Zeta Mahler measures

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Abstract

We introduce the zeta Mahler measure with a complex parameter, whose derivative is a generalization of the classical Mahler measure. We study a fundamental theory of the zeta Mahler measure, including holomorphic regions and transformation formulas. We also express some specific examples of zeta Mahler measures in terms of hypergeometric functions.

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1 Introduction

For a nonzero Laurent polynomial $f(X_1, \ldots, X_r) \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$, the associated (logarithmic) Mahler measure $m(f)$ is defined to be

$$m(f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r})| dt_r \cdots dt_1.$$

It is known that the Mahler measure has interesting connections with zeta/L values, (multiple) polylogarithms and multiple sine functions, see [B, D, L1, L2, O, RV, S, V] and the references therein.

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In this paper we introduce the following *zeta Mahler measure*. For a nonzero Laurent polynomial $f(X_1, \ldots, X_r) \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$, the associated zeta Mahler measure is defined by

$$Z(s, f) := \int_0^1 \cdots \int_0^1 |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r})|^s dt_r \cdots dt_1. \tag{1.1}$$

The integral converges absolutely in $\mathrm{Re}(s) > \sigma_0(f)$ for some $\sigma_0(f) < 0$, as explained in §2 below. Since

$$\frac{dZ}{ds}(0, f) = m(f), \tag{1.2}$$

$\frac{dZ}{ds}$ can be regarded as a generalization of the Mahler measure.[1] The first purpose of this paper is to investigate fundamental properties of the zeta Mahler measure, including convergent domains of the integral (1.1) and transformation formulas. The second purpose is to express some specific examples of zeta Mahler measures in terms of (generalized) hypergeometric functions. We will explain the fundamental properties in §2. In this section we state our results on specific examples of zeta Mahler measures. From Jensen's formula, Mahler measures for one variable polynomials $f(X) = a\prod_{j=1}^{d}(X - \alpha_j) \in \mathbb{C}[X] \setminus \{0\}$ is evaluated as

$$m(f) = \log|a| + \sum_{j=1}^{d} \log^+ |\alpha_j|, \tag{1.3}$$

where $\log^+ x := \max\{\log x, 0\}$ for $x \geq 0$. Since $m(X^n f) = m(f)$ for any $n \in \mathbb{Z}$, Mahler measures for one variable Laurent polynomials can be evaluated in terms of their zeros lying in $\{\alpha \in \mathbb{C} : |\alpha| > 1\}$. On the other hand, it is difficult to calculate zeta Mahler measures for general one variable Laurent polynomials. But we can calculate two examples of zeta Mahler measures for one variable Laurent polynomials as explained below.

Theorem 1. *Let* $a \in \mathbb{C}$ *and put* $f(X) = X + a$. *Then,*
(1) *when* $|a| = 1$, *for* $\mathrm{Re}(s) > -1$ *we have*

$$Z(s, f) = 2^s \pi^{-1/2} \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2} + 1)}.$$

[1] When we interchange differentiation and integration in (1.2), we have to pay attention to singularities of the integrand. See §2 for a rigorous treatment.

(2) *when* $|a| \neq 1$, *for* $s \in \mathbb{C}$ *we have*

$$Z(s,f) = (|a|^2+1)^{s/2} F\left(-\frac{s}{4}, -\frac{s}{4}+\frac{1}{2}; 1; \left(\frac{2|a|}{|a|^2+1}\right)^2\right),$$

where $F(\alpha,\beta;\gamma;z) = {}_2F_1(\alpha,\beta;\gamma;z)$ *is the hypergeometric function given by*

$$F(\alpha,\beta;\gamma;z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \qquad (|z|<1), \tag{1.4}$$

and $(\alpha)_0 := 1$, $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ $(n \in \mathbb{Z}_{\geq 1})$ *is the Pochhammer symbol.*

In the case $|a| \neq 1$ it is not easy to see that Theorem 1 is compatible with (1.3). For $|a| \neq 1$, $Z(s, X+a)$ also has the following expression, from which we easily understand the compatibility.

Theorem 2. *Suppose that* $a \in \mathbb{C}$ *satisfies* $|a| \neq 1$. *Then we have*

$$Z(s, X+a) = \begin{cases} |a|^s F(-\frac{s}{2}, -\frac{s}{2}; 1; |a|^{-2}) & \text{if } |a|>1, \\ F(-\frac{s}{2}, -\frac{s}{2}; 1; |a|^2) & \text{if } |a|<1. \end{cases}$$

Remark 1.1. From Theorem 1 (1) and Theorem 2 together with (1.4), we easily recover (1.3) for $f(X) = X + a$, that is, $m(f) = Z'(0,f) = \log^+ |a|$.

In the case $|a| \neq 1$, $Z(s, X+a)$ has the following functional equation between s and $-s-2$:

Theorem 3. *For* $f(X) = X + a$ *with* $|a| \neq 1$ *we have the functional equation*

$$Z(-s-2, f) = ||a|^2 - 1|^{-s-1} Z(s,f).$$

We also treat zeta Mahler measures for $f(X) = X + X^{-1} + k$ with $k \in \mathbb{R}$.

Theorem 4. *Let* $k \in \mathbb{R}$ *and put* $f(X) = X + X^{-1} + k$. *Then*
(1) *when* $|k| > 2$, *for any* $s \in \mathbb{C}$ *we have*

$$Z(s,f) = \left(\frac{|k| + \sqrt{k^2-4}}{2}\right)^s F\left(-s, -s; 1; \left(\frac{|k| - \sqrt{k^2-4}}{2}\right)^2\right).$$

(2) *when* $|k|=2$, *for* $\mathrm{Re}(s)>-1/2$ *we have*

$$Z(s,f)=4^s\pi^{-1/2}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}.$$

(3) *when* $|k|<2$, *for* $\mathrm{Re}(s)>-1$ *we have*

$$\begin{aligned} Z(s,f) &= \frac{1}{2\pi^{1/2}}\frac{\Gamma(s+1)}{\Gamma(s+\frac{3}{2})}\Bigg((2-k)^{s+\frac{1}{2}}F\left(\frac{1}{2},\frac{1}{2};s+\frac{3}{2};\frac{2-k}{4}\right) \\ &\quad + (2+k)^{s+\frac{1}{2}}F\left(\frac{1}{2},\frac{1}{2};s+\frac{3}{2};\frac{2+k}{4}\right)\Bigg). \end{aligned}$$

Remark 1.2. When $k\in\mathbb{R}$ satisfies $|k|\geq 2$, from Theorem 4 together with (1.4) we recover (1.3) for $f(X)=X+X^{-1}+k$, that is, $m(f)=Z'(0,f)=\log(\frac{|k|+\sqrt{k^2-4}}{2})$. On the other hand, it is difficult to recover (1.3) in the case $-2<k<2$. But (1.3) and Theorem 4 (3) produce the following nontrivial formula for $k\in\mathbb{R}$, $|k|<2$:

$$\begin{aligned} & (2+k)^{1/2}\sum_{\substack{m,n\in\mathbb{Z}\\ 0\leq m\leq n}}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})(m+\frac{1}{2})}\left(\frac{2+k}{4}\right)^n \\ + & (2-k)^{1/2}\sum_{\substack{m,n\in\mathbb{Z}\\ 0\leq m\leq n}}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})(m+\frac{1}{2})}\left(\frac{2-k}{4}\right)^n \\ = & 4\pi\log 2+4\arcsin\left(\frac{\sqrt{2+k}}{2}\right)\log(2+k)+4\arcsin\left(\frac{\sqrt{2-k}}{2}\right)\log(2-k), \end{aligned} \quad (1.5)$$

where arcsin takes a value in $(-\pi/2,\pi/2)$. See §3.2 for the proof of (1.5).

In the case $k\in\mathbb{R}$, $|k|>2$, $Z(s,X+X^{-1}+k)$ has the following functional equation:

Theorem 5. *Suppose that* $k\in\mathbb{R}$ *satisfies* $|k|>2$ *and put* $f(X)=X+X^{-1}+k$. *Then* $Z(s,f)$ *satisfies the following functional equation:*

$$Z(-s-1,f)=(k^2-4)^{-s-\frac{1}{2}}Z(s,f).$$

We also treat the 2-variable Laurent polynomials as follows:

Theorem 6. *Suppose that* $k\in\mathbb{R}$ *satisfies* $|k|>4$ *and put* $f(X,Y):=X+X^{-1}+Y+Y^{-1}+k$. *Then we have*

$$Z(s,f)=|k|^s{}_3F_2\left(\frac{1}{2},-\frac{s}{2},\frac{-s+1}{2};1,1;\left(\frac{4}{k}\right)^2\right),$$

where ${}_3F_2$ is the generalized hypergeometric function defined by

$$ {}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{(\beta_1)_n(\beta_2)_n}\frac{z^n}{n!} \qquad (|z|<1). \tag{1.6} $$

Remark 1.3. The Mahler measure for $f(X,Y) := X + X^{-1} + Y + Y^{-1} + k$ was studied by Rodriguez-Villegas [RV, §15]. His method [RV, §11] is extendable to our cases, see [KLO, §6.1] for the proof of Theorem 6 along with his method. But in the case $-4 < k < 4$, in which f has zeros on $\mathbb{T}^2$, his method is not applicable. Our proof has potentialities to treat such a case.

We end the introduction by mentioning j-higher Mahler measure

$$ m_j(f) := \int_0^1 \cdots \int_0^1 (\log|f(e^{2\pi i t_1},\ldots,e^{2\pi i t_r})|)^j \, dt_r \cdots dt_1 $$

recently introduced and studied by Kurokawa-Lalín-Ochiai [KLO]. According to [KLO], $m_j(f)$ are related to (multiple) zeta values for some polynomials f. For example, they obtained

$$ m_j(X-1) = (-1)^j j! \sum_{h\geq 1} \frac{1}{2^{2h}} \sum_{\substack{b_1,\ldots,b_h\geq 2\\ b_1+\cdots+b_h=j}} \zeta(b_1,\ldots,b_h), $$

where the $\zeta(b_1,\ldots,b_h)$ are multiple zeta values defined by

$$ \zeta(b_1,\ldots,b_h) = \sum_{0<n_1<\cdots<n_h} \frac{1}{n_1^{b_1}\cdots n_h^{b_h}}. $$

As was pointed out in [KLO], $m_j(f)$ are the Taylor coefficients of $Z(s,f)$ as follows:

$$ Z(s,f) = \sum_{j=0}^{\infty} \frac{m_j(f)}{j!} s^j. $$

From Theorems 2 and 4 together with results on generating functions for sums of multiple polylogarithms obtained by Ohno-Zagier [OZ] (see (3.14)), we can express $m_j(X+a)$ and $m_j(X+X^{-1}+k)$ with $|k|>2$ in terms of multiple polylogarithms. For example, we have

Theorem 7. *For $j \geq 2$ and $a \in \mathbb{C}$ satisfying $|a|<1$ we have*

$$ m_j(X+a) = (-1)^j j! \sum_{\frac{j}{2}-1\leq n\leq j-2} \frac{1}{2^{2(j-n-1)}} \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_n)\in\{1,2\}^n\\ \#\{i:\varepsilon_i=2\}=j-n-2}} L_{(\varepsilon_1,\ldots,\varepsilon_n,2)}(|a|^2), $$

where

$$L_{(b_1,\dots,b_h)}(t) := \sum_{0<n_1<\cdots<n_h} \frac{t^{n_h}}{n_1^{b_1}\cdots n_h^{b_h}}.$$

This paper is organized as follows. In §2 we develop a fundamental theory for the zeta Mahler measure, including absolutely convergent and holomorphic regions, transformation formulas and the validity of (1.2). In §3 we prove Theorems 1–5, (1.5) and Theorem 7. In §4 we treat zeta Mahler measures for $(X_1+X_1^{-1})+\cdots+(X_r+X_r^{-1})+k$, including the proof of Theorem 6.

Notation For convenience we collect the notation frequently used in this paper.

$F = {}_2F_1$: the hypergeometric function given by the analytic continuation of (1.4) to $z \in \mathbb{C}\setminus[1,\infty)$.

${}_3F_2$: the generalized hypergeometric function given by the analytic continuation of (1.6) to $z \in \mathbb{C}\setminus[1,\infty)$.

$\mathfrak{S}_r$: the symmetric group on $\{1,\dots,r\}$.

μ_r: the Lebesgue measure on $\mathbb{R}^r$.

$\mathbb{T}^r$: the r-dimensional torus given by $\{(z_1,\dots,z_r)\in\mathbb{C}^r : |z_1|=\cdots=|z_r|=1\}$.

2 Fundamental properties of the zeta Mahler measure

In this section we will give fundamental properties of the zeta Mahler measure. In some parts of this section we refer to [EW, Chapter 3], which establishes fundamental properties of the classical Mahler measure.

First we consider absolutely convergent and holomorphic regions of (1.1) and the validity of (1.2). For Laurent polynomials $f \in \mathbb{C}[X_1^{\pm1},\dots,X_r^{\pm1}]\setminus\{0\}$ we define $\sigma_0(f)$ by

$$\sigma_0(f) := \inf\left\{\sigma\in\mathbb{R} : \int_0^1\cdots\int_0^1 |f(e^{2\pi i t_1},\dots,e^{2\pi i t_r})|^\sigma dt_r\cdots dt_1 < \infty\right\} \in \mathbb{R}\cup\{-\infty\}.$$

We remark that $\sigma_0(f)\le 0$ because $\int_0^1\cdots\int_0^1 |f(e^{2\pi i t_1},\dots,e^{2\pi i t_r})|^0 dt_r\cdots dt_1 = 1 < \infty$.

Proposition 2.1. *Let* $f \in \mathbb{C}[X_1^{\pm 1}, \dots, X_r^{\pm 1}] \setminus \{0\}$. *Then the integral in (1.1) converges absolutely and locally uniformly in* $\mathrm{Re}(s) > \sigma_0(f)$. *In particular, in* $\mathrm{Re}(s) > \sigma_0(f)$, $Z(s, f)$ *is holomorphic and we have*

$$\frac{d^k Z}{ds^k}(s, f) = \int_0^1 \cdots \int_0^1 |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|^s (\log |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|)^k dt_r \cdots dt_1.$$

Proof. Let $\varepsilon > 0$, $R \geq \max\{10,\ \sigma_0(f) + \varepsilon\}$ and $\sigma_0(f) + \varepsilon \leq \mathrm{Re}(s) \leq R$. Then, by the definition of $\sigma_0(f)$, there exists $\delta = \delta(f, \varepsilon) \in [0, \varepsilon)$ such that

$$\int_0^1 \cdots \int_0^1 |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|^{\sigma_0(f)+\delta} dt_r \cdots dt_1 < \infty. \tag{2.1}$$

We divide $[0, 1]^r$ into

$$X_r^+(f) := \{(t_1, \dots, t_r) \in [0, 1]^r : |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})| \geq 1\}, \tag{2.2}$$

$$X_r^-(f) := \{(t_1, \dots, t_r) \in [0, 1]^r : |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})| < 1\}. \tag{2.3}$$

If $(t_1, \dots, t_r) \in X_r^+(f)$, then we have

$$\begin{aligned} ||f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|^s| &\leq |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|^{\mathrm{Re}(s)} \\ &\leq |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|^R \leq M^R, \end{aligned}$$

where $M := \max_{(z_1, \dots, z_r) \in \mathbb{T}^r} |f(z_1, \dots, z_r)|$. Note that the maximal value M exists because $\mathbb{T}^r$ is compact. On the other hand, if $(t_1, \dots, t_r) \in X_r^-(f)$, then we have

$$||f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|^s| \leq |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|^{\sigma_0(f)+\delta}.$$

From $\mu_r(X_r^+(f)) \leq 1$ and (2.1) we have

$$\int \cdots \int_{X_r^+(f)} M^R dt_r \cdots dt_1 + \int \cdots \int_{X_r^-(f)} |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_r})|^{\sigma_0(f)+\delta} dt_r \cdots dt_1 < \infty.$$

This completes the proof. □

We give an estimate of $\sigma_0(f)$. First, we easily see that

$$\sigma_0(X_1^{m_1} \cdots X_r^{m_r} f) = \sigma_0(f) \text{ for any } (m_1, \dots, m_r) \in \mathbb{Z}^r, \tag{2.4}$$

$$\sigma_0(f^\tau) = \sigma_0(f) \text{ for any } \tau \in \mathfrak{S}_r, \tag{2.5}$$

where for $f(X_1,\dots,X_r) := \sum_{\underline{n}=(n_1,\dots,n_r)\in\mathbb{Z}^r} c(\underline{n})X_1^{n_1}\cdots X_r^{n_r}$ we define $f^\tau(X_1,\dots,X_r) := \sum_{\underline{n}=(n_1,\dots,n_r)\in\mathbb{Z}^r} c(\underline{n})X_{\tau(1)}^{n_1}\cdots X_{\tau(r)}^{n_r}$. From (2.4) it is sufficient to consider $\sigma_0(f)$ for polynomials only. In order to state the results, we introduce some more notations. As usual, for $f(X_1,\dots,X_r) = \sum_{n_1,\dots,n_r\geq 0} c(n_1,\dots,n_r)X_1^{n_1}\cdots X_r^{n_r} \in \mathbb{C}[X_1,\dots,X_r]\setminus\{0\}$ we denote $\deg(f)$, $\deg_{X_j}(f)$ by $\deg(f) := \max\{n_1+\cdots+n_r : c(n_1,\dots,n_r)\neq 0\}$, $\deg_{X_j}(f) := \max\{n_j^0 : c(n_1,\dots,n_r)\neq 0 \text{ and } n_j = n_j^0 \text{ for some } (n_1,\dots,n_r)\in(\mathbb{Z}_{\geq 0})^r\}$, respectively.

Definition 2.2. For $f(X_1,\dots,X_r)\in\mathbb{C}[X_1,\dots,X_r]\setminus\{0\}$ we define $d_r(f)$ inductively by

$$\begin{cases} d_1(f) := \deg(f) & \text{if } r = 1,\\ d_r(f) := \deg_{X_r}(f) + d_{r-1}(g) & \text{if } r\geq 2,\end{cases}$$

where $g(X_1,\dots,X_{r-1})\in\mathbb{C}[X_1,\dots,X_{r-1}]\setminus\{0\}$ is the coefficient of $X_r^{\deg_{X_r}(f)}$ for f. We also define $d_r^{\min}(f)$ by

$$d_r^{\min}(f) := \min_{\tau\in\mathfrak{S}_r} d(f^\tau).$$

We note that $d_r^{\min}(f)\leq d_r(f)\leq\deg(f)$. Estimates of $\sigma_0(f)$ are given as follows:

Theorem 8. *Let $f(X_1,\dots,X_r)\in\mathbb{C}[X_1,\dots,X_r]\setminus\{0\}$. Then*

(1) $\sigma_0(f)\geq -1/d_r^{\min}(f)$.

(2) *If f does not vanish on $\mathbb{T}^r$, then $\sigma_0(f) = -\infty$.*

Remark 2.3. Theorem 8 (1) is a crude bound because $\sigma_0(f)$ should be determined not by the degree of f but on the behavior of f near its zeros on $\mathbb{T}^r$.

Combining Proposition 2.1 and Theorem 8 (1), we obtain

Corollary 2.4. *Equation (1.1) is valid for any $f(X_1,\dots,X_r)\in\mathbb{C}[X_1^{\pm1},\dots,X_r^{\pm1}]\setminus\{0\}$.*

For the proof of Theorem 8, the following lemma is crucial.

Lemma 2.5. *Let $f\in\mathbb{C}[X_1^{\pm1},\dots,X_r^{\pm r}]\setminus\{0\}$. Then there exists $C = C_r(f) > 0$ such that*

$$\mu_r(\{(t_1,\dots,t_r)\in[0,1]^r : |f(e^{2\pi i t_1},\dots,e^{2\pi i t_r})|\leq\varepsilon\})\leq C\varepsilon^{1/d_r^{\min}(f)} \tag{2.6}$$

for any $\varepsilon > 0$.

Remark 2.6. This lemma is essentially due to Everest-Ward [EW, p.58, Lemma 3.8]. But they did not give an explicit exponent for ε. We will give the exponent $1/d_r^{\min}(f)$ using their method.

Remark 2.7. For one variable (Laurent) polynomials, there is a stronger bound due to Lawton [Law, Theorem 1] than (2.6).

Proof of Lemma 2.5. From (2.5), it is sufficient to show that the left hand side of (2.6) is bounded above by $C\varepsilon^{1/d_r(f)}$ for some $C = C_r(f) > 0$. We prove this by induction on r. We consider the case $r = 1$. Let $f(X) \in \mathbb{C}[X]$ be nonzero polynomials with degree $d = d_1(f)$. We factorize $f(X)$ as

$$f(X) = a\prod_{j=1}^{d}(X - g_j)$$

with $a \in \mathbb{C}^\times$ and $g_j \in \mathbb{C}$. Take $z \in \mathbb{C}$ with $|f(z)| \leq \varepsilon$. Then, since $|\prod_{j=1}^{d}(z - g_j)| \leq |a|^{-1}\varepsilon$, there exists $j \in \{1, \ldots, d\}$ such that $|z - g_j| \leq (|a|^{-1}\varepsilon)^{1/d}$. Hence, we have

$$\begin{aligned} & \mu_1(\{t \in [0,1] : |f(e^{2\pi it})| \leq \varepsilon\}) \\ \leq\ & \mu_1\left(\bigcup_{j=1}^{d}\{t \in [0,1] : |e^{2\pi it} - g_j| \leq (|a|^{-1}\varepsilon)^{1/d}\}\right) \\ \leq\ & \sum_{j=1}^{d}\mu_1(\{t \in [0,1] : |e^{2\pi it} - g_j| \leq (|a|^{-1}\varepsilon)^{1/d}\}) \\ =\ & \sum_{j=1}^{d}\mu_1(\{t \in [0,1] : |e^{2\pi it} - |g_j|| \leq (|a|^{-1}\varepsilon)^{1/d}\}). \end{aligned} \tag{2.7}$$

In the last equality we used the periodicity of $e^{2\pi it}$. We assume that $t \in [0,1]$ satisfies $|e^{2\pi it} - |g_j|| \leq (|a|^{-1}\varepsilon)^{1/d}$. Then we get $||g_j| - 1| \leq (|a|^{-1}\varepsilon)^{1/d}$ by the triangle inequality. By the triangle inequality again, we obtain $|e^{2\pi it} - 1| \leq |e^{2\pi it} - |g_j|| + ||g_j| - 1| \leq 2(|a|^{-1}\varepsilon)^{1/d}$. Hence, (2.7) is estimated above as

$$\begin{aligned} \leq\ & \sum_{j=1}^{d}\mu_1(\{t \in [0,1] : |e^{2\pi it} - 1| \leq 2(|a|^{-1}\varepsilon)^{1/d}\}) \\ =\ & \sum_{j=1}^{d}\mu_1(\{t \in [0,1] : \sin(\pi t) \leq (|a|^{-1}\varepsilon)^{1/d}\}) \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{j=1}^{d}\mu_1(\{t\in[0,1/2]:\sin(\pi t)\le(|a|^{-1}\varepsilon)^{1/d}\})\\
&\le 2\sum_{j=1}^{d}\mu_1(\{t\in[0,1/2]:t\le(|a|^{-1}\varepsilon)^{1/d}/2\})\\
&\le d|a|^{-1/d}\varepsilon^{1/d}.
\end{aligned}
$$

Here, in the fourth line we used $\sin(\pi t)\ge 2t$ for any $t\in[0,1/2]$. Hence we obtain the lemma in the case $r=1$.

Let $r\ge 2$ and suppose that the lemma is true for $r-1$. Let f be r-variable nonzero polynomials. For $(z_1,\dots,z_{r-1})\in\mathbb{C}^{r-1}$ we factorize f as

$$
f(z_1,\dots z_{r-1},X_r)=a(z_1,\dots,z_{r-1})\prod_{j=1}^{m}(X_r-g_j(z_1,\dots,z_{r-1}))
$$

where $m=\deg_{X_r}(f)$, $a(X_1,\dots,X_{r-1})\in\mathbb{C}[X_1,\dots,X_{r-1}]\setminus\{0\}$ is the coefficient of X_r^m for f and g_j are suitable branches of algebraic functions. Let $\varepsilon'>0$. We divide the left hand side of (2.6) into two parts according as $|a(e^{2\pi it_1},\dots,e^{2\pi it_{r-1}})|\le\varepsilon'$ or $>\varepsilon'$. From the assumption of the induction we estimate the former part as follows:

$$
\begin{aligned}
&\mu_r(\{(t_1,\dots,t_r)\in[0,1]^r:|f(e^{2\pi it_1},\dots,e^{2\pi it_r})|\le\varepsilon,\ |a(e^{2\pi it_1},\dots,e^{2\pi it_{r-1}})|\le\varepsilon'\})\\
\le\ & C_{r-1}(a)(\varepsilon')^{1/d_r(a)}.
\end{aligned}
$$

On the other hand, the latter part is estimated by

$$
\begin{aligned}
&\mu_r(\{(t_1,\dots,t_r)\in[0,1]^r:|f(e^{2\pi it_1},\dots,e^{2\pi it_r})|\le\varepsilon,\ |a(e^{2\pi it_1},\dots,e^{2\pi it_{r-1}})|>\varepsilon'\})\\
\le\ & \mu_r\left(\left\{(t_1,\dots,t_r)\in[0,1]^r:\prod_{j=1}^{m}|e^{2\pi it_r}-g_j(e^{2\pi it_1},\dots,e^{2\pi it_{r-1}})|\le\frac{\varepsilon}{\varepsilon'}\right\}\right). \qquad (2.8)
\end{aligned}
$$

In the same manner as the case $r=1$, (2.8) is bounded above by $m(\varepsilon/\varepsilon')^{1/m}$. Hence the left hand side of (2.6) is

$$
\le C_{r-1}(a)(\varepsilon')^{1/d_{r-1}(a)}+m(\varepsilon/\varepsilon')^{1/m}.
$$

Taking $\varepsilon'=\varepsilon^{\frac{d_{r-1}(a)}{d_{r-1}(a)+m}}$, we obtain the desired result. □

Proof of Theorem 8. (2) Since $\mathbb{T}^r$ is compact, there exist m and M such that $0 < m \leq |f(z_1, \ldots, z_r)| \leq M$ for any $(z_1, \ldots, z_r) \in \mathbb{T}^r$. This implies $\sigma_0(f) = -\infty$.
(1) It is sufficient to prove

$$\int_0^1 \cdots \int_0^1 |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r})|^\sigma dt_r \cdots dt_1 < \infty$$

for any $\sigma \in (-1/d_r^{\min}(f), 0)$. We divide $[0,1]^r$ into $X_r^+(f)$ and $X_r^-(f)$, which are given by (2.2) and (2.3), respectively. We first consider the integral on $X_r^+(f)$. Since f is bounded on $\mathbb{T}^r$, we have

$$\int_{X_r^+(f)} |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r})|^\sigma dt_r \cdots dt_1 < \infty.$$

On the other hand, from Lemma 2.5, the integral on $X_r^-(f)$ is estimated as follows:

$$\begin{aligned} & \int_{X_r^-(f)} |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r})|^\sigma dt_r \cdots dt_1 \\ = & \sum_{n=0}^\infty \int_{\substack{(t_1,\ldots,t_r) \in X_r^-(f), \\ 2^{-(n+1)} \leq |f(e^{2\pi i t_1},\ldots,e^{2\pi i t_r})| < 2^{-n}}} |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r})|^\sigma dt_r \cdots dt_1 \\ \leq & \sum_{n=0}^\infty 2^{-\sigma(n+1)} \mu_r(\{(t_1, \ldots, t_r) \in [0,1]^r : |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r})| < 2^{-n}\}) \\ \leq & C_r(f) \sum_{n=0}^\infty 2^{-\sigma(n+1)} \cdot 2^{-n/d_r^{\min}(f)} < \infty. \end{aligned}$$

Hence we obtain Theorem 8 (1). □

Next we give transformation formulas for zeta Mahler measures. Let $A \in M_r(\mathbb{Z}) \cap GL_r(\mathbb{Q})$ and $f(\underline{X}) := \sum_{\underline{n}=(n_1,\ldots,n_r) \in \mathbb{Z}^r} c(\underline{n}) \underline{X}^{\underline{n}}$ be Laurent polynomials, where $\underline{X} := (X_1, \ldots, X_r)$, $\underline{X}^{\underline{n}} := X_1^{n_1} \cdots X_r^{n_r}$ and $c(\underline{n}) \in \mathbb{C}$ such that $c(\underline{n}) = 0$ except for at most finitely many $\underline{n} \in \mathbb{Z}^r$. Then $f^{(A)}(\underline{X}) \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ is defined by

$$f^{(A)}(\underline{X}) := \sum_{\underline{n}=(n_1,\ldots,n_r) \in \mathbb{Z}^r} c(\underline{n}) \underline{X}^{\underline{n}A},$$

where $\underline{n}A$ is the usual product of matrices. Then zeta Mahler measures have the following properties:

Theorem 9. *(cf. [EW, Exercise 3.1.]) Let $f(X_1, \ldots, X_r) \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}] \setminus \{0\}$. Then*
(1) $Z(s, 1) = 1$ *for any* $s \in \mathbb{C}$.

(2) $Z(s, af) = |a|^s Z(s, f)$ *for any* $a \in \mathbb{C}^\times$ *and* $\mathrm{Re}(s) > \sigma_0(f)$.

(3) $Z(s, f^k) = Z(ks, f)$ *for any* $k \in \mathbb{Z}_{\geq 1}$ *and* $\mathrm{Re}(s) > \sigma_0(f)/k$.

(4) $Z(s, f^{(A)}) = Z(s, f)$ *for any* $A \in M_r(\mathbb{Z}) \cap GL_r(\mathbb{Q})$ *and* $\mathrm{Re}(s) > \max\{\sigma_0(f), \sigma_0(f^{(A)})\}$.

Remark 2.8. A property corresponding to $m(fg) = m(f) + m(g)$ seems absent for zeta Mahler measures.

Proof. It is easy to show (1)-(3). We prove (4). We restrict s to $\mathrm{Re}(s) > 0$ and finally we relax this restriction by analytic continuation (see Proposition 2.1). First, we easily see that for any $A, B \in M_r(\mathbb{Z}) \cap GL_r(\mathbb{Q})$ we have $f^{(AB)} = (f^{(A)})^{(B)}$, in particular,

$$Z(s, f^{(AB)}) = Z(s, (f^{(A)})^{(B)}).$$

We also note that any nonsingular matrices $A \in M_r(\mathbb{Z}) \cap GL_r(\mathbb{Q})$ can be expressed as the product of some matrices of the following three types (i), (ii), (iii): (i) $r \times r$ lower triangle nonsingular matrices with integer entries, (ii) $r \times r$ upper triangle nonsingular matrices with integer entries, (iii) $(\delta_{i,\tau(j)})_{1 \leq i,j \leq r}$ for transpositions $\tau = (k\ l) \in \mathfrak{S}_r$, where δ_{ij} is the Kronecker's delta. This fact follows from elementary row operations of matrices combined with the Euclidean algorithm; see [M, p.33, Theorem 22.3]. From the above facts, it is sufficient to show (4) for type (i)-(iii) matrices. We easily see that (4) holds when A are type (iii) matrices. We treat type (i) matrices. Suppose that $A = (a_{ij})_{1 \leq i,j \leq r} \in M_r(\mathbb{Z}) \cap GL_r(\mathbb{Q})$ satisfies $a_{ij} = 0$ for any $i < j$ and put $f(\underline{X}) := \sum_{\underline{n} \in \mathbb{Z}^r} c(\underline{n}) \underline{X}^{\underline{n}}$. Then we prove $Z(s, f^{(A)}) = Z(s, f)$ by induction on r. When $r = 1$, we have $A = (a) \in M_1(\mathbb{Z})$ with $a \in \mathbb{Z} \setminus \{0\}$. Then we have $f^{(A)}(X) = \sum_{n \in \mathbb{Z}} c(n) X^{an}$. If $a > 0$, then we have

$$\begin{aligned} Z(s, f^{(A)}) &= \int_0^1 \left| \sum_{n \in \mathbb{Z}} c(n) e(nat) \right|^s dt = \frac{1}{a} \int_0^a \left| \sum_{n \in \mathbb{Z}} c(n) e(nu) \right|^s du \\ &= \frac{1}{a} \sum_{k=0}^{a-1} \int_k^{k+1} \left| \sum_{n \in \mathbb{Z}} c(n) e(nu) \right|^s du = \frac{1}{a} a \int_0^1 \left| \sum_{n \in \mathbb{Z}} c(n) e(nu) \right|^s du = Z(s, f), \end{aligned}$$

where $e(x) := e^{2\pi i x}$. If $a < 0$, changing the variable $t' = 1 - t$ reduces to the case $a > 0$. Hence, we obtain the desired result in the case $r = 1$. Let $r \in \mathbb{Z}_{\geq 2}$. Then we have $f^{(A)}(\underline{X}) =$

$\sum_{\underline{n}=(n_1,\ldots,n_r)\in\mathbb{Z}^r} c(\underline{n}) X_1^{a_{11}n_1}(X_1^{a_{21}}X_2^{a_{22}})^{n_2}\cdots(X_1^{a_{r1}}\cdots X_r^{a_{rr}})^{n_r}$. Hence we have

$$Z(s,f^{(A)}) = \int_0^1 \cdots \int_0^1 \left| \sum_{\underline{n}\in\mathbb{Z}^r} c(\underline{n}) e\left(\sum_{j=1}^r n_j(a_{j1}t_1+\cdots+a_{jj}t_j)\right)\right|^s dt_r\cdots dt_1.$$

Changing the variable t_r by $u_r = a_{r1}t_1+\cdots+a_{rr}t_r$ together with the same argument as the case $r=1$, we have

$$\begin{aligned} Z(s,f^{(A)}) &= \int_0^1 \cdots \int_0^1 \int_0^1 \\ &\left| \sum_{\underline{n}'=(n_1,\ldots,n_{r-1})\in\mathbb{Z}^{r-1}} \left(\sum_{n_r\in\mathbb{Z}} c(\underline{n}',n_r)e(n_r u_r)\right) e\left(\sum_{j=1}^{r-1} n_j(a_{j1}t_1+\cdots+a_{jj}t_j)\right)\right|^s du_r dt_{r-1}\cdots dt_1 \\ &= \int_0^1 Z(s,(\widetilde{f}_{u_r})^{(A')})du_r, \end{aligned}$$

where $A' := (a_{ij})_{1\le i,j\le r-1}$ and

$$\widetilde{f}_{u_r}(X_1,\ldots,X_{r-1}) := \sum_{\underline{n}'=(n_1,\ldots,n_{r-1})\in\mathbb{Z}^{r-1}} \left(\sum_{n_r\in\mathbb{Z}} c(\underline{n}',n_r)e(n_r u_r)\right) X_1^{n_1}\cdots X_{r-1}^{n_{r-1}}.$$

Applying the assumption of the induction to $Z(s,(\widetilde{f}_{u_r})^{(A')})$, we obtain the desired result.

In the same manner, we obtain Theorem 9 (4) for (ii) type matrices A. □

3 Examples of zeta Mahler measures for one variable

In this section we show Theorems 1–5, (1.5) and Theorem 7.

3.1 Zeta Mahler measures of $X+a$ for $a\in\mathbb{C}$

Proof of Theorem 1. When $a=0$, by definition we have $Z(s,f)=1=1^{s/2}F(1/2,0;1;0)$, that is, Theorem 1 holds. We consider the case $a\in\mathbb{C}\setminus\{0\}$. For $t\in[0,1]$ we have

$$\begin{aligned} |e^{2\pi i t}+a|^2 &= (\cos(2\pi t)+\mathrm{Re}(a))^2+(\sin(2\pi t)+\mathrm{Im}(a))^2 \\ &= |a|^2+1+2(\mathrm{Re}(a)\cos(2\pi t)+\mathrm{Im}(a)\sin(2\pi t)). \end{aligned}$$

Here, there exists $\theta = \theta(a) \in [0,1]$ such that $\cos(2\pi\theta) = \mathrm{Re}(a)/|a|$ and $\sin(2\pi\theta) = \mathrm{Im}(a)/|a|$. Hence we have

$$|e^{2\pi it} + a|^2 = |a|^2 + 1 + 2|a|\cos(2\pi(t-\theta)).$$

Therefore, we have

$$\begin{aligned} Z(s,f) &= \int_0^1 (|a|^2 + 1 + 2|a|\cos(2\pi(t-\theta)))^{s/2} dt \\ &= (|a|^2+1)^{s/2} \int_0^1 \left(1 + \frac{2|a|}{|a|^2+1}\cos(2\pi t)\right)^{s/2} dt. \end{aligned} \tag{3.1}$$

When $|a| = 1$, for $\mathrm{Re}(s) > -1$, (3.1) becomes

$$\begin{aligned} Z(s,f) &= 2^{s/2} \int_0^1 (1+\cos(2\pi t))^{s/2} dt \\ &= 2^s \int_0^1 |\cos(\pi t)|^s dt = 2^{s+1} \int_0^{1/2} (\cos(\pi t))^s dt \\ &= \frac{2^s}{\pi} \int_0^1 u^{(s-1)/2}(1-u)^{-1/2} du = \frac{2^s}{\pi} B\left(\frac{s+1}{2}, \frac{1}{2}\right) \\ &= \frac{2^s}{\pi} \frac{\Gamma(\frac{s+1}{2})\Gamma(1/2)}{\Gamma(\frac{s}{2}+1)} = 2^s \pi^{-1/2} \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2}+1)}. \end{aligned}$$

Here, in the fourth equality we put $u = \cos^2(\pi t)$. Hence we obtain Theorem 1 (1).

We turn to the remaining case $|a| \neq 0, 1$. From the binomial expansion, (3.1) is calculated as follows:

$$Z(s,f) = (|a|^2+1)^{s/2} \sum_{n=0}^{\infty} \binom{s/2}{n} \left(\frac{2|a|}{|a|^2+1}\right)^n \int_0^1 \cos^n(2\pi t) dt. \tag{3.2}$$

Here, the interchange between the sum and the integral is justified from $2|a|/(|a|^2+1) < 1$. Recall that

$$\int_0^1 \cos^n(2\pi t) dt = \begin{cases} 0 & n\text{: odd}, \\ \dfrac{(n-1)!!}{n!!} & n\text{: even}, \end{cases}$$

which follows from integration by parts. Here $(2k)!! := 2k(2k-2)\cdots 2$, $(2k-1)!! := (2k-1)(2k-3)\cdots 1$ and $0!! = (-1)!! := 1$. Applying this to (3.2), we obtain

$$Z(s,f) = (|a|^2+1)^{s/2} \sum_{n=0}^{\infty} \binom{s/2}{2n} \frac{(2n-1)!!}{(2n)!!} \left(\frac{2|a|}{|a|^2+1}\right)^{2n}. \tag{3.3}$$

For any $n \in \mathbb{Z}_{\geq 1}$ we have

$$\begin{aligned}
& \binom{s/2}{2n} \frac{(2n-1)!!}{(2n)!!} \\
= \ & \frac{\frac{s}{2}(\frac{s}{2}-1)\cdots(\frac{s}{2}-2n+1)}{(2n)!} \frac{(2n-1)!!}{(2n)!!} \\
= \ & \frac{(-1)^{2n}(-\frac{s}{2})(-\frac{s}{2}+1)\cdots(-\frac{s}{2}+2n-1)}{((2n)!!)^2} \\
= \ & \frac{\{(-\frac{s}{2})(-\frac{s}{2}+2)\cdots(-\frac{s}{2}+2n-2)\}\{(-\frac{s}{2}+1)(-\frac{s}{2}+3)\cdots(-\frac{s}{2}+2n-1)\}}{2^{2n}(1)_n n!} \\
= \ & \frac{(2^n(-\frac{s}{4})_n)(2^n(-\frac{s}{4}+\frac{1}{2})_n)}{2^{2n}(1)_n n!} = \frac{(-\frac{s}{4})_n(-\frac{s}{4}+\frac{1}{2})_n}{(1)_n n!}.
\end{aligned}$$

Applying this to (3.3), we obtain Theorem 1 (2). □

Proof of Theorem 2. According to [Le, p.251, (9.6.5)], we have

$$F\left(\alpha, \alpha+\frac{1}{2}; \gamma; z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(2\alpha, 2\alpha-\gamma+1; \gamma; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$$
$$(|\arg(1-z)| < \pi).$$

Put $\alpha = -s/4$, $\gamma = 1$ and $z = (\frac{2|a|}{|a|^2+1})^2$ and use $\sqrt{1-z} = ||a|^2-1|/(|a|^2+1)$. □

Proof of Theorem 3. Applying

$$F(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; z) \qquad (|\arg(1-z)| < \pi) \tag{3.4}$$

[Le, p.248, (9.5.3)] with $\alpha = \beta = -s/2$, $\gamma = 1$ and $z = \min\{|a|^2, |a|^{-2}\}$ to Theorem 2. □

3.2 Zeta Mahler measures of $X + X^{-1} + k$ for $k \in \mathbb{R}$

Proof of Theorem 4. Let $k \in \mathbb{R}$ and put $f(X) = X + X^{-1} + k$. Then we have

$$Z(s, f) = \int_0^1 |2\cos(2\pi t) + k|^s dt.$$

From this, we easily see $Z(s, X + X^{-1} + k) = Z(s, X + X^{-1} - k)$. Hence, it is sufficient to prove Theorem 4 for $k \geq 0$. We continue to calculate $Z(s, f)$ as follows:

$$Z(s, f) = \int_0^1 |2(2\cos^2(\pi t) - 1) + k|^s dt = 2\int_0^{1/2} |4\cos^2(\pi t) + k - 2|^s dt.$$

We replace the variable t by $u = \cos^2(\pi t)$. Then, since $dt = -du/(2\pi u^{1/2}(1-u)^{1/2})$, we have

$$Z(s,f) = \frac{1}{\pi}\int_0^1 u^{-1/2}(1-u)^{-1/2}|4u+k-2|^s du. \tag{3.5}$$

First we treat the case $k > 2$. Since in this case $4u + k - 2 > 0$ holds for any $u \in [0,1]$, we have

$$Z(s,f) = \frac{(k-2)^s}{\pi}\int_0^1 u^{-1/2}(1-u)^{-1/2}\left(1+\frac{4}{k-2}u\right)^s du. \tag{3.6}$$

Applying

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt \tag{3.7}$$
$$(\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0,\ |\arg(1-z)| < \pi)$$

[Le, p.240, (9.1.6)] with $\alpha = -s$, $\beta = 1/2$, $\gamma = 1$, $z = -4/(k-2)$ to (3.6), we get

$$Z(s,f) = (k-2)^s F\left(-s, \frac{1}{2}; 1; -\frac{4}{k-2}\right).$$

The formula [Le, p.253, (9.6.12)]

$$F(\alpha,\beta;2\beta;z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(\alpha, \alpha-\beta+\frac{1}{2}; \beta+\frac{1}{2}; \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right)$$
$$(|\arg(1-z)| < \pi,\ 2\beta \neq -1, -3, -5, \ldots)$$

with $\alpha = -s$, $\beta = 1/2$, $z = -4/(k-2)$ leads to Theorem 4 (1).

Next we deal with the case $k = 2$. Since $Z(s,f) = Z(s, X^2+2X+1) = Z(2s, X+1)$, we immediately obtain Theorem 4 (2) from Theorem 1 (1). Finally we treat the case $0 \le k < 2$. From (3.5) we have

$$\begin{aligned} Z(s,f) &= \frac{1}{\pi}\int_0^{\frac{2-k}{4}} u^{-1/2}(1-u)^{-1/2}(2-k-4u)^s du \\ &+ \frac{1}{\pi}\int_{\frac{2-k}{4}}^1 u^{-1/2}(1-u)^{-1/2}(4u+k-2)^s du \end{aligned} \tag{3.8}$$

We calculate the first integral. Replacing u by $\frac{2-k}{4}u$, we have

$$\int_0^{\frac{2-k}{4}} u^{-1/2}(1-u)^{-1/2}(2-k-4u)^s du$$

$$= \frac{(2-k)^{s+\frac{1}{2}}}{2}\int_0^1 u^{-1/2}(1-u)^s\left(1-\frac{2-k}{4}u\right)^{-1/2}du$$

$$= \frac{\pi^{1/2}(2-k)^{s+\frac{1}{2}}}{2}\frac{\Gamma(s+1)}{\Gamma(s+\frac{3}{2})}F\left(\frac{1}{2},\frac{1}{2};s+\frac{3}{2};\frac{2-k}{4}\right). \tag{3.9}$$

Here, in the last equality we used (3.7). Next we calculate the second integral in (3.8). Replacing u by $1-\frac{k+2}{4}u$, we have

$$\int_{\frac{2-k}{4}}^1 u^{-1/2}(1-u)^{-1/2}(4u+k-2)^s du$$

$$= \frac{(2+k)^{s+\frac{1}{2}}}{2}\int_0^1 u^{-1/2}(1-u)^s\left(1-\frac{2+k}{4}u\right)^{-1/2}du$$

$$= \frac{\pi^{1/2}(2+k)^{s+\frac{1}{2}}}{2}\frac{\Gamma(s+1)}{\Gamma(s+\frac{3}{2})}F\left(\frac{1}{2},\frac{1}{2};s+\frac{3}{2};\frac{2+k}{4}\right). \tag{3.10}$$

Here, we used (3.7) in the last equality. Applying (3.9) and (3.10) to (3.8), we obtain Theorem 4 (3). □

Proof of Theorem 5. Suppose that $k\in\mathbb{R}$ satisfies $|k|>2$. Then, applying (3.4) with $\alpha=\beta=-s$, $\gamma=1$, $z=\{(|k|-\sqrt{k^2-4})/2\}^2$ to Theorem 4, we obtain the desired result. □

Proof of (1.5). Let $f(X)=X+X^{-1}+k$ with $-2<k<2$. Then, from (1.3) and $|\frac{-k\pm\sqrt{k^2-4}}{2}|=1$ we have

$$Z'(0,f)=m(f)=0. \tag{3.11}$$

We calculate the derivative of Theorem 4 (3) at $s=0$. For this purpose, for $0<x<1$ we evaluate $\Gamma(s+\frac{3}{2})^{-1}F(\frac{1}{2},\frac{1}{2};s+\frac{3}{2};x)$ and its derivative at $s=0$. We easily see that

$$\Gamma\left(s+\frac{3}{2}\right)^{-1}F\left(\frac{1}{2},\frac{1}{2};s+\frac{3}{2};x\right)\Bigg|_{s=0} = \frac{2}{\pi^{1/2}}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x\right)$$

$$= \frac{2}{(\pi x)^{1/2}}\arcsin(x^{1/2}). \tag{3.12}$$

Here, in the last equality we used the formula $\arcsin(z)=zF(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2)$ (see [Le, p.259, (9.8.5)]). From (1.4) we have

$$\frac{\partial}{\partial s}\left(\Gamma\left(s+\frac{3}{2}\right)^{-1}F\left(\frac{1}{2},\frac{1}{2};s+\frac{3}{2};x\right)\right)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial s}\left(\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{n!\Gamma(s+\frac{3}{2}+n)}x^n\right)\\
&= -\sum_{n=0}^{\infty}\frac{1}{\Gamma(s+\frac{3}{2}+n)}\frac{\Gamma'}{\Gamma}\left(s+\frac{3}{2}+n\right)\frac{(\frac{1}{2})_n^2}{n!}x^n.
\end{aligned}
$$

Using the formulas $\Gamma(n+\frac{3}{2}) = (\frac{1}{2})_{n+1}\pi^{1/2}$ and $\frac{\Gamma'}{\Gamma}(n+\frac{3}{2}) = \sum_{m=0}^{n}\frac{1}{m+\frac{1}{2}} - \gamma - 2\log 2$ (for the latter formula see [Le, p.6, (1.3.9)]), where γ is the Euler constant, we have

$$
\begin{aligned}
&\frac{\partial}{\partial s}\left(\Gamma\left(s+\frac{3}{2}\right)^{-1}F\left(\frac{1}{2},\frac{1}{2};s+\frac{3}{2};x\right)\right)\Bigg|_{s=0}\\
&= -\frac{1}{\pi^{1/2}}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})}x^n\left(\sum_{m=0}^{n}\frac{1}{m+\frac{1}{2}}-\gamma-2\log 2\right)\\
&= \frac{2(\gamma+2\log 2)}{\pi^{1/2}}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x\right) - \frac{1}{\pi^{1/2}}\sum_{0\le m\le n}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})(m+\frac{1}{2})}x^n\\
&= \frac{2(\gamma+2\log 2)}{(\pi x)^{1/2}}\arcsin(x^{1/2}) - \frac{1}{\pi^{1/2}}\sum_{0\le m\le n}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})(m+\frac{1}{2})}x^n. \qquad (3.13)
\end{aligned}
$$

Hence, applying (3.12), (3.13) and $Z(0,f) = 1$ to Theorem 4 (3), together with routine calculations, we have

$$
\begin{aligned}
Z'(0,f) &= 2\log 2 + \frac{2}{\pi}\left(\arcsin\left(\frac{\sqrt{2+k}}{2}\right)\log(2+k) + \arcsin\left(\frac{\sqrt{2-k}}{2}\right)\log(2-k)\right)\\
&- \frac{1}{2\pi}\left((2+k)^{1/2}\sum_{0\le m\le n}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})(m+\frac{1}{2})}\left(\frac{2+k}{4}\right)^n\right.\\
&\qquad \left. + (2-k)^{1/2}\sum_{0\le m\le n}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})(m+\frac{1}{2})}\left(\frac{2-k}{4}\right)^n\right).
\end{aligned}
$$

This together with (3.11) completes the proof. □

3.3 Higher Mahler measures —Proof of Theorem 7

Proof of Theorem 7. According to [OZ, p.485, eighth formula] with suitable specializations (taking $\alpha = \beta$ and $x = 0$ in [OZ]), for $|t| < 1$ we have

$$
F(\alpha,\alpha;1;t) = 1 + \alpha^2\sum_{\substack{n,s\ge 0\\ n\ge s}} 2^{n-s}\left(\sum_{\substack{(\varepsilon_1,\dots,\varepsilon_n)\in\{1,2\}^n\\ \#\{i:\varepsilon_i=2\}=s}} L_{(\varepsilon_1,\dots,\varepsilon_n,2)}(t)\right)\alpha^{n+s}. \qquad (3.14)
$$

From this and Theorem 2, we reach the desired result. □

Remark 3.1. In the same manner we can evaluate $m_j(X+a)$ for $a \in \mathbb{C}$ satisfying $|a|>1$ and $m_j(X+X^{-1}+k)$ for $k \in \mathbb{R}$, $|k|>2$ in terms of multiple polylogarithms.

4 Examples of zeta Mahler measures for multivariable f

In this section we treat zeta Mahler measures for

$$f_r(X_1,\ldots,X_r) := (X_1+X_1^{-1})+\cdots+(X_r+X_r^{-1})+k$$

with $k \in \mathbb{C}$.

Proposition 4.1. *For $k \in \mathbb{C}$ and $\mathrm{Re}(s)>-1$ we have*

$$Z(s,f_r) = \frac{2^{2s}}{\pi^r}\int_0^r \left| t+\frac{k-2r}{4}\right|^s g^{*r}(t)dt,$$

where

$$g(t) := \begin{cases} \dfrac{1}{t^{1/2}(1-t)^{1/2}} & \text{if } 0<t<1, \\ 0 & \text{otherwise,} \end{cases}$$

the symbol $*$ is the usual convolution given by

$$(F*G)(t) := \int_{\mathbb{R}} F(u)G(t-u)du$$

and $g^{*r}(t)$ is inductively defined by $g^{*1}(t):=g(t)$, $g^{*r}(t):=(g*g^{*(r-1)})(t)$ for $r \in \mathbb{Z}_{\geq 2}$.

For the proof we need the following lemma:

Lemma 4.2. *For $m \in \mathbb{Z}_{\geq 1}$ and $t \in \mathbb{R}$ we have*

$$\begin{aligned} &\int\cdots\int_{\substack{(u_1,\ldots,u_m)\in[0,1]^m,\\ t-1\leq u_1+\cdots+u_m\leq t}} \frac{1}{u_1^{1/2}\cdots u_m^{1/2}(t-u_1-\cdots-u_m)^{1/2}} \\ &\times\frac{du_m\cdots du_1}{(1-u_1)^{1/2}\cdots(1-u_m)^{1/2}(1-t+u_1+\cdots+u_m)^{1/2}} \\ = \ & g^{*(m+1)}(t). \end{aligned}$$

Proof. We prove this lemma by induction on m. We first consider the case $m = 1$. Then the left hand side equals

$$
\begin{aligned}
&\int_{\substack{u\in[0,1] \\ t-1\le u\le t}} \frac{du}{u^{1/2}(1-u)^{1/2}(t-u)^{1/2}(1-t+u)^{1/2}} \\
&= \begin{cases} \displaystyle\int_0^t \frac{du}{u^{1/2}(1-u)^{1/2}(t-u)^{1/2}(1-t+u)^{1/2}} & \text{if } 0 < t \le 1, \\ \displaystyle\int_{t-1}^1 \frac{du}{u^{1/2}(1-u)^{1/2}(t-u)^{1/2}(1-t+u)^{1/2}} & \text{if } 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

On the other hand, since $\mathrm{supp}(g) = [0,1]$, the right hand side equals

$$
\begin{aligned}
g^{*2}(t) &= \int_{\mathbb{R}} g(u)g(t-u)du \\
&= \begin{cases} \displaystyle\int_0^t g(u)g(t-u)du & \text{if } 0 < t \le 1, \\ \displaystyle\int_{t-1}^1 g(u)g(t-u)du & \text{if } 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

From these equalities we obtain Lemma 4.2 in the case $m = 1$.

Let $m \ge 2$ and assume that the lemma holds for $m-1$. Then, from the assumption of the induction, the left hand side equals

$$
\begin{aligned}
&\int_0^1 \frac{1}{u_1^{1/2}(1-u_1)^{1/2}} \Bigg(\int \cdots \int_{\substack{(u_2,\ldots,u_m)\in[0,1]^{m-1} \\ (t-u_1)-1\le u_2+\cdots+u_m\le t-u_1}} \frac{1}{u_2^{1/2}\cdots u_m^{1/2}((t-u_1)-u_2-\cdots-u_m)^{1/2}} \\
&\times \frac{du_m\cdots du_2}{(1-u_2)^{1/2}\cdots(1-u_m)^{1/2}(1-(t-u_1)+u_2+\cdots+u_m)^{1/2}} \Bigg) du_1 \\
&= \int_0^1 g(u_1)g^{*m}(t-u_1)du_1 = g^{*(m+1)}(t).
\end{aligned}
$$

Here, in the last equality we used $\mathrm{supp}(g) = [0,1]$. We obtain the desired result. □

Proof of Proposition 4.1. We have

$$
\begin{aligned}
Z(s, f_r) &= \int_0^1 \cdots \int_0^1 |2\cos(2\pi t_1) + \cdots + 2\cos(2\pi t_r) + k|^s dt_r \cdots dt_1 \\
&= \int_0^1 \cdots \int_0^1 |4\cos^2(\pi t_1) + \cdots + 4\cos^2(\pi t_r) + k - 2r|^s dt_r \cdots dt_1
\end{aligned}
$$

$$= \quad 2^{2s}2^r \int_0^{1/2} \cdots \int_0^{1/2} \left| \cos^2(\pi t_1) + \cdots + \cos^2(\pi t_r) + \frac{k-2r}{4} \right|^s dt_r \cdots dt_1.$$

Here, we used the duplication formula for the cosine function in the second equality and $\cos(\pi - t) = -\cos(\pi t)$ in the third equality. Changing the variables $u_j = \cos^2(\pi t_j)$, we get

$$Z(s, f_r) = \frac{2^{2s}}{\pi^r} \int_0^1 \cdots \int_0^1 \frac{|u_1 + \cdots + u_r + \frac{k-2r}{4}|^s}{u_1^{1/2} \cdots u_r^{1/2} (1-u_1)^{1/2} \cdots (1-u_r)^{1/2}} du_r \cdots du_1.$$

When $r = 1$, this gives the desired result. We concentrate on the case $r \geq 2$ below. Changing u_r by $t := u_1 + \cdots + u_r$, we obtain

$$\begin{aligned} Z(s, f_r) = & \frac{2^{2s}}{\pi^r} \int_0^1 \cdots \int_0^1 \int_{u_1 + \cdots + u_{r-1}}^{u_1 + \cdots + u_{r-1} + 1} \frac{|t + \frac{k-2r}{4}|^s}{u_1^{1/2} \cdots u_{r-1}^{1/2} (t - u_1 - \cdots - u_{r-1})^{1/2}} \\ & \times \frac{dt du_{r-1} \cdots du_1}{(1-u_1)^{1/2} \cdots (1-u_{r-1})^{1/2} (1 - t + u_1 + \cdots + u_{r-1})^{1/2}} \\ = & \frac{2^{2s}}{\pi^r} \int_0^r \left| t + \frac{k-2r}{4} \right|^s \Bigg(\int \cdots \int_{\substack{(u_1, \ldots, u_{r-1}) \in [0,1]^{r-1}, \\ t-1 \leq u_1 + \cdots + u_{r-1} \leq t}} \frac{1}{u_1^{1/2} \cdots u_{r-1}^{1/2} (t - u_1 - \cdots - u_{r-1})^{1/2}} \\ & \times \frac{du_{r-1} \cdots du_1}{(1-u_1)^{1/2} \cdots (1-u_{r-1})^{1/2} (1 - t + u_1 + \cdots + u_{r-1})^{1/2}} \Bigg) dt. \end{aligned}$$

Applying Lemma 4.2 with $m = r - 1$ completes the proof. □

Below we show Theorem 6.

Lemma 4.3. *For $t \in \mathbb{R}$ we have*

$$g^{*2}(t) = \begin{cases} \pi F\left(\frac{1}{2}, \frac{1}{2}; 1; t(2-t)\right) & \text{if } 0 < t < 1 \text{ or } 1 < t < 2, \\ 0 & \text{if } t \leq 0 \text{ or } t \geq 2. \end{cases}$$

Proof. We have

$$g^{*2}(t) = \int_{\substack{0<u<1, \\ t-1<u<t}} \frac{du}{u^{1/2}(1-u)^{1/2}(t-u)^{1/2}(1-t+u)^{1/2}}.$$

When $t \leq 0$ or $t \geq 2$, this equals 0 because $(0,1) \cap (t-1, t) = \emptyset$. We consider the case $0 < t < 1$. Since $(0,1) \cap (t-1, t) = (0, t)$, we have

$$g^{*2}(t) \quad = \quad \int_0^t \frac{du}{u^{1/2}(1-u)^{1/2}(t-u)^{1/2}(1-t+u)^{1/2}}$$

$$= \int_0^1 \frac{du}{u^{1/2}(1-u)^{1/2}(1-tu)^{1/2}(1-t(1-u))^{1/2}}.$$

Here, in the second equality we replaced u by tu. Expanding $(1-tu)^{-1/2}$ and $(1-t(1-u))^{-1/2}$, we have

$$\begin{aligned} g^{*2}(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m}{m!}\frac{(\frac{1}{2})_n}{n!} t^{m+n} \int_0^1 u^{m-\frac{1}{2}}(1-u)^{n-\frac{1}{2}} du \\ &= \pi \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m^2 (\frac{1}{2})_n^2}{(m+n)!m!n!} t^{m+n} \\ &= \pi F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};1;t,t\right), \end{aligned}$$

where F_3 is the Apell series of the third kind given by

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m!n!} \qquad (|x|<1, |y|<1).$$

Applying the formula [GR, p.1009, 9.182.4]

$$F_3(\alpha,\gamma-\alpha,\beta,\gamma-\beta;\gamma;x,y) = (1-y)^{\alpha+\beta-\gamma} F(\alpha,\beta;\gamma;x+y-xy)$$

with $\alpha=\beta=1/2$, $\gamma=1$, $x=y=t$, we obtain Lemma 4.3 in the case $0<t<1$. Finally we deal with the case $1<t<2$. Since $g(u)=g(1-u)$ for any $u\in\mathbb{R}$, we easily obtain $g^{*2}(2-u)=g^{*2}(u)$. This together with Lemma 4.3 for $0<t<1$ implies the desired result. □

Proof of Theorem 6. It is sufficient to prove Theorem 6 for $k>4$ since $Z(s,X+X^{-1}+Y+Y^{-1}+k)=Z(s,X+X^{-1}+Y+Y^{-1}-k)$. Let $f(X,Y):=X+X^{-1}+Y+Y^{-1}+k$ with $k>4$. From Proposition 4.1 and Lemma 4.3 we have

$$\begin{aligned} Z(s,f) &= \frac{2^{2s}}{\pi}\int_0^2 \left|t+\frac{k-4}{4}\right|^s F\left(\frac{1}{2},\frac{1}{2};1;t(2-t)\right) dt \\ &= \frac{2^{2s}}{\pi}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{(n!)^2}\int_0^2 t^n(2-t)^n\left(t+\frac{k-4}{4}\right)^s dt. \end{aligned} \tag{4.1}$$

We calculate the integral in (4.1). Replacing t by $2t$ and applying (3.7) with $\alpha=-s$, $\beta=n+1$, $\gamma=2n+2$, $z=-8/(k-4)$, we have

$$\int_0^2 t^n(2-t)^n\left(t+\frac{k-4}{4}\right)^s dt$$

$$
\begin{aligned}
&= 2^{2n+1}\left(\frac{k-4}{4}\right)^s \int_0^1 t^n(1-t)^n\left(1+\frac{8}{k-4}t\right)^s dt \\
&= 2^{2n+1}\left(\frac{k-4}{4}\right)^s \frac{(n!)^2}{(2n+1)!}F\left(-s, n+1; 2n+2, -\frac{8}{k-4}\right).
\end{aligned}
$$

Applying

$$
F(\alpha,\beta;2\beta;z) = \left(1-\frac{z}{2}\right)^{-\alpha} F\left(\frac{\alpha}{2},\frac{\alpha+1}{2};\beta+\frac{1}{2};\left(\frac{z}{2-z}\right)^2\right)
$$
$$
(|\arg(1-z)|<\pi,\ 2\beta\neq -1,-3,-5,\ldots)
$$

[Le, p.255,(9.6.17)] with $\alpha=-s$, $\beta=n+1$, $z=-8/(k-4)$, we have

$$
\begin{aligned}
&\int_0^2 t^n(2-t)^n\left(t+\frac{k-4}{4}\right)^s dt \\
&= 2^{2n+1}\left(\frac{k-4}{4}\right)^s \frac{(n!)^2}{(2n+1)!}\left(\frac{k}{k-4}\right)^s F\left(-\frac{s}{2},\frac{-s+1}{2};n+\frac{3}{2};\left(\frac{4}{k}\right)^2\right) \\
&= \left(\frac{k}{4}\right)^s 2^{2n+1}\frac{(n!)^2}{(2n+1)!}F\left(-\frac{s}{2},\frac{-s+1}{2};n+\frac{3}{2};\left(\frac{4}{k}\right)^2\right).
\end{aligned}
$$

Applying this to (4.1), we get

$$
\begin{aligned}
Z(s,f) &= \frac{2k^s}{\pi}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!(2n+1)}F\left(-\frac{s}{2},\frac{-s+1}{2};n+\frac{3}{2};\left(\frac{4}{k}\right)^2\right) \\
&= \frac{2k^s}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_n}{n!(2n+1)}\frac{(-\frac{s}{2})_m(\frac{-s+1}{2})_m}{(n+\frac{3}{2})_m m!}\left(\frac{4}{k}\right)^{2m} \\
&= \frac{k^s}{\pi}\sum_{m=0}^{\infty}\frac{(-\frac{s}{2})_m(\frac{-s+1}{2})_m}{m!}\left(\frac{4}{k}\right)^{2m}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})_{m+1}}.
\end{aligned}
$$

Here, in the last equality we used $(n+\frac{3}{2})_m(n+\frac{1}{2})=(n+\frac{1}{2})_{m+1}$ and interchanged the sums. Hence, to show Theorem 6, it is sufficient to prove

$$
\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!(n+\frac{1}{2})_{m+1}}=\pi\frac{(\frac{1}{2})_m}{(m!)^2} \tag{4.2}
$$

for any $m\in\mathbb{Z}_{\geq 0}$. First we remark that the sum in (4.2) converges for any $m\in\mathbb{Z}_{\geq 0}$ since $(\frac{1}{2})_n/n!\sim 1/(\pi n)^{1/2}$ as $n\to\infty$ from Stirling's formula. To prove (4.2), we consider

$$
I_m := \int_0^1\int_0^{u_m}\cdots\int_0^{u_1}\frac{1}{u_0^{1/2}(1-u_0)^{1/2}}du_0\cdots du_{m-1}du_m. \tag{4.3}
$$

Expanding $(1-u_0)^{-1/2}$, we have

$$\begin{aligned} I_m &= \int_0^1 \int_0^{u_m} \cdots \int_0^{u_1} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} u_0^{-\frac{1}{2}+n} du_0 \cdots du_{m-1} du_m \\ &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} \int_0^1 \int_0^{u_m} \cdots \int_0^{u_1} u_0^{-\frac{1}{2}+n} du_0 \cdots du_{m-1} du_m \\ &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} \frac{1}{(n+\frac{1}{2})_{m+1}}. \end{aligned} \tag{4.4}$$

On the other hand, applying Fubini's theorem to (4.3), we have

$$\begin{aligned} I_m &= \int_0^1 \frac{1}{u_0^{1/2}(1-u_0)^{1/2}} \left(\int \cdots \int_{\substack{(u_1,\dots,u_m)\in[0,1]^m \\ u_0 \le u_1 \le u_2 \le \cdots \le u_m \le 1}} du_1 \cdots du_m \right) du_0 \\ &= \int_0^1 \frac{1}{u_0^{1/2}(1-u_0)^{1/2}} \left(\frac{1}{\#\mathfrak{S}_m} \int \cdots \int_{(u_1,\dots,u_m)\in[u_0,1]^m} du_1 \cdots du_m \right) du_0 \\ &= \frac{1}{m!} \int_0^1 u_0^{-1/2}(1-u_0)^{m-\frac{1}{2}} du_0 \\ &= \frac{1}{m!} B\left(\frac{1}{2}, m+\frac{1}{2}\right) = \frac{1}{m!} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} = \pi \frac{(\frac{1}{2})_m}{(m!)^2}. \end{aligned} \tag{4.5}$$

Comparing (4.4) and (4.5), we obtain (4.2). This completes the proof of Theorem 6. □

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