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Supplement to the paper "Asymptotic expansions of the distributions of the chi-square statistic based on the asymptotically distribution-free theory in covariance structures"

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This note gives asymptotic moments as applications of Ogasawara (2009, Lemma 2) required for the Bartlett correction of Ogasawara (2009, Theorem 2). The results in Subsections 1.2 and 1.3 of this note are not required for the Bartlett correction but shown here for completeness. In this note, σ_{0abcd} is denoted by σ_{abcd} for simplicity of notation with other similar expressions. The notation $\overset{k}{\Sigma}$ is synonymous with Σ^k (the sum of k terms with similar patterns). In the following, subscripts a, b, c, d, e, f, g, h, i, j, k, l, w, x, y and z $=1,\ldots,p$.

Errata will be given at the end of this note.

1. Higher-order asymptotic variances

1.1 $\mathrm{E}\{(s_{ab}-\sigma_{ab})(s_{cd}-\sigma_{cd})\}$ (see e.g., Kaplan, 1952, Equation (3))

$$
\begin{aligned}
&=\frac{1}{N}\kappa_{abcd}+\frac{1}{N-1}(\sigma_{ac}\sigma_{bd}+\sigma_{ad}\sigma_{bc})\\
&=\frac{1}{N}(\sigma_{abcd}-\sigma_{ab}\sigma_{cd}-\sigma_{ac}\sigma_{bd}-\sigma_{ad}\sigma_{bc})+\frac{1}{N-1}(\sigma_{ac}\sigma_{bd}+\sigma_{ad}\sigma_{bc})\\
&=\frac{1}{N}(\sigma_{abcd}-\sigma_{ab}\sigma_{cd})+\frac{1}{N^2}(\sigma_{ac}\sigma_{bd}+\sigma_{aa}\sigma_{bc})+O(N^{-3})\\
&=\frac{1}{n}(\sigma_{abcd}-\sigma_{ab}\sigma_{cd})-\frac{1}{n^2}(\sigma_{abcd}-\sigma_{ab}\sigma_{cd}-\sigma_{ac}\sigma_{bd}-\sigma_{ad}\sigma_{bc})+O(n^{-3})\\
&=\frac{1}{n}(\sigma_{abcd}-\sigma_{ab}\sigma_{cd})-\frac{1}{n^2}\kappa_{abcd}+O(n^{-3}),
\end{aligned}
$$

where κ_{abcd} denotes the fourth-order multivariate cumulant for variables X_a, X_b, X_c and X_d.

1.2 $\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(s_{ef}-\sigma_{ef})\}$

Recall

$$m_{abcd}=\frac{1}{N}S_{abcd}-\frac{1}{N^2}\sum^{4}S_aS_{bcd}+\frac{1}{N^3}\sum^{6}S_aS_bS_{cd}-\frac{3}{N^4}S_aS_bS_cS_d,$$

$$s_{ef}=\frac{1}{N-1}S_{ef}-\frac{1}{N(N-1)}S_eS_f.$$

Using Ogasawara (2008, Lemmas 1 and 2),

$$\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(s_{ef}-\sigma_{ef})\}=\mathrm{E}(m_{abcd}s_{ef})-\mathrm{E}(m_{abcd})\sigma_{ef}$$

$$=\left(\frac{1}{N}-\frac{4}{N^2}\right)\sigma_{abcdef}+\left(1-\frac{5}{N}+\frac{10}{N^2}\right)\sigma_{abcd}\sigma_{ef}+\frac{2}{N^2}\sum^{6}\sigma_{abef}\sigma_{cd}$$

$$+\frac{1}{N^2}\sum^{8}\sigma_{bcde}\sigma_{af}+\left(-\frac{1}{N}+\frac{4}{N^2}\right)\sum^{4}\sigma_{aef}\sigma_{bcd}+\left(\frac{2}{N}-\frac{9}{N^2}\right)\left(\sum^{3}\sigma_{ab}\sigma_{cd}\right)\sigma_{ef}$$

$$-\frac{1}{N^2}\sum^{6}(\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}$$

$$-\left(1-\frac{4}{N}+\frac{6}{N^2}\right)\sigma_{abcd}\sigma_{ef}-\left(\frac{2}{N}-\frac{5}{N^2}\right)\left(\sum^{3}\sigma_{ab}\sigma_{cd}\right)\sigma_{ef}+O(N^{-3})$$

$$=\frac{1}{N}(\sigma_{abcdef}-\sigma_{abcd}\sigma_{ef}-\sum^{4}\sigma_{aef}\sigma_{bcd})$$

$$+\frac{1}{N^2}\{-4\sigma_{abcdef}+4\sigma_{abcd}\sigma_{ef}+2\sum^{6}\sigma_{abef}\sigma_{cd}+\sum^{8}\sigma_{bcde}\sigma_{af}$$

$$+4\sum^{4}\sigma_{aef}\sigma_{bcd}-4\left(\sum^{3}\sigma_{ab}\sigma_{cd}\right)\sigma_{ef}-\sum^{6}(\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}\}+O(N^{-3})$$

$$=\frac{1}{n}(\sigma_{abcdef}-\sigma_{abcd}\sigma_{ef}-\sum^{4}\sigma_{aef}\sigma_{bcd})$$

$$+\frac{1}{n^2}\{-5\sigma_{abcdef}+5\sigma_{abcd}\sigma_{ef}+2\sum^{6}\sigma_{abef}\sigma_{cd}+\sum^{8}\sigma_{bcde}\sigma_{af}$$

$$+5\sum^{4}\sigma_{aef}\sigma_{bcd}-4\left(\sum^{3}\sigma_{ab}\sigma_{cd}\right)\sigma_{ef}-\sum^{6}(\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}\}+O(n^{-3}).$$

1.3 $\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(m_{efgh}-\sigma_{efgh})\}$

In the following the subscripts $[\cdot]$ are used to identify terms for confirmation of correspondence in equations.

$$\mathrm{E}(m_{abcd}m_{efgh}) = {}_{[1]}\frac{1}{N}\sigma_{abcdefgh} + {}_{[2AB]}\frac{N-1}{N}\sigma_{abcd}\sigma_{efgh} - \frac{1}{N^3}\{{}_{[3]}8N\sigma_{abcdefgh}$$

$$+ {}_{[4AB]}8(N^2-N)\sigma_{abcd}\sigma_{efgh} + {}_{[5AB]}(N^2-N)\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})\}$$

$$+\frac{1}{N^4}\{{}_{[6]}(N^2-N)2\sum^{6}(\sigma_{abcdef}\sigma_{gh}+\sigma_{efghab}\sigma_{cd}) + {}_{[7]}(N^2-N)3$$

$$\times\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd}) + {}_{[8]}12(N^2-N)\sigma_{abcd}\sigma_{efgh}$$

$$+ {}_{[9AB]}N(N-1)(N-2)2\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})\}$$

$$-{}_{[10]}\frac{3}{N^5}N(N-1)(N-2)\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})$$

$$+\frac{1}{N^4}\{{}_{[11]}N(N-1)\sum^{16}(\sigma_{bcdfgh}\sigma_{ae}+\sigma_{bcde}\sigma_{fgha})$$

$$+ {}_{[12]}N(N-1)16\sigma_{abcd}\sigma_{efgh} + {}_{[13]}N(N-1)4\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})$$

$$+ {}_{[14AB]}N(N-1)(N-2)\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}\}$$

$$-\frac{1}{N^5}N(N-1)(N-2)\{{}_{[15]}8\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})$$

$$+ {}_{[16]}\sum^{48}(\sigma_{bcde}\sigma_{af}\sigma_{gh}+\sigma_{fgha}\sigma_{eb}\sigma_{cd}) + {}_{[17]}6\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}$$

$$+ {}_{[18]}2\sum^{24}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{eab}\sigma_{fgh}\sigma_{cd})\}$$

$$+\frac{1}{N^6}N(N-1)(N-2)(N-3)\Big(_{[19]}\underbrace{4\sum^{9}\sigma_{ab}\sigma_{cd}\sigma_{ef}\sigma_{gh}}+_{[20]}\underbrace{\sum^{72}\sigma_{ae}\sigma_{bf}\sigma_{cd}\sigma_{gh}}\Big)$$

$$+O(N^{-3})$$

$$=_{[2A]}\underbrace{\sigma_{abcd}\sigma_{efgh}}+\frac{1}{N}\Big\{_{[1]}\underbrace{\sigma_{abcdefgh}}-_{[5A]}\underbrace{\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})}$$

$$-_{[2B,4A]}\underbrace{9\sigma_{abcd}\sigma_{efgh}}+_{[9A]}\underbrace{2\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})}+_{[14A]}\underbrace{\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}}\Big\}$$

$$+\frac{1}{N^2}\Big\{_{[3]}\underbrace{-8\sigma_{abcdefgh}}+_{[11]}\underbrace{\sum^{16}(\sigma_{bcdfgh}\sigma_{ae}+\sigma_{bcde}\sigma_{fgha})}$$

$$+_{[4B,8,12]}\underbrace{36\sigma_{abcd}\sigma_{efgh}}+_{[5B,7,13]}\underbrace{8\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})}$$

$$+_{[6]}\underbrace{2\sum^{6}(\sigma_{abcdef}\sigma_{gh}+\sigma_{efghab}\sigma_{cd})}-_{[16]}\underbrace{\sum^{48}(\sigma_{bcde}\sigma_{af}\sigma_{gh}+\sigma_{fgha}\sigma_{eb}\sigma_{cd})}$$

$$-_{[9B,10,15]}\underbrace{17\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})}-_{[14B,17]}\underbrace{9\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}}$$

$$-_{[18]}\underbrace{2\sum^{24}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{eab}\sigma_{fgh}\sigma_{cd})}+_{[19]}\underbrace{4\sum^{9}\sigma_{ab}\sigma_{cd}\sigma_{ef}\sigma_{gh}}$$

$$+_{[20]}\underbrace{\sum^{72}\sigma_{ae}\sigma_{bf}\sigma_{cd}\sigma_{gh}}\Big\}+O(N^{-3})$$

$$=\sigma_{abcd}\sigma_{efgh}+\frac{1}{n}\{\sigma_{abcdefgh}-\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})$$

$$-9\sigma_{abcd}\sigma_{efgh}+2\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})+\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}\}$$

$$+\frac{1}{n^2}\{-9\sigma_{abcdefgh}+\sum^{16}(\sigma_{bcdfgh}\sigma_{ae}+\sigma_{bcde}\sigma_{fgha})$$

$$+45\sigma_{abcd}\sigma_{efgh}+9\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})$$

$$+2\sum^{6}(\sigma_{abcdef}\sigma_{gh}+\sigma_{efghab}\sigma_{cd})-\sum^{48}(\sigma_{bcde}\sigma_{af}\sigma_{gh}+\sigma_{fgha}\sigma_{eb}\sigma_{cd})$$

$$-19\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})-10\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}$$

$$-2\sum^{24}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{eab}\sigma_{fgh}\sigma_{cd})+4\sum^{9}\sigma_{ab}\sigma_{cd}\sigma_{ef}\sigma_{gh}+\sum^{72}\sigma_{ae}\sigma_{bf}\sigma_{cd}\sigma_{gh}\}$$

$$+O(n^{-3}).$$

On the other hand, from Ogasawara (2008, Lemma 1),

$$-\mathrm{E}(m_{abcd})\sigma_{efgh}-\mathrm{E}(m_{efgh})\sigma_{abcd}=-2\left(1-\frac{4}{N}+\frac{6}{N^2}\right)\sigma_{abcd}\sigma_{efgh}$$

$$-\left(\frac{2}{N}-\frac{5}{N^2}\right)\sum^{3}(\sigma_{ab}\sigma_{cd}\sigma_{efgh}+\sigma_{ef}\sigma_{gh}\sigma_{abcd})+O(N^{-3})$$

$$=-2\left(1-\frac{4}{n}+\frac{10}{n^2}\right)\sigma_{abcd}\sigma_{efgh}-\left(\frac{2}{n}-\frac{7}{n^2}\right)\sum^{3}(\sigma_{ab}\sigma_{cd}\sigma_{efgh}+\sigma_{ef}\sigma_{gh}\sigma_{abcd})$$

$$+O(n^{-3}).$$

From the above results,

$$\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(m_{efgh}-\sigma_{efgh})\}$$

$$=\frac{1}{N}\{\sigma_{abcdefgh}-\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})-\sigma_{abcd}\sigma_{efgh}+\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}\}$$

$$+\frac{1}{N^2}\{-8\sigma_{abcdefgh}+\sum^{16}(\sigma_{bcdfgh}\sigma_{ae}+\sigma_{bcde}\sigma_{fgha})$$

$$+24\sigma_{abcd}\sigma_{efgh}+8\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})$$

$$+2\sum^{6}(\sigma_{abcdef}\sigma_{gh}+\sigma_{efghab}\sigma_{cd})-\sum^{48}(\sigma_{bcde}\sigma_{af}\sigma_{gh}+\sigma_{fgha}\sigma_{eb}\sigma_{cd})$$

$$-12\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})-9\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}$$

$$-2\sum^{24}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{eab}\sigma_{fgh}\sigma_{cd})+4\sum^{9}\sigma_{ab}\sigma_{cd}\sigma_{ef}\sigma_{gh}+\sum^{72}\sigma_{ae}\sigma_{bf}\sigma_{cd}\sigma_{gh}\}$$

$$+O(N^{-3})$$

$$=\frac{1}{n}\{\sigma_{abcdefgh}-\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})-\sigma_{abcd}\sigma_{efgh}+\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}\}$$

$$+\frac{1}{n^2}\{-9\sigma_{abcdefgh}+\sum^{16}(\sigma_{bcdfgh}\sigma_{ae}+\sigma_{bcde}\sigma_{fgha})$$

$$+25\sigma_{abcd}\sigma_{efgh}+9\sum^{4}(\sigma_{abcde}\sigma_{fgh}+\sigma_{efgha}\sigma_{bcd})$$

$$+2\sum^{6}(\sigma_{abcdef}\sigma_{gh}+\sigma_{efghab}\sigma_{cd})-\sum^{48}(\sigma_{bcde}\sigma_{af}\sigma_{gh}+\sigma_{fgha}\sigma_{eb}\sigma_{cd})$$

$$-12\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})-10\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}$$

$$-2\sum^{24}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{eab}\sigma_{fgh}\sigma_{cd})+4\sum^{9}\sigma_{ab}\sigma_{cd}\sigma_{ef}\sigma_{gh}+\sum^{72}\sigma_{ae}\sigma_{bf}\sigma_{cd}\sigma_{gh}\}$$

$$+O(n^{-3}).$$

2. The third moments

2.1 $\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh})\}$

Ogasawara (2008, Lemma 5) gave

$$\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh})\}$$

$$=\frac{1}{N^2}\Bigg[\ \sigma_{abcdefgh}-(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef})$$

$$-\sum^{4}(\sigma_{bcdef}\sigma_{agh}+\sigma_{bcdgh}\sigma_{aef}+\sigma_{aefgh}\sigma_{bcd})$$

$$-\sum^{4}\sigma_{abcde}\sigma_{fgh}-5\sigma_{abcd}\sigma_{efgh}+6\sigma_{abcd}\sigma_{ef}\sigma_{gh}-\sum^{4}(\sigma_{aef}\sigma_{gh}+\sigma_{agh}\sigma_{ef})\sigma_{bcd}$$

$$+\sum^{4}(\sigma_{ag}\sigma_{efh}+\sigma_{ah}\sigma_{efg}+\sigma_{ae}\sigma_{ghf}+\sigma_{af}\sigma_{ghe})\sigma_{bcd}$$

$$+\sum^{{}_4C_2=6}\{(\sigma_{aef}\sigma_{bgh}+\sigma_{agh}\sigma_{bef})\sigma_{cd}+(\sigma_{acd}\sigma_{bgh}+\sigma_{agh}\sigma_{bcd})\sigma_{ef}$$

$$+(\sigma_{acd}\sigma_{bef}+\sigma_{aef}\sigma_{bcd})\sigma_{gh}\}+2\sum^{3}\sigma_{ab}\sigma_{cd}(\sigma_{efgh}-\sigma_{ef}\sigma_{gh})\Bigg]+O(N^{-3}),$$

where N can be replaced by n.

2.2 $\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(m_{efgh}-\sigma_{efgh})(s_{ij}-\sigma_{ij})\}$

Write,

$$m_{abcd}=\frac{1}{N}S_{abcd}-\frac{1}{N^2}\sum^{4}S_aS_{bcd}+\frac{1}{N^3}\sum^{6}S_aS_bS_{cd}-\frac{3}{N^4}S_aS_bS_cS_d,$$

$$m_{efgh}=\frac{1}{N}S_{efgh}-\frac{1}{N^2}\sum^{4}S_eS_{fgh}+\frac{1}{N^3}\sum^{6}S_eS_fS_{gh}-\frac{3}{N^4}S_eS_fS_gS_h,$$

$$s_{ij}=\frac{1}{N-1}S_{ij}-\frac{1}{N(N-1)}S_iS_j.$$

Then, first we have

$$\mathrm{E}(m_{abcd}m_{efgh}s_{ij})$$

$$=\frac{1}{N(N-1)}\sigma_{abcdefghij}+\frac{1}{N}\Big({}_{[2-1]}\underrightarrow{\sigma_{abcdefgh}\sigma_{ij}}+{}_{[2-2]}\underrightarrow{\sum^{2}\sigma_{abcdij}\sigma_{efgh}}\Big)$$

$$+{}_{[3AB]}\underrightarrow{\frac{N-2}{N}\sigma_{abcd}\sigma_{efgh}\sigma_{ij}}$$

$$-\frac{1}{N^2}\left[8\sigma_{abcdefgh}\sigma_{ij}+\sum^{2}\{8\sigma_{abcd}\sigma_{efghij}+\sum^{4}(\sigma_{aefgh}\sigma_{bcdij}+\sigma_{aij}\sigma_{bcdefgh}+\sigma_{aefghij}\sigma_{bcd})\}\right]$$

$$-_{[5AB]}\underline{\frac{N-2}{N^2}\{8\sigma_{abcd}\sigma_{efgh}\sigma_{ij}+\sum^{2}\sum^{4}(\sigma_{aefgh}\sigma_{bcd}\sigma_{ij}+\sigma_{aij}\sigma_{bcd}\sigma_{efgh})\}}$$

$$+\frac{N-2}{N^3}\Bigg[12\sigma_{abcd}\sigma_{efgh}\sigma_{ij}+\sum_{(abcd)}^{2}\sum^{6}[\,(\sigma_{abefgh}\sigma_{ij}+\sigma_{abij}\sigma_{efgh})\sigma_{cd}+\sigma_{ab}(\sigma_{cdefgh}\sigma_{ij}$$

$$+\sigma_{cdij}\sigma_{efgh}+\sigma_{cd}\sigma_{efghij})+\sum_{(ab)}^{2}\{\,\sigma_{acd}(\sigma_{befgh}\sigma_{ij}+\sigma_{efgh}\sigma_{bij})$$

$$+\sigma_{aefgh}(\sigma_{bcd}\sigma_{ij}+\sigma_{bij}\sigma_{cd})+\sigma_{aij}(\sigma_{bcd}\sigma_{efgh}+\sigma_{befgh}\sigma_{cd})\,\}\,]\Bigg]$$

$$+_{[7AB]}\underline{\frac{(N-2)(N-3)}{N^3}\sum_{(abcd)}^{2}\sum^{3}2\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ij}}$$

$$-\frac{(N-2)(N-3)}{N^4}3\sum_{(abcd)}^{2}\sum^{3}\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ij}$$

$$+\frac{N-2}{N^3}\{\,\sum^{16}\sigma_{ae}(\sigma_{bcdfgh}\sigma_{ij}+\sigma_{bcdij}\sigma_{fgh}+\sigma_{fghij}\sigma_{bcd})+16\sigma_{abcd}\sigma_{efgh}\sigma_{ij}$$

$$+4\sum^{2}\sigma_{abcd}\sum^{4}\sigma_{eij}\sigma_{fgh}+\sum^{16}(\sigma_{afgh}\sigma_{ebcd}\sigma_{ij}+\sigma_{afgh}\sigma_{eij}\sigma_{bcd}+\sigma_{ebcd}\sigma_{aij}\sigma_{fgh})$$

$$+4\sum^{2}\sum^{4}\sigma_{abcde}\sigma_{fgh}\sigma_{ij}+\sum^{16}\sigma_{aeij}\sigma_{bcd}\sigma_{fgh}\,\}$$

$$+_{[10AB]}\underline{\frac{(N-2)(N-3)}{N^3}\sum^{16}\sigma_{ae}\sigma_{bcd}\sigma_{fgh}\sigma_{ij}}$$

$$-\frac{(N-2)(N-3)}{N^4}\sum^{2}\sum^{6}\sum^{4}[\quad\sigma_{ab}(\sigma_{cde}\sigma_{fgh}\sigma_{ij}+\sigma_{cd}\sigma_{efgh}\sigma_{ij}+\sigma_{cd}\sigma_{eij}\sigma_{fgh})$$

$$+\sum_{(ab)}^{2}\{\sigma_{acd}\sigma_{be}\sigma_{fgh}\sigma_{ij}+\sigma_{ae}(\sigma_{bcd}\sigma_{fgh}\sigma_{ij}+\sigma_{bfgh}\sigma_{cd}\sigma_{ij}+\sigma_{bij}\sigma_{cd}\sigma_{fgh})$$

$$+\sigma_{afgh}\sigma_{be}\sigma_{cd}\sigma_{ij}+\sigma_{aij}\sigma_{be}\sigma_{cd}\sigma_{fgh}\}+\sigma_{abe}\sigma_{cd}\sigma_{fgh}\sigma_{ij}\quad]$$

$$+\frac{(N-2)(N-3)(N-4)}{N^5}\sum^{36}(\sigma_{ab}\sigma_{ef}+\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}\sigma_{gh}\sigma_{ij}$$

$$-\frac{1}{N^2}(\sigma_{abcdefgh}\sigma_{ij}+\sum^{2}\sigma_{abcdi}\sigma_{efghj})-_{[14AB]}\underrightarrow{\frac{N-2}{N^2}\sigma_{abcd}\sigma_{efgh}\sigma_{ij}}$$

$$+\frac{N-2}{N^3}\sum^{2}\sum^{4}\{\ (\sigma_{abcd}\sigma_{efgh}+\sigma_{aefgh}\sigma_{bcd})\sigma_{ij}+\sum^{2}\sigma_{ai}(\sigma_{bcd}\sigma_{efghj}+\sigma_{bcdj}\sigma_{efgh})$$

$$+\sigma_{aij}\sigma_{bcd}\sigma_{efgh}\ \}$$

$$-\frac{(N-2)(N-3)}{N^4}\sum^{2}\sum^{6}(\sigma_{ab}\sigma_{ij}+\sigma_{ai}\sigma_{bj}+\sigma_{aj}\sigma_{bi})\sigma_{cd}\sigma_{efgh}$$

$$-\frac{(N-2)(N-3)}{N^4}\sum^{16}(\sigma_{ae}\sigma_{ij}+\sigma_{ai}\sigma_{ej}+\sigma_{aj}\sigma_{ei})\sigma_{bcd}\sigma_{fgh}+O(N^{-3})$$

$$=_{[3A]}\underrightarrow{\sigma_{abcd}\sigma_{efgh}\sigma_{ij}}$$

$$+\frac{1}{N}\{\ \underrightarrow{_{[2-1]}\sigma_{abcdefgh}\sigma_{ij}}-_{[3B,5A,14A]}\underrightarrow{11\sigma_{abcd}\sigma_{efgh}\sigma_{ij}}+_{[2-2]}\underrightarrow{\sum^{2}\sigma_{abcdij}\sigma_{efgh}}$$

$$-_{[5A]}\underrightarrow{\sum^{2}\sum^{4}(\sigma_{aefgh}\sigma_{bcd}\sigma_{ij}+\sigma_{aij}\sigma_{bcd}\sigma_{efgh})}$$

$$+_{[7A]}\underrightarrow{\sum^{2}\sum_{(abcd)}^{3}2\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ij}}+_{[10A]}\underrightarrow{\sum^{16}\sigma_{ae}\sigma_{bcd}\sigma_{fgh}\sigma_{ij}}\ \}$$

$$+\frac{1}{N^2}\Big[\sigma_{abcdefghij}-\{\ 8\sigma_{abcdefgh}\sigma_{ij}+\sum^{2}\{8\sigma_{abcd}\sigma_{efghij}+\sum^{4}(\sigma_{aefgh}\sigma_{bcdij}+\sigma_{aij}\sigma_{bcdefgh}$$

$$+\sigma_{aefghij}\sigma_{bcd})\}\ \}+16\sigma_{abcd}\sigma_{efgh}\sigma_{ij}+2\sum^{2}\sum^{4}(\sigma_{aefgh}\sigma_{bcd}\sigma_{ij}+\sigma_{aij}\sigma_{bcd}\sigma_{efgh})$$

$$+12\sigma_{abcd}\sigma_{efgh}\sigma_{ij}+\sum_{(abcd)}^{2}\sum^{6}\{\ (\sigma_{abefgh}\sigma_{ij}+\sigma_{abij}\sigma_{efgh})\sigma_{cd}+\sigma_{ab}(\sigma_{cdefgh}\sigma_{ij}$$

$$+\sigma_{cdij}\sigma_{efgh}+\sigma_{cd}\sigma_{efghij})+\sum_{(ab)}^{2}\{\sigma_{acd}(\sigma_{befgh}\sigma_{ij}+\sigma_{efgh}\sigma_{bij})+\sigma_{aefgh}(\sigma_{bcd}\sigma_{ij}+\sigma_{bij}\sigma_{cd})$$

$$+\sigma_{aij}(\sigma_{bcd}\sigma_{efgh}+\sigma_{befgh}\sigma_{cd})\}\ \}-13\sum_{(abcd)}^{2}\sum^{3}\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ij}$$

$$+\sum^{16}\sigma_{ae}(\sigma_{bcdfgh}\sigma_{ij}+\sigma_{bcdij}\sigma_{fgh}+\sigma_{fghij}\sigma_{bcd})+16\sigma_{abcd}\sigma_{efgh}\sigma_{ij}+4\sum^{2}\sigma_{abcd}\sum^{4}\sigma_{eij}\sigma_{fgh}$$

$$+\sum^{16}(\sigma_{afgh}\sigma_{ebcd}\sigma_{ij}+\sigma_{afgh}\sigma_{eij}\sigma_{bcd}+\sigma_{ebcd}\sigma_{aij}\sigma_{fgh})$$

$$+4\sum^{2}\sum^{4}\sigma_{abcde}\sigma_{fgh}\sigma_{ij}+\sum^{16}\sigma_{aeij}\sigma_{bcd}\sigma_{fgh}-5\sum^{16}\sigma_{ae}\sigma_{bcd}\sigma_{fgh}\sigma_{ij}$$

$$-\sum^{2}\sum^{6}\sum^{4}\{\ \sigma_{ab}(\sigma_{cde}\sigma_{fgh}\sigma_{ij}+\sigma_{cd}\sigma_{efgh}\sigma_{ij}+\sigma_{cd}\sigma_{eij}\sigma_{fgh})$$

$$+\sum_{(ab)}^{2}\{\sigma_{acd}\sigma_{be}\sigma_{fgh}\sigma_{ij}+\sigma_{ae}(\sigma_{bcd}\sigma_{fgh}\sigma_{ij}+\sigma_{bfgh}\sigma_{cd}\sigma_{ij}+\sigma_{bij}\sigma_{cd}\sigma_{fgh})$$

$$+\sigma_{afgh}\sigma_{be}\sigma_{cd}\sigma_{ij}+\sigma_{aij}\sigma_{be}\sigma_{cd}\sigma_{fgh}\}+\sigma_{abe}\sigma_{cd}\sigma_{fgh}\sigma_{ij}\ \}$$

$$+\sum^{36}(\sigma_{ab}\sigma_{ef}+\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}\sigma_{gh}\sigma_{ij}$$

$$-(\sigma_{abcdefgh}\sigma_{ij}+\sum^{2}\sigma_{abcdi}\sigma_{efghj})+2\sigma_{abcd}\sigma_{efgh}\sigma_{ij}$$

$$+\sum^{2}\sum^{4}\{\ (\sigma_{abcd}\sigma_{efgh}+\sigma_{aefgh}\sigma_{bcd})\sigma_{ij}+\sum^{2}\sigma_{ai}(\sigma_{bcd}\sigma_{efghj}+\sigma_{bcdj}\sigma_{efgh})$$

$$+\sigma_{aij}\sigma_{bcd}\sigma_{efgh}\ \}$$

$$-\sum^{2}\sum^{6}(\sigma_{ab}\sigma_{ij}+\sigma_{ai}\sigma_{bj}+\sigma_{aj}\sigma_{bi})\sigma_{cd}\sigma_{efgh}$$

$$-\sum^{16}(\sigma_{ae}\sigma_{ij}+\sigma_{ai}\sigma_{ej}+\sigma_{aj}\sigma_{ei})\sigma_{bcd}\sigma_{fgh}\Big]+O(N^{-3}).$$

The final results are given from the above ones and the known ones (Subsection 1.3 of this supplement; Ogasawara, 2008, Lemma 2).

2.3 $\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(m_{efgh}-\sigma_{efgh})(m_{ijkl}-\sigma_{ijkl})\}$

Write,

$$m_{abcd}=\frac{1}{N}S_{abcd}-\frac{1}{N^2}\sum^{4}S_aS_{bcd}+\frac{1}{N^3}\sum^{6}S_aS_bS_{cd}-\frac{3}{N^4}S_aS_bS_cS_d,$$

$$m_{efgh}=\frac{1}{N}S_{efgh}-\frac{1}{N^2}\sum^{4}S_eS_{fgh}+\frac{1}{N^3}\sum^{6}S_eS_fS_{gh}-\frac{3}{N^4}S_eS_fS_gS_h,$$

$$m_{ijkl}=\frac{1}{N}S_{ijkl}-\frac{1}{N^2}\sum^{4}S_iS_{jkl}+\frac{1}{N^3}\sum^{6}S_iS_jS_{kl}-\frac{3}{N^4}S_iS_jS_kS_l.$$

Then,

$$\begin{aligned}
&\mathrm{E}(m_{abcd}m_{efgh}m_{ijkl})\\
&=\frac{1}{N^2}\sigma_{abcdefghijkl}+\frac{N-1}{N^2}\sum^{3}\sigma_{abcdefgh}\sigma_{ijkl}+\frac{(N-1)(N-2)}{N^2}\sigma_{abcd}\sigma_{efgh}\sigma_{ijkl}\\
&-\frac{N-1}{N^3}\sum^{3}\{12\sigma_{abcdefgh}\sigma_{ijkl}+\sum^{4}(\sigma_{aefgh}\sigma_{bcdijkl}+\sigma_{aijkl}\sigma_{bcdefgh}+\sigma_{aefghijkl}\sigma_{bcd})\}\\
&-\frac{(N-1)(N-2)}{N^3}(12\sigma_{abcd}\sigma_{efgh}\sigma_{ijkl}+\sum^{3}\sum^{4}\sum^{2}\sigma_{aefgh}\sigma_{bcd}\sigma_{ijkl})\\
&+\frac{(N-1)(N-2)}{N^4}\Bigg[18\sigma_{abcd}\sigma_{befgh}\sigma_{ijkl}+\sum^{3}\sum^{6}\{\sigma_{ab}(\sum^{2}\sigma_{cdefgh}\sigma_{ijkl}+\sigma_{cd}\sigma_{efghijkl})\\
&+\sum_{(ab)}^{2}\sigma_{acd}\sum^{2}\sigma_{befgh}\sigma_{ijkl}+\sum_{(ab)}^{2}\sum^{2}\sigma_{aefgh}(\sigma_{bcd}\sigma_{ijkl}+\sigma_{bijkl}\sigma_{cd})+\sum^{2}\sigma_{abefgh}\sigma_{cd}\sigma_{ijkl}\ \}\Bigg]\\
&+\frac{(N-1)(N-2)(N-3)}{N^4}\sum^{3}\sum^{3}2\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ijkl}\\
&-\frac{3(N-1)(N-2)(N-3)}{N^5}\sum^{3}\sum^{3}\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ijkl}\\
&+\frac{(N-1)(N-2)}{N^4}\sum^{3}\sum^{16}\{\sigma_{ae}(\sigma_{bcdfgh}\sigma_{ijkl}+\sigma_{bcdijkl}\sigma_{fgh}+\sigma_{bcd}\sigma_{fghijkl})\\
&\qquad+\sigma_{abcd}(\sigma_{efgh}\sigma_{ijkl}+\sigma_{eijkl}\sigma_{fgh})+\sigma_{afgh}(\sigma_{bcde}\sigma_{ijkl}+\sigma_{bcd}\sigma_{eijkl})\\
&\quad+\sigma_{aijkl}(\sigma_{bcde}\sigma_{fgh}+\sigma_{bcd}\sigma_{efgh})+\sigma_{aeijkl}\sigma_{bcd}\sigma_{fgh}+\sum^{2}\sigma_{aebcd}\sigma_{fgh}\sigma_{ijkl}\}\\
&+\frac{(N-1)(N-2)(N-3)}{N^4}\sum^{3}\sum^{16}\sigma_{ae}\sigma_{bcd}\sigma_{fgh}\sigma_{ijkl}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(N-1)(N-2)(N-3)}{N^5}\sum^{6}\sum^{6}\sum^{4}[\ \sigma_{ab}\{\sigma_{cd}(\sigma_{efgh}\sigma_{ijkl}+\sigma_{eijkl}\sigma_{fgh})+\sigma_{cde}\sigma_{fgh}\sigma_{ijkl}\}\\
&\quad+\sum_{(ab)}^{2}\{\ \sigma_{acd}\sigma_{be}\sigma_{fgh}\sigma_{ijkl}+\sigma_{ae}(\sigma_{bcd}\sigma_{fgh}\sigma_{ijkl}+\sigma_{bfgh}\sigma_{cd}\sigma_{ijkl}+\sigma_{bijkl}\sigma_{cd}\sigma_{fgh})\\
&\qquad+\sigma_{afgh}\sigma_{be}\sigma_{cd}\sigma_{ijkl}+\sigma_{aijkl}\sigma_{be}\sigma_{cd}\sigma_{fgh}\ \}+\sigma_{abe}\sigma_{cd}\sigma_{fgh}\sigma_{ijkl}\ ]\\
&+\frac{(N-1)(N-2)(N-3)(N-4)}{N^6}\sum^{3}\sum^{36}(\sigma_{ab}\sigma_{ef}+\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}\sigma_{gh}\sigma_{ijkl}\\
&-\frac{(N-1)(N-2)(N-3)}{N^5}\sum^{64}\{\ \sigma_{aei}\sigma_{bcd}\sigma_{fgh}\sigma_{jkl}+\sum^{3}\sigma_{abcd}\sigma_{ei}\sigma_{fgh}\sigma_{jkl}\\
&\qquad+\sum^{3}(\sigma_{afgh}\sigma_{jkl}+\sigma_{ajkl}\sigma_{fgh})\sigma_{bcd}\sigma_{ei}\ \}\\
&+\frac{(N-1)(N-2)(N-3)(N-4)}{N^6}\sum^{3}\sum^{96}(\sigma_{ab}\sigma_{ei}+\sigma_{ae}\sigma_{bi}+\sigma_{ai}\sigma_{be})\sigma_{cd}\sigma_{fgh}\sigma_{jkl}\\
&+O(N^{-3})\\
&=\sigma_{abcd}\sigma_{efgh}\sigma_{ijkl}\\
&+\frac{1}{N}(\sum^{3}\sigma_{abcdefgh}\sigma_{ijkl}-15\sigma_{abcd}\sigma_{efgh}\sigma_{ijkl}-\sum^{3}\sum^{4}\sum^{2}\sigma_{aefg}\sigma_{bcd}\sigma_{ijkl}\\
&\qquad+2\sum^{3}\sum^{3}\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ijkl}+\sum^{3}\sum^{16}\sigma_{ae}\sigma_{bcd}\sigma_{fgh}\sigma_{ijkl})\\
&+\frac{1}{N^2}\Bigg[\sigma_{abcdefghijkl}-13\sum^{3}\sigma_{abcdefgh}\sigma_{ijkl}+104\sigma_{abcd}\sigma_{efgh}\sigma_{ijkl}\\
&-\sum^{3}\sum^{4}(\sigma_{aefgh}\sigma_{bcdijkl}+\sigma_{aijkl}\sigma_{bcdefgh}+\sigma_{aefghijkl}\sigma_{bcd})+3\sum^{3}\sum^{4}\sum^{2}\sigma_{aefgh}\sigma_{bcd}\sigma_{ijkl}\\
&+\sum^{3}\sum^{6}\{\ \sigma_{ab}(\sum^{2}\sigma_{cdefgh}\sigma_{ijkl}+\sigma_{cd}\sigma_{efghijkl})+\sum_{(ab)}^{2}\sigma_{acd}\sum^{2}\sigma_{befgh}\sigma_{ijkl}\\
&+\sum_{(ab)}^{2}\sum^{2}\sigma_{aefgh}(\sigma_{bcd}\sigma_{ijkl}+\sigma_{bijkl}\sigma_{cd})+\sum^{2}\sigma_{abefgh}\sigma_{cd}\sigma_{ijkl}\ \}\\
&-15\sum^{3}\sum^{3}\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ijkl}
\end{aligned}
$$

$$+\sum^{3}\sum^{16}\{\ \sigma_{ae}(\sigma_{bcdfgh}\sigma_{ijkl}+\sigma_{bcdijkl}\sigma_{fgh}+\sigma_{bcd}\sigma_{fghijkl})+\sigma_{abcd}\sigma_{eijkl}\sigma_{fgh}$$

$$+\sigma_{afgh}(\sigma_{bcde}\sigma_{ijkl}+\sigma_{bcd}\sigma_{eijkl})+\sigma_{aijkl}(\sigma_{bcde}\sigma_{fgh}+\sigma_{bcd}\sigma_{efgh})$$

$$+\sigma_{aeijkl}\sigma_{bcd}\sigma_{fgh}+\sum^{2}\sigma_{aebcd}\,\sigma_{fgh}\sigma_{ijkl}\ \}-6\sum^{3}\sum^{16}\sigma_{ae}\sigma_{bcd}\sigma_{fgh}\sigma_{ijkl}$$

$$-\sum^{6}\sum^{6}\sum^{4}[\ \sigma_{ab}\{\sigma_{cd}(\sigma_{efgh}\sigma_{ijkl}+\sigma_{eijkl}\sigma_{fgh})+\sigma_{cde}\sigma_{fgh}\sigma_{ijkl}\}$$

$$+\sum_{(ab)}^{2}\{\ \sigma_{acd}\sigma_{be}\sigma_{fgh}\sigma_{ijkl}+\sigma_{ae}(\sigma_{bcd}\sigma_{fgh}\sigma_{ijkl}+\sigma_{bfgh}\sigma_{cd}\sigma_{ijkl}+\sigma_{bijkl}\sigma_{cd}\sigma_{fgh})$$

$$+\sigma_{afgh}\sigma_{be}\sigma_{cd}\sigma_{ijkl}+\sigma_{aijkl}\sigma_{be}\sigma_{cd}\sigma_{fgh}\ \}+\sigma_{abe}\sigma_{cd}\sigma_{fgh}\sigma_{ijkl}\ ]$$

$$+\sum^{3}\sum^{36}(\sigma_{ab}\sigma_{ef}+\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}\sigma_{gh}\sigma_{ijkl}$$

$$-\sum^{64}\{\ \sigma_{aei}\sigma_{bcd}\sigma_{fgh}\sigma_{jkl}+\sum^{3}\sigma_{abcd}\sigma_{ei}\sigma_{fgh}\sigma_{ikl}$$

$$+\sum^{3}(\sigma_{afgh}\sigma_{jkl}+\sigma_{ajkl}\sigma_{fgh})\sigma_{bcd}\sigma_{ei}\ \}$$

$$+\sum^{3}\sum^{96}(\sigma_{ab}\sigma_{ei}+\sigma_{ae}\sigma_{bi}+\sigma_{ai}\sigma_{be})\sigma_{cd}\sigma_{fgh}\sigma_{jkl}\Big]+O(N^{-3}).$$

Using the results of Subsection 2.2 of this supplement and Ogasawara (2008, Lemma 1), we have the final results.

3. The fourth moments

From Ogasawara (2009, Equation (2.12)),

$$\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh})(s_{ij}-\sigma_{ij})\}$$

$$=\sum^{3}\mathrm{acov}(m_{abcd},s_{ef})\mathrm{acov}(s_{gh},s_{ij})+O(N^{-3})$$

with $N\,\mathrm{acov}(m_{abcd},s_{ef})=\sigma_{abcdef}-\sigma_{abcd}\sigma_{ef}-\sum^{4}\sigma_{aef}\sigma_{bcd}$ and

$N\,\mathrm{acov}(s_{gh},s_{ij})=\sigma_{ghij}-\sigma_{gh}\sigma_{ij}$;

and

$$\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(m_{efgh}-\sigma_{efgh})(s_{ij}-\sigma_{ij})(s_{kl}-\sigma_{kl})\}$$

$$=\mathrm{acov}(m_{abcd},m_{efgh})\mathrm{acov}(s_{ij},s_{kl})+\sum^{2}\mathrm{acov}(m_{abcd},s_{ij})\,\mathrm{acov}(m_{efgh},s_{kl})$$

$$+O(N^{-3}),$$

with

$$N\,\mathrm{acov}(m_{abcd},m_{efgh})=\sigma_{abcdefgh}-\sum^{4}(\sigma_{efgha}\sigma_{bcd}+\sigma_{abcde}\sigma_{fgh})-\sigma_{abcd}\sigma_{efgh}$$

$$+\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae},$$

where N can be replaced by n.

The results

$\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(m_{efgh}-\sigma_{efgh})(m_{ijkl}-\sigma_{ijkl})(s_{wx}-\sigma_{wx})\}$ and

$\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(m_{efgh}-\sigma_{efgh})(m_{ijkl}-\sigma_{ijkl})(m_{wxyz}-\sigma_{wxyz})\}$

are similarly given.

Errata

The term $-2\omega^{AB}(\mathbf{R}^{*})_{CD}$ in Equations (4.6) and (4.7) should be $-\omega^{AB}(\mathbf{R}^{*})_{CD}-\omega^{CD}(\mathbf{R}^{*})_{AB}$.

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