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Supplement to the paper "Asymptotic expansions in multi-group analysis of moment structures with an application to linearized estimators" and errata for the paper on the ADF chi-square statistic.

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This note is to supplement Ogasawara (in press), and gives errata for Ogasawara (2009).

1. Supplement to Ogasawara (in press)

Let

$$\mathbf{\Omega}^{-1} = (\omega^{AB}) = \begin{pmatrix} \mathbf{\Omega}^{(k)} & \mathbf{\Omega}^{(k,2k)} \\ \mathbf{\Omega}^{(2k,k)} & \mathbf{\Omega}^{(2k)} \end{pmatrix} = (\mathbf{\Omega}^{(\cdot k)} \ \ \mathbf{\Omega}^{(\cdot 2k)}) = \begin{pmatrix} \mathbf{\Omega}^{(k\cdot)} \\ \mathbf{\Omega}^{(2k\cdot)} \end{pmatrix}$$

$(A, B = 1, \ldots, G^* p^{**})$,

$\mathbf{\Omega}_{(2k|k)}^{-1} = \mathbf{\Omega}^{(2k)}$ and $\mathrm{E}_{\mathbf{u}|\cdot}(\cdot) = \mathrm{E}_{\mathbf{u}_{(k)}|\mathbf{\Gamma}} \mathrm{E}_{\mathbf{u}_{(2k)}|\mathbf{\Omega}_{(2k|k)}, \mathbf{u}_{(k)}}(\cdot)$. Let $(\cdot)_{(A-D)} = (\cdot)_{(ABCD)}$ and $(\cdot)_{(A-F)} = (\cdot)_{(ABCDEF)}$. Then, the expectation of $\mathbf{h}_{(2)}$ with the redefinition of $\mathbf{u}_{(2k)} \sim N(\mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{u}_{(k)}, \mathbf{\Omega}_{(2k|k)})$ is

$$\mathrm{E}_{\mathbf{u}|\cdot}(\mathbf{h}_{(2)})_{(AB)} = \{\mathbf{\Omega}^{-1}\mathrm{E}_{\mathbf{u}|\cdot}(\mathbf{u}\mathbf{u}')\mathbf{\Omega}^{-1}\}_{AB} - \omega^{AB}$$

$$= \left\{ \mathbf{\Omega}^{-1} \mathrm{E}_{\mathbf{u}(k)|\mathbf{\Gamma}} \begin{pmatrix} \mathbf{u}_{(k)}\mathbf{u}_{(k)}' & \mathbf{u}_{(k)}\mathbf{u}_{(k)}'\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Omega}_{(k,2k)} \\ \mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{u}_{(k)}\mathbf{u}_{(k)}' & \mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{u}_{(k)}\mathbf{u}_{(k)}'\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Omega}_{(k,2k)} + \mathbf{\Omega}_{(2k|k)} \end{pmatrix} \mathbf{\Omega}^{-1} \right\}_{AB} - \omega^{AB}$$

$$= \left\{ \mathbf{\Omega}^{-1} \begin{pmatrix} \mathbf{\Gamma} & \mathbf{\Gamma}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Omega}_{(k,2k)} \\ \mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Gamma} & \mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Gamma}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Omega}_{(k,2k)} + \mathbf{\Omega}_{(2k|k)} \end{pmatrix} \mathbf{\Omega}^{-1} \right\}_{AB} - \omega^{AB}.$$

Define $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}_1, \mathbf{Q}_2$ and $\boldsymbol{\Phi}$ as follows:

$$\boldsymbol{\Gamma} = \boldsymbol{\Omega}_\pi^{1/2}(\mathbf{I}_{G^*p^*} - 2it\boldsymbol{\Omega}_\pi^{1/2}\mathbf{J}_0\boldsymbol{\Omega}_\pi^{1/2})^{-1}\boldsymbol{\Omega}_\pi^{1/2} = \boldsymbol{\Omega}_\pi^{1/2}(\phi\mathbf{P}_1\mathbf{P}_1' + \mathbf{P}_2\mathbf{P}_2')\boldsymbol{\Omega}_\pi^{1/2}$$
$$= \boldsymbol{\Omega}_\pi^{1/2}\mathbf{P}\boldsymbol{\Phi}\mathbf{P}'\boldsymbol{\Omega}_\pi^{1/2} = \boldsymbol{\Omega}_\pi^{1/2}(\phi\mathbf{Q}_1 + \mathbf{Q}_2)\boldsymbol{\Omega}_\pi^{1/2},$$
$$\mathbf{P} = (\mathbf{P}_1 \ \mathbf{P}_2),\ \mathbf{P}\mathbf{P}' = \mathbf{I}_{G^*p^*}, \boldsymbol{\Phi} = \begin{pmatrix} \phi\mathbf{I}_{p^+} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_q \end{pmatrix}, \mathbf{Q}_1 = \mathbf{P}_1\mathbf{P}_1', \mathbf{Q}_2 = \mathbf{P}_2\mathbf{P}_2'.$$

Then,

$$\mathrm{E}_{\mathbf{u}|\cdot}(\mathbf{h}_{(2)})_{(AB)} = (\boldsymbol{\Omega}^{(\cdot k)}\boldsymbol{\Gamma}\boldsymbol{\Omega}^{(k\cdot)})_{AB} + (\boldsymbol{\Omega}^{(\cdot 2k)}\boldsymbol{\Omega}_{(2k,k)}\boldsymbol{\Omega}_\pi^{-1}\boldsymbol{\Gamma}\boldsymbol{\Omega}^{(k\cdot)})_{AB}$$
$$+(\boldsymbol{\Omega}^{(\cdot k)}\boldsymbol{\Gamma}\boldsymbol{\Omega}_\pi^{-1}\boldsymbol{\Omega}_{(k,2k)}\boldsymbol{\Omega}^{(2k\cdot)})_{AB} + (\boldsymbol{\Omega}^{(\cdot 2k)}\boldsymbol{\Omega}_{(2k,k)}\boldsymbol{\Omega}_\pi^{-1}\boldsymbol{\Gamma}\boldsymbol{\Omega}_\pi^{-1}\boldsymbol{\Omega}_{(k,2k)}\boldsymbol{\Omega}^{(2k\cdot)})_{AB}$$
$$+ (\boldsymbol{\Omega}^{(\cdot 2k)}\boldsymbol{\Omega}_{(2k|k)}\boldsymbol{\Omega}^{(2k\cdot)})_{AB} - \omega^{AB}$$
$$= \phi\{\boldsymbol{\Omega}^{(\cdot k)}\boldsymbol{\Omega}_\pi^{1/2}\mathbf{Q}_1\boldsymbol{\Omega}_\pi^{1/2}\boldsymbol{\Omega}^{(k\cdot)} + \boldsymbol{\Omega}^{(\cdot 2k)}\boldsymbol{\Omega}_{(2k,k)}\boldsymbol{\Omega}_\pi^{-1/2}\mathbf{Q}_1\boldsymbol{\Omega}_\pi^{1/2}\boldsymbol{\Omega}^{(k\cdot)}$$
$$+ \boldsymbol{\Omega}^{(\cdot k)}\boldsymbol{\Omega}_\pi^{1/2}\mathbf{Q}_1\boldsymbol{\Omega}_\pi^{-1/2}\boldsymbol{\Omega}_{(k,2k)}\boldsymbol{\Omega}^{(\cdot 2k)}$$
$$+ \boldsymbol{\Omega}^{(\cdot 2k)}\boldsymbol{\Omega}_{(2k,k)}\boldsymbol{\Omega}_\pi^{-1/2}\mathbf{Q}_1\boldsymbol{\Omega}_\pi^{-1/2}\boldsymbol{\Omega}_{(k,2k)}\boldsymbol{\Omega}^{(\cdot 2k)}\}_{AB}$$
$$+ \{\boldsymbol{\Omega}^{(\cdot k)}\boldsymbol{\Omega}_\pi^{1/2}\mathbf{Q}_2\boldsymbol{\Omega}_\pi^{1/2}\boldsymbol{\Omega}^{(k\cdot)} + \boldsymbol{\Omega}^{(\cdot 2k)}\boldsymbol{\Omega}_{(2k,k)}\boldsymbol{\Omega}_\pi^{-1/2}\mathbf{Q}_2\boldsymbol{\Omega}_\pi^{1/2}\boldsymbol{\Omega}^{(k\cdot)}$$
$$+ \boldsymbol{\Omega}^{(\cdot k)}\boldsymbol{\Omega}_\pi^{1/2}\mathbf{Q}_2\boldsymbol{\Omega}_\pi^{-1/2}\boldsymbol{\Omega}_{(k,2k)}\boldsymbol{\Omega}^{(\cdot 2k)}$$
$$+ \boldsymbol{\Omega}^{(\cdot 2k)}\boldsymbol{\Omega}_{(2k,k)}\boldsymbol{\Omega}_\pi^{-1/2}\mathbf{Q}_1\boldsymbol{\Omega}_\pi^{-1/2}\boldsymbol{\Omega}_{(k,2k)}\boldsymbol{\Omega}^{(\cdot 2k)}$$
$$+ \boldsymbol{\Omega}^{(\cdot 2k)}\boldsymbol{\Omega}_{(2k|k)}\boldsymbol{\Omega}^{(\cdot 2k)}\}_{AB} - \omega^{AB}$$
$$\equiv \phi(\mathbf{Q}_1^*)_{AB} + (\mathbf{Q}_2^*)_{AB} + (\boldsymbol{\Omega}^*)_{AB} - \omega^{AB}$$
$$\equiv \phi(\mathbf{Q}_1^*)_{AB} + (\mathbf{R}^*)_{AB} - \omega^{AB}.$$

Similarly, we have

$$\mathrm{E}_{\mathbf{u}|\cdot}(\mathbf{h}_{(4)})_{(A-D)} = \phi^2\sum^3(\mathbf{Q}_1^*)_{AB}(\mathbf{Q}_1^*)_{CD} + \phi\sum^6\{(\mathbf{R}^*)_{AB} - \omega^{AB}\}(\mathbf{Q}_1^*)_{CD}$$
$$+\sum^3(\mathbf{R}^*)_{AB}(\mathbf{R}^*)_{CD} - \sum^6\omega^{AB}(\mathbf{R}^*)_{CD} + \sum^3\omega^{AB}\omega^{CD},$$

$$\mathrm{E}_{\mathbf{u}|\cdot}(\mathbf{h}_{(6)})_{(A-F)} = \phi^3 \sum^{15} (\mathbf{Q}_1^*)_{AB} (\mathbf{Q}_1^*)_{CD} (\mathbf{Q}_1^*)_{EF}$$
$$+\phi^2 \sum^{45} \{(\mathbf{R}^*)_{AB} - \omega^{AB}\} (\mathbf{Q}_1^*)_{CD} (\mathbf{Q}_1^*)_{EF}$$
$$+\phi \sum^{45} \{(\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} - \omega^{AB} (\mathbf{R}^*)_{CD} - \omega^{CD} (\mathbf{R}^*)_{AB} + \omega^{AB}\omega^{CD}\} (\mathbf{Q}_1^*)_{EF}$$
$$+\sum^{15} (\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} (\mathbf{R}^*)_{EF} - \sum^{45} \omega^{AB} (\mathbf{R}^*)_{CD} (\mathbf{R}^*)_{EF}$$
$$+\sum^{45} \omega^{AB}\omega^{CD} (\mathbf{R}^*)_{EF} - \sum^{15} \omega^{AB}\omega^{CD}\omega^{EF}$$
$$(A, B, C, D, E, F = 1, \ldots, G^* p^{**}).$$

Define

$$A^{(1)}_{A\text{-}F} = \sum^{15} (\mathbf{Q}_1^*)_{AB} (\mathbf{Q}_1^*)_{CD} (\mathbf{Q}_1^*)_{EF}, \quad A^{(2)}_{A\text{-}D} = \sum^{3} (\mathbf{Q}_1^*)_{AB} (\mathbf{Q}_1^*)_{CD},$$
$$A^{(3)}_{A\text{-}F} = \sum^{45} \{(\mathbf{R}^*)_{AB} - \omega^{AB}\} (\mathbf{Q}_1^*)_{CD} (\mathbf{Q}_1^*)_{EF},$$
$$A^{(4)}_{A\text{-}D} = \sum^{6} \{(\mathbf{R}^*)_{AB} - \omega^{AB}\} (\mathbf{Q}_1^*)_{CD},$$
$$A^{(5)}_{A\text{-}F} = \sum^{45} \{(\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} - \omega^{AB} (\mathbf{R}^*)_{CD} - \omega^{CD} (\mathbf{R}^*)_{AB} + \omega^{AB}\omega^{CD}\} (\mathbf{Q}_1^*)_{EF},$$
$$A^{(6)}_{AB} = (\mathbf{R}^*)_{AB} - \omega^{AB},$$
$$A^{(7)}_{A\text{-}D} = \sum^{3} (\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} - \sum^{6} \omega^{AB} (\mathbf{R}^*)_{CD} + \sum^{3} \omega^{AB}\omega^{CD},$$
$$A^{(8)}_{A\text{-}F} = \sum^{15} (\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} (\mathbf{R}^*)_{EF} - \sum^{45} \omega^{AB} (\mathbf{R}^*)_{CD} (\mathbf{R}^*)_{EF}$$
$$+\sum^{45} \omega^{AB}\omega^{CD} (\mathbf{R}^*)_{EF} - \sum^{15} \omega^{AB}\omega^{CD}\omega^{EF},$$
$$B^{(1)}_{A\text{-}D} = \sum^{3} (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{AD} (\boldsymbol{\Omega}\mathbf{Q}_1^*\boldsymbol{\Omega})_{BC},$$
$$B^{(2)}_{A\text{-}D} = \sum^{3} \left\{ (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{AD} (\boldsymbol{\Omega}\mathbf{R}^*\boldsymbol{\Omega})_{BC} + (\boldsymbol{\Omega}\mathbf{R}^*)_{AD} (\boldsymbol{\Omega}\mathbf{Q}_1^*\boldsymbol{\Omega})_{BC} \right\},$$
$$B^{(3)}_{A\text{-}D} = \sum^{3} (\boldsymbol{\Omega}\mathbf{R}^*)_{AD} (\boldsymbol{\Omega}\mathbf{R}^*\boldsymbol{\Omega})_{BC},$$
$$B^{(4)}_{A\text{-}F} = \sum^{6} (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{AD} (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{BE} (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{CF} + \sum^{9} (\boldsymbol{\Omega}\mathbf{Q}_1^*\boldsymbol{\Omega})_{AB} (\mathbf{Q}_1^*)_{DE} (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{CF},$$
$$B^{(5)}_{A\text{-}F} = \sum^{18} (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{AD} (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{BE} (\boldsymbol{\Omega}\mathbf{R}^*)_{CF} + \sum^{27} (\boldsymbol{\Omega}\mathbf{Q}_1^*\boldsymbol{\Omega})_{AB} (\mathbf{Q}_1^*)_{DE} (\boldsymbol{\Omega}\mathbf{R}^*)_{CF}$$
$$-\sum^{9} (\boldsymbol{\Omega}\mathbf{Q}_1^*)_{AD} (\boldsymbol{\Omega}\mathbf{Q}_1^*\boldsymbol{\Omega})_{BC}\, \omega^{EF},$$

$$B^{(6)}_{A-F}=\sum\nolimits^{18}(\mathbf{\Omega Q}_1^*)_{AD}(\mathbf{\Omega R}^*)_{BE}(\mathbf{\Omega R}^*)_{CF}+\sum\nolimits^{27}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{AB}(\mathbf{R}^*)_{DE}(\mathbf{\Omega R}^*)_{CF}$$
$$-\sum\nolimits^{18}(\mathbf{\Omega Q}_1^*)_{AD}(\mathbf{\Omega R}^*\mathbf{\Omega})_{BC}\omega^{EF},$$
$$B^{(7)}_{A-F}=\sum\nolimits^{6}(\mathbf{\Omega R}^*)_{AD}(\mathbf{\Omega R}^*)_{BE}(\mathbf{\Omega R}^*)_{CF}+\sum\nolimits^{9}(\mathbf{\Omega R}^*\mathbf{\Omega})_{AB}(\mathbf{R}^*)_{DE}(\mathbf{\Omega R}^*)_{CF}$$
$$-\sum\nolimits^{9}(\mathbf{\Omega R}^*)_{AD}(\mathbf{\Omega R}^*\mathbf{\Omega})_{BC}\omega^{EF},$$
$$B^{(8)}_{A-D}=\sum\nolimits^{3}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{AB}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{CD},B^{(9)}_{A-D}=\sum\nolimits^{6}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{AB}(\mathbf{\Omega R}^*\mathbf{\Omega})_{CD},$$
$$B^{(10)}_{A-D}=\sum\nolimits^{3}(\mathbf{\Omega R}^*\mathbf{\Omega})_{AB}(\mathbf{\Omega R}^*\mathbf{\Omega})_{CD},$$
$$C^{(1)}_{A-F}=\sum\nolimits^{15}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{AB}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{CD}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{EF},$$
$$C^{(2)}_{A-F}=\sum\nolimits^{45}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{AB}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{CD}(\mathbf{\Omega R}^*\mathbf{\Omega})_{EF},$$
$$C^{(3)}_{A-F}=\sum\nolimits^{45}(\mathbf{\Omega Q}_1^*\mathbf{\Omega})_{AB}(\mathbf{\Omega R}^*\mathbf{\Omega})_{CD}(\mathbf{\Omega R}^*\mathbf{\Omega})_{EF},$$
$$C^{(4)}_{A-F}=\sum\nolimits^{15}(\mathbf{\Omega R}^*\mathbf{\Omega})_{AB}(\mathbf{\Omega R}^*\mathbf{\Omega})_{CD}(\mathbf{\Omega R}^*\mathbf{\Omega})_{EF}$$
$$(A,...,F=1,...,G*p^{**}).$$

Then, after some algebra (see Ogasawara, 2009),

$$C_T(t^*)=\phi^{p^+/2}\{1+N_\cdot^{-1}(a^{(3)}\phi^3+a^{(2)}\phi^2+a^{(1)}\phi+a^{(0)}$$
$$+\mathrm{i}t^*b^{(1)}+(\mathrm{i}t^*)^2b^{(2)})\}+O(N_\cdot^{-2}),$$

where

$$a^{(3)}=\sum_{A-F}\left[\frac{1}{72}\left\{(\boldsymbol{\beta}^{<2>}_{(3)})_{(A-F)}A^{(1)}_{A-F}+\frac{\partial^3F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(3)})_{(DEF)}B^{(4)}_{A-F}\right\}\right.$$
$$\left.+\frac{1}{288}\frac{\partial^3F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}\frac{\partial^3F_0}{\partial\tau_D\partial\tau_E\partial\tau_F}C^{(1)}_{A-F}\right],$$

$$a^{(2)} = \sum_{A-D}\left\{\left(\frac{1}{24}\boldsymbol{\beta}_{(4)} + \frac{1}{6}\boldsymbol{\beta}_{(1)} \otimes \boldsymbol{\beta}_{(3)}\right)_{(A-D)} A^{(2)}_{A-D}\right.$$

$$\left.+\frac{1}{12}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(1)})_D B^{(1)}_{A-D} + \frac{1}{48}\frac{\partial^4 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C\partial\tau_D}B^{(8)}_{A-D}\right\}$$

$$+\sum_{A-F}\left[\frac{1}{72}\left\{(\boldsymbol{\beta}^{<2>}_{(3)})_{(A-F)}A^{(3)}_{A-F} + \frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(3)})_{(DEF)}(-B^{(4)}_{A-F} + B^{(5)}_{A-F})\right\}\right.$$

$$\left.+\frac{1}{288}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}\frac{\partial^3 F_0}{\partial\tau_D\partial\tau_E\partial\tau_F}(-2C^{(1)}_{A-F} + C^{(2)}_{A-F})\right],$$

$$a^{(1)} = \frac{1}{2}\sum_{AB}(\boldsymbol{\beta}_{(\Delta 2)} + \boldsymbol{\beta}^{<2>}_{(1)})_{(AB)}(\mathbf{Q}^{*}_{1})_{AB} + \sum_{A-D}\left\{\left(\frac{1}{24}\boldsymbol{\beta}_{(4)} + \frac{1}{6}\boldsymbol{\beta}_{(1)} \otimes \boldsymbol{\beta}_{(3)}\right)_{(A-D)} A^{(4)}_{A-D}\right.$$

$$\left.+\frac{1}{12}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(1)})_D(-B^{(1)}_{A-D} + B^{(2)}_{A-D}) + \frac{1}{48}\frac{\partial^4 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C\partial\tau_D}(-B^{(8)}_{A-D} + B^{(9)}_{A-D})\right\}$$

$$+\sum_{A-F}\left[\frac{1}{72}\left\{(\boldsymbol{\beta}^{<2>}_{(3)})_{(A-F)}A^{(5)}_{A-F} + \frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(3)})_{(DEF)}(-B^{(5)}_{A-F} + B^{(6)}_{A-F})\right\}\right.$$

$$\left.+\frac{1}{288}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}\frac{\partial^3 F_0}{\partial\tau_D\partial\tau_E\partial\tau_F}(C^{(1)}_{A-F} - 2C^{(2)}_{A-F} + C^{(3)}_{A-F})\right],$$

$$a^{(0)} = \frac{1}{2}\sum_{AB}(\boldsymbol{\beta}_{(\Delta 2)} + \boldsymbol{\beta}^{<2>}_{(1)})_{(AB)}A^{(6)}_{AB} + \sum_{A-D}\left\{\left(\frac{1}{24}\boldsymbol{\beta}_{(4)} + \frac{1}{6}\boldsymbol{\beta}_{(1)} \otimes \boldsymbol{\beta}_{(3)}\right)_{(A-D)} A^{(7)}_{A-D}\right.$$

$$\left.-\frac{1}{12}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(1)})_D B^{(2)}_{A-D} - \frac{1}{48}\frac{\partial^4 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C\partial\tau_D}B^{(9)}_{A-D}\right\}$$

$$+\sum_{A-F}\left[\frac{1}{72}\left\{(\boldsymbol{\beta}^{<2>}_{(3)})_{(A-F)}A^{(8)}_{A-F} - \frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(3)})_{(DEF)}B^{(6)}_{A-F}\right\}\right.$$

$$\left.+\frac{1}{288}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}\frac{\partial^3 F_0}{\partial\tau_D\partial\tau_E\partial\tau_F}(C^{(2)}_{A-F} - C^{(3)}_{A-F})\right],$$

$$b^{(1)} = \sum_{A-D} \left\{ \frac{1}{6} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} (\boldsymbol{\beta}_{(1)})_D B_{A-D}^{(3)} + \frac{1}{24} \frac{\partial^4 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C \partial \tau_D} B_{A-D}^{(10)} \right\}$$

$$+ \sum_{A-F} \left\{ \frac{1}{36} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} (\boldsymbol{\beta}_{(3)})_{(DEF)} B_{A-F}^{(7)} - \frac{1}{144} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} \frac{\partial^3 F_0}{\partial \tau_D \partial \tau_E \partial \tau_F} C_{A-F}^{(3)} \right\},$$

$$b^{(2)} = \frac{1}{72} \sum_{A-F} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} \frac{\partial^3 F_0}{\partial \tau_D \partial \tau_E \partial \tau_F} C_{A-F}^{(4)}.$$

2. Errata for Ogasawara (2009)

Ogasawara's (2009, Equations (4.4) and (4.5)) result with k=2 and without mean structures corresponding to the result in the previous section is a special case of this note, and should be corrected by using the definitions of $\mathbf{Q}_1^*$, $\mathbf{Q}_2^*$, $\boldsymbol{\Omega}^*$ and $\mathbf{R}^*$ given earlier. See also Ogasawara (2010) for an additional minor erratum in Ogasawara (2009, Equations (4.6) and (4.7)).

References

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