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A supplementary note on the paper "Asymptotic expansion and asymptotic robustness of the normal-theory estimators in the random regression model"

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This note is to supplement Ogasawara (2007), and gives (1) the matrix expressions of the partial derivatives of the parameter estimators with respect to sample variances and covariances, and (2) the direct proof of Theorem 1 using cumulants.

1. Matrix expressions of the partial derivatives

1.1 The regression coefficients

(a) The first partial derivatives

Let $\mathbf{s}_X = \mathrm{v}(\mathbf{S}_{XX})$, then

$$\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X{}'} = \frac{\partial(\mathbf{s}_{XY}{}'\otimes \mathbf{I}_p)\mathrm{vec}\mathbf{S}_{XX}^{-1}}{\partial \mathbf{s}_X{}'}$$

$$= -(\mathbf{s}_{XY}{}'\otimes \mathbf{I}_p)(\mathbf{S}_{XX}^{-1}\otimes \mathbf{S}_{XX}^{-1})\frac{\partial\, \mathrm{vec}\mathbf{S}_{XX}}{\partial \mathbf{s}_X{}'} = -\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p, \qquad \text{(A1.1)}$$

$$\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_{XY}{}'} = \mathbf{S}_{XX}^{-1},$$

where for $\mathrm{v}(\cdot)$, $\mathrm{vec}(\cdot)$, $\otimes$ and $\mathbf{D}_p$ see Section 4 of Ogasawara (2007).

(b) The nonzero second partial derivatives

Let $\mathbf{A}$ and $\mathbf{C}$ be $u\times v$ and $p\times q$ matrices, respectively. Then, the commutation matrix $\mathbf{K}_{uv}$ is defined as $\mathrm{vec}(\mathbf{A}')=\mathbf{K}_{uv}\mathrm{vec}(\mathbf{A})$ with $\mathbf{K}_u \equiv \mathbf{K}_{uu}$, which yields

$$\mathrm{vec}(\mathbf{A}\otimes\mathbf{C}) = (\mathbf{I}_v \otimes \mathbf{K}_{qu} \otimes \mathbf{I}_p)(\mathrm{vec}\mathbf{A}\otimes \mathrm{vec}\mathbf{C}), \qquad \text{(A1.2)}$$

(Magnus & Neudecker, 1999, p.47), where $\mathbf{I}_v$ is the $v\times v$ identity matrix. Let $\mathbf{A}^{<k>} = \mathbf{A}\otimes\cdots\otimes\mathbf{A}$ (k times) and $\mathrm{vec}\mathbf{A}\otimes\cdots = (\mathrm{vec}\mathbf{A})\otimes\cdots$, then

$$\partial \mathrm{vec}\left(\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X{}'}\right) / \partial \mathbf{s}_X{}' = -(\mathbf{D}_p{}' \otimes \mathbf{s}_{XY}{}' \otimes \mathbf{I}_p) \frac{\partial \mathrm{vec}(\mathbf{S}_{XX}^{-1} \otimes \mathbf{S}_{XX}^{-1})}{\partial \mathbf{s}_X{}'}$$

$$= -(\mathbf{D}_p{}' \otimes \mathbf{s}_{XY}{}' \otimes \mathbf{I}_p)(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p) \frac{\partial}{\partial \mathbf{s}_X{}'} (\mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1}) \qquad \text{(A1.3)}$$

$$= (\mathbf{D}_p{}' \otimes \mathbf{s}_{XY}{}' \otimes \mathbf{I}_p)(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)$$

$$\times \{(\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p) \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + \mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes (\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p)\},$$

$$\partial \mathrm{vec}\left(\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X{}'}\right) / \partial \mathbf{s}_{XY}{}' = -(\mathbf{D}_p{}' \otimes \mathbf{I}_p) \frac{\partial \mathrm{vec}\{(\mathbf{s}_{XY}{}' \mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}}{\partial \mathbf{s}_{XY}{}'}$$

$$= -(\mathbf{D}_p{}' \otimes \mathbf{I}_p) \left\{ \frac{\partial \mathrm{vec}(\mathbf{s}_{XY}{}' \mathbf{S}_{XX}^{-1})}{\partial \mathbf{s}_{XY}{}'} \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} \right\}$$

$$= -(\mathbf{D}_p{}' \otimes \mathbf{I}_p)(\mathbf{S}_{XX}^{-1} \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1}).$$

(c) The nonzero third partial derivatives

Let $p^* = p(p+1)/2$, then

$$\partial \mathrm{vec}\left\{ \partial \mathrm{vec}\left(\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X{}'}\right) / \partial \mathbf{s}_X{}' \right\} / \partial \mathbf{s}_X{}' = \partial \mathrm{vec}\left\{ \frac{\partial^2 \hat{\boldsymbol{\beta}}}{(\partial \mathbf{s}_X{}')^{<2>}} \right\} / \partial \mathbf{s}_X{}'$$

$$= [\mathbf{I}_{p^*} \otimes \{(\mathbf{D}_p{}' \otimes \mathbf{s}_{XY}{}')(\mathbf{I}_p \otimes \mathbf{K}_p)\} \otimes \mathbf{I}_p] \qquad \text{(A1.4)}$$

$$\times \partial \mathrm{vec}\{(\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p) \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + \mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes (\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p)\} / \partial \mathbf{s}_X{}',$$

where

$$\partial \mathrm{vec}\{(\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p) \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + \mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes (\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p)\} / \partial \mathbf{s}_X{}'$$

$$= \partial[\mathrm{vec}(\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p) \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1}$$

$$+ (\mathbf{K}_{p^*,p^2} \otimes \mathbf{I}_{p^2})\{\mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes \mathrm{vec}(\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p)\}] / \partial \mathbf{s}_X{}'$$

$$= \partial[\{(\mathbf{D}_p{}' \otimes \mathbf{I}_{p^2})\mathrm{vec}\mathbf{S}_{XX}^{-1<2>}\} \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + (\mathbf{K}_{p^*,p^2} \otimes \mathbf{I}_{p^2})$$

$$\times \{\mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes \{(\mathbf{D}_p{}' \otimes \mathbf{I}_{p^2})\mathrm{vec}\mathbf{S}_{XX}^{-1<2>}\}\}] / \partial \mathbf{s}_X{}' \qquad \text{(A1.5)}$$

$$= \partial[\{(\mathbf{D}_p{}' \otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)(\mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1})\}$$

$$\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + (\mathbf{K}_{p^*,p^2} \otimes \mathbf{I}_{p^2})\{\mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes \{(\mathbf{D}_p{}' \otimes \mathbf{I}_{p^2})$$

$$\times (\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)(\mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1})\}\}] / \partial \mathbf{s}_X{}'$$

$$
\begin{aligned}
&= -[(\mathbf{D}_p{}'\otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)\{(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} \\
&\quad + \mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes (\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\}] \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} \\
&\quad -\{(\mathbf{D}_p{}'\otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)(\mathrm{vec}\mathbf{S}_{XX}^{-1})^{<2>}\} \otimes (\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p) \\
&\quad -(\mathbf{K}_{p*,p^2} \otimes \mathbf{I}_{p^2})[(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\otimes \{(\mathbf{D}_p{}'\otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p) \\
&\qquad\qquad \times (\mathrm{vec}\mathbf{S}_{XX}^{-1})^{<2>}\}] \\
&\quad -(\mathbf{K}_{p*,p^2} \otimes \mathbf{I}_{p^2})[\mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes \{(\mathbf{D}_p{}'\otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p) \\
&\quad \times ((\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + \mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes (\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p))\}],
\end{aligned}
$$

$$
\begin{aligned}
&\partial \mathrm{vec}\left\{\partial \mathrm{vec}\left(\frac{\partial \hat{\boldsymbol\beta}}{\partial \mathbf{s}_X{}'}\right)/\partial \mathbf{s}_X{}'\right\}/\partial \mathbf{s}_{XY}{}' = \partial \mathrm{vec}\left\{\frac{\partial^2 \hat{\boldsymbol\beta}}{(\partial \mathbf{s}_X{}')^{<2>}}\right\}/\partial \mathbf{s}_{XY}{}' \\
&= ([\{(\mathbf{D}_p{}'\mathbf{S}_{XX}^{-1<2>})\otimes \mathrm{vec}'\mathbf{S}_{XX}^{-1} + \mathrm{vec}'\mathbf{S}_{XX}^{-1} \otimes (\mathbf{D}_p{}'\mathbf{S}_{XX}^{-1<2>})\} \\
&\quad \times (\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)] \otimes \mathbf{I}_{p*\cdot p})\partial \mathrm{vec}(\mathbf{D}_p{}'\otimes \mathbf{s}_{XY}{}'\otimes \mathbf{I}_p)/\partial \mathbf{s}_{XY}{}',
\end{aligned} \tag{A1.6}
$$

where

$$
\begin{aligned}
&\partial \mathrm{vec}(\mathbf{D}_p{}'\otimes \mathbf{s}_{XY}{}'\otimes \mathbf{I}_p)/\partial \mathbf{s}_{XY}{}' \\
&= \partial(\mathbf{I}_{p^2} \otimes \mathbf{K}_{p^2,p*} \otimes \mathbf{I}_p)\{\mathrm{vec}(\mathbf{D}_p{}')\otimes \mathrm{vec}(\mathbf{s}_{XY}{}'\otimes \mathbf{I}_p)\}/\partial \mathbf{s}_{XY}{}' \\
&= (\mathbf{I}_{p^2} \otimes \mathbf{K}_{p^2,p*} \otimes \mathbf{I}_p)[\mathrm{vec}(\mathbf{D}_p{}')\otimes \{(\partial \mathbf{s}_{XY}/\partial \mathbf{s}_{XY}{}')\otimes \mathrm{vec}\mathbf{I}_p\}] \\
&= (\mathbf{I}_{p^2} \otimes \mathbf{K}_{p^2,p*} \otimes \mathbf{I}_p)\{\mathrm{vec}(\mathbf{D}_p{}')\otimes \mathbf{I}_p \otimes \mathrm{vec}\mathbf{I}_p\}.
\end{aligned} \tag{A1.7}
$$

1.2 The partial derivatives of the residual variance

(a) The first partial derivatives

$$
\frac{\partial \hat{\psi}}{\partial \mathbf{s}_X} = \mathbf{D}_p{}'(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})^{<2>}, \quad \frac{\partial \hat{\psi}}{\partial \mathbf{s}_{XY}} = -2\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}. \tag{A1.8}
$$

(b) The nonzero second partial derivatives

Using (A1.1),

$$
\begin{aligned}
\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_X{}'} &= \mathbf{D}_p{}'[\{\partial\, \mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_X{}'\}\otimes \mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}) \\
&\quad + \mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes \{\partial\, \mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_X{}'\}] \\
&= -\mathbf{D}_p{}'[\,(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)\otimes (\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}) \\
&\quad + (\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes (\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)\,].
\end{aligned} \tag{A1.9}
$$

Similarly, we have

$$\frac{\partial^2\hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_{XY}'} = -2\frac{\partial \mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}}{\partial \mathbf{s}_X} = 2\mathbf{D}_p'\{(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes\mathbf{S}_{XX}^{-1}\},$$
$$\frac{\partial^2\hat{\psi}}{\partial \mathbf{s}_{XY} \partial \mathbf{s}_{XY}'} = -2\mathbf{S}_{XX}^{-1}. \qquad \text{(A1.10)}$$

(c) The nonzero third partial derivatives

From (A1.9),

$$\partial\,\mathrm{vec}\left(\frac{\partial^2\hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_X'}\right)/\partial \mathbf{s}_X' = \partial\left\{\frac{\partial^2\hat{\psi}}{(\partial \mathbf{s}_X)^{<2>}}\right\}/\partial \mathbf{s}_X'$$
$$= -(\mathbf{I}_{p*}\otimes\mathbf{D}_p')$$
$$\times[\ \{\partial\,\mathrm{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)/\partial \mathbf{s}_X'\}\otimes\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})$$
$$+\mathrm{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)\otimes\{\partial\,\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_X'\}$$
$$+(K_{p*,p}\otimes\mathbf{I}_p)\{\ (\partial\,\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_X')$$
$$\otimes(\mathrm{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p))$$
$$+\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes(\partial\,\mathrm{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)/\partial \mathbf{s}_X')\ \}\ ]. \qquad \text{(A1.11)}$$

In (A1.11), let $\mathbf{U}*$ and $\mathbf{V}*$ be defined as follows:

$$\partial\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_X' = -\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p \equiv -\mathbf{U}* \qquad \text{(A1.12)}$$

used before, and

$$\frac{\partial\,\mathrm{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)}{\partial \mathbf{s}_X'}$$
$$= (\mathbf{D}_p'\otimes\mathbf{I}_p)\frac{\partial\,\mathrm{vec}\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}}{\partial \mathbf{s}_X'}$$
$$= -(\mathbf{D}_p'\otimes\mathbf{I}_p)[\ (\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)\otimes\mathrm{vec}\mathbf{S}_{XX}^{-1}$$
$$+(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\ ] \equiv -\mathbf{V}*. \qquad \text{(A1.13)}$$

Using (A1.12) and (A1.13), (A1.11) becomes

$$\partial\,\mathrm{vec}\left(\frac{\partial^2\hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_X'}\right)/\partial \mathbf{s}_X'$$
$$= (\mathbf{I}_{p*}\otimes\mathbf{D}_p')[\ \mathbf{V}*\otimes(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}) + (\mathrm{vec}\mathbf{U}*)\otimes\mathbf{U}*$$
$$+(K_{p*,p}\otimes\mathbf{I}_p)\{\mathbf{U}*\otimes\mathrm{vec}\mathbf{U}* + (\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes\mathbf{V}*\}]. \qquad \text{(A1.14)}$$

Similarly,

$$\partial\,\mathrm{vec}\left(\frac{\partial^2\hat{\psi}}{\partial\mathbf{s}_X\partial\mathbf{s}_X{}'}\right)/\partial\mathbf{s}_{XY}{}' = \partial\left\{\frac{\partial^2\hat{\psi}}{(\partial\mathbf{s}_X)^{<2>}}\right\}/\partial\mathbf{s}_{XY}{}'$$
$$= -(\mathbf{I}_{p*}\otimes\mathbf{D}_p{}')$$
$$\times[\ \{\partial(\{\mathrm{vec}(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)/\partial\mathbf{s}_{XY}{}'\}\otimes(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})$$
$$+\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)\otimes\{\partial(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial\mathbf{s}_{XY}{}'\} \tag{A1.15}$$
$$+(K_{p*,p}\otimes\mathbf{I}_p)\{\ (\partial(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial\mathbf{s}_{XY}{}')\otimes(\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p))$$
$$+(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes(\partial\,\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)/\partial\mathbf{s}_{XY}{}')\ \}\ ].$$

Using

$$\partial\mathrm{vec}[\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p]/\partial\mathbf{s}_{XY}{}'$$
$$=(\mathbf{D}_p{}'\otimes\mathbf{I}_p)\frac{\partial\mathrm{vec}\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}}{\partial\mathbf{s}_{XY}{}'} \tag{A1.16}$$
$$=(\mathbf{D}_p{}'\otimes\mathbf{I}_p)(\mathbf{S}_{XX}^{-1}\otimes\mathrm{vec}\mathbf{S}_{XX}^{-1}),$$

(A1.15) becomes

$$\partial\,\mathrm{vec}\left(\frac{\partial^2\hat{\psi}}{\partial\mathbf{s}_X\partial\mathbf{s}_X{}'}\right)/\partial\mathbf{s}_{XY}{}'$$
$$=-(\mathbf{I}_{p*}\otimes\mathbf{D}_p{}')[\ \{(\mathbf{D}_p{}'\otimes\mathbf{I}_p)(\mathbf{S}_{XX}^{-1}\otimes\mathrm{vec}\mathbf{S}_{XX}^{-1})\}\otimes(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})$$
$$+\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)\otimes\mathbf{S}_{XX}^{-1} \tag{A1.17}$$
$$+(K_{p*,p}\otimes\mathbf{I}_p)\{\ \mathbf{S}_{XX}^{-1}\otimes\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)$$
$$+(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes((\mathbf{D}_p{}'\otimes\mathbf{I}_p)(\mathbf{S}_{XX}^{-1}\otimes\mathrm{vec}\mathbf{S}_{XX}^{-1}))\ \}\ ].$$

From (A1.10),

$$\partial\,\mathrm{vec}\left(\frac{\partial^2\hat{\psi}}{\partial\mathbf{s}_X\partial\mathbf{s}_{XY}{}'}\right)/\partial\mathbf{s}_{XY}{}'$$
$$=2\{\mathbf{I}_p\otimes(\mathbf{D}_p{}'\mathbf{S}_{XX}^{-1<2>})\}(\mathbf{K}_p\otimes\mathbf{I}_p)\left(\frac{\partial\mathbf{s}_{XY}}{\partial\mathbf{s}_{XY}{}'}\otimes\mathrm{vec}\mathbf{I}_p\right) \tag{A1.18}$$
$$=2\{\mathbf{I}_p\otimes(\mathbf{D}_p{}'\mathbf{S}_{XX}^{-1<2>})\}(\mathbf{K}_p\otimes\mathbf{I}_p)(\mathbf{I}_p\otimes\mathrm{vec}\mathbf{I}_p),$$

which becomes, when the order of derivatives is changed,

$$\partial \operatorname{vec}\left(\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_{XY} \partial \mathbf{s}_{XY}'}\right) / \partial \mathbf{s}_X' = 2\mathbf{S}_{XX}^{-1<2>} \mathbf{D}_p. \tag{A1.19}$$

2. The direct proof of Theorem 1

(a) Asymptotic robustness of $\mathrm{avar}_{\mathrm{NT}}(\hat{\beta}_i)$

It is known that

$$(\mathbf{\Omega})_{ab,cd} = \kappa_{abcd} + (\mathbf{\Omega}_{\mathrm{NT}})_{ab,cd} = \kappa_{abcd} + \omega_{\mathrm{NT}ab,cd}, \quad (p+1 \geq a \geq b \geq 1; p+1 \geq c \geq d \geq 1), \tag{A2.1}$$

(see e.g., Stuart & Ort, 1994, Equation (13.44)). As was derived in (4.2), we have

$$n \ \mathrm{avar}(\hat{\beta}_i) = \sum_U \sum_V \frac{\partial \beta_i}{\partial \sigma_U} (\mathbf{\Omega})_{U,V} \frac{\partial \beta_i}{\partial \sigma_V}, \ (i = 1,.., p), \tag{A2.2}$$

where $\sum_U$ and $\sum_V$ denote the summations over the ranges:

$$U \in \{(a,b), (e,Y);\ p \geq a \geq b \geq 1,\ e = 1,..., p\},$$
$$V \in \{(c,d), (g,Y);\ p \geq c \geq d \geq 1,\ g = 1,..., p\},$$

respectively. Using (A2.1), (A2.2) becomes

$$= -\sum_{a,b=1}^{p} \sum_V \beta_a \sigma^{bi} (\kappa_{abV} + \omega_{\mathrm{NT}ab,V}) \frac{\partial \beta_i}{\partial \sigma_V} + \sum_{e=1}^{p} \sum_V \sigma^{ei} (\kappa_{YeV} + \omega_{\mathrm{NT}Ye,V}) \frac{\partial \beta_i}{\partial \sigma_V}. \tag{A2.3}$$

In (A2.3), κ_{YeV} is the multivariate cumulant of Y, x_e and the two variables in V, say v_1 and v_2, which is equivalent to that of $\boldsymbol{\beta}'\mathbf{x}$, x_e, v_1 and v_2 due to the property of cumulants and the assumption of the independence of $\mathbf{x}$ and $\boldsymbol{\varepsilon}$. That is, using the notation $\mu(\,\cdot\,,...,\,\cdot\,)$ for the multivariate central moment of the argument variables,

$$\kappa_{YeV} = \mu\left(\sum_{k=1}^{p}\beta_k x_k, x_e, v_1, v_2\right) - \mu\left(\sum_{k=1}^{p}\beta_k x_k, x_e\right)\mu(v_1, v_2)$$
$$-\mu\left(\sum_{k=1}^{p}\beta_k x_k, v_1\right)\mu(x_e, v_2) - \mu\left(\sum_{k=1}^{p}\beta_k x_k, v_2\right)\mu(x_e, v_1)$$
$$= \sum_{k=1}^{p}\beta_k\{\mu(x_k, x_e, v_1, v_2) - \mu(x_k, x_e)\mu(v_1, v_2)$$
$$- \mu(x_k, v_1)\mu(x_e, v_2) - \mu(x_k, v_2)\mu(x_e, v_1)\} \qquad \text{(A2.4)}$$
$$= \sum_{k=1}^{p}\beta_k \kappa_{keV}.$$

Using (A2.3) and (A2.4), we have

$$n\ \text{avar}(\hat{\beta}_i) = -\sum_{a,b=1}^{p}\sum_{V}\beta_a\sigma^{bi}\omega_{\text{NT}ab,V}\frac{\partial\beta_i}{\partial\sigma_V} + \sum_{e=1}^{p}\sum_{V}\sigma^{ei}\omega_{\text{NT}Ye,V}\frac{\partial\beta_i}{\partial\sigma_V}$$
$$= n\ \text{avar}_{\text{NT}}(\hat{\beta}_i), \quad (i = 1,\dots,p). \qquad \text{(A2.5)}$$

(b) Asymptotic robustness of $\text{abis}_{\text{NT}}(\hat{\beta}_i)$

In a similar manner, we have

$$n\ \text{abis}(\hat{\beta}_i) = \frac{1}{2}\sum_{U}\sum_{V}\frac{\partial^2\beta_i}{\partial\sigma_U\partial\sigma_V}(\boldsymbol{\Omega})_{U,V}$$
$$= \frac{1}{2}\sum_{p\ge a\ge b\ge 1}\ \sum_{p\ge c\ge d\ge 1}\frac{\partial^2\beta_i}{\partial\sigma_{ab}\partial\sigma_{cd}}(\boldsymbol{\Omega})_{ab,cd} + \sum_{p\ge c\ge d\ge 1}\ \sum_{e=1}^{p}\frac{\partial^2\beta_i}{\partial\sigma_{ab}\partial\sigma_{Ye}}(\boldsymbol{\Omega})_{ab,Ye}$$
$$= \sum_{a,b,c,d=1}^{p}\beta_c\sigma^{da}\sigma^{bi}(\kappa_{abcd} + \omega_{\text{NT}ab,cd}) - \sum_{a,b,e=1}^{p}\sigma^{ea}\sigma^{bi}(\kappa_{abeY} + \omega_{\text{NT}ab,eY}), \qquad \text{(A2.6)}$$
$$(i = 1,\dots,p).$$

Since $\kappa_{abeY} = \sum_{k=1}^{p}\beta_k\kappa_{abek}$ (see (A2.4)), (A2.6) becomes

$$= \sum_{a,b,c,d=1}^{p}\beta_c\sigma^{da}\sigma^{bi}\omega_{\text{NT}ab,cd} - \sum_{a,b,e=1}^{p}\sigma^{ea}\sigma^{bi}\omega_{\text{NT}ab,eY}$$
$$= n\ \text{abis}_{\text{NT}}(\hat{\beta}_i) = 0.$$

(c) Asymptotic robustness of $\text{abis}_{\text{NT}}(\hat{\psi})$

Noting that s_{YY} in $\hat{\psi}$ does not contribute to the asymptotic/exact bias of

$\hat{\psi}$, we can use the same ranges of U and V as before:

$$n\ \mathrm{abis}(\hat{\psi}) = \frac{1}{2}\sum_U\sum_V \frac{\partial^2\psi}{\partial\sigma_U\partial\sigma_V}(\boldsymbol{\Omega})_{U,V}$$

$$= \frac{1}{2}\sum_{p\ge a\ge b\ge 1}\sum_{p\ge c\ge d\ge 1}\frac{\partial^2\psi}{\partial\sigma_{ab}\partial\sigma_{cd}}(\boldsymbol{\Omega})_{ab,cd} + \sum_{p\ge c\ge d\ge 1}\sum_{e=1}^{p}\frac{\partial^2\psi}{\partial\sigma_{ab}\partial\sigma_{Ye}}(\boldsymbol{\Omega})_{ab,Ye}$$

$$+\frac{1}{2}\sum_{e,f=1}^{p}\frac{\partial^2\psi}{\partial\sigma_{Ye}\partial\sigma_{Yf}}(\boldsymbol{\Omega})_{Ye,Yf}$$

$$= -\sum_{a,b,c,d=1}^{p}\beta_c\sigma^{da}\beta_b(\kappa_{abcd}+\omega_{\mathrm{NT}ab,cd}) \tag{A2.7}$$

$$+2\sum_{a,b,e=1}^{p}\sigma^{ea}\beta_b(\kappa_{abYe}+\omega_{\mathrm{NT}ab,Ye}) - \sum_{e,f=1}^{p}\sigma^{ef}(\kappa_{YeYf}+\omega_{\mathrm{NT}Ye,Yf})$$

$$= n\ \mathrm{abis}_{\mathrm{NT}}(\hat{\psi}),$$

where $\kappa_{YeYf} = \sum_{l,m=1}^{p}\beta_l\kappa_{lemf}\beta_m$ is used. Q. E. D.

References

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