 we have

$$\begin{aligned} S_3(z, \underline{\omega}) &= e^{c_0 + c_1 z + c_2 z^2 + c_3 z^3} z \prod_{n_1, n_2, n_3 \ge 0}{}' P_3\left(-\frac{z}{n_1\omega_1 + n_2\omega_2 + n_3\omega_3}\right) \\ &\quad \times \prod_{n_1, n_2, n_3 \ge 1} P_3\left(\frac{z}{n_1\omega_1 + n_2\omega_2 + n_3\omega_3}\right). \end{aligned} \tag{3.2}$$

Hence we have

$$\begin{aligned} &\frac{d^3}{dz^3}(\log S_3(z, \underline{\omega})) \\ &= C_{\underline{\omega}} + \frac{2}{z^3} \\ &+2 \sum_{n_1, n_2, n_3 \ge 1} \left(\frac{1}{(z + (n_1\omega_1 + n_2\omega_2 + n_3\omega_3))^3} + \frac{1}{(z - (n_1\omega_1 + n_2\omega_2 + n_3\omega_3))^3}\right) \\ &+2 \left(\sum_{n_1, n_2 \ge 1} \frac{1}{(z + (n_1\omega_1 + n_2\omega_2))^3} + \sum_{n_2, n_3 \ge 1} \frac{1}{(z + (n_2\omega_2 + n_3\omega_3))^3}\right. \\ &\quad \left. + \sum_{n_1, n_3 \ge 1} \frac{1}{(z + (n_1\omega_1 + n_3\omega_3))^3}\right) \\ &+2 \left(\sum_{n_1 \ge 1} \frac{1}{(z + n_1\omega_1)^3} + \sum_{n_2 \ge 1} \frac{1}{(z + n_2\omega_2)^2} + \sum_{n_3 \ge 1} \frac{1}{(z + n_3\omega_3)^3}\right) \\ &= C_{\underline{\omega}} + 2F_3(z, \underline{\omega}) + F_2(z, (\omega_1, \omega_2)) + F_2(z, (\omega_2, \omega_3)) + F_2(z, (\omega_3, \omega_1)) \\ &+ \frac{1}{2} F_1(z, \omega_1) + \frac{1}{2} F_1(z, \omega_2) + \frac{1}{2} F_1(z, \omega_3), \end{aligned} \tag{3.3}$$

where $C_{\underline{\omega}}$ is a constant depending on $\underline{\omega}$ and

$$F_r(z, (\omega_1, \ldots, \omega_r)) := \sum_{n_1, \ldots, n_r \ge 0} \varepsilon_{n_1, \ldots, n_r} \left(\frac{1}{(z + n_1\omega_1 + \cdots + n_r\omega_r)^3} + \frac{(-1)^{r-1}}{(z - n_1\omega_1 - \cdots - n_r\omega_r)^3}\right)$$

for $r = 1, 2, 3$.

First we transform $F_3(z, \underline{\omega})$ by using Theorem 3. We put

$$H(t) = \frac{1}{(z+t)^3} + \frac{1}{(z-t)^3} \quad (\mathrm{Im}(z) > 0).$$

As we have

$$\widetilde{H}(x) = 2\pi i \operatorname{Res}_{t=z}(H(t)e^{itx}) = \pi i x^2 e^{ixz} \quad (x \geq 0)$$

and $H(t) = O(t^{-4})$ as $|t| \to \infty$, $H(t)$ satisfies the assumptions of Theorem 3. Putting $a = \frac{2\pi}{\omega_1},\ b = \frac{2\pi}{\omega_2},\ c = \frac{2\pi}{\omega_3}$, by Theorem 3 we obtain

$$\begin{aligned}
F_3(z, \underline{\omega}) = & -\frac{a}{8}\sum_{n_1=1}^{\infty} \cot\left(\pi\frac{n_1 a}{b}\right)\cot\left(\pi\frac{n_1 a}{b}\right) i(n_1 a)^2 e^{in_1 az} \\
& -\frac{b}{8}\sum_{n_2=1}^{\infty} \cot\left(\pi\frac{n_2 b}{a}\right)\cot\left(\pi\frac{n_2 b}{c}\right) i(n_2 b)^2 e^{in_2 bz} \\
& -\frac{c}{8}\sum_{n_3=1}^{\infty} \cot\left(\pi\frac{n_3 c}{a}\right)\cot\left(\pi\frac{n_3 c}{b}\right) i(n_3 c)^2 e^{in_3 cz} - \frac{abc}{16\pi^2} i \\
= & \frac{d^3}{dz^3}\left(\frac{1}{8}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right) e^{2\pi i n_1\frac{z}{\omega_1}}\right. \\
& +\frac{1}{8}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right) e^{2\pi i n_2\frac{z}{\omega_2}} \\
& \left.+\frac{1}{8}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right) e^{2\pi i n_3\frac{z}{\omega_3}}\right) - \frac{\pi i}{2\omega_1\omega_2\omega_3}. \quad (3.4)
\end{aligned}$$

Next we transform $F_2(z, (\omega_{j_1}, \omega_{j_2}))$ for $1 \leq j_1, j_2 \leq 3,\ j_1 \neq j_2$ by using Lemma 3.2. We put

$$I(t) = \frac{1}{(z+t)^3} - \frac{1}{(z-t)^3}.$$

Then $I(t)$ is an odd, absolutely integrable function on $\mathbb{R}$ and satisties $I(t) = O(t^{-3})\quad (|t| \to \infty)$. As we have

$$\widetilde{I}(x) = 2\pi i \operatorname{Res}_{t=z}(I(t)e^{itx}) = -\pi i x^2 e^{ixz} \quad (x \geq 0).$$

Hence $I(t)$ satisfies the assumptions of Lemma 3.2. Putting $a = \frac{2\pi}{\omega_{j_1}},\ b = \frac{2\pi}{\omega_{j_2}}$, from Lemma 3.2 we get

$$\begin{aligned}
& F_2(z, (\omega_{j_1}, \omega_{j_2})) \\
= & -\frac{a}{4}\sum_{k=1}^{\infty}\cot\left(\pi\frac{ka}{b}\right)(ka)^2 e^{ikaz} - \frac{b}{4}\sum_{n=1}^{\infty}\cot\left(\pi\frac{nb}{a}\right)(nb)^2 e^{inbz} \\
= & \frac{d^3}{dz^3}\left(\frac{1}{4i}\sum_{k=1}^{\infty}\frac{1}{k}\cot\left(\pi\frac{k\omega_{j_2}}{\omega_{j_1}}\right) e^{2\pi i k\frac{z}{\omega_{j_1}}} + \frac{1}{4i}\sum_{n=1}^{\infty}\frac{1}{n}\cot\left(\pi\frac{n\omega_{j_1}}{\omega_{j_2}}\right) e^{2\pi i n\frac{z}{\omega_{j_2}}}\right). \quad (3.5)
\end{aligned}$$

It follows from Lemma 3.1 that

$$F_1(z,\omega_j) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n\omega_j)^3} = \frac{d^3}{dz^3}\left(\frac{1}{2}\log(1-e^{2\pi i \frac{z}{\omega_j}})\right). \tag{3.6}$$

By (3.3)-(3.6),

$$\begin{aligned}
Q(z) := \log S_3(z,\underline{\omega}) &- \frac{1}{4}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)e^{2\pi i n_1\frac{z}{\omega_1}} \\
&- \frac{1}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}} \\
&- \frac{1}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}} \\
&- \frac{1}{4i}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\left(\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)+\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)\right)e^{2\pi i n_1\frac{z}{\omega_1}} \\
&- \frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\left(\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)+\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)\right)e^{2\pi i n_2\frac{z}{\omega_2}} \\
&- \frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\left(\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)+\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)\right)e^{2\pi i n_3\frac{z}{\omega_3}} \\
&- \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_1}}) - \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}}) - \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}})
\end{aligned} \tag{3.7}$$

is a polynomial of degree at most three. Thus we put $Q(z) = \alpha + \beta z + \gamma z^2 + \delta z^3$ and will compute α, β, γ and δ. First we compute β, γ and δ by considering

$$Q(z+\omega_1) - Q(z) = (\beta\omega_1 + \gamma\omega_1^2 + \delta\omega_1^3) + (2\gamma\omega_1 + 3\delta\omega_1^2)z + 3\delta\omega_1 z^2. \tag{3.8}$$

By (3.7) we have

$$\begin{aligned}
&Q(z+\omega_1) - Q(z) \\
&= \log\frac{S_3(z+\omega_1,\underline{\omega})}{S_3(z,\underline{\omega})} - \frac{1}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}(e^{2\pi i n_2\frac{\omega_1}{\omega_2}}-1) \\
&-\frac{1}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}(e^{2\pi i n_3\frac{\omega_1}{\omega_3}}-1) \\
&-\frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\left(\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)+\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)\right)e^{2\pi i n_2\frac{z}{\omega_2}}(e^{2\pi i n_2\frac{\omega_1}{\omega_2}}-1)
\end{aligned}$$

$$-\frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\left(\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)+\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)\right)e^{2\pi i n_3\frac{z}{\omega_3}}(e^{2\pi i n_3\frac{\omega_1}{\omega_3}}-1)$$
$$-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_2}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_3}})$$
$$+\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}})+\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}}). \tag{3.9}$$

Using the formula

$$\cot(\pi x)=i\frac{e^{2\pi i x}+1}{e^{2\pi i x}-1}, \tag{3.10}$$

(3.9) turns to

$$Q(z+\omega_1)-Q(z)$$
$$=\log\frac{S_3(z+\omega_1,\underline{\omega})}{S_3(z,\underline{\omega})}$$
$$-\frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}(e^{2\pi i n_2\frac{\omega_1}{\omega_2}}-1)+\frac{1}{2i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}$$
$$-\frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}(e^{2\pi i n_3\frac{\omega_1}{\omega_3}}-1)+\frac{1}{2i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}$$
$$-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_2}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_3}})+\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}})+\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}}).$$

By Lemma 3.5 (1) we get

$$S_3(z+\omega_1,\underline{\omega})S_3(z,\underline{\omega})^{-1}=S_2(z,(\omega_2,\omega_3))^{-1}. \tag{3.11}$$

By Lemma 3.3 and (3.11), we have

$$Q(z+\omega_1)-Q(z)$$
$$=-\frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}(e^{2\pi i n_2\frac{\omega_1}{\omega_2}}-1)$$
$$-\frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}(e^{2\pi i n_3\frac{\omega_1}{\omega_3}}-1)$$
$$-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_2}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_3}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}})$$
$$-\frac{\pi i z^2}{2\omega_2\omega_3}+\frac{\pi i}{2}\left(\frac{1}{\omega_2}+\frac{1}{\omega_3}\right)z-\frac{\pi i}{12}\left(\frac{\omega_3}{\omega_2}+\frac{\omega_2}{\omega_3}+3\right). \tag{3.12}$$

By (3.10) we have

$$Q(z+\omega_1)-Q(z)=-\frac{\pi i z^2}{2\omega_2\omega_3}+\frac{\pi i}{2}\left(\frac{1}{\omega_2}+\frac{1}{\omega_3}\right)z-\frac{\pi i}{12}\left(\frac{\omega_3}{\omega_2}+\frac{\omega_2}{\omega_3}+3\right). \tag{3.13}$$

Comparing the coefficients of (3.13), we have

$$
\begin{aligned}
\beta &= -\frac{\pi i}{12}\left(\frac{3}{\omega_1}+\frac{3}{\omega_2}+\frac{3}{\omega_3}+\frac{\omega_1}{\omega_2\omega_3}+\frac{\omega_2}{\omega_3\omega_1}+\frac{\omega_3}{\omega_1\omega_2}\right),\\
\gamma &= \frac{\pi i}{4}\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_2\omega_3}+\frac{1}{\omega_3\omega_1}\right),\\
\delta &= -\frac{\pi i}{6\omega_1\omega_2\omega_3}.
\end{aligned}
$$

Next we will treat α by considering

$$\sum_{k_1,k_2,k_3=0}^{1} Q(z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})-Q(2z) \tag{3.14}$$

The constant term of (3.14) is

$$7\alpha-\frac{7\pi i}{24\omega_1\omega_2\omega_3}(\omega_1^2\omega_2+\omega_1\omega_2^2+\omega_2^2\omega_3+\omega_2\omega_3^2+\omega_3^2\omega_1+\omega_3\omega_1^2+3\omega_1\omega_2\omega_3). \tag{3.15}$$

On the other hand we will compute (3.14) by using (3.7). Putting

$$
\begin{aligned}
A_0 &:= \log\frac{\prod_{k_1,k_2,k_3=0}^{1} S_3\left(z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2},\underline{\omega}\right)}{S_3(2z,\underline{\omega})},\\
A_1 &:= -\frac{1}{4}\sum_{n_1=1}^{\infty}\left\{\frac{1}{n_1}\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)\right.\\
&\qquad\left.\times\left(\sum_{k_1,k_2,k_3=0}^{1} e^{\frac{2\pi i n_1(z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_1}}-e^{\frac{4\pi i n_1 z}{\omega_1}}\right)\right\}\\
&\quad-\frac{1}{4i}\sum_{n_1=1}^{\infty}\left\{\frac{1}{n_1}\left(\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)+\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)\right)\right.\\
&\qquad\left.\times\left(\sum_{k_1,k_2,k_3=0}^{1} e^{\frac{2\pi i n_1(z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_1}}-e^{\frac{4\pi i n_1 z}{\omega_1}}\right)\right\}\\
&\quad(=: G(z;\omega_1,\omega_2,\omega_3)),\\
A_2 &:= G(z;\omega_2,\omega_3,\omega_1)\\
A_3 &:= G(z;\omega_3,\omega_1,\omega_2)\\
A_{3+j} &:= -\frac{1}{4}\log\frac{\prod_{k_1,k_2,k_3=0}^{1}\left(1-e^{\frac{2\pi i(z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_j}}\right)}{1-e^{\frac{4\pi i z}{\omega_j}}} \qquad (j=1,2,3),
\end{aligned}
$$

we write (3.14) as $\sum_{j=0}^{6} A_j$. Lemma 3.5 (2) gives $A_0 = 0$. Computing A_1 by (3.10), we have

$$\begin{aligned} A_1 &= -\frac{1}{4}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\left(e^{\frac{4\pi i n_1(z+\frac{\omega_2+\omega_3}{2})}{\omega_1}} + e^{\frac{4\pi i n_1(z+\frac{\omega_2}{2})}{\omega_1}} + e^{\frac{4\pi i n_1(z+\frac{\omega_3}{2})}{\omega_1}}\right) \\ &= \frac{1}{4}\log\{(1-e^{\frac{4\pi i(z+\frac{\omega_2+\omega_3}{2})}{\omega_1}})(1-e^{\frac{4\pi i(z+\frac{\omega_2}{2})}{\omega_1}})(1-e^{\frac{4\pi i(z+\frac{\omega_3}{2})}{\omega_1}})\}. \end{aligned}$$

A_4 is easily computed as

$$A_4 = -\frac{1}{4}\log\{(1-e^{\frac{4\pi i(z+\frac{\omega_2+\omega_3}{2})}{\omega_1}})(1-e^{\frac{4\pi i(z+\frac{\omega_2}{2})}{\omega_1}})(1-e^{\frac{4\pi i(z+\frac{\omega_3}{2})}{\omega_1}})\}.$$

Therefore $A_1 + A_4 = 0$. Similarly computing we have $A_2 + A_5 = A_3 + A_6 = 0$. Hence $\sum_{j=0}^{6} A_j = 0$. Therefore its constant term (3.15) vanishes, which leads to

$$\alpha = \frac{\pi i}{24}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} + \frac{\omega_3}{\omega_1} + \frac{\omega_1}{\omega_3} + 3\right).$$

□

4 $\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)$ for distinct primes p,q,r

In this section we show Theorem 1 from Theorem 2.

Let $Z_j (j = 1,\ldots,r)$ be meromorphic functions of order μ_j. We put the Hadamard product as

$$Z_j(s) = s^{k_j} e^{Q_j(s)} \prod_{\rho\in\mathbb{C}}{}' P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)},$$

where $P_r(u) := (1-u)\exp(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r})$, m_j denotes the multiplicity function for Z_j, $k_j := m_j(0)$ and Q_j is a polynomial with $\deg Q_j(s) \le \mu_j$. The product $\prod_{\rho\in\mathbb{C}}{}'$ means $\lim_{R\to\infty}\prod_{0<|\rho|<R}$. Then, we have

$$Z_1(s)\otimes\cdots\otimes Z_r(s) = s^{k_1\cdots k_r} e^{Q(s)} \prod_{\rho_1,\ldots,\rho_r\in\mathbb{C}}{}' P_{\mu_1+\cdots+\mu_r}\left(\frac{s}{\rho_1+\cdots+\rho_r}\right)^{m(\rho_1,\ldots,\rho_r)},$$

where $Q(s)$ is a polynomial with $\deg Q(s) \le \mu_1 + \cdots + \mu_r$.

Lemma 4.1. *The absolute tensor product of Hasse zeta function for finite fields is given as follows:*

$$\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)\cong S_3\left(is,\left(\frac{2\pi}{\log p},\frac{2\pi}{\log q},\frac{2\pi}{\log r}\right)\right)^{-1}.$$

Proof. We compute that the Hadamard product for Hasse zeta function is given by

$$\zeta(s,\mathbb{F}_p)\cong s^{-1}\prod_{n=-\infty}^{\infty}{}' P_1\left(\frac{s}{\frac{2\pi i}{\log p}n}\right)^{-1}.$$

Thus by the definition of the absolute tensor product,

$$\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)\cong s^{-1}\prod_{n_1,n_2,n_3\in\mathbb{Z}}{}' P_3\left(\frac{s}{\frac{2\pi i}{\log p}n_1+\frac{2\pi i}{\log q}n_2+\frac{2\pi i}{\log r}n_3}\right)^{m_{n_1,n_2,n_3}},$$

where

$$\begin{aligned} m_{n_1,n_2,n_3} &= m\left(\frac{2\pi i}{\log p}n_1,\frac{2\pi i}{\log q}n_2,\frac{2\pi i}{\log r}n_3\right)\\ &= \begin{cases} -1 & \text{if } n_1,n_2,n_3\geq 0 \text{ or } n_1,n_2,n_3<0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence

$$\begin{aligned} &\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)\\ \cong\ & s^{-1}\prod_{n_1,n_2,n_3\geq 0}{}' P_3\left(\frac{s}{\frac{2\pi i}{\log p}n_1+\frac{2\pi i}{\log q}n_2+\frac{2\pi i}{\log r}n_3}\right)^{-1}\prod_{n_1,n_2,n_3\geq 1} P_3\left(-\frac{s}{\frac{2\pi i}{\log p}n_1+\frac{2\pi i}{\log q}n_2+\frac{2\pi i}{\log r}n_3}\right)^{-1}.\end{aligned} \tag{4.1}$$

By (3.2) the result follows. □

Proof of Theorem 1. Applying Theorem 2 with $z=is$ and $(\omega_1,\omega_2,\omega_3)=(\frac{2\pi}{\log p},\frac{2\pi}{\log q},\frac{2\pi}{\log r})$, Theorem 1 follows. □

5 $\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)$

In this section we deal with $\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)$.

From [6] we recall the primitive r-ple sine function $\mathscr{S}_r(z)$ and the relation between $S_r(z)$ and $\mathscr{S}_r(z)$. We define $\mathscr{S}_r(z)$ as

$$\mathscr{S}_r(z) := \begin{cases} \exp\left(\dfrac{z^{r-1}}{r-1}\right) \displaystyle\prod_{n=-\infty}^{\infty}{}' P_r\left(\frac{z}{n}\right)^{n^{r-1}} & \text{if } r = 2, 3, \ldots, \\ 2\sin(\pi z) & \text{if } r = 1. \end{cases}$$

Lemma 5.1. *([6, Theorem 2.8]) Let $r = 2, 3, \ldots$. Then it holds that*

$$\mathscr{S}_r(z) = \exp\left(-\frac{(r-1)!}{(-2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(-2\pi i z)^k}{k!} \operatorname{Li}_{r-k}(e^{2\pi i z}) - \frac{\pi i}{r} z^r + \frac{(r-1)!}{(-2\pi i)^{r-1}} \zeta(r) \right) \tag{5.1}$$

for $\operatorname{Im}(z) > 0$.

Lemma 5.2. *([6, Example 3.6]) The following formula holds:*

$$S_3(z) = e^{-\zeta'(2)} \mathscr{S}_3(z)^{\frac{1}{2}} \mathscr{S}_2(z)^{-\frac{3}{2}} \mathscr{S}_1(z). \tag{5.2}$$

Lemma 5.3. *([6, Lemma 3.2]) Let $a(r, k) \in \mathbb{Q}$ satisfy*

$$\frac{(X + r - 2)(X + r - 3) \cdots X}{(r-1)!} = \sum_{k=1}^{r-1} a(r, k) X^k.$$

Then we have

$$S_r(1) = \exp\left(-2 \sum_{\substack{2 \le k \le r-1 \\ k:\text{even}}} a(r, k) \zeta'(-k) \right).$$

Proof of Theorem 5. We have

$$\mathscr{S}_1(z) = \frac{e^{\pi i z} - e^{-\pi i z}}{i} = \exp\left(-\operatorname{Li}_1(e^{2\pi i z}) - \pi i z + \frac{\pi i}{2} \right). \tag{5.3}$$

Applying (5.1) and (5.3) to (5.2) we get

$$\begin{aligned} S_3(z)^2 = \exp\Bigg(&\frac{1}{2\pi^2} \operatorname{Li}_3(e^{2\pi i z}) + \frac{i}{2\pi}(-2z + 3) \operatorname{Li}_2(e^{2\pi i z}) - (z-1)(z-2) \operatorname{Li}_1(e^{2\pi i z}) \\ &- \frac{\pi i}{3} z^3 + \frac{3\pi i}{2} z^2 - 2\pi i z + 2\left(-\frac{1}{4\pi^2}\zeta(3) + \frac{3\pi i}{8} - \zeta'(2) \right) \Bigg). \end{aligned}$$

By the fuctional equation for $\zeta(s)$ we get

$$\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}. \tag{5.4}$$

Hence we have $S_3(z) = R(z)$ or $-R(z)$, where $R(z)$ is the right hand side of (1.1). Next we determine the sign. We consider the limit $t \downarrow 0$ when $z = 1 + it$. By Lemma 5.3 and (5.4) we have

$$\lim_{t\downarrow 0} S_3(1+it) = S_3(1) = \exp(-\zeta'(2)) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right).$$

On the other hand we easily calculate

$$\lim_{t\downarrow 0} R(1+it) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right).$$

Hence we get $S_3(z) = R(z)$. □

Proof of Theorem 4. By (3.2) and (4.1) we have

$$\zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_p) \cong S_3\left(-\frac{s \log p}{2\pi i}\right)^{-1}. \tag{5.5}$$

Applying Theorem 5 to (5.5), we reach the desired result. □

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