Conditional estimates for error terms related to the distribution of zeros of $\zeta'(s)$

Hirotaka Akatsuka

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Abstract

Berndt and Levinson-Montgomery investigated the distribution of nonreal zeros of derivatives of the Riemann zeta function, including the number of zeros up to a height T and the distribution of the real part of nonreal zeros. In this paper we obtain sharper estimates for the error terms of their results in the case of the first derivative of the Riemann zeta function, under the truth of the Riemann hypothesis.

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1 Introduction

Zeros of the first derivative $\zeta'(s)$ of the Riemann zeta function $\zeta(s)$ have been investigated for a long time. For example, Speiser [Spe] showed that the Riemann hypothesis (RH) is equivalent to $\zeta'(s)$ having no nonreal zeros in $\mathrm{Re}(s) < 1/2$. In 1970s the distribution of zeros of $\zeta'(s)$ was investigated statistically by Berndt [B] and Levinson-Montgomery [LM]. Here we recall a part of their results. Let $N_1(T)$ be the number of zeros of $\zeta'(s)$ with $0 < \mathrm{Im}(s) \le T$, counted with multiplicity. Berndt [B, Theorem] proved

$$N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi} - \frac{T}{2\pi} + O(\log T). \tag{1.1}$$

Later, Levinson and Montgomery [LM, Theorem 10] showed

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ 0<\gamma'\le T}} \left(\beta'-\frac{1}{2}\right) = \frac{T}{2\pi}\log\log\frac{T}{2\pi}+\frac{1}{2\pi}\left(\frac{1}{2}\log 2-\log\log 2\right)T -\mathrm{li}\left(\frac{T}{2\pi}\right)+O\left(\log T\right), \tag{1.2}$$

where $\rho'=\beta'+i\gamma'$ runs over the zeros of $\zeta'(s)$ with $0<\gamma'\le T$, counted with multiplicity, and $\mathrm{li}(x):=\int_2^x \frac{dt}{\log t}$. We remark that (1.1) and (1.2) hold without any hypothesis. We also note that Berndt and Levinson-Montgomery treated higher derivatives of the Riemann zeta function as well as $\zeta'(s)$. After [B, LM], zeros of $\zeta'(s)$ near the critical line $\mathrm{Re}(s)=1/2$ were studied by many specialists, for example in [CG, So].

The aim of this paper is to improve the error terms of (1.1) and (1.2) under RH. Assuming RH, we improve the error term for (1.2) as follows:

Theorem 1. *Assume RH. Then we have*

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ 0<\gamma'\le T}} \left(\beta'-\frac{1}{2}\right) = \frac{T}{2\pi}\log\log\frac{T}{2\pi}+\frac{1}{2\pi}\left(\frac{1}{2}\log 2-\log\log 2\right)T -\mathrm{li}\left(\frac{T}{2\pi}\right)+O\left((\log\log T)^2\right).$$

This immediately gives

Corollary 2. (cf. [LM, Theorem 3]) *Assume RH. Then for $0<U<T$ we have*

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ T<\gamma'\le T+U}} \left(\beta'-\frac{1}{2}\right) = \frac{U}{2\pi}\log\log\frac{T}{2\pi}+\frac{1}{2\pi}\left(\frac{1}{2}\log 2-\log\log 2\right)U +O\left(\frac{U^2}{T\log T}\right)+O\left((\log\log T)^2\right).$$

It may be interesting to compare Theorem 1 with the following two formulas expressing the distribution of zeros of $\zeta(s)$. The first formula is

$$\sum_{\substack{\rho=\beta+i\gamma,\\ 0<\gamma\le T}} \left(\beta-\frac{1}{2}\right)=0, \tag{1.3}$$

where $\rho = \beta + i\gamma$ runs over zeros of $\zeta(s)$ in $0 < \gamma \leq T$. This is an immediate consequence of the functional equation for $\zeta(s)$ and $\overline{\zeta(\overline{s})} = \zeta(s)$. The second formula is on the number $N(T)$ of zeros of $\zeta(s)$ with $0 < \mathrm{Im}(s) \leq T$. That is, we know

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + S(T) + O(1), \tag{1.4}$$

where $S(T) = \pi^{-1} \arg \zeta(\frac{1}{2} + iT)$ with a standard branch (see [T2, §9.3]). The following bounds are well-known (see [T2, Theorems 9.4 and 14.13]):

$$S(T) = \begin{cases} O(\log T) & \text{unconditional,} \\ O\left(\frac{\log T}{\log\log T}\right) & \text{under RH.} \end{cases} \tag{1.5}$$

It is not expected that for $\sum_{0<\gamma'\leq T}(\beta' - \frac{1}{2})$ there exists a formula without error terms such as (1.3). On the other hand, the error term $O((\log\log T)^2)$ for $\sum_{0<\gamma'\leq T}(\beta' - \frac{1}{2})$ is much smaller than (1.5). Furthermore, we keep in mind that

$$S(T) = \begin{cases} \Omega_{\pm}\left(\frac{(\log T)^{1/3}}{(\log\log T)^{7/3}}\right) & \text{unconditional [S, Theorem 9],} \\ \Omega_{\pm}\left(\frac{(\log T)^{1/2}}{(\log\log T)^{1/2}}\right) & \text{under RH [M, Theorem 2].} \end{cases}$$

In particular, $S(T)$ cannot be estimated above by $O((\log\log T)^2)$.

We also give a modest improvement of (1.1) under RH as follows:

Theorem 3. *Assume RH. Then we have*

$$N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{(\log\log T)^{1/2}}\right).$$

It is desirable to replace $O(\log T/(\log\log T)^{1/2})$ by $O(\log T/\log\log T)$ similarly to the conditional estimate (1.5) of $S(T)$. However, we do not reach $O(\log T/\log\log T)$ at present and Theorem 3 is the best conditional estimate that we know.

We outline the proofs of our results. To prove Theorem 1, we treat

$$\sum_{0<\gamma'\leq T} (\beta' - b)$$

uniformly for $0 \leq b < 1/2$, using zero-free regions of $\zeta'(s)$. Note that Levinson and Montgomery [LM, §3] deal with it for b away from $1/2$ in the proof of (1.2). As a result of the uniform estimate we obtain (2.15). After taking the limit $b \uparrow 1/2$, we see that the error term for $\sum_{0<\gamma'\leq T}(\beta' - \frac{1}{2})$ is nearly given by

$$\frac{1}{2\pi} \int_{1/2}^{\infty} \arg\left(-\frac{2^{\sigma+iT}}{\log 2} \frac{\zeta'}{\zeta}(\sigma + iT)\right) d\sigma$$

(see Proposition 2.2). Next we give bounds for the integrand by two ways. When σ is away from 1/2, we estimate the integrand, using a bound (2.19) for $(\zeta'/\zeta)(s)$ (see Lemma 2.3). On the other hand, when σ is near 1/2, we divide the integrand into $\arg\zeta(s)$ and $\arg\zeta'(s)$. We know a well-known bound (2.23) for $\arg\zeta(s)$. We estimate $\arg\zeta'(s)$, using a bound for $\zeta'(s)$ (see Lemmas 2.4 and 2.6). Combining these, we reach Theorem 1. To show Theorem 3, roughly speaking, we differentiate (2.1), which follows from Littlewood's lemma, with respect to b at $b = 1/2$. Then we see that the error term for $N_1(T)$ is given in terms of $\arg\zeta(s)$ and $\arg\zeta'(s)$ on $\mathrm{Re}(s) = 1/2$ (see Proposition 3.1). Standard bounds for $\arg\zeta(s)$ and $\arg\zeta'(s)$ (see (2.23) and Lemma 2.4) give Theorem 3.

Throughout this paper we assume RH and use the following notation. We denote a complex variable by $s = \sigma + it$. $\rho = \frac{1}{2} + i\gamma$ denotes the nontrivial zeros of $\zeta(s)$ and $\rho' = \beta' + i\gamma'$ denotes the zeros of $\zeta'(s)$, counted according to multiplicity.

2 Proof of Theorem 1

In this section we prove Theorem 1. First of all, we prepare a lemma, which is essentially a collection of well-known facts related to zero-free regions for $\zeta'(s)$. Put

$$F(s) := 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s), \qquad G(s) := -\frac{2^s}{\log 2}\zeta'(s).$$

Then we have

Lemma 2.1. *Assume RH. Then there exist $\sigma_0 \le -1$, $t_0 \ge 10$ and $a \ge 10$ such that they satisfy the following conditions:*

1. $|G(s) - 1| \le \frac{1}{2}(\frac{2}{3})^{\sigma/2}$ *for any* $\sigma \ge a$.
2. $|\frac{1}{\frac{F'}{F}(s)} \frac{\zeta'}{\zeta}(1-s)| \le 2^\sigma$ *for any* $\sigma \le \sigma_0$ *and* $t \ge 2$.
3. $|(F'/F)(s)| \ge 1$ *and* $(5\pi)/6 \le \arg(F'/F)(s) \le (7\pi)/6 \mod 2\pi\mathbb{Z}$ *hold for any* $s = \sigma + it$ *with* $\sigma_0 \le \sigma \le 1/2$ *and* $t \ge t_0 - 1$, *where* $\alpha \le x \le \beta \mod 2\pi\mathbb{Z}$ *means* $x \in \bigcup_{n\in\mathbb{Z}}[\alpha + 2\pi n, \beta + 2\pi n]$.
4. $\mathrm{Re}(\zeta'/\zeta)(s) < 0$ *for* $\sigma < 1/2$ *and* $t \ge t_0 - 1$.
5. $\zeta'(\sigma + it_0) \ne 0$ *for any* $\sigma \in \mathbb{R}$.
6. $\zeta(\sigma + it_0) \ne 0$ *for any* $\sigma \in \mathbb{R}$.
7. $t_0 \ge -\sigma_0$.

Proof. First of all, we look for a constant a satisfying the first condition. Since $\zeta'(s) = -\sum_{n=1}^{\infty} n^{-s}\log n$ for $\mathrm{Re}(s) > 1$, we have $G(s) = 1 + O((2/3)^\sigma)$ as $\sigma \to \infty$ uniformly on $t \in \mathbb{R}$. Hence there exists $a \ge 10$ such that the first condition holds.

Next we seek σ_0 satisfying the second condition. Let $s = \sigma + it$ with $\sigma \le -1$ and $t \ge 2$. Then from Stirling's formula we have

$$\frac{F'}{F}(s) = \log(2\pi) + \frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right) - \frac{\Gamma'}{\Gamma}(1-s) = -\log|1-s| + O(1).$$

Since $|1-s| \ge 1-\sigma$, there exists $A \le -1$ satisfying $|(F'/F)(s)| \ge \frac{1}{2}\log(1-\sigma)$ for any $\sigma \le A$ and $t \ge 2$. Together with $(\zeta'/\zeta)(1-s) = -\sum_{n=1}^{\infty}\Lambda(n)n^{-(1-s)}$, where $\Lambda(n)$ is the von Mangoldt function, we have $\frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s) = O(2^{\sigma}(\log(1-\sigma))^{-1})$ as $\sigma \to -\infty$ uniformly on $t \ge 2$. Therefore there exists $\sigma_0 \le -1$ such that the second condition holds.

Finally for the above a and σ_0 we look for t_0 satisfying the third to seventh conditions. From Stirling's formula we have $(F'/F)(s) = -\log t + O(1)$ as $t \to \infty$ uniformly for $\sigma_0 \le \sigma \le 1/2$. Hence there exists t_1 such that $|(F'/F)(s)| \ge 1$ and $(5\pi)/6 \le \arg(F'/F)(s) \le (7\pi)/6 \mod 2\pi\mathbb{Z}$ hold for any $\sigma_0 \le \sigma \le 1/2$, $t \ge t_1$. Concerning the fourth condition Spira [Spi2, p.149] showed that $\mathrm{Re}(\zeta'/\zeta)(s) < 0$ holds for $\sigma < 1/2$, $t \ge 164$. Put $t_2 := \max\{|\sigma_0|, t_1, 164\}$. We take $t_0 \in [t_2+1, t_2+2]$ such that $\zeta'(\sigma + it_0) \ne 0$ for any $\sigma \in [\sigma_0, a]$ and $\zeta(\sigma + it_0) \ne 0$ for any $\sigma \in [0,1]$. Then the third to seventh conditions hold for the above t_0.

When we choose a, σ_0 and t_0 as above, all the conditions in Lemma 2.1 are satisfied. This completes the proof of Lemma 2.1. □

Proposition 2.2. *Assume RH. Take t_0 and a which satisfy all the conditions of Lemma 2.1. Then for $T \ge t_0$, which satisfies $\zeta'(\sigma + iT) \ne 0$ and $\zeta(\sigma + iT) \ne 0$ for any $\sigma \in \mathbb{R}$, we have*

$$\begin{aligned}\sum_{0<\gamma'\le T}\left(\beta' - \frac{1}{2}\right) &= \frac{T}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - \log\log 2\right)T - \mathrm{li}\left(\frac{T}{2\pi}\right)\\ &\quad + \frac{1}{2\pi}\int_{1/2}^{a}(-\arg\zeta(\sigma+iT) + \arg G(\sigma+iT))d\sigma + O(1),\end{aligned}$$

where the implied constant depends only on t_0 and a. Here we take the logarithmic branches so that $\log\zeta(s)$ and $\log G(s)$ tend to 0 as $\sigma \to \infty$ and are holomorphic in $\mathbb{C}\setminus\{\rho+\lambda : \zeta(\rho) = 0 \text{ or } \infty, \lambda \le 0\}$, $\mathbb{C}\setminus\{\rho'+\lambda : \zeta'(\rho') = 0 \text{ or } \infty, \lambda \le 0\}$, respectively.

Proof. We take σ_0, t_0 and a as Lemma 2.1 and fix them. Take $T \ge t_0$ which satisfies $\zeta'(\sigma + iT) \ne 0$ and $\zeta(\sigma + iT) \ne 0$ for $\sigma \in \mathbb{R}$. Let $\delta \in (0, 1/2]$ and put $b := \frac{1}{2} - \delta$. We consider the rectangle with vertices at $a + it_0$, $a + iT$, $b + iT$ and $b + it_0$. Then applying Littlewood's lemma (see [T1, §3.8]) to $G(s)$ on the rectangle, we have

$$\begin{aligned} 2\pi \sum_{t_0<\gamma'\le T} (\beta'-b) &= \int_{t_0}^{T} \log|G(b+it)|dt - \int_{t_0}^{T} \log|G(a+it)|dt \\ &\quad - \int_b^a \arg G(\sigma+it_0)d\sigma + \int_b^a \arg G(\sigma+iT)d\sigma \\ &=: I_1+I_2+I_3+I_4. \end{aligned} \tag{2.1}$$

Here we remark that, assuming RH, $\zeta'(s)$ has no nonreal zeros for $\mathrm{Re}(s)<1/2$ (see [Spe, p.520] or [LM, §2]). We estimate I_j uniformly for δ as well as for T. Since $|\arg G(\sigma+it_0)|$ is continuous on the interval $[0,a]$, we see that

$$I_3 = O(1).$$

As was shown by Levinson and Montgomery [LM, p.54], we have

$$I_2 = O(1).$$

Next we treat I_1. From the functional equation $\zeta(s)=F(s)\zeta(1-s)$ we have

$$\begin{aligned} \zeta'(s) &= F'(s)\zeta(1-s) - F(s)\zeta'(1-s) \\ &= F(s)\frac{F'}{F}(s)\zeta(1-s)\left(1-\frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s)\right). \end{aligned}$$

Therefore we have

$$\begin{aligned} I_1 &= \int_{t_0}^{T} \log\frac{2^b}{\log 2}dt + \int_{t_0}^{T}\log|\zeta'(b+it)|dt \\ &= (b\log 2-\log\log 2)(T-t_0) + \int_{t_0}^{T}\log|F(b+it)|dt \\ &\quad + \int_{t_0}^{T}\log\left|\frac{F'}{F}(b+it)\right|dt + \int_{t_0}^{T}\log\left|1-\frac{1}{\frac{F'}{F}(b+it)}\frac{\zeta'}{\zeta}(1-b-it)\right|dt \\ &\quad + \int_{t_0}^{T}\log|\zeta(1-b-it)|dt. \end{aligned} \tag{2.2}$$

Stirling's formula gives

$$\log|F(b+it)| = \left(\frac{1}{2}-b\right)\log\frac{t}{2\pi} + O\left(\frac{1}{t^2}\right), \tag{2.3}$$

$$\frac{F'}{F}(b+it) = -\log\frac{t}{2\pi} + \frac{\frac{1}{2}-b}{it} + O\left(\frac{1}{t^2}\right).$$

Hence we obtain

$$\int_{t_0}^{T} \log|F(b+it)|dt = \left(\frac{1}{2}-b\right)\left(T\log\frac{T}{2\pi}-T\right)+O(1), \tag{2.4}$$

$$\begin{aligned}\int_{t_0}^{T} \log\left|\frac{F'}{F}(b+it)\right| dt &= \int_{t_0}^{T} \mathrm{Re}\left(\log\frac{F'}{F}(b+it)\right) dt \\ &= \int_{t_0}^{T} \log\log\frac{t}{2\pi}dt + O\left(\int_{t_0}^{T}\frac{dt}{t^2\log t}\right) \\ &= T\log\log\frac{T}{2\pi} - 2\pi \mathrm{li}\left(\frac{T}{2\pi}\right) + O(1). \end{aligned} \tag{2.5}$$

Next we treat the fourth term in (2.2). To do this, we consider $1-\frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s)$. It follows from the second condition in Lemma 2.1 that it is holomorphic and has no zeros in the region including $\sigma \le \sigma_0$, $t \ge 2$. We note that the functional equation $\zeta(s) = F(s)\zeta(1-s)$ gives

$$1-\frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s) = \frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(s). \tag{2.6}$$

By RH and the third and fourth conditions in Lemma 2.1, (2.6) is holomorphic and has no zeros in $\sigma_0 < \sigma < 1/2$, $t > t_0 - 1$. Thus we determine $\log(1-\frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s))$ so that it tends to 0 as $\sigma \to -\infty$ uniformly for $t > t_0 - 1$ and is holomorphic in $\sigma < 1/2$, $t > t_0 - 1$. Cauchy's theorem gives

$$\int_C \log\left(1-\frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s)\right) ds = 0, \tag{2.7}$$

where C is the trapezoid joining $b+it_0$, $b+iT$, $-T+iT$ and $-t_0+it_0$. From the second condition in Lemma 2.1 we have

$$\left|\int_{\sigma_0+iT}^{-T+iT} \log\left(1-\frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s)\right) ds\right| \ll \int_{-T}^{\sigma_0} 2^{\sigma} d\sigma \ll 1, \tag{2.8}$$

$$\left|\left(\int_{-T+iT}^{-t_0+it_0} + \int_{-t_0+it_0}^{\sigma_0+it_0}\right) \log\left(1-\frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s)\right) ds\right| \ll 1. \tag{2.9}$$

Applying (2.8) and (2.9) to (2.7), estimating the integral from $\sigma_0 + it_0$ to $b + it_0$ trivially and taking the imaginary part, we obtain

$$\begin{aligned}&\int_{t_0}^{T} \log\left|1-\frac{1}{\frac{F'}{F}(b+it)}\frac{\zeta'}{\zeta}(1-b-it)\right| dt \\ &= \int_{\sigma_0}^{b} \arg\left(\frac{1}{\frac{F'}{F}(\sigma+iT)}\frac{\zeta'}{\zeta}(\sigma+iT)\right) d\sigma + O(1). \end{aligned} \tag{2.10}$$

Here we used (2.6). From the third and fourth conditions in Lemma 2.1 we get

$$-\frac{2}{3}\pi \leq \arg\left(\frac{1}{\frac{F'}{F}(\sigma+iT)}\frac{\zeta'}{\zeta}(\sigma+iT)\right) \leq \frac{2}{3}\pi \mod 2\pi\mathbb{Z}$$

for $\sigma_0 \leq \sigma < 1/2$. It follows from the choice of the logarithmic branch, the second condition in Lemma 2.1 and (2.6) that $\arg(\frac{1}{\frac{F'}{F}(\sigma_0+iT)}\frac{\zeta'}{\zeta}(\sigma_0+iT)) \in (-\pi/2, \pi/2)$. Since $[\sigma_0, 1/2)$ is connected and $\sigma \mapsto \arg(\frac{1}{\frac{F'}{F}(\sigma+iT)}\frac{\zeta'}{\zeta}(\sigma+iT))$ is continuous in $\sigma \in [\sigma_0, 1/2)$, the image of this map is also connected. The connected component of $\bigcup_{n\in\mathbb{Z}}[-\frac{2}{3}\pi + 2\pi n, \frac{2}{3}\pi + 2\pi n]$ which $\arg(\frac{1}{\frac{F'}{F}(\sigma_0+iT)}\frac{\zeta'}{\zeta}(\sigma_0+iT))$ belongs to is $[-(2\pi)/3, (2\pi)/3]$. Hence for $\sigma_0 \leq \sigma < 1/2$ we have

$$-\frac{2}{3}\pi \leq \arg\left(\frac{1}{\frac{F'}{F}(\sigma+iT)}\frac{\zeta'}{\zeta}(\sigma+iT)\right) \leq \frac{2}{3}\pi. \tag{2.11}$$

Applying this to (2.10), we obtain

$$\int_{t_0}^{T} \log\left|1 - \frac{1}{\frac{F'}{F}(b+it)}\frac{\zeta'}{\zeta}(1-b-it)\right| dt = O(1). \tag{2.12}$$

Finally we treat the fifth term of (2.2). We note that $|\zeta(1-b-it)| = |\zeta(1-b+it)|$ because $\overline{\zeta(\overline{s})} = \zeta(s)$. Since $1-b > 1/2$, Cauchy's theorem gives

$$\int_{C'} \log\zeta(s)ds = 0, \tag{2.13}$$

where C' is the rectangle joining $1-b+it_0$, $a+it_0$, $a+iT$, $1-b+iT$. Here the logarithmic branch is determined so that $\log\zeta(s) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\log n}$ holds for $\mathrm{Re}(s) > 1$, and it is holomorphic in $\mathbb{C}\setminus\{\rho+\lambda : \zeta(\rho) = 0 \text{ or } \infty, \lambda \leq 0\}$. We have

$$\int_{1-b+it_0}^{a+it_0} \log\zeta(s)ds = O(1),$$
$$\int_{a+it_0}^{a+iT} \log\zeta(s)ds = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^a\log^2 n}(n^{-it_0} - n^{-iT}) = O(1).$$

Applying these to (2.13) and taking the imaginary part, we obtain

$$\int_{t_0}^{T} \log|\zeta(1-b-it)|dt = -\int_{1-b}^{a} \arg\zeta(\sigma+iT)d\sigma + O(1). \tag{2.14}$$

Applying (2.4), (2.5), (2.12) and (2.14) to (2.2), we have

$$\begin{aligned} &2\pi \sum_{0<\gamma'\le T} (\beta' - b) \\ = \ & \left(\frac{1}{2} - b\right) T \log \frac{T}{2\pi} + T \log\log \frac{T}{2\pi} + \left(b\log 2 - \log\log 2 - \left(\frac{1}{2} - b\right)\right) T \\ &-2\pi \mathrm{li}\left(\frac{T}{2\pi}\right) - \int_{1-b}^{a} \arg\zeta(\sigma+iT)d\sigma + \int_{b}^{a} \arg G(\sigma+iT)d\sigma + O(1). \quad (2.15)\end{aligned}$$

Taking the limit $\delta \downarrow 0$, we complete the proof of Proposition 2.2. □

In view of Proposition 2.2 we need bounds for

$$-\arg\zeta(\sigma+iT) + \arg G(\sigma+iT) = \arg\left(-\frac{2^{\sigma+iT}}{\log 2}\frac{\zeta'}{\zeta}(\sigma+iT)\right). \qquad (2.16)$$

Here the argument in the right-hand side is taken so that $\log(-\frac{2^s}{\log 2}\frac{\zeta'}{\zeta}(s))$ tends to 0 as $\sigma \to \infty$ and is holomorphic in $\mathbb{C} \setminus \{z+\lambda : (\zeta'/\zeta)(z) = 0 \text{ or } \infty, \lambda \le 0\}$. Below we give two bounds for (2.16).

Lemma 2.3. *Assume RH. Then for* $1/2 < \sigma \le a$ *we have*

$$\arg\left(-\frac{2^{\sigma+iT}}{\log 2}\frac{\zeta'}{\zeta}(\sigma+iT)\right) = O\left(\frac{\log\log T}{\sigma-\frac{1}{2}}\right),$$

where the implied constant depends only on a.

Proof. Since $G(s)/\zeta(s) = -\frac{2^s}{\log 2}\frac{\zeta'}{\zeta}(s) \to 1$ as $\sigma \to \infty$ uniformly for $t \in \mathbb{R}$, we can take $c \ge a+1$ satisfying $1/2 \le \mathrm{Re}(G(s)/\zeta(s)) \le 3/2$ for $\mathrm{Re}(s) \ge c$. Let $\sigma \in (1/2, a]$. If $\mathrm{Re}(G(u+iT)/\zeta(u+iT))$ vanishes $q_{G/\zeta} = q_{G/\zeta}(\sigma, T)$ times on $u \in [\sigma, c]$, then $|\arg(G(\sigma+iT)/\zeta(\sigma+iT))| \le (q_{G/\zeta} + \frac{3}{2})\pi$. To estimate $q_{G/\zeta}$, we put $H(z) = H_T(z) := (\frac{G(z+iT)}{\zeta(z+iT)} + \frac{G(z-iT)}{\zeta(z-iT)})/2$ and $n_H(r) := \#\{z \in \mathbb{C} : H(z) = 0, |z-c| \le r\}$. Since $H(x) = \mathrm{Re}(G(x+iT)/\zeta(x+iT))$ for $x \in \mathbb{R}$, we have $q_{G/\zeta} \le n_H(c-\sigma)$ for $1/2 < \sigma \le a$. For each $\sigma \in (1/2, a]$ we take $\varepsilon = \varepsilon_{\sigma,T}$ satisfying $0 < \varepsilon < \sigma - \frac{1}{2}$. Then, since $\sigma - \varepsilon > 1/2$, $H(z)$ is holomorphic in a region including $|z-c| \le c - \sigma + \varepsilon$. Thus Jensen's theorem (see [T1, §3.61]) gives

$$\int_0^{c-\sigma+\varepsilon} \frac{n_H(r)}{r}dr = \frac{1}{2\pi}\int_0^{2\pi} \log|H(c+(c-\sigma+\varepsilon)e^{i\theta})|d\theta - \log|H(c)|. \qquad (2.17)$$

We estimate the left-hand side as follows:

$$\begin{aligned}\int_0^{c-\sigma+\varepsilon} \frac{n_H(r)}{r}dr &\ge \int_{c-\sigma}^{c-\sigma+\varepsilon} \frac{n_H(r)}{r}dr \ge n_H(c-\sigma)\log\left(1+\frac{\varepsilon}{c-\sigma}\right) \\ &\ge n_H(c-\sigma)\log\left(1+\frac{\varepsilon}{c-\frac{1}{2}}\right) \ge C_1\varepsilon n_H(c-\sigma), \qquad (2.18)\end{aligned}$$

where $C_1 > 0$ is a constant depending only on c. Next we treat the right-hand side of (2.17). It follows from [T2, Theorems 9.2 and 9.6 (A)] that RH implies

$$\left|\frac{\zeta'}{\zeta}(x \pm it)\right| = O\left(\frac{\log T}{x - \frac{1}{2}}\right) \tag{2.19}$$

for $T/2 \leq t \leq 2T$ and $1/2 < x \leq 2c$. Thus we have

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \log|H(c + (c - \sigma + \varepsilon)e^{i\theta})| d\theta \\ \leq\ & \frac{1}{2\pi}\int_0^{2\pi} \log\left(C_2 \frac{\log T}{c + (c - \sigma + \varepsilon)\cos\theta - \frac{1}{2}}\right) d\theta \\ =\ & \log(C_2 \log T) - \frac{1}{2\pi}\int_0^{2\pi} \log\left(c + (c - \sigma + \varepsilon)\cos\theta - \frac{1}{2}\right) d\theta. \end{aligned} \tag{2.20}$$

We estimate the integral. From Jensen's theorem again we have

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \log\left(c + (c - \sigma + \varepsilon)\cos\theta - \frac{1}{2}\right) d\theta \\ =\ & \log\frac{c - \sigma + \varepsilon}{2} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{i\theta} + e^{-i\theta} + 2\frac{c - \frac{1}{2}}{c - \sigma + \varepsilon}\right| d\theta \\ =\ & \log\frac{c - \sigma + \varepsilon}{2} + \log^+|\alpha| + \log^+|\beta| \\ \geq\ & \log\frac{c - \sigma + \varepsilon}{2} \geq \log\frac{c - a}{2}, \end{aligned}$$

where α and β are the solutions of $X + X^{-1} + 2\frac{c - \frac{1}{2}}{c - \sigma + \varepsilon} = 0$ and $\log^+ x := \max\{\log x, 0\}$. Applying this to (2.20), we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log|H(c + (c - \sigma + \varepsilon)e^{i\theta})| d\theta \leq C_3 \log\log T, \tag{2.21}$$

where C_3 depends only on a and c. Applying (2.18), (2.21) and $H(c) = \mathrm{Re}(G(c + iT)/\zeta(c + iT)) \in [1/2, 3/2]$ to (2.17), we obtain $n_H(c - \sigma) \leq C_4\varepsilon^{-1} \log\log T$. Putting $\varepsilon = (\sigma - \frac{1}{2})/2$, we establish Lemma 2.3. □

Lemma 2.4. *Assume RH. Then for $1/2 \leq \sigma \leq 3/4$ we have*

$$\arg G(\sigma + iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{(\log\log T)^{1/2}}\right).$$

Remark 2.5. For $\frac{1}{2} + \frac{(\log\log T)^2}{\log T} \leq \sigma \leq 1 - \delta$ with any given $\delta > 0$ we can replace Lemma 2.4 by

$$\arg G(\sigma + iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right) \tag{2.22}$$

though an estimate for σ near $1/2$ is important and (2.22) is not needed for our purpose. To prove (2.22), we note that RH implies

$$\arg\zeta(\sigma+iT)=O\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right) \tag{2.23}$$

uniformly for $1/2\le\sigma\le 1-\delta$ (see [T2, (14.14.3) and (14.14.5)]). Applying Lemma 2.3 and (2.23) to (2.16) gives (2.22).

To show Lemma 2.4, we prepare the following lemma.

Lemma 2.6. *Assume RH. Let $A\ge 2$ be fixed. Then there exists $C>0$ such that*

$$|\zeta'(\sigma+it)|\le\exp\left(C\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}+(\log T)^{1/10}\right)\right)$$

hold for $T\ge 10$, $T/2\le t\le 2T$ and $\frac{1}{2}-\frac{1}{\log\log T}\le\sigma\le A$.

Proof. We first prove

$$|\zeta(\sigma+it)|\le\exp\left(C_5\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}+(\log T)^{1/10}\right)\right) \tag{2.24}$$

for $\frac{1}{2}-\frac{2}{\log\log T}\le\sigma\le A+1$, $T/3\le t\le 3T$. According to [T2, (14.14.2), (14.14.5) and the first equation in p.384], (2.24) holds for $1/2\le\sigma\le A+1$, $T/3\le t\le 3T$. Hence it is sufficient to prove (2.24) in the case $\frac{1}{2}-\frac{2}{\log\log T}\le\sigma\le 1/2$. From the functional equation $\zeta(s)=F(s)\zeta(1-s)$, (2.3), (2.24) with $1/2\le\sigma\le A+1$ and $|\zeta(\overline{s})|=|\zeta(s)|$ we have

$$\begin{aligned}|\zeta(\sigma+it)| &= |F(\sigma+it)||\zeta(1-\sigma+it)|\\ &\le \exp\left(\left(\frac{1}{2}-\sigma\right)\log\frac{t}{2\pi}+\frac{C_6}{t^2}+C_5\left(\frac{(\log T)^{2\sigma}}{\log\log T}+(\log T)^{1/10}\right)\right)\\ &\le \exp\left(C_7\frac{\log T}{\log\log T}\right)\\ &\le \exp\left(C_8\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}+(\log T)^{1/10}\right)\right).\end{aligned}$$

Here in the third and last lines we used $0\le\frac{1}{2}-\sigma\le 2/\log\log T$. Thus (2.24) also holds for $\frac{1}{2}-\frac{2}{\log\log T}\le\sigma\le 1/2$, $T/3\le t\le 3T$.

We prove the lemma. Cauchy's integral formula says

$$\zeta'(s)=\frac{1}{2\pi i}\int_{|z-s|=\varepsilon}\frac{\zeta(z)}{(z-s)^2}dz$$

for $\varepsilon>0$. Taking $\varepsilon=1/\log\log T$ and applying (2.24), we obtain Lemma 2.6. □

Proof of Lemma 2.4. Let $\sigma \in [1/2, 3/4]$. If $\operatorname{Re} G(u+iT)$ vanishes $q_G = q_G(\sigma, T)$ times on $u \in [\sigma, a]$, then we have $|\arg G(\sigma+iT)| \le (q_G + \frac{3}{2})\pi$. To estimate q_G, put $X(z) = X_T(z) := (G(z+iT)+G(z-iT))/2$ and $n_X(r) := \#\{z \in \mathbb{C} : X(z) = 0, |z-a| \le r\}$. Since $\overline{G(\overline{s})} = G(s)$, we have $X(x) = \operatorname{Re} G(x+iT)$ for any $x \in \mathbb{R}$. Hence we have $q_G \le n_X(a-\sigma)$. We estimate $n_X(a-\sigma)$. Let $0 < \varepsilon = \varepsilon_{\sigma,T} \le \sigma - \frac{1}{2} + \frac{1}{\log\log T}$. From Jensen's theorem we have

$$\int_0^{a-\sigma+\varepsilon} \frac{n_X(r)}{r} dr = \frac{1}{2\pi}\int_0^{2\pi} \log|X(a+(a-\sigma+\varepsilon)e^{i\theta})| d\theta - \log|X(a)|. \tag{2.25}$$

In the same manner as (2.18) we have

$$\int_0^{a-\sigma+\varepsilon} \frac{n_X(r)}{r} dr \ge C_9 \varepsilon n_X(a-\sigma). \tag{2.26}$$

On the other hand, from Lemma 2.6 and $\overline{G(\overline{s})} = G(s)$ we see that

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \log|X(a+(a-\sigma+\varepsilon)e^{i\theta})| d\theta \\ \le\ & C_{10}\int_0^{2\pi} \left(\frac{(\log T)^{2-2(a+(a-\sigma+\varepsilon)\cos\theta)}}{\log\log T} + (\log T)^{1/10} \right) d\theta. \end{aligned}$$

Using

$$\int_0^{2\pi} e^{-x\cos\theta} d\theta = 2\pi I_0(x),$$

where I_ν is the Bessel function, and $I_0(x) \sim e^x/\sqrt{2\pi x}$ as $x \to \infty$ (see [GR, 8.431.3 and 8.451.5]), we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log|X(a+(a-\sigma+\varepsilon)e^{i\theta})| d\theta \le C_{11}\left(\frac{(\log T)^{2(1-\sigma+\varepsilon)}}{(\log\log T)^{3/2}} + (\log T)^{1/10} \right). \tag{2.27}$$

Applying (2.26), (2.27) and $1/2 \le X(a) = \operatorname{Re} G(a+iT) \le 3/2$ (see the first condition in Lemma 2.1) to (2.25), we have

$$n_X(a-\sigma) \ll \frac{1}{\varepsilon}\left(\frac{(\log T)^{2(1-\sigma+\varepsilon)}}{(\log\log T)^{3/2}} + (\log T)^{1/10} \right).$$

Taking $\varepsilon = 1/\log\log T (\le \sigma - \frac{1}{2} + \frac{1}{\log\log T})$, we reach Lemma 2.4. □

Proof of Theorem 1. Let $0 < \varepsilon \le 1/4$. From Lemma 2.3 we have

$$\int_{\frac{1}{2}+\varepsilon}^{a} \arg\left(-\frac{2^{\sigma+iT}}{\log 2}\frac{\zeta'}{\zeta}(\sigma+iT) \right) d\sigma = O\left(\log\frac{1}{\varepsilon} \log\log T \right).$$

It follows from Lemma 2.4 and (2.23) that

$$\int_{1/2}^{\frac{1}{2}+\varepsilon} (-\arg\zeta(\sigma+iT)+\arg G(\sigma+iT))d\sigma = O\left(\varepsilon\frac{\log T}{(\log\log T)^{1/2}}\right).$$

Applying these to Proposition 2.2 and taking $\varepsilon = 1/\log T$, we obtain Theorem 1. □

Proof of Corollary 2. This is an immediate consequence of Theorem 1; see [LM, Theorem 10]. □

3 Proof of Theorem 3

In this section we prove Theorem 3. We keep the notation in §2. Then,

Proposition 3.1. *Assume RH. Then for $T \geq 2$, which satisfies $\zeta(\sigma+iT) \neq 0$ and $G(\sigma+iT) \neq 0$ for any $\sigma \in \mathbb{R}$, we have*

$$N_1(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{2\pi}\arg G\left(\frac{1}{2}+iT\right) + \frac{1}{2\pi}\arg\zeta\left(\frac{1}{2}+iT\right) + O(1),$$

where the arguments are determined in the same manner as Proposition 2.2.

Proof. We take σ_0, t_0, a, δ, T and b as in the beginning of the proof of Proposition 2.2. Replacing b by $b' = \frac{1}{2} - \frac{\delta}{2}$ in (2.1), we have

$$\begin{aligned} 2\pi\sum_{t_0<\gamma'\leq T}(\beta'-b') &= \int_{t_0}^{T}\log|G(b'+it)|dt - \int_{t_0}^{T}\log|G(a+it)|dt \\ &\quad - \int_{b'}^{a}\arg G(\sigma+it_0)d\sigma + \int_{b'}^{a}\arg G(\sigma+iT)d\sigma. \end{aligned}$$

Subtracting this from (2.1), we have

$$\begin{aligned} \pi\delta(N_1(T)-N_1(t_0)) &= \int_{t_0}^{T}\log|G(b+it)|dt - \int_{t_0}^{T}\log|G(b'+it)|dt \\ &\quad - \int_{b}^{b'}\arg G(\sigma+it_0)d\sigma + \int_{b}^{b'}\arg G(\sigma+iT)d\sigma \\ &=: J_1+J_2+J_3+J_4. \end{aligned} \tag{3.1}$$

Clearly we have

$$J_3 = O(\delta). \tag{3.2}$$

Next we treat $J_1 + J_2$. From (2.2) we have

$$\begin{aligned}
& J_1 + J_2 \\
= \ & (b-b')(T-t_0)\log 2 + \left(\int_{t_0}^{T} \log|F(b+it)|dt - \int_{t_0}^{T} \log|F(b'+it)|dt \right) \\
& + \left(\int_{t_0}^{T} \log\left|\frac{F'}{F}(b+it)\right| dt - \int_{t_0}^{T} \log\left|\frac{F'}{F}(b'+it)\right| dt \right) \\
& + \left(\int_{t_0}^{T} \log\left| 1 - \frac{1}{\frac{F'}{F}(b+it)} \frac{\zeta'}{\zeta}(1-b-it) \right| dt \right. \\
& \qquad \left. - \int_{t_0}^{T} \log\left| 1 - \frac{1}{\frac{F'}{F}(b'+it)} \frac{\zeta'}{\zeta}(1-b'-it) \right| dt \right) \\
& + \left(\int_{t_0}^{T} \log|\zeta(1-b-it)|dt - \int_{t_0}^{T} \log|\zeta(1-b'-it)|dt \right) \\
=: \ & -\frac{\delta}{2}(T-t_0)\log 2 + K_1 + K_2 + K_3 + K_4, \qquad (3.3)
\end{aligned}$$

where K_j denotes the j-th brace. We deal with K_1. We define a branch of $\log F(s)$ for $0 < \sigma < 1$ and $t > 0$ by

$$\log F(s) := s \log 2 + (s-1)\log \pi + \log\left(\sin\left(\frac{\pi s}{2}\right)\right) + \log \Gamma(1-s), \qquad (3.4)$$

where

$$\log\left(\sin\left(\frac{\pi s}{2}\right)\right) := -\frac{\pi i s}{2} - \log 2 + \frac{\pi i}{2} - \sum_{n=1}^{\infty} \frac{e^{\pi i n s}}{n}$$

and $\log \Gamma(1-s)$ is a holomorphic function in the strip $0 < \sigma < 1$ satisfying $\log \Gamma(1-\sigma) \in \mathbb{R}$ for any $\sigma \in (0,1)$. It follows from Cauchy's theorem that

$$\int_C \log F(s) ds = 0,$$

where C is a path joining $b' + it_0$, $b' + iT$, $b + iT$ and $b + it_0$. Taking the imaginary part, we have

$$K_1 = -\int_b^{b'} \arg F(\sigma + iT) d\sigma + \int_b^{b'} \arg F(\sigma + it_0) d\sigma.$$

Applying Stirling's formula to (3.4) and taking the imaginary part, we have

$$\arg F(\sigma + iT) = -T \log \frac{T}{2\pi} + T + O(1)$$

uniformly for $0 < \sigma < 1$. This, together with $\arg F(\sigma + it_0) = O(1)$, gives

$$K_1 = \frac{\delta}{2}\left(T\log\frac{T}{2\pi} - T\right) + O(\delta). \tag{3.5}$$

Next we treat K_2. Since all the zeros and poles of $F(s)$ lie on $\mathbb{R}$, $(F'/F)(s)$ has no poles in $t > 0$. From the third condition in Lemma 2.1 we can define a branch of $\log(F'/F)(s)$ for $0 < \sigma < 1/2$ and $t > t_0 - 1$ by $\arg(F'/F)(s) \in [(5\pi)/6, (7\pi)/6]$. Applying Cauchy's theorem to $\log(F'/F)(s)$ on the path C and taking the imaginary part, we have

$$K_2 = -\int_b^{b'} \arg\left(\frac{F'}{F}(\sigma + iT)\right) d\sigma + \int_b^{b'} \arg\left(\frac{F'}{F}(\sigma + it_0)\right) d\sigma = O(\delta). \tag{3.6}$$

Here in the last equality we used the choice of the branch.

Next we deal with K_3. We define a branch of $\log(1 - \frac{1}{\frac{F'}{F}(s)}\frac{\zeta'}{\zeta}(1-s))$ in the same manner as (2.7). Then it is holomorphic in the $\{\sigma + it : 0 < \sigma < 1/2, t > t_0 - 1\}$. Applying Cauchy's theorem and taking the imaginary part, we have

$$\begin{aligned} K_3 \quad = \quad & -\int_b^{b'} \arg\left(1 - \frac{1}{\frac{F'}{F}(\sigma + iT)}\frac{\zeta'}{\zeta}(1 - \sigma - iT)\right) d\sigma \\ & + \int_b^{b'} \arg\left(1 - \frac{1}{\frac{F'}{F}(\sigma + it_0)}\frac{\zeta'}{\zeta}(1 - \sigma - it_0)\right) d\sigma. \end{aligned}$$

Applying (2.6) and (2.11), we obtain

$$K_3 = O(\delta). \tag{3.7}$$

Next we treat K_4. We define a branch of $\log\zeta(s)$ in the same manner as (2.13). We note that $1 - b > 1 - b' > 1/2$ and $|\zeta(\overline{s})| = |\zeta(s)|$. Applying Cauchy's theorem to $\log\zeta(s)$ on the rectangle with vertices at $1 - b + it_0$, $1 - b + iT$, $1 - b' + iT$ and $1 - b' + it_0$ and taking the imaginary part, we obtain

$$K_4 = \int_{1-b'}^{1-b} \arg\zeta(\sigma + iT)d\sigma + O(\delta). \tag{3.8}$$

Applying (3.5)–(3.8) to (3.3), we obtain

$$J_1 + J_2 = \frac{\delta}{2}\left(T\log\frac{T}{4\pi} - T\right) + \int_{1-b'}^{1-b} \arg\zeta(\sigma + iT)d\sigma + O(\delta). \tag{3.9}$$

Applying (3.2) and (3.9) to (3.1), we get

$$\begin{aligned} N_1(T) &= \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{\pi\delta}\int_b^{b'} \arg G(\sigma+iT)d\sigma \\ &\quad + \frac{1}{\pi\delta}\int_{1-b'}^{1-b} \arg\zeta(\sigma+iT)d\sigma + O(1). \end{aligned}$$

Taking the limit $\delta \downarrow 0$, we complete the proof of Proposition 3.1. □

Proof of Theorem 3. Applying Lemma 2.4 and (2.23) to Proposition 3.1, we immediately obtain the desired result. □

Remark 3.2. From computational analysis Skorokhodov [SK, §7.6] conjectured

$$N(T) \overset{?}{=} N_1(T) + \frac{T\log 2}{2\pi} + O(1), \tag{3.10}$$

which is a modification to a conjecture by Spira [Spi1, §3]. From (1.4) and Proposition 3.1, RH implies

$$N(T) = N_1(T) + \frac{T\log 2}{2\pi} - \frac{1}{2\pi}\arg\left(-\frac{2^{\frac{1}{2}+iT}}{\log 2}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iT\right)\right) + O(1)$$

for $T \geq 2$. Here we take a branch in the same manner as (2.16). Under RH, (3.10) is equivalent that $\arg(-\frac{2^{\frac{1}{2}+iT}}{\log 2}\frac{\zeta'}{\zeta}(\frac{1}{2}+iT))$ is bounded. However, at present under RH we only have

$$\arg\left(-\frac{2^{\frac{1}{2}+iT}}{\log 2}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iT\right)\right) = O\left(\frac{\log T}{(\log\log T)^{1/2}}\right). \tag{3.11}$$

In fact, separating $\arg(-\frac{2^{\frac{1}{2}+iT}}{\log 2}\frac{\zeta'}{\zeta}(\frac{1}{2}+iT))$ into $\arg\zeta(\frac{1}{2}+iT)$ and $\arg(-\frac{2^{\frac{1}{2}+iT}}{\log 2}\zeta'(\frac{1}{2}+iT))$ and applying Lemma 2.4 and (2.23), we obtain (3.11).

With some more efforts, we might replace the error term in (3.11) (and in Theorem 3) by $O(\log T/\log\log T)$ under RH. However, in view of (2.23), there is a barrier to further improvement. To overcome this difficulty, we will need a new method of giving a bound for $\arg(-\frac{2^{\frac{1}{2}+iT}}{\log 2}\frac{\zeta'}{\zeta}(\frac{1}{2}+iT))$ by properties of $(\zeta'/\zeta)(s)$ instead of those of $\zeta(s)$ and $\zeta'(s)$.

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Department of Mathematics, Tokyo Institute of Technology,
2-12-1 Ookayama, Meguro-ku, Tokyo, 152-8551, Japan.
E-mail: akatsuka@math.titech.ac.jp

Current Address:
Faculty of Mathematics, Kyushu University,
744, Motooka, Nishi-ku, Fukuoka, 819-0395, Janan.
E-mail: akatsuka@math.kyushu-u.ac.jp