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TIME DISCOUNT AND CONVEX HIRING COST

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When rebargaining on wages is allowed after the worker-firm match is formed, search equilibrium with multiple hiring does not necessarily exhausts labor resources when hiring cost is convex. The level of output depends on the time discount factor of the consumer. Properties of the resulting demand-driven business cycles are studied using periodic steady state technique with comparison with the productivity cycles. Wage rate generally exhibits phase shift against marginal productivity for discount factor fluctuation in contrast to the synchronization of the productivity cycle case.

1. INTRODUCTION

Search models typically assume constant hiring cost per vacancy as a direct extension of one-to-one production by a firm and a worker in the tradition of Mortensen and Pissarides. However, Yashiv (2000) and Blatter et al. (2012) show that the hiring function is empirically highly convex and there is a good reason that convexity should be assumed in the function. Because of search friction, hiring activities are forced to rely on the internal resources if information needed for hiring is accessible only by insiders. More vigorous hiring activities cause congestions over the internal resources, therefore it raises the hiring cost in convexity. Yashiv (2006, 2007) analyzes the dynamics of the economy with capital when hiring cost is convex, which focuses on the most efficient path. However, the convex hiring cost implies that the dynamic path and the steady state can generically be different from the one in the Mortensen-Pissarides model, depending on the time preference of the consumer. In contrast to linear hiring cost, transition is time-consuming with convex hiring cost, therefore future discounting matters. Higher discounting by the consumer raises equilibrium interest rate, which in turn lowers the value of the firm and therefore lowers the steady state output. The lower value of production coalition makes the firms become unwilling to expand their sizes.

In this paper, business cycles caused by the fluctuation in intertemporal consumer preference is analyzed, not only the ones caused by the cycles in productivity. Note that the former source of business cycles becomes possible due to convex hiring cost which is inevitable requirement under the existence of labor friction with the reasons mentioned above. The paper also presents useful formulation and tools to analyze business cycles in the economy in which discrete jumps are regulated.

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2. THE MODEL

2.1. *The firm*

We focus on the case in which all existing firms are homogeneous and have strictly positive employment. Firms are competitive and continuously many with measure one, each run by an entrepreneur. Their production function is given by $f(l)$ with property $f' > 0$, $f'' \leq 0$ where labor l is assumed to be the only input. The firms face frictions in the labor market and wages are allowed to be bargained at any moment after workers are employed. Thus, in general, wage rate is a function of employment in present and future as well as other market conditions affecting the bargaining power, so we denote it by $w(\mathscr{L})$ where script letter $\mathscr{L}$ represents *function* $l(\cdot)$ the domain of which is time period $[t, \infty)$. We use this notation in general: $\mathscr{L}$ is a function the value of which as of time t is $l(t)$ and function $\mathscr{R}$ gives value $r(t)$ at time t. A script letter interprets the corresponding lower letter variable as a function of time. The hiring cost function $\kappa : \mathbb{R}_+ \to \mathbb{R}$ is a convex function of vacancy post $v \geq 0$. It is convex since, under the existence of friction, any hiring activities *must* use internal resources. In other words, the firm is producing two products, the output good and the filled vacancies with fixed resources. The convexity arises from the decreasing return of the hiring activities. The function is assumed to have the property $\kappa' > 0$, $\kappa'' > 0$, $\kappa(0) > 0$ and that κ' is unbounded above. The return of the posted vacancies depends on the tightness of the labor market. Let the vu ratio be denoted by $\theta := \hat{v}/\hat{u}$ where $\hat{u}$ and $\hat{v}$ are unemployed workers and vacancies in the market, respectively. Let $g : \mathbb{R}_+ \to \mathbb{R}$ denote a function which relates the tightness of the labor market θ to the arrival rate of applicants per posted vacancy. We keep a general representation of the matching function as far as possible since its functional form is arguable, however, when specification is necessary, we assume a Lagos (2000) type function as an underlying matching function. Namely, when matching sessions open n times per unit of time, we assume that the number of match per session is given by $\bar{\Phi}_n \min\{\hat{u}, \Phi_n/\bar{\Phi}_n\, \hat{v}\}$ where $\bar{\Phi}_n$ denotes the efficiency of the matching process and Φ_n captures that of hiring activities, and they are assumed to hold $\bar{\Phi}_1, \Phi_1 \leq 1$. This functional form has a desirable property that match is less than $\min\{\hat{u}, \hat{v}\}$ especially in the neighborhood of the origin. The hazard rate of vacancy is given by $\phi := \Phi_\infty = -\log(1 - \Phi_1)$ if $\theta \leq \bar{\phi}/\phi$, whereas that of unemployment is $\bar{\phi} := \bar{\Phi}_\infty = -\log(1 - \bar{\Phi}_1)$ for $\theta \geq \bar{\phi}/\phi$. Since the latter translates to $\bar{\phi}/\theta$ in terms of vacancy hazard rate, the increase of employment for vacancy post v is given by $g(\theta)v$ where

$$g(\theta) := \begin{cases} \phi & \text{if } \theta \leq \bar{\phi}/\phi \\ \bar{\phi}/\theta & \text{if } \theta \geq \bar{\phi}/\phi. \end{cases}$$

On the other hand, employed workers separate from the firm at rate σ and the firm can potentially fire workers at rate $x \in [0, \bar{x}]$ where $\bar{x}$ is taken sufficiently large so that we never need to consider the binding case. Then, the change of employment of the representative firm is given by

$$\dot{l}_t = g(\theta_t)v_t - \sigma_t l_t - x_t. \tag{1}$$

where subscripts denote the time of variables. The representative firm is risk neutral. Denoting the interest rate by r, it maximizes intertemporal profits given below subject to transition equation (1) to obtain the value function $J(l_t, r)$:

$$J(l_t, \mathscr{R}) = \max_{v,x} \int_t^\infty \Big(\min\left\{f(l_\xi), y^d(\mathscr{R})\right\} - w(\mathscr{L})\, l_\xi - \kappa(v_\xi)\Big)\, e^{-\int_t^\xi r_\tau \mathrm{d}\tau} \mathrm{d}\xi. \tag{2}$$

where $y^d(\mathscr{R})$ is the demand for output. In the first term of the integrand, the condition that production should not exceed the demand is required to keep consistency of the model, since a temporary plunge of output demand can lead to optimal labor hoarding. Instantaneous level of output demand is affected by interest rate, whereas that of supply is influenced by the value of labor which is an *asset* for the firm. Sufficiently short positive shock of discount rate and the derived fall of output demand will not accompany firing, since subsequent hiring after the shock becomes too costly compared to the cost of labor hoarding.

2.2. *Goods demand by households and equilibrium interest rate*

We assume households to have linear utility functions so that their utility value simply becomes discounted value of their income. This assumption to fit the tradition of search theory significantly simplifies the following analysis. The intertemporal utility function of consumers then becomes $U = \int_t^\infty \mathsf{E}_t c_\xi \, e^{-\int_t^\xi \beta_\nu \mathrm{d}\nu} \mathrm{d}\xi$ with budget constraint $\dot{a}_t = r_t a_t + y_t - c_t$ where c is consumption, β discount rate, a asset, r interest rate, E_t expectation operator with information set as of time t and subscripts denote the time related. There are three statuses for consumers: employed, unemployed and entrepreneur. Each earns instantaneous income $y = w$, $y = 0$ and $y = f(l) - wl - \kappa(m)$, respectively. An entrepreneur never loses its status so it can be understood as an independent type of consumer. On the other hand, employed and unemployed workers switch their status stochastically. An employed worker becomes the unemployed at next moment with separation rate σ and the flow in the opposite direction occurs with matching rate μ. Consumers can choose not only the level of consumption at any instance but also level of β at any moment.[1] This is so because future income levels are *not* data but they are simply the outcome of a game between consumers. If a consumer expects different β, or different future income schedule, when all other consumers forecasts unanimously different level, his income will turn out to be different from his initial expectation and his consumption plan becomes suboptimal. Note that this game is brought about from the fact that a firm can grow only continuously when the hiring cost shows convexity. The outcome in

[1] By doing so, consumers are choosing equilibrium income schedule.

the case of linear utility is simple: optimal condition requires $\beta_t = r_t$ for all t and for all players where any level of β can be supported as a Nash equilibrium. The linear utility function together with the above condition implies all consumers become indifferent between any intertemporal allocation of consumption as far as expected lifetime consumption equates that of income, thus clearing market.

2.3. *Labor value*

Total labor force in the economy is fixed at W. Workers are in either state, employed or unemployed. An unemployed worker receives non-transferential instantaneous unemployment benefit b_t at time t. An employed worker will be paid instantaneous wage w_t. The value of unemployment as of time t is denoted by U_t and the value of employment as of time t is denote by E_t. Matching sessions between job-seekers and vacancies open at any moment. Since matching probability of an unemployed worker in one shot is $\min\{\bar{\Phi}_n, \Phi_n\theta\}$, the matching hazard rate is given by

$$\mu(\theta) := \begin{cases} \phi\theta & \text{if } \theta \leq \bar{\phi}/\phi \\ \bar{\phi} & \text{if } \theta \geq \bar{\phi}/\phi. \end{cases}$$

When an unemployed worker is successfully matched, he shifts to the employment status and receives capital gain $E_t - U_t$. The Bellman equation for unemployment status is

$$r_t\, U_t = b_t + \mu(\theta_t)\ [E_t - U_t] + \dot{U}_t \tag{3}$$

where r is interest rate. Similarly, the employment value is given by

$$r_t\, E_t = w_t - \tilde{\sigma}_t\, [E_t - U_t] + \dot{E}_t \tag{4}$$

where $\tilde{\sigma}_t := \sigma_t + x_t/l_t$ is an instantaneous separation probability of a worker.

3. WAGE BARGAINING

When there exists friction in the labor market, pseudo-rent arises in an existing firm-workers group. The rent comes from the fact that any firms or workers who have not formed a group yet must enter a costly process of search. We allow the existing firm-worker group to bargain over the rent. The bargaining is made in each moment after the firm has undertaken an optimal policy. This assumption on timing can potentially affect wage rate when x is strictly positive. If the firm can decide x after bargaining is made, it can threat workers to swap with the planned dismissed. Although this alternative assumption is also plausible, we adopt the former since the alternative complicates the bargaining process and also makes wage rate formulation depend on future history. Coalition of a firm and continuum of workers with measure l will get intertemporal payoff F which is the value of

$$F(l_t) = \int_t^{\infty} \left(\min \left\{ f\left(l_\xi^*\right), \, y_\xi^d(\mathscr{R}) \right\} - \kappa\left(v_\xi^*\right) \right) e^{-\int_t^\xi r_\nu \mathrm{d}\nu} \mathrm{d}\xi \tag{5}$$

where l^* and v^* are on the optimal choice by the firm reflecting the fact that the decision of hiring is at its discretion. Our model corresponds to game $\Game_\infty^3(\Omega, v)$ defined in Appendix A, and we obtain the following result:

Theorem 1 *In* $\Game_\infty^3(\Omega, v)$*, the imputation*

$$E = \frac{1}{2}\left(U + \frac{\partial F}{\partial l_-} \right) \tag{6}$$

is supported by Shapley value and nucleolus where $\partial F / \partial l_-$ *is a derivative from left.*

Proof: See Appendix A. *Q.E.D.*

As the required assumptions show to obtain the theorem, this result is expected to hold in fairly general environments.[2] Equation (6) is Shapley value at the limit if the bargaining game between the firm and workers satisfies the properties of 1) essentiality, 2) anonymity, 3) indispensability and 4) existence of non-degenerate player. See Appendix A for the definition of these properties. With additional property of 5) essential concavity which requires concavity only for coalitions with more than *two* players, equation (6) is also nucleolus at the limit.

This approximate result is obtained by making number of workers go infinity while measure of a worker goes infinitesimally small so that total measure of labor input is fixed. On the other hand, the firm, or the entrepreneur, has massive influence on coalitional payoff in the sense that production will not be undertaken without him. The property that workers get only partial contribution depends on the assumption that the entrepreneur does not degenerate, rather than the particular value assumption that the number of entrepreneurs is one. Solving the value function, wage bargaining outcome is given by the next theorem.

[2]We require a singleton solution to proceed with the model. Pissarides (1985) assumed that, in the case that production is undertaken by a pair of a firm and a worker, they divide the rent by a Nash bargaining solution. There would have been an option to generalize it by adopting n-player Nash equilibrium. However, since the present model contains significant asymmetry between a firm and workers, it seems more natural to take coalitional rationality into account.

Also, we seek for the limit of the sequence of bargaining solutions and do not pursue uniformity as in the solution concept of uniform approximate core by Wooders (1992). Indeed, the pregame which generates the game presented here does not satisfy uniform inessentiality of large groups.

THEOREM 2 *Let $H_1 := \{t : y_t^d(\mathscr{R}) < f(l_t^*)\}$. Wage rate at time t is given by*

$$w_t = b_t + \frac{\mathrm{MP}_t - \mathrm{MC}_t}{2} + \left(\tilde{\sigma}_t + \frac{\mu_t}{2}\right) \int_t^\infty \frac{\mathrm{MP}_\xi - \mathrm{MC}_\xi}{2} e^{-\int_t^\xi (r+\mu/2)} \mathrm{d}\xi$$

where

$$\mathrm{MP}_\xi = \begin{cases} f'(l_\xi^*) & \text{if } \xi \notin H_1 \\ 0 & \text{if } \xi \in H_1, \end{cases}$$

$\mathrm{MC}_t = \kappa'(v_\xi^*)\mathrm{d}v_\xi^*/\mathrm{d}l_t < 0$ *shows the change of hiring cost when l hypothetically moves to a neighboring optimal path and v^* and l^* are along the optimal path.*

PROOF: See Appendix B. *Q.E.D.*

Wage rate is the sum of unemployment benefit and half of the discounted cost which arises when a worker quits. The above theorem implies that in general wage rate shows discontinuity at transition points $\partial \bar{H}_1$. Note that MC_t is the change of hiring cost when a worker *hypothetically* quits from the optimal path and actually does not.

COROLLARY 3 *At steady states, the wage rate w is given by*

$$w - b = \frac{1}{2} k(r, \mu, \sigma) \left(\frac{\partial f}{\partial l} - \frac{\mathrm{d}\kappa}{\mathrm{d}l} \right)$$

where $k(r, \mu, \sigma) := 1 + (\sigma + \mu/2)/(r + \mu/2)$.

PROOF: Directly derived from the theorem noting that $\mathrm{MP} = 0$ and $x > 0$ are impossible at steady states. *Q.E.D.*

COROLLARY 4 *Suppose $x_t = 0$ for all t. If $f'(l) - b$ and $r + \mu/2$ share common cycle T, wage rate at time t is given by*

$$w_t = b_t + \frac{\mathrm{MP}_t - \mathrm{MC}_t}{2} + \left(\tilde{\sigma}_t + \frac{\mu_t}{2}\right) \left(g(t) + \frac{e^{-\int_t^T (r+\mu/2)}}{1 - e^{-\int_0^T (r+\mu/2)}} g(0) \right).$$

where $g(t) := \int_t^T \frac{\mathrm{MP}_\xi - \mathrm{MC}_\xi}{2} e^{-\int_t^\xi (r+\mu/2)} \mathrm{d}\xi$.

Note that the separation is assumed to be exogenous both for the firm and for workers, and both sides are not responsible for the separation. Thus, the risk of the separation is taken into account in the wage bargaining, and it will be compensated by the firm. Corollary 3 shows that higher steady state output is achieved through lower real wage rate.

4. OPTIMAL POLICY OF FIRM

The optimization problem for the firm is given by equations (1) and (2) and Theorem 2. Suppose that the firm is about to make decision at time t. Wage rate derived in Theorem 2 is translated into a state variable by introducing a new variable "accumulated turnover of labor" L defined by $L_\xi := \int_t^\xi (\tilde{\sigma}_\tau + \mu_\tau/2)\, l_\tau e^{-\int_\tau^\xi \frac{\mu}{2}} \mathrm{d}\tau$, the transition of which is given by

$$\dot{L}_\xi = \left(\sigma_\xi + \frac{\mu_\xi}{2}\right) l_\xi - \frac{\mu_\xi}{2} L_\xi + x_\xi \tag{7}$$

and $L_t = 0$. Then, integration of $w(\mathscr{L})$ can be expressed in terms of l and L so that the discounted wage payment is given by

$$\int_t^\infty w(\mathscr{L})\, l e^{-\int r} = \int_t^\infty \left[\big(b + X(l,v)\big)l + X(l,v)L\right] e^{-\int r}$$

applying Dirichlet transform to Theorem 2 where we denote $X(l_\xi, v_\xi) := \big(\mathrm{MP}(l_\xi) - \mathrm{MC}(l_\xi, v_\xi)\big)/2$. On the other hand, indifferentiability of equation (2) is dissolved by introducing actual working labor m as a control variable, allowing for labor hoarding:

$$\begin{aligned} J(l_t, r) = \max_{v,m,x} \int_t^\infty &\Big[f(m_\xi) - (b_\xi + X(l_\xi, v_\xi))\, l_\xi \\ &- X(l_\xi, v_\xi) L_\xi - \kappa(v_\xi)\Big] e^{-\int_t^\xi r_\nu \mathrm{d}\nu} \mathrm{d}\xi \\ \text{s.t. } & f(m) \le y^d(\mathscr{R}),\ m \le l,\ x \ge 0. \end{aligned} \tag{2'}$$

Transition of l and L is given by equations (1) and (7).[3] Then, the optimal condition is given by

$$v \begin{cases} \text{solves } \kappa'(v) = g(\theta)\lambda_1 - X_v(l+L) & (\text{if } \lambda_1 > 0) \\ = 0 & (\text{if } \lambda_1 = 0) \end{cases} \tag{8}$$

$$x \begin{cases} = 0 & (\text{if } \lambda_1 > \lambda_2) \\ \ge 0 & (\text{if } \lambda_1 = \lambda_2) \end{cases} \tag{9}$$

$$m = \begin{cases} l & \text{if } \xi \notin H_1 \\ f^{-1}(y^d) & \text{if } \xi \in H_1 \end{cases} \tag{10}$$

$$\dot{\lambda}_1 = (r+\sigma)\,\lambda_1 - \left(\sigma + \frac{\mu}{2}\right)\lambda_2 + b + X + X_l(l+L) - \mathrm{MP}(l) \tag{11}$$

$$\dot{\lambda}_2 = (r + \mu/2)\,\lambda_2 + X \tag{12}$$

[3] Halkin (1964) provides the proofs for the case where the transition equation and the integrand of the evaluation function are once continuously differentiable in state variables, continuous in control variables and piecewise continuous in time.

where λ_1, λ_2 are the Hamiltonian conjugates for l and L respectively. The labor hoarding occurs when negative shock in $y^d(\mathscr{R})$ is sufficiently large so that reducing v to zero cannot catch up the move and still the value of labor is greater than λ_2 reflecting good conditions in future. Suppose the plunge of $y^d(\mathscr{R})$ is sufficiently large in size and expected to continue only for a short period so that its effect on the value of labor is negligible and kept above λ_2. From the production function, we have $\dot{y} = f'(l)\big(g(\theta)v - \sigma l\big)$ where v is determined by (8). Note that $x = 0$ due to $\lambda_1 > \lambda_2$. For $\dot{y} < 0$ sufficiently large in absolute value, the right hand side of the equation cannot match the decrease on the left side since the optimal control is $v^*(\lambda_1) > 0$ and $x = 0$ from equations (8) and (9). This fact requests the model to incorporate possible discrepancy between labor and actual working labor. By doing so, we acquire an additional control variable to match the right hand side of the equation to meet the change of y. The argument also shows that the length of recession matters for unemployment. If the recession is expected to be short enough, firms do not fire workers and its effect remains only in income and will not propagate to unemployment. It raises the growth rate of the value of labor as shown in equation (11), since the aim of the labor hoarding policy is to preserve the value of labor. Contrarily, a long recession raises unemployment due to reduced value of labor. Note that $\bar{x}$ is assumed to be sufficiently large, so firing/rejection occurs with $\lambda_1 = \lambda_2$.

The optimal path derived from equation (11) implies that $\lambda_1 = \partial J/\partial l_t$ is positive and decreasing in time as far as the initial labor is lower than the steady state. Namely, $\partial J/\partial l > 0$ and $\partial^2 J/\partial l \partial t < 0$ along the optimal path. It implies that the value of firm J is greater for incumbent firms compared to possible entrants. This is due to the fact that new entrants must pay higher wage rate due to higher marginal productivity. The no-entrance condition $J(0, r) \leq 0$ is satisfied even when the incumbent firms are willing to hire, i.e. $J(l, r) > 0$ for some $l > 0$. It justifies the assumption of fixed measure of firms in the model for sufficiently small fluctuations around the steady state. It also suggests that, when the economy is extended to allow for heterogeneous firms in size, downward swing of interest rate or rise of productivity in boom sufficiently large to make $J(0, r) > 0$ is the only chance to enter. Note that the current model only applies to the economy in which firms already have some size even if small. It does not mention how the universe has been created.

5. Static steady states

Suppose that parameters are stationary. From now on, the steady state with stationary parameters is called a *static steady state* when it needs to be clearly distinguished from a periodic steady state. Eliminating λ_2, it is characterized by

$$l = \frac{g(\bar{\theta})v}{\sigma} = \begin{cases} [\bar{\phi}/(\bar{\phi}+\sigma)]W & \text{if } \bar{\theta} \geq \bar{\phi}/\phi \\ (\phi/\sigma)v & \text{otherwise} \end{cases} \tag{1'}$$

$$\bar{v} \text{ solves} \begin{cases} \bar{v}\left(\kappa'(\bar{v}) - \frac{\mu+\sigma}{\mu} l \frac{\mathrm{d}v}{\mathrm{d}l}\kappa''(\bar{v})\right) = \bar{\phi}(W-l)\bar{\lambda}_1 & \text{if } \bar{\theta} \geq \bar{\phi}/\phi \\ \kappa'(\bar{v}) - \frac{\mu+\sigma}{\mu} l \frac{\mathrm{d}v}{\mathrm{d}l}\kappa''(\bar{v}) = \phi\bar{\lambda}_1 & \text{otherwise} \end{cases} \tag{8'}$$

$$\bar{\lambda}_1 = h(r, \bar{v}, \bar{l}) := \frac{\mathscr{V}(r, \bar{v}, \bar{l})}{r+\sigma} \tag{11'}$$

$$r = \beta \tag{13}$$

where

$$\mathscr{V}(r, \bar{v}, \bar{l}) := f'(\bar{l}) - b - k\left(r, \mu(\bar{\theta}), \sigma\right) X(\bar{l}, \bar{v}) - \frac{\sigma + \mu(\bar{\theta})}{\mu(\bar{\theta})/2} X_l(\bar{l}, \bar{v})\, \bar{l}$$

and the variables with a bar denote the solution at the static steady state. First three equations are derived from firm's problem and the last comes from that of consumers and at the same time it is an equilibrium condition. Equation (1') is the Beveridge curve showing negative relationship between u and v at static steady states. Equation (8') implies that $\bar{v}$ is an increasing function of $\bar{\lambda}_1$ as far as the neighboring path effect is not pathologically large. From equations (8), (11') and (13), we obtain a relation

$$\bar{v} = v(\bar{l}; \beta) \tag{14}$$

with $\partial v/\partial l < 0$ and $\partial v/\partial \beta < 0$. It is drawn as curve (b) in Figure 1. On the other hand, as far as $\bar{\theta} \leq \bar{\phi}/\phi$ holds, equation (1') is drawn as an upward sloping curve (a). Then, an equilibrium is given by its intersection with curve (b). The rise of β appears as the downward shift of (b), which pushes the equilibrium employment downward. Note that this happens with a *flexible* intertemporal relative price r.

On the other hand, if $\bar{\theta} \geq \bar{\phi}/\phi$ holds, equation (1') implies that $\bar{l}$ becomes constant at $\bar{l} = \bar{\phi}W/(\bar{\phi}+\sigma)$ regardless of v. It now turns out to be a vertical line (c) in Figure 1. Since the firm cannot increase employment further, it can be interpreted as the state of natural unemployment. It holds for $\beta \leq \bar{\beta}$ where the threshold $\bar{\beta}$ solves

$$\kappa'\left(\frac{\sigma}{\phi}\frac{\bar{\phi}}{\bar{\phi}+\sigma}W\right) - \frac{\mu+\sigma}{\mu}\frac{\bar{\phi}}{\bar{\phi}+\sigma}W\frac{\mathrm{d}v}{\mathrm{d}l}\kappa''\left(\frac{\sigma}{\phi}\frac{\bar{\phi}}{\bar{\phi}+\sigma}W\right) = \phi\, h\left(\bar{\beta}, \frac{\sigma}{\phi}\frac{\bar{\phi}}{\bar{\phi}+\sigma}W, \frac{\bar{\phi}}{\bar{\phi}+\sigma}W\right).$$

When the matching function is Lagos type, steady state natural unemployment rate is strictly positive whereas it is not the case for Cobb-Douglas matching function.

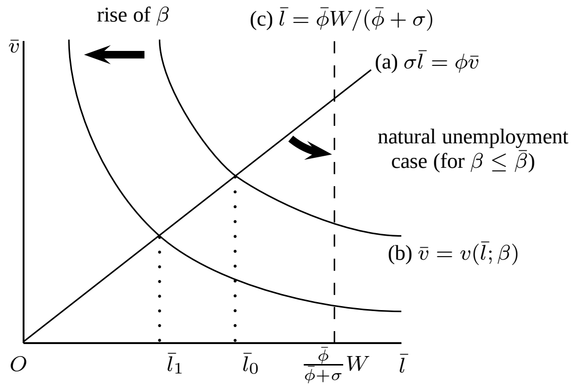


Figure 1: The effect of the change in β

Equation (14) or curve (b) represents the firm's demand for labor, which equates the expected value of additional vacancy post to its marginal cost. Bearish view for future good prices by consumers is represented by a higher discount rate. Resulting higher equilibrium interest rate reduces attractiveness of labor since hiring activity is time consuming. Thus, it shifts curve (b) down reducing hiring and employment. This process can be interpreted as a source of the effective demand principle, a variant represented in our model. Viewed as a whole intertemporal system, everything is in equilibrium. However, if it is cut out at a given moment abstracting the cost of vacancy posting, the goods market is in excess supply in the sense that the marginal productivity exceeds the marginal cost, and the same holds for the labor market in excess demand. The degree of the hiring cost anchors this divergent force determining where to stop. Note that, with a traditional linear hiring cost function, equation (14) becomes free from determining v. Since the initial jump of employment to the static steady state level is weakly optimum in that specification, v exceeds the natural employment level so that $\theta \geq \bar{\phi}/\phi$ to hold. Note that our results obtained here does not depend on the particular specification of the matching function. As far as equation (14) determines the optimal vacancy posting, the effective demand principle is derived.

There has been a controversy on the effectiveness of minimum wage policy since Card and Krueger (1994). Its outcome has been mixed. Studying the impact of the change of minimum wage in New Jersey in 1992 on the regional employment, Card and Krueger concluded that there is no indication that the rise in minimum wage reduced employment. However, reexamination by Neumark and Wascher (2000) using payroll data derives an opposite conclusion to them. With newer data, Dube et al. (2007) studied the minimum wage policy change in San Francisco in 2000's and found that its negative impact on employment can be rejected with greater confidence. Our model suggests that when data includes different markets or periods, the effect can vary depending on whether the market is in the state of natural employment or not. The minimum wage policy can be incorporated by setting the outside option b for the worker to minimum wages in our model. Comparing its effects at steady states, sufficiently small change in minimum wages at natural employment has no impacts on other variables than wages. However, a rise of b in underemployment *does* lower the value of labor via equation (11'), causing the leftward shift of curve (b) in Figure 1. It implies that measurement of minimum wage policy requires distinction between the states of natural employment and underemployment. Note that when external forces are present, the natural employment level is not fixed but generally fluctuates.

6. IMPACT OF THE CHANGE IN PRODUCTIVITY AND THE DISCOUNT FACTOR

Steady state solutions in Section 5 suggest that difference in the discount factor brings different level of output. This is still true out of static steady states. It means that there are two different sources of business cycles possible in our model: change in productivity and that in the discount factor. The former is the supply side cycle and the latter is the one on the demand side. Since a change in business cycles sources propagates with time lag due to lagged response of wage rates, we are inclined to analyze the response of the economy to forced oscillation in these sources. This approach fits our model better than the traditional shock analysis. A persistent change in external forces in future needs to be expected *ex ante* to have real upward effects on employment, since any unexpected instantaneous shocks upward are completely absorbed by the change of interest rate due to convex hiring cost. Only expected and persistent shocks can have real positive effects. In other words, for an unexpected positive shock to have real effects, it must accompany coordinated change of expectation after it becomes apparent. The shock must be persuasive enough to make people believe that the source of the shock *will* continue to have impacts at least for some time. An exception is the case of downward panic. Firms are allowed to dismiss workers with no cost. To mimic the traditional shock analysis in our framework, a pulse shock with a short period can be used, which is easily incorporated using Fourier approximation.

Our approach seeks for *periodic steady states* brought by forced oscillation. This bears in mind the application for general non-periodic inputs. A sufficiently long cycle in inputs should provide a good approximation for them since distant future is exponentially discounted. The approach may seem to remove the "transition" effect when the initial value is out of the periodic steady states. However, since any level of output is supported as an equilibrium by a corresponding discount factor and even after an unexpected shock, new coordinated expectation needs to be formed, it is not the case as a tool for a researcher. Advantage of this approach is, after linearizing the model around a static steady state, any oscillation can be approximated by sine and cosine curves. As far as we have a knowledge on the behavior of the system in reaction to those basis functions, its response to a general external force can be easily composed by the superposition principle.

Now, we modify the original model to be able to see the implication of policies on wages as well as linearization. Let the production function explicitly depend on the technological parameter α such that $f(l, \alpha)$ and $X(l, L, \lambda_1, \alpha)$. From Theorem 2, dynamics of wage rate is given by

$$\begin{aligned}\dot{w} = &\left(X_l(l, L, \lambda_1, \alpha) + \frac{\phi\theta}{2(W-l)} C \right) \dot{l} + \frac{\phi^2}{2(W-l)\kappa''(v(l, L, \lambda_1, \alpha))} C \dot{\lambda}_1 \\ &+ \left[\left(r + \frac{\mu}{2} \right) \left(\tilde{\sigma} + \frac{\mu}{2} \right) + \dot{x} \right] C - \left(\tilde{\sigma} + \frac{\mu}{2} \right) X(l, L, \lambda_1, \alpha)\end{aligned}$$

where $C := (w - b - X(l, L, \lambda_1, \alpha))/(\tilde{\sigma} + \mu/2)$. Then, the dynamics of the optimal path is described by $\dot{z} = F(z, \zeta)$ using the equation above and the transition equations of state variables, where $z = {}^t(l, L, \lambda_1, \lambda_2, w)$ and $\zeta = {}^t(1, \alpha, \beta)$. Although w is redundant for the dynamics of the whole system, its inclusion makes feasible the stability analysis of economic policies on wages. Since wages are dynamically determined and thus depend on state variables, if one wants to employ eigenvalue analysis, this sort of specification is necessary. Let the variables of researcher's interest be denoted by q which is generally a function of z and ζ: $q = H(z, \zeta)$, so that the economy becomes an element having input vector ζ and output vector q. Let $q = {}^t(y, w)$ *pro tempore*. The specification is only tentative and can be arranged to fit the purpose of analysis. The linearized system around the steady state is expressed by

$$\dot{z} = \mathrm{D}_z F(\bar{z}, \bar{\zeta}) z + \mathrm{D}_\zeta F(\bar{z}, \bar{\zeta}) \zeta \tag{15}$$

$$q = \mathrm{D}_z H(\bar{z}, \bar{\zeta}) z + \mathrm{D}_\zeta H(\bar{z}, \bar{\zeta}) \zeta \tag{16}$$

where

$$\mathrm{D}_z F = \begin{pmatrix} -\sigma & 0 & \phi \frac{\mathrm{d}v}{\mathrm{d}\lambda_1} & 0 & 0 \\ \sigma + \frac{\mu}{2} \frac{W-L}{W-l} & -\frac{\mu}{2} & \phi \frac{\mathrm{d}v}{\mathrm{d}\lambda_1} \frac{W}{2(W-l)} (l - L) & 0 & 0 \\ a_{31}(\alpha) & X_l & \beta + \sigma + \phi \frac{\mathrm{d}v}{\mathrm{d}\lambda_1} \frac{W}{2(W-l)} \lambda_2 & \sigma + \frac{\mu}{2} & 0 \\ \frac{\mu}{2} \frac{1}{W-l} \lambda_2 + X_l & 0 & -\phi \frac{\mathrm{d}v}{\mathrm{d}\lambda_1} \frac{W}{2(W-l)} \lambda_2 & \beta + \frac{\mu}{2} & 0 \\ \text{omit.} & \text{omit.} & \text{omit.} & \text{omit.} & \text{omit.} \end{pmatrix}$$

$$\mathrm{D}_\zeta F = \begin{pmatrix} C_1 & 0 & 0 \\ C_2 & 0 & 0 \\ C_3 & X_\alpha + X_{l\alpha}(l + L) - f_{l\alpha} & \lambda_1 \\ C_4 & X_\alpha & \lambda_2 \\ \text{omit.} & \text{omit.} & \text{omit.} \end{pmatrix}$$

$$\mathrm{D}_z H = \begin{pmatrix} f_l & 0 & 0 & 0 & 0 \\ \frac{LX}{l} \left(\frac{X_l}{X} - \frac{1}{l} \right) & X & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{D}_\zeta H = \begin{pmatrix} f_\alpha & 0 \\ \left(1 + \frac{L}{l}\right) X_\alpha & 0 \end{pmatrix},$$

$\mu = \phi W/(W-l)\,\kappa'^{-1}(\phi\lambda_1)$, $\mathrm{d}v/\mathrm{d}\lambda_1 = \phi/\kappa''(\kappa'^{-1}(\phi\lambda_1))$ and $a_{31}(\alpha) = \mu/[2(W-l)]\lambda_2 + 2X_l(l,\alpha) + X_{ll}(l,\alpha)(l+L) - f_{ll}(l,\alpha)$. C_i's in $\mathrm{D}_\varsigma F$ are the non-homogeneous part of the linearized system. Coefficients for $\dot{w}$ are omitted since they are too lengthy to be filled in this space. In the above specification, the autonomous economy is set as a MIMO (multiple-input-multiple-output) element which is ready to be included in a more comprehensive system accompanying policy controls or feedbacks to the recognition of agents.[4]

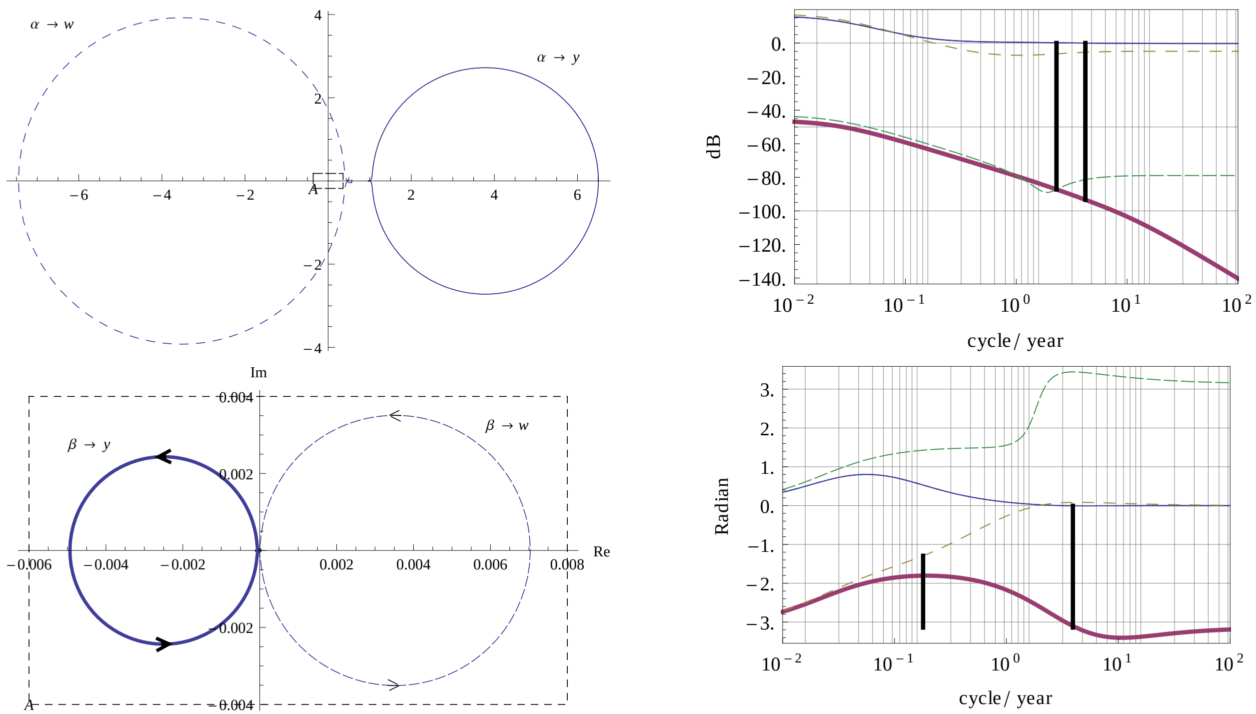
We are going to specify moderate parameter values to see the behavior of the above system. For the moment, we only handle a simple case with a fixed gross separation rate $\tilde{\sigma}$ with $x = 0$. To incorporate the fact that the observed separation rate is countercyclical, we need to consider the case where employees are fired or rejected, i.e. $x > 0$, in recession. However, it requires to distinguish dynamics in boom and recession and will not undertaken in this paper. Therefore, the following analysis is for an economy changing in a moderate speed especially when it goes downward. Monthly correspondents of Φ and $\bar{\Phi}$ are set to 0.4 and 0.65 respectively and monthly separation rate is set to 0.034. Data for Φ is adopted from Shimer (2005) as the average of the sample period. Data for $\bar{\Phi}$ is its historical high record of worker's matching probability in the same period and the separation rate takes the trend value in 2003 in the same data resource. They imply $\phi = 6.13$, $\bar{\phi} = 12.60$ and $\sigma = -12\log(1-0.034) = 0.415$. The values imply that *natural employment requires vu ratio to be higher than* $\bar{\phi}/\phi = 2.06$ *at static steady states* according to the results obtained in Section 5. Total workforce W is normalized to 1. Since the unemployment insurance payment can be approximated by the half of average wage rate, the calculated instantaneous unemployment insurance from its present value at steady state is $b = \int_0^{1/2} \bar{b}\mathrm{e}^{-rt} / \int_0^\infty \mathrm{e}^{-rt} = 0.0097w$ for $r = 0.039$ where $\bar{b}$ is the actual payment done in six months and w is an equilibrium wage rate. The production function is assumed to be Cobb-Douglas with other inputs fixed, using information about labor share to settle down the coefficient. It is formulated as αl^γ with normalization $\alpha = 1$ at steady states. On the other hand, the hiring cost function is assumed to be ηv^ψ. Corollary 3 implies that the share of labor is approximated by $bl/y + k(r,\mu,\sigma)\gamma/2$ approximating the second variation effect $\mathrm{d}v/\mathrm{d}l$ to be a constant -1. Using $b = 0.0097w$, we numerically set parameters to target the steady-state labor share to 0.64 which is the average of Japanese economy from 1980 to 2007. On the other hand, the parameters need to attain moderate steady-state labor market values, which are set to $\theta = 1.01$ and $u = 0.064$. Then, marginal productivity of labor should be set at $\gamma = 0.518$ and the hiring cost to have $\eta = 2.05$ and $\psi = 2.5$. The last two parameters specify the curvature of hiring cost function which affect both v and the uv ratio. The targeted vu ratio $1.01 < \bar{\phi}/\phi = 2.06$ justifies the state of underemployment. Also, resulting value $\lambda_1 = 0.000329 > 0$ satisfies the no-firing condition.

[4]The modularization implemented in this way also enables utilization of extensive range of computer applications for signal analysis.

With the parameter values given above, the economy bears reasonable outcomes at a static steady state. In below, we are going to add cyclical perturbations to external inputs around the static steady state to obtain periodic steady state outcomes. Before providing actual perturbations, the linearized system (15) and (16) already gives some insights on the "stability" when it is integrated in a feedback mechanism. The stability implied here is the property that the whole system is free from resonance. Since our system produces a phase shift, there is always a risk that a badly designed economic policy amplifies the fluctuation of the original business cycle source. Although it is unlikely for the resonance to persist since an economic policy tend to react observations, a rational policy maker will not adopt it from the start. As it is normal for an economy that outcomes are contaminated by external errors, any economic policy *rules* must contain a feedback mechanism.

Let $G(s)$ denote a transfer function which is defined to be a Laplace transform of a unit impulse response, which equals the ratio of a Laplace transform of the output function to that of the input function both in time domain. Then, function $G(\mathrm{i}\omega)$ where ω denotes the angular frequency shows the steady state response of the system to a unit sine input since $\mathrm{i}\omega$ and its complex conjugate are poles of the input function. Figure 2a shows the *vector loci* of our open-loop system with above parameter values in which the loci of the frequency response $G(\mathrm{i}\omega)$ is drawn in the Gaussian plane when ω is changed from zero to infinity. The lower graph in Figure 2a magnifies the dashed square labeled A in the upper graph. For example, the locus $\alpha \to y$ shows how the transfer function from α to y changes as ω moves in the above domain. For given ω, the distance from the origin shows the amplitude of outputs and the argument shows the forward shift of the phase. The figure shows that both amplitude and phase of output changes as the frequency of input does. When economic policies are brought under consideration, the above open-loop system is extended to a closed loop by defining an economic policy as a negative feedback mechanism, sometimes including a dead time element. Suppose a particular economic policy that is designed to have negative output directly fed back to the input as pass-through. Then, if the output phase shift of the open-loop element is $-\pi$ and its amplitude is greater than or equal to one, it causes resonance. It corresponds to the case when the vector locus of the pass-through feedback crosses the real axis in the region smaller than -1. Of course, a feedback mechanism can have different amplitude and phase. In such a case, a similar consideration must be made for the composite effect of the original system and the feedback mechanism. The vector loci of the system crosses the negative real axis for the transfer from α to y and β to y. It implies that when constructing an economic policy which affects α or β based on the information of y, attention must be paid to the risk of resonance.

Now, we compare the response of the economy to fluctuations in two different sources of business cycles: fluctuation in productivity on the supply side and that in the discount factor of the consumer on the demand side. We refer to these as a



(a) Vector locus

(b) Bode plot

Figure 2: Characteristics of the economic system

productivity cycle case and a demand cycle case, respectively, in short. Imposed assumptions are $\alpha = 1 + 0.0072 \sin[2\pi t/(\text{period})]$ and $\beta = 3.92$ in the former and $\alpha = 1$ and $\beta = 3.92(1 + 0.87 \sin[2\pi t/(\text{period}) - \pi])$ in the latter. The amplitude of the oscillation in the former makes the standard deviation of output in the ten year cycle case imitate the observed data. Although Canova (1998) shows that its value is susceptible to detrending method, we adopt the data obtained from HP filter, specifically s.d.$(y) = 1.76$ by Hansen (1985). This value is around the median value of the variety of data presented in Canova (1998). Some of Hansen's estimation is listed in Table I as a reference. The amplitude of oscillation in β in the latter case is chosen so that its annual rate is four percent and the lower bound is above zero. The given coefficient satisfies these conditions and is tweaked to imitate the standard deviation of working hours, i.e. s.d.$(l) = 1.66$. Table II summarizes variation of variables of the model. Focusing on the ten year cycle, labor shows smaller fluctuation than it should be for appropriate amplitude in outputs in the case of productivity cycles, while fluctuation of outputs is smaller for appropriate fluctuation in labor in the demand cycle case. It suggests either that actual data mixes fluctuation in these two sources or that capital not incorporated in the model plays a critical role. Hansen (1985) called large variability of working hours compared to that of productivity an open problem in equilibrium business cycle theory. The observed ratio s.d.(l)/s.d.(y/l) in Table I is 1.4. Corresponding ratio in our model is listed in the last column of Table II. Though the ratio for productivity fluctuation is small, that for discount fluctuation exceeds the observed ratio. This fact also supports the above suggestion that actual economy mixes two sources of cycles.

Table D.1 and Table D.2 draw dynamics of major variables. Note that with the values calibrated above, the economy is at the state of underemployment and no labor hoarding will occur with the cycles examined. Although distortion is hardly observed in most of variables in the tables, composite variables such as share of labor can have significant distortion depending on the cycles of external forces. Table D.3 shows correlation between variables and Table D.4 provides the "peak-shift matrices" to show the degree of phase shift, which lists the time shift maximizing cross-correlation between two variables. Correlation between outputs and labor is 0.73 in the case of ten-year productivity cycle whereas 1.00 in the demand cycle case. This is natural since in the latter case labor and outputs are directly connected via the production function whereas in the former case there exists a wedge due to fluctuating productivity. Note that the correlation of the productivity cycle case is close to the observed data in Table I. On the other hand, correlation between outputs and productivity is 0.98 in the productivity cycle case and -1.00 in the demand cycle case.

Table D.4 shows that interest rate is nearly countercyclical in the demand cycle case. In the one-year cycle case, it proceeds income cycle by 0.54 year which is 54 percent shift of the cycle. Its shift is 2.10 year (42 percent) in the five-year cycle case and 3.50 year (35 percent) in the ten-year cycle case. Interest rate is acyclical in the productivity case by the definition that the discount

TABLE I

STANDARD DEVIATION OF U.S. DATA ESTIMATED BY HANSEN (1985)

	Output (y)	Working hours (l)	Productivity (y/l)
standard deviation	1.76	1.66	1.18
correlation with y	1.00	0.76	0.42

TABLE II

STANDARD DEVIATION OF MAJOR VARIABLES

(a) Productivity cycle case

cycle	α	y	l	y/l, $\mathrm{d}y/\mathrm{d}l$	λ_1	w	wl/y	θ	$\frac{\text{s.d.}(l)}{\text{s.d.}(y/l)}$
1 year	0.51	0.51	0.01	0.50	0.09	0.36	0.19	0.24	0.02
5 year	1.14	1.21	0.15	1.07	0.26	1.10	0.09	2.40	0.14
10 year	1.61	1.76	0.41	1.49	0.41	1.66	0.31	6.48	0.28

(b) Demand cycle case

cycle	β	y	l	y/l, $\mathrm{d}y/\mathrm{d}l$	λ_1	w	wl/y	θ	$\frac{\text{s.d.}(l)}{\text{s.d.}(y/l)}$
1 year	86.5	0.02	0.03	0.02	0.27	0.44	0.45	0.73	2.08
5 year	193.3	0.29	0.55	0.27	0.95	1.78	1.96	8.80	2.07
10 year	273.4	0.86	1.66	0.80	1.63	3.17	3.23	26.12	2.07

factor is constant. Other variables show different lags to react. Among others, labor productivity and wage rate show sharp contrast between two cases. They are almost procyclical in the productivity cycle case, whereas they have larger phase shift in the other case. From the wage function given by Theorem 2, any factors which affect coalitional marginal profit more than momentarily raises a phase shift. Wage rate shows synchronization in the former case since a quarter-cycle shift raised by the future consideration is canceled out by the fluctuation of $\mu = \phi\theta$ in the third term of the wage equation in Theorem 2. It mitigates the shift of employment which in turn brings the synchronization of labor productivity. Demand cycle gives perfect countercyclicality in labor productivity which arises from the production function and the fact that $0 < \gamma < 1$. Wage rate shift is a mixed effect of both the half-cycle shift raised by the current coalitional marginal productivity and the nearly synchronized wave brought by the future consideration where the effect of μ is relatively small since it is almost procyclical. The countercyclicality of labor productivity in the pure demand cycle case seems counterfactual. Whereas a mixed source of forced oscillation can mitigate this property, introduction of labor hoarding is another possibility as it has procyclical impact on average productivity.

Finally, Figure D.5 shows the dynamics in the uv plane. The Beveridge curve is represented by a dashed line in the graph. It is obtained by simply putting $\dot{l} = 0$ in equation (1), which becomes $\phi v = \sigma(W - u)$ in the state of underemployment. Since the Beveridge curve is a collection of static steady states, it is invisible in data. Cycles in productivity and demand both raises counterclockwise trajectories in the uv plane as empirically observed. A longer cycle in the sources gives a larger fluctuation in unemployment.

7. CONCLUSION: SOME POLITICAL IMPLICATIONS

In our model, the government sector can be formulated as another "big" consumer who has a mass measure. Its discount rate is considered as a reflection of political stance and the budget is required to satisfy the transversality condition. Due to the mass measure of the government, its change of the discount rate affects that of the representative consumer. As already observed, it in turn affects hiring activity of firms via interest rate and changes the level of output in long run, not only the short run substitution between consumers and the government. Namely, in the state of underemployment, Ricardian equivalence does not hold. Although wage/price is flexible, the modeled economy does not show unconditional stability towards the natural unemployment even in long run. This property can be relieved once house production is introduced in the economy. It requires relaxation of the assumption that technical knowledge is monopolized by entrepreneurs.

The paper showed that if labor is heterogeneous so that the usage of internal resources is inevitable for the firm to hire workers, due to required hiring activities such as interviews and trainings, the economy exhibits the principle of effective demand. This is due to the fact that increasing employment is time consuming, therefore the market discount rate of the future affects the long-run production level. It shows that, with a frictional labor market, demand-driven business cycles should not be excluded from analysis.

APPENDIX A: BARGAINING OUTCOME IN TERMS OF VALUE

This appendix provides the proof of Theorem 1. Let Ω be a set of all players. Players are partitioned by groups S_0 and S_1 such that $S_0 \cup S_1 = \Omega$ and $S_0 \cap S_1 = \emptyset$. Each group consists of $N_i \in \mathbb{N} \cup \{\infty\}$ players ($i = 0, 1$). The j-th player in group S_i is denoted by $s_i(j)$ ($j = 1, \ldots, N_i$). $s_1(j)$ has measure dl for all j and there exists a fixed number $l \in \mathbb{R}_+$ such that $N_1\, dl = l$. Characteristic function is denoted by $v : 2^\Omega \to \mathbb{R}$. We require following assumptions:

1. (Essential game) The game is essential, i.e. $v(\Omega) > \sum_{s \in \Omega} v\big(\{s\}\big)$.
2. (Anonymity) Players in the same group are anonymous, i.e. for any S and j, $v\Big(S \cup \big\{s_1(j)\big\}\Big) - v(S)$ is common.
3. (Indispensability) Missing groups make coalition unproductive, i.e. $v(S) = \sum_{i,j \in S} v(\{s_i(j)\})$ if there exists i such that $S \cap S_i = \emptyset$.
4. (Existence of a non-degenerate player) S_0 is a special group which consists of only one player, i.e. $N_0 \equiv 1$.

Refer to this game with symbol $\partial^1_{N_1}(\Omega, v)$. Next, define a more specific game within the class of $\partial^1_{N_1}(\Omega, v)$ which possesses essential concavity defined below as an additional assumption.

Namely, it requires concavity only for coalitions with more than *two* players so that essentiality of the game will not be lost.[5] Namely, we put the following additional property:

5. (Essential concavity) The game is *essentially concave*, i.e. its characteristic function v has the property that, if $S, T \subseteq \Omega$ satisfies $\emptyset \subset S \subset T$, then
$$v(S \cup \{s\}) - v(S) \geq v(T \cup \{s\}) - v(T)$$
for any $s \in \Omega \setminus T$.

Refer to this game with symbol $\partial^2_{N_1}(\Omega, v)$. Finally, define an even more specific game $\partial^3_{N_1}(\Omega, v)$ with characteristic function

$$v(S) = \begin{cases} U\tilde{l} & \text{if there exists } i \text{ such that } S \cap S_i = \emptyset \\ F(\tilde{l}) & \text{otherwise} \end{cases} \tag{17}$$

where $\tilde{l} := \|S \cap S_1\|$ and F is increasing and concave.[6]

Our objective is to obtain a bargaining solution of the above games when $N \to \infty$ keeping l fixed.[7] Note that, by doing so, the firm $s_0(1)$ keeps discrete influence on coalitional payoff. The property that workers get only partial contribution depends on the assumption that players in S_0 does not degenerate, rather than the particular value assumption $N_0 \equiv 1$. Also, note that concavity of v and F is sufficient to hold *only from below* at Ω and $\boldsymbol{l}$, respectively, i.e. the concavity need not hold for supersets of Ω or any $\hat{l} \geq l$. Denote the density imputation to player s by $\iota(s)$, i.e. imputation of player s with measure $\mathrm{d}l$ becomes $\iota(s)\mathrm{d}l$.

Lemma 5 *In $\partial^1_\infty(\Omega, v)$, the imputation to allocate*

$$\iota\big(s_1(j)\big)\,\mathrm{d}l = \frac{1}{2} v\Big(\big\{s_1(j)\big\}\Big) + \frac{1}{2}\Big[v(\Omega) - v\Big(\Omega \setminus \big\{s_1(j)\big\}\Big)\Big] \tag{18}$$

for any j is supported by Shapley value.

Proof: Choose a player $s_{\hat{\imath}}(\hat{\jmath})$ for some $\hat{\imath}$ and $\hat{\jmath}$. Consider any coalition S such that $s_{\hat{\imath}}(\hat{\jmath}) \in S$ containing n_i players from group S_i such that $n_i \geq 0$ and $n_{\hat{\imath}} \geq 1$. The contribution of $s_{\hat{\imath}}(\hat{\jmath})$ to coalition S is $v(\{s_{\hat{\imath}}(\hat{\jmath})\})$ if there exists i such that $S \cap S_i = \emptyset$ from the indispensability assumption. In other cases, it is $v(S) - v(S \setminus \{s_{\hat{\imath}}(\hat{\jmath})\})$. The Shapley's weight $\gamma(S)$ for the contribution of $s_{\hat{\imath}}(\hat{\jmath})$ to coalition S is given by

$$\begin{aligned} \gamma(S) &= \frac{(\sum_{i=0}^1 n_i - 1)!(\sum_{i=0}^1 N_i - \sum_{i=0}^1 n_i)!}{(\sum_{i=0}^1 N_i)!} \\ &= \left(\sum_{i=0}^1 N_i\right)^{-1} \binom{\sum_{i=0}^1 N_i - 1}{\sum_{i=0}^1 n_i - 1}^{-1}. \end{aligned}$$

Without loss of generality, let us assume $\hat{\imath} = 1$ below for concise notations. From the anonymity assumption, any S with same (n_0, n_1) has the same $\gamma(S)$. The number of cases to form coalition S containing $s_{\hat{\imath}}(\hat{\jmath}) = s_1(\hat{\jmath})$ with same (n_0, n_1) is given by

$$\binom{N_0}{n_0} \cdot \binom{N_1 - 1}{n_1 - 1}.$$

[5] A globally concave game always violates zero-additivity, thus formation of non-trivial coalitions cannot be expected.

[6] The assumption of global concavity in F is actually asking too much than necessary. It is sufficient if F satisfies $F(\tilde{l}) \leq F(l) - \partial F(l)/\partial l \cdot (l - \tilde{l})$, $\forall \tilde{l} \leq l$ for the current level of employment l.

[7] We seek for the limit of the sequence of bargaining solutions and do not pursue uniformity as in uniform approximate core by Wooders (1992). Indeed, the pregame which generates the game presented here does not satisfy uniform inessentiality of large groups.

Then, Shapley value is given by

$$\iota\big(s_1(\hat{\jmath})\big)\,dl_1 = v\Big(\{s_1(\hat{\jmath})\}\Big) \sum_{\{S:\prod_{i=0}^{1} n_i=0\}} \Gamma(S) + \sum_{\{S:\prod_{i=0}^{1} n_i\geq 1\}} \Gamma(S)\left[v(S) - v\Big(S\setminus\{s_1(\hat{\jmath})\}\Big)\right] \tag{19}$$

where

$$\Gamma(S) := \gamma(S)\cdot\binom{N_0}{n_0}\cdot\binom{N_1-1}{n_1-1} = \frac{\binom{N_0}{n_0}\cdot\binom{N_1-1}{n_1-1}}{\left(\sum_{i=0}^{1} N_i\right)\binom{\sum_{i=0}^{1} N_i-1}{\sum_{i=0}^{1} n_i-1}}.$$

Proposition 11 in Appendix C show that coefficient $\Gamma(S)$ is a probability mass function such that $\Gamma(S) = \Upsilon(n_0, n_1-1; N_0, N_1-1)$ where distribution Υ is defined in Appendix C. Note that the distribution possesses point symmetry $\Upsilon(n_1;\zeta_1) = \Upsilon(\zeta_1 - n_1;\zeta_1)$. Using these two facts,

$$\sum_{\{S:n_0=0\}} \Gamma(S) = \sum_{\{S:n_0=1\}} \Gamma(S)$$

Now, in either case of $n_0 = 0, 1$, $\sum_{\{S:\prod_{i=1}^{1} n_i=0\}} \Gamma(S) \to 0$ as $N_1 \to \infty$. $\sum_{\{S:\prod_{i=1}^{1} n_i=0\}} \Gamma(S)$ can be written as

$$\begin{aligned}
\sum_{\{S:\prod_{i=1}^{1} n_i=0\}} \Gamma(S) &= \frac{\prod_{\{i:n_i=0,n_1=1\}}\binom{N_i}{0}}{\sum_{i=0}^{1} N_i} \sum_{n_i} \frac{\prod_{\{i:n_i\geq 1,n_1\geq 2\}}\binom{N_i}{n_i}}{\binom{\sum_{i=0}^{1} N_i-1}{\sum_{i=0}^{1} n_i-1}} \\
&= \frac{1}{\sum_{i=0}^{1} N_i} \sum_{n_i} \frac{\prod_{\{i:n_i\geq 1,n_1\geq 2\}}\binom{N_i}{n_i}}{\binom{\sum_{i=0}^{1} N_i-1}{\sum_{i=0}^{1} n_i-1}}
\end{aligned}$$

where the right hand side converges to zero as $N_1 \to \infty$. Therefore,

$$\sum_{\{S:\prod_{i=0}^{1} n_i=0\}} \Gamma(S) \approx \sum_{\{S:n_0=0\}} \Gamma(S) = \sum_{\{S:n_0=1\}} \Gamma(S) \approx \sum_{\{S:\prod_{i=0}^{1} n_i\geq 1\}} \Gamma(S) \approx \frac{1}{2}.$$

It shows the coefficient of $v\Big(\{s_{\hat{\imath}}(\hat{\jmath})\}\Big)\,dl_{\hat{\imath}}$ in (19) converges to $1/2$ as $N_1 \to \infty'$. From Proposition 12 in Appendix C, we obtain equation (18) for all j. *Q.E.D.*

The above derivation critically depends on the indispensability and the existence assumption of a non-degenerate player, which enable for the firm to keep discrete influence on payoffs whereas that of individual workers becomes negligible as $N_1 \to \infty$. On the other hand, characterizing bargaining solution as nucleolus requires an additional assumption that the game should be essentially concave. At the outset, the following lemma shows that core is non-empty if and only if the production process is more productive for the last marginal worker than unemployment in terms of value.

Lemma 6 *$\partial^1_{N_1}(v,\Omega)$ has non-empty core if it is zero-additive. So does $\partial^2_{N_1}(v,\Omega)$ iff*

$$v(\Omega) - v\big(\Omega\setminus\{s_1(j)\}\big) \geq v\Big(\{s_1(j)\}\Big) \tag{20}$$

for all j. In $\partial^3_{N_1}(v,\Omega)$, the condition (20) is replaced by $\partial F/\partial l \geq U$.

Proof: We start from the necessary condition of ∂^1. Consider imputation such that any $s \in \Omega\setminus S_0$ is allocated by $\iota(s) = v\big(\{s\}\big)$ and player $s_0(1)$ is allocated by $\iota\big(s_0(1)\big) = v(\Omega) - \sum_{s\in\Omega\setminus S_0} v(s)$. This is feasible by essentiality of the game. Obviously, any S such that $s_0(1) \notin S$ satisfies coalitional rationality since $\sum_{s\in S}\iota(s) \geq v(S) = \sum_{s\in S} v\big(\{s\}\big)$. So does any coalition

S such that $s_0(1) \in S$ since its imputation yields $\sum_{s \in S} \iota(s) = v(\Omega) - \sum_{s \notin S} v(s) \geq v(S)$ by zero-additivity. which implies that this imputation is located in core. If (20) holds for $\mathfrak{D}^2$, zero-additivity holds from the essential concavity, which shows that (20) is a necessary condition for $\mathfrak{D}^2$. The case for $\mathfrak{D}^3$ is direct from this since $\mathfrak{D}^3$ is a special case of $\mathfrak{D}^2$.

To show (20) is a sufficient condition for $\mathfrak{D}^2$, suppose $v(\Omega) - v\Big(\Omega \setminus \{s_i(j)\}\Big) < v\Big(\{s_i(j)\}\Big)$ for some i, j. The individual rationality of $s_i(j)$ requires $\iota\Big(s_i(j)\Big) \geq v\Big(\{s_i(j)\}\Big)$. Also, coalition of the rest requires $\sum_{s \in \Omega \setminus \{s_i(j)\}} \iota\Big(s\Big) \geq v\Big(\Omega \setminus \{s_i(j)\}\Big)$, which implies $\iota\Big(s_i(j)\Big) \leq v(\Omega) - v\Big(\Omega \setminus \{s_i(j)\}\Big) < v\Big(\{s_i(j)\}\Big)$. These two equations are not satisfied at the same time, thus core is empty. It shows zero-additivity is also sufficient. The result for $\mathfrak{D}^3$ is derived from this. *Q.E.D.*

Following the context of our model in which workers and the firm are all rational in participating in production, the bargaining solution must be in core. Otherwise, at least one player will leave the coalition, which implies that the current coalition is not actually on the optimal path. The above lemma means that the problem can be restricted to the case of $\partial F / \partial l_{ij} \geq U_i$ on the optimal path.

LEMMA 7 *If game (Ω, v) is essentially concave in which players are partitioned by groups such that $\Omega = \sum_{i=1}^{M} S_i$, then for any $S, T \subseteq \Omega$ such that $S \subset T$, the following inequality holds.*

$$v(T) - v(T \setminus S) \geq \sum_{i=1}^{M} \|S \cap S_i\| \Big[v(T) - v\Big(T \setminus \{s_i(j)\}\Big)\Big]$$

PROOF: We use the fact that $v(T) - v(T \setminus S)$ has common value regardless of how players of S are removed from T. Define $\mathfrak{S}_{i_1 i_2 \cdots i_m}(n_{i_1}, \ldots, n_{i_m}) := \bigcap_{k=\{1,\ldots,m: n_{i_k} \neq 0\}} \bigcap_{j=1}^{n_{i_k}} \Big\{s_{i_k}(j)\Big\}$ where $m = 1$ and $1 \leq n_i \leq N_i$. When $n_{i_k} = 0$ for all k, define $\mathfrak{S}_{i_1 i_2 \cdots i_m}(0 \cdots 0) = \emptyset$ for convenience. Then, for given $k = 1$,

$$\begin{aligned} v(T) - v(T \setminus S) &= \sum_{n_k=1}^{N_k} \Big[v\Big(T \setminus \mathfrak{S}_k(n_k - 1)\Big) - v\Big(T \setminus \mathfrak{S}_k(n_k)\Big)\Big] \\ &\quad + \sum_{n_{i_2}=1}^{N_{i_2}} \Big[v\Big(T \setminus \mathfrak{S}_{k i_2}(N_k, n_{i_2} - 1)\Big) - v\Big(T \setminus \mathfrak{S}_{k i_2}(N_k, n_{i_2})\Big)\Big] \\ &\quad + \cdots + \sum_{n_{i_M}=1}^{N_{i_M}} \Bigg[v\Big(T \setminus \mathfrak{S}_{k i_2 \cdots i_M}(N_k, \ldots, N_{i_{M-1}}, n_{i_M} - 1)\Big) \\ &\qquad\qquad - v\Big(T \setminus \mathfrak{S}_{k i_2 \cdots i_M}(N_1, \ldots, N_{i_{M-1}}, n_{i_M})\Big)\Bigg] \\ &\geq M \sum_{n_k=1}^{N_k} \Big[v\Big(T \setminus \mathfrak{S}_k(n_k - 1)\Big) - v\Big(T \setminus \mathfrak{S}_k(n_k)\Big)\Big] \end{aligned}$$

where $i_2, \ldots, i_M$ are taken in an arbitrary order so that $i_j \neq k$ and $i_j \neq i_{j'}$ if $j \neq j'$. The last line comes from the essential concavity. Summing up the above inequality for all $k = 1, \ldots, M$,

$$\begin{aligned} v(T) - v(T \setminus S) &\geq \sum_{k=1}^{M} \sum_{n_k=1}^{N_k} \Big[v\Big(T \setminus \mathfrak{S}_k(n_k - 1)\Big) - v\Big(T \setminus \mathfrak{S}_k(n_k)\Big)\Big] \\ &\geq \sum_{k=1}^{M} \|S \cap S_k\| \Big[v(T) - v\Big(T \setminus \Big\{s_k(j)\Big\}\Big)\Big] \end{aligned}$$

for any $j = 1, \ldots, N_k$ again from the essential concavity. *Q.E.D.*

This lemma is analogous to the property of an ordinary concave function: $f(x_1+\Delta x_1, \ldots, x_n+\Delta x_n) \leq f_{x_1}\Delta x_1 + \cdots + f_{x_n}\Delta x_n$. In the current particular model, set $M = 1$.

Theorem 8 *In* $\partial^2_{N_1}(\Omega, v)$ *for any* N_1*, the imputation* (18) *is supported by nucleolus.*[8]

Proof: Consider imputation in the ε-core for given excess ε. In the payoff space $X \subseteq \mathbb{R}^{N_0+N_1}$ such that $\{\iota(s_0(1)), \iota(s_1(1)) \ldots, \iota(s_1(N_1))\} \in X$, consider domain $\mathsf{B}(S, \varepsilon) \subseteq X$ which satisfies coalitional rationality of player set S and its complement $\Omega \setminus S$ and total rationality $\Delta := \{X \in \mathbb{R}^{N_0+N_1} : \sum_{s\in\Omega} \iota(s) = v(\Omega)\}$. Since it defines the same domain for any player set and its complement, $s_0(1) \notin S$ can be assumed by symmetricity without loss of generality. We seek for the least core by finding out $\min_\varepsilon\{\varepsilon : \bigcap_{S\in 2^\Omega} \mathsf{B}(S, \varepsilon) \neq \emptyset\}$ where $\mathsf{B}(S, \varepsilon)$ has the form

$$\mathsf{B}(S, \varepsilon) = \left\{ X \in \Delta : v(S) - \varepsilon \leq \sum_{s\in S} \iota(s) \leq v(\Omega) - v(\Omega \setminus S) + \varepsilon \right\}. \tag{21}$$

Consider a special case where $S = \{s_1(j)\}$ for any j in equation (21) and call it equation (21'). Any point in the intersection of B's for all such j's in some coalition S such that $s_0(1) \notin S$, i.e. $\bigcap_{s\in S} \mathsf{B}(\{s\}, \varepsilon)$, satisfies the sum of the conditioning inequalities in equation (21'). Namely,

$$\begin{aligned} \bigcap_{s\in S} \mathsf{B}(\{s\}, \varepsilon) \subseteq \Bigg\{ X \in \Delta : \sum_{s\in S} v(\{s\}) - \|S\|\,\varepsilon &\leq \sum_{s\in S} \iota(s) \\ &\leq \sum_{s\in S} \left[v(\Omega) - v(\Omega \setminus \{s\})\right] + \|S\|\,\varepsilon \Bigg\} \end{aligned} \tag{22}$$

Now, we show

$$\min_\varepsilon\{\varepsilon : \bigcap_{S\in 2^\Omega} B(S, \varepsilon) \neq \emptyset\} = \min_\varepsilon\{\varepsilon : \mathsf{B}(\{s\}, \varepsilon) \neq \emptyset\}. \tag{23}$$

Since $\varepsilon \leq 0$ from Lemma 6, indispensability implies

$$v(S) - \varepsilon \leq v(S) - m\varepsilon = \sum_{s\in S} v(\{s\}) - m\varepsilon \tag{24}$$

for any $m \in \mathbb{N}$. On the other hand, from Lemma 7,

$$\begin{aligned} v(\Omega) - v(\Omega \setminus S) + \varepsilon &\geq v(\Omega) - v(\Omega \setminus S) + m\varepsilon \\ &\geq \sum_{s\in S} \Big[v(\Omega) - v(\Omega \setminus \{s\})\Big] + m\varepsilon \end{aligned} \tag{25}$$

holds for any $m \in \mathbb{N}$. From (24) and (25), equations (21) and (22) imply $B(S, \varepsilon) \supseteq \bigcap_{s\in S} \mathsf{B}(\{s\}, \varepsilon)$ for any $S \in 2^\Omega$. Then, together with anonymity, equation (23) is derived.

From equation (21'), setting $v(s_1(j)) - \varepsilon = v(\Omega) - v(\Omega \setminus \{s_1(j)\}) + \varepsilon$ degenerates the range of $\iota(s_1(j))$ to a point, i.e.

$$\varepsilon^* = -\frac{1}{2}\left[v(\Omega) - v\Big(\Omega \setminus \{s_1(j)\}\Big) - v\Big(\{s_1(j)\}\Big)\right] \tag{26}$$

[8]This result coincides with Stole and Zwiebel (1996) by extending its result to a case of infinite number of agents.

which is common for all $j \in S$. Smaller ε than ε^* makes $\bigcap_{s\in S} \mathsf{B}(\{s\},\varepsilon)$ empty. Therefore, we have $\arg\min_\varepsilon\{\varepsilon : \bigcap_{s\in\Omega} \mathsf{B}(\{s\},\varepsilon) \neq \emptyset\} = \varepsilon^*$ and the ε-core it defines becomes a nucleolus since it degenerates to a point. It implies payoff $\iota(s_1(j))\mathrm{d}l$ becomes

$$\iota(s_1(j))\mathrm{d}l = v\Big(\{s_1(j)\}\Big) - \varepsilon = \frac{1}{2}v\Big(\{s_1(j)\}\Big) + \frac{1}{2}\Big[v(\Omega) - v\Big(\Omega \setminus \{s_1(j)\}\Big)\Big]$$

for all j. *Q.E.D.*

Finally, we obtain our objective theorem.

THEOREM 9 *In $\eth^3_\infty(\Omega, v)$, the imputation*

$$E(n) = \frac{1}{2}\left(U + \frac{\partial F}{\partial l_-}\right)$$

is supported by Shapley value and nucleolus where $\partial F/\partial l_-$ is a derivative from left.

PROOF: The result follows from Theorem 5 and Theorem 8. For the latter, it is sufficient to show that F satisfies essential concavity. From concavity of F, $\partial F(l^1)/\partial l\,\mathrm{d}l \leq \partial F(l^2)/\partial l\,\mathrm{d}l$ for $l^1 > l^2$.Then, it implies essential concavity of F since $F(l^1 + \mathrm{d}l) - F(l^1) \leq F(l^2 + \mathrm{d}l) - F(l^2)$. *Q.E.D.*

We labeled S_0 as a set of a firm or an entrepreneur above. However, if there is any player who exerts non-degenerate influence on productivity or, in other words, those who embodies critical knowledge for production as rent, this player will receive non-marginal part of coalitional rent. In this section, we derived bargaining solution in terms of value function. Its distribution is actually done through wage payment. Bargaining outcome in terms of wages is derived in section B.

APPENDIX B: BARGAINING OUTCOME IN TERMS OF WAGE

This appendix provides the proof of Theorem 2. By defining $z := E - U$ in Bellman equations (3) and (4), the dimension of the dynamics is reduced by one:

$$\dot{z}_t = (r_t + \mu_t + \tilde{\sigma}_t)\, z_t - (w_t - b_t)\,, \tag{27}$$

which solves to

$$z_t = \int_t^\infty \left(w_\xi - b_\xi\right) e^{-\int_t^\xi (r_\nu + \tilde{\sigma}_\nu + \mu_\nu) d\nu} \mathrm{d}\xi. \tag{28}$$

Solving differential equation (3) for U using (28),

$$U_t = \int_t^\infty b_s e^{-\int_t^s r} \mathrm{d}s + \int_t^\infty \mathrm{d}s \int_s^\infty \mu_s \left[\left(w_\xi - b_\xi\right) e^{-\int_s^\xi (r + \tilde{\sigma} + \mu)}\right] e^{-\int_t^s r} \mathrm{d}\xi \tag{29}$$

Similarly, we obtain the value function of employment for each type.

$$E_t = \int_t^\infty w_s e^{-\int_t^s r} \mathrm{d}s - \int_t^\infty \mathrm{d}s \int_s^\infty \tilde{\sigma}_s \left[\left(w_\xi - b_\xi\right) e^{-\int_s^\xi (r + \tilde{\sigma} + \mu)}\right] e^{-\int_t^s r} \mathrm{d}\xi \tag{30}$$

The unemployment value is the discounted series of unemployment benefit and capital gain arising from matching. The employment value is the discounted series of wage rate, expected change of capital gain in new jobs and capital gain (loss) of dismissal.

Theorem 10 *Let $H_1 := \{t : y^d_t(\mathscr{R}) < f(l_t)\}$. Wage rate at time t is given by*

$$w_t = b_t + \frac{\mathrm{MP}_t - \mathrm{MC}_t}{2} + \left(\tilde{\sigma}_t + \frac{\mu_t}{2}\right) \int_t^\infty \frac{\mathrm{MP}_\xi - \mathrm{MC}_\xi}{2} e^{-\int_t^\xi (r+\mu/2)} \mathrm{d}\xi$$

where

$$\mathrm{MP}_\xi = \begin{cases} f'(l^*_\xi) & \text{if } \xi \notin H_1 \\ 0 & \text{if } \xi \in H_1, \end{cases}$$

$\mathrm{MC}_t = \kappa'(v^*_\xi)\mathrm{d}v^*_\xi/\mathrm{d}l_t < 0$ *shows the change of hiring cost when l hypothetically moves to a neighboring optimal path and v^* and l^* are along the optimal path.*

Proof: Theorem 1 implies $\partial^2 F/\partial t_+ \partial l_- = 2\dot{E} - \dot{U}$. Note that the bargaining is made for the next moment, therefore time derivative must be taken from right. Applying (3), (4), (29) and (30) on it,

$$(w_t - b_t) - \left(\tilde{\sigma}_t + \frac{\mu_t}{2}\right) \int_t^\infty (w_\xi - b_\xi)\, e^{-\int_t^\xi (r+\mu+\tilde{\sigma})} \mathrm{d}\xi = \frac{\mathfrak{F}_t}{2} \tag{31}$$

where $\mathfrak{F}_t := r\,\partial F/\partial l_- - \partial^2 F/\partial t_+ \partial l_-$. Since $\partial^2 F/\partial t_+ \partial l_- = r\partial F/\partial l_- - \mathrm{MP} + \mathrm{MC}$ from (5), it actually becomes $\mathfrak{F}_t = \mathrm{MP}_t - \mathrm{MC}_t$ which is the cost of losing marginal labor. Note that, when $x_\xi > 0$ for some ξ, the shock of the resignation of a worker can be absorbed by the cancellation of dismissal. Due to dependency to initial value, neighboring paths coincide afterwards since x_ξ is adjusted to match the external force y^d. Thus, $\mathfrak{F}_\tau = 0$ for all $\tau \geq \xi$. Equation (31) is an integral equation concerning to $Y_t := w_t - b_t$. Namely,

$$Y_t - \int_t^\infty K(t,\xi) Y_\xi \mathrm{d}\xi = \frac{1}{2}\mathfrak{F}_t$$

where kernel $K(t,\xi)$ is defined by $K(t,\xi) := (\tilde{\sigma}_t + \mu_t/2)\, e^{-\int_t^\xi (r+\tilde{\sigma}+\mu)}$. The solution to this equation is given by

$$Y_t = \frac{1}{2}\mathfrak{F}_t - \frac{1}{2}\int_t^\infty G(t,\xi)\mathfrak{F}_\xi \mathrm{d}\xi$$

where $G(t,\xi) := -\sum_{\zeta=1}^\infty \overset{*}{K}{}^\zeta(t,\xi)$. The iterated kernel $\overset{*}{K}{}^n$ is defined by $\overset{*}{K}{}^n := \underbrace{K * K * \cdots * K}_{n}$ and $K{*}L$ denotes the composition of the first kind defined by $K(t,\xi){*}L(t,\xi) = \int_t^\xi K(t,\tau)\, L(\tau,\xi)\, d\tau$. Since

$$\overset{*}{K}{}^n(t,\xi) \quad = \quad \left(\tilde{\sigma}_t + \frac{\mu_t}{2}\right) e^{-\int_t^\xi (r+\tilde{\sigma}+\mu)} \frac{\left[\int_t^\xi \left(\tilde{\sigma}_s + \frac{\mu_s}{2}\right) \mathrm{d}s\right]^{n-1}}{(n-1)!},$$

we obtain

$$G(t,\xi) = -\left(\tilde{\sigma}_t + \frac{\mu_t}{2}\right) e^{-\int_t^\xi (r+\mu/2)}$$

and the solution for Y_t,

$$Y_t = \frac{1}{2}\mathfrak{F}_t + \left(\tilde{\sigma}_t + \frac{\mu_t}{2}\right) \int_t^\infty \frac{\mathfrak{F}_\xi}{2} e^{-\int_t^\xi (r+\mu/2)} \mathrm{d}\xi.$$

which brings the proposition. *Q.E.D.*

APPENDIX C: ON A PROBABILITY DISTRIBUTION

Proof that the coefficients of (19) are equivalent to probability.

Proposition 11 *Let* $(\zeta_1, \ldots, \zeta_N) \in \mathbb{N}^N$ *be a vector of parameters. Define*

$$\Upsilon(y_1, \ldots, y_N; \zeta_1, \ldots, \zeta_N) := \frac{1}{1 + \sum_{i=1}^N \zeta_i} \frac{\prod_{i=1}^M \binom{\zeta_i}{y_i}}{\binom{\sum_{i=1}^M \zeta_i}{\sum_{i=1}^M y_i}} \tag{32}$$

for any $y_i \in \mathbb{N}$ *such that* $0 \le y_i \le \zeta_i$*. Then, equation* (32) *is a probability mass function.*

Proof: $\Upsilon \ge 0$ is obvious. If we sum it up for all x_i, it becomes

$$\sum \Upsilon = \frac{1}{1 + \sum_{i=1}^N \zeta_i} \sum_{k=0}^{\sum \zeta_i} \sum_{\sum y_i = k} \text{Mult.Hypg.}(y_1, \ldots, y_N; k; \zeta_1, \ldots, \zeta_N) \; = \; 1$$

where Mult.Hypg.$(y_1, \ldots, y_N; k; \zeta_1, \ldots, \zeta_N)$ is a multivariate hypergeometric distribution with parameter $(k; \zeta_1, \ldots, \zeta_N)$. It sums up to one if all n_i's are summed up keeping $\sum n_i = k$. *Q.E.D.*

Proposition 12 *Define a density function* $\tilde{\Upsilon} : \mathbb{R}^N \to \mathbb{R}$ *characterized by* Υ *such that*

$$\tilde{\Upsilon}(x_1, \ldots, x_N)\, dl_1 \cdots dl_N = \Upsilon(y_1, \ldots, y_N) \tag{33}$$

where $x_i = y_i dl_i$ *and* $0 \le x_i \le l_i$ *where* l_i *is fixed for any* ζ_i *and* dl_i *keeping* $l_i = \zeta_i dl_i$ $(i = 1, \ldots, N)$*. Then, the functional form of* $\tilde{\Upsilon}$ *is given by* $\tilde{\Upsilon}(x_1, \ldots, x_N) = \delta(1 - x_1/l_1, \ldots, 1 - x_N/l_N)$ *as* $\zeta_i \to \infty$ *for all* i *where* δ *denotes Dirac's delta, i.e.* $\delta(z_1, \ldots, z_N) = \infty$ *if* $z_i = 0$, $\forall i$ *and* $\delta(z_1, \ldots, z_N) = 0$ *otherwise, and*

$$\int_0^1 \cdots \int_0^1 \delta(z_1, \ldots, z_N)\, dz_1 \cdots dz_N = 1. \tag{34}$$

Proof: From Proposition 11, $\sum_{y_1=1}^{\zeta_1} \cdots \sum_{y_N=1}^{\zeta_N} \Upsilon(y_1, \ldots, y_N) = 1$. Using (33), it means $\sum_{x_1=dl_1}^{l_1} \cdots \sum_{x_N=dl_N}^{l_N} \tilde{\Upsilon}(x_1, \ldots, x_N)\, dl_1 \cdots dl_N = 1$ which leads to show $\tilde{\Upsilon}$ satisfies property (34) as $\zeta_i \to \infty$, i.e. $dl_i \to 0$, for all i. Note that

$$\left(\prod_{i=1}^M \zeta_i\right) \left(\prod_{i=1}^M \binom{\zeta_i}{y_i}\right) = o\left(\binom{\sum_{i=1}^M \zeta_i}{\sum_{i=1}^M y_i}\right)$$

if there exists i such that $y_i < \zeta_i$. Then, (33) becomes

$$\begin{aligned} \tilde{\Upsilon}(l_1, \ldots, l_N) &= \frac{1}{1 + \sum_{i=1}^N \zeta_i} \frac{\prod_{i=1}^M \binom{\zeta_i}{y_i}}{\binom{\sum_{i=1}^M \zeta_i}{\sum_{i=1}^M y_i}} \frac{1}{dl_1 \cdots dl_N} \\ &= \frac{1}{\prod_{i=1}^N l_i} \frac{1}{1 + \sum_{i=1}^N \zeta_i} \frac{\left(\prod_{i=1}^N \zeta_i\right)\left(\prod_{i=1}^M \binom{\zeta_i}{y_i}\right)}{\binom{\sum_{i=1}^M \zeta_i}{\sum_{i=1}^M y_i}} \;\to\; 0 \end{aligned}$$

as $\zeta_i \to \infty$ for all i if there exists i such that $y_i < \zeta_i$. On the other hand, if $y_i = \zeta_i$ for all i, we have $\prod_{i=1}^M \binom{\zeta_i}{y_i} = 1$ and thus $\Upsilon(\zeta_1, \ldots, \zeta_N) = 1/(1 + \sum_{i=1}^N \zeta_i)$. Then, from (33),

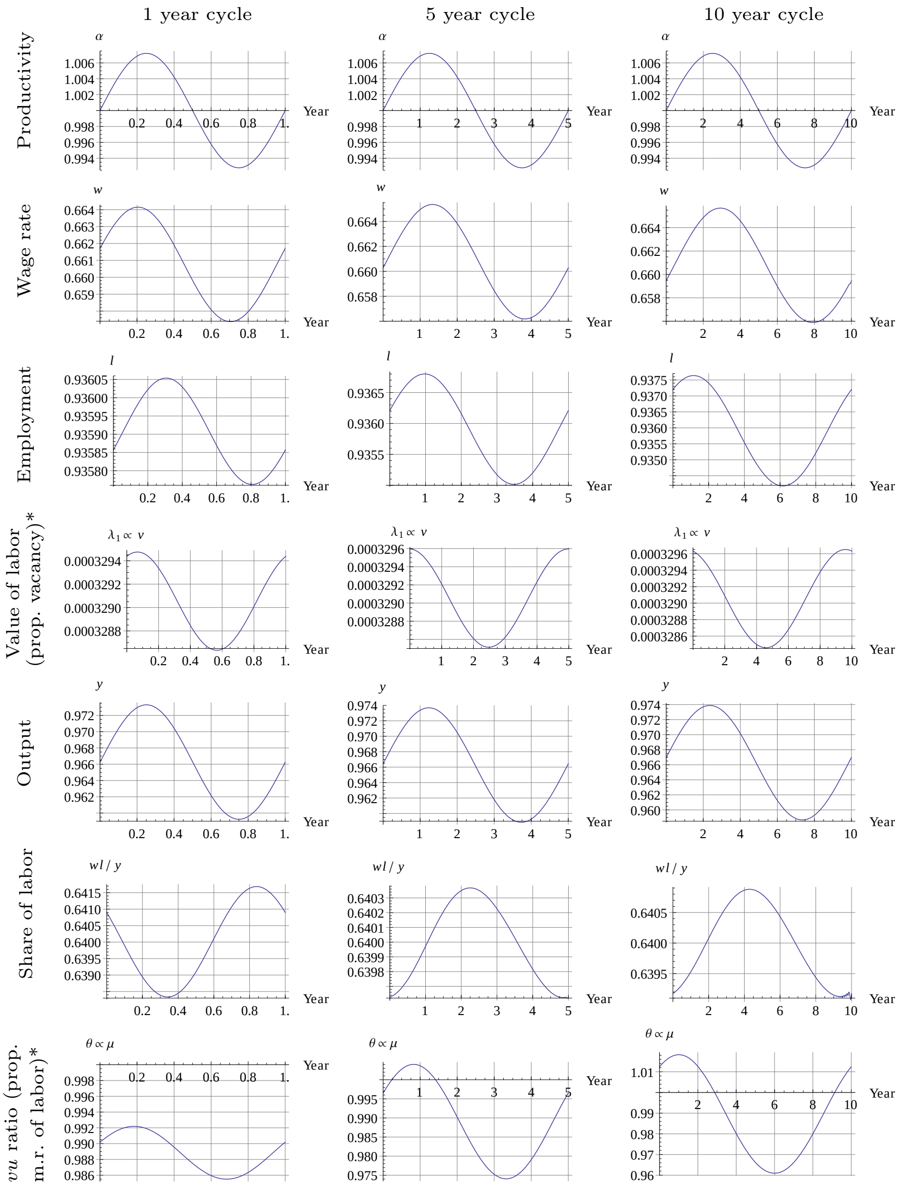
$$\tilde{\Upsilon}(l_1, \ldots, l_N) = \frac{1}{1 + \sum_{i=1}^N \zeta_i} \frac{1}{dl_1 \cdots dl_N} = \frac{1}{\prod_{i=1}^N l_i} \frac{\prod_{i=1}^N \zeta_i}{1 + \sum_{i=1}^N \zeta_i}.$$

The second fraction diverges as ζ_i's become large, therefore $\tilde{\Upsilon}(l_1, \ldots, l_N) \to \infty$ as $\zeta_i \to \infty$ for all i. *Q.E.D.*

APPENDIX D: PROPERTIES OF DYNAMICS

D.1. *Dynamics of major variables: productivity cycle case*

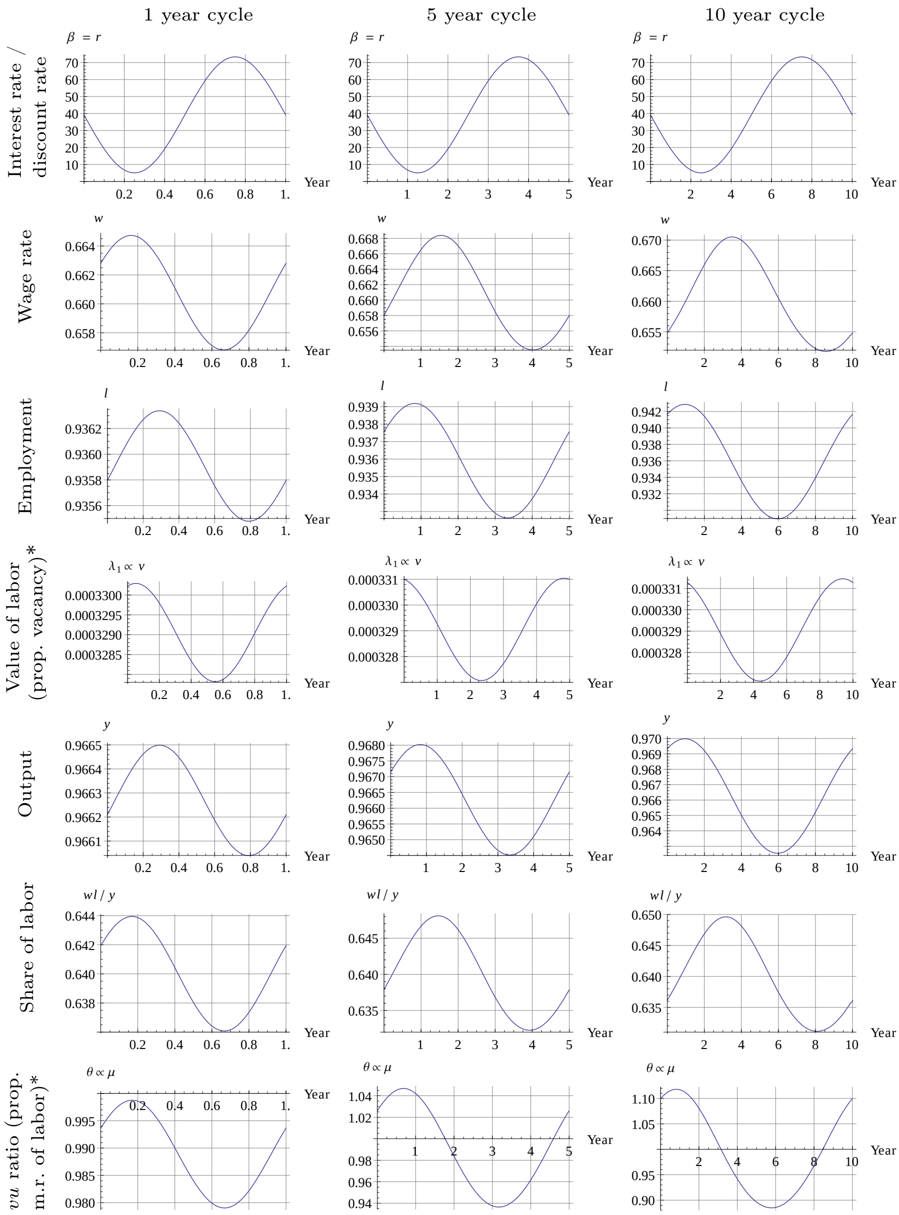
External input is given by $\alpha = 1 + 0.0072 \sin[2\pi t/(\text{period})]$ and $\beta = 3.92$ below.



* prop.: proportionate to, m.r.: matching rate

D.2. *Dynamics of major variables: demand cycle case*

External input is given by $\alpha = 1$ and $\beta = 3.92(1 + 0.87 \sin[2\pi t/(\text{period}) - \pi])$ below.



* prop.: proportionate to, m.r.: matching rate

1. Productivity cycle case

(a) 1 year cycle

	α	y	l	$y/l, y'$	λ_1	w	wl/y	θ
α	1.0000	1.0000	0.9369	1.0000	0.4106	0.9602	−0.8436	0.9159
y	1.0000	1.0000	0.9382	1.0000	0.4071	0.9591	−0.8457	0.9143
l	0.9369	0.9382	1.0000	0.9356	0.06592	0.8020	−0.9781	0.7177
$y/l, y'$	1.0000	1.0000	0.9356	1.0000	0.4140	0.9612	−0.8416	0.9174
λ_1	0.4106	0.4071	0.06592	0.4140	1.0000	0.6490	0.1432	0.7421
w	0.9602	0.9591	0.8020	0.9612	0.6490	1.0000	−0.6601	0.9916
wl/y	−0.8436	−0.8457	−0.9781	−0.8416	0.1432	−0.6601	1.0000	−0.5571
θ	0.9159	0.9143	0.7177	0.9174	0.7421	0.9916	−0.5571	1.0000

(b) 5 year cycle

	α	y	l	$y/l, y'$	λ_1	w	wl/y	θ
α	1.0000	0.9998	0.9416	0.9997	−0.02437	0.9946	0.2357	0.8602
y	0.9998	1.0000	0.9487	0.9990	−0.002671	0.9921	0.2146	0.8711
l	0.9416	0.9487	1.0000	0.9337	0.3137	0.9016	−0.1049	0.9817
$y/l, y'$	0.9997	0.9990	0.9337	1.0000	−0.04722	0.9967	0.2579	0.8484
λ_1	−0.02437	−0.002671	0.3137	−0.04722	1.0000	−0.1279	−0.9765	0.4888
w	0.9946	0.9921	0.9016	0.9967	−0.1279	1.0000	0.3353	0.8027
wl/y	0.2357	0.2146	−0.1049	0.2579	−0.9765	0.3353	1.0000	−0.2923
θ	0.8602	0.8711	0.9817	0.8484	0.4888	0.8027	−0.2923	1.0000

(c) 10 year cycle

	α	y	l	$y/l, y'$	λ_1	w	wl/y	θ
α	1.0000	0.9959	0.6656	0.9950	−0.2557	0.9655	0.3917	0.5905
y	0.9959	1.0000	0.7306	0.9818	−0.1670	0.9378	0.3067	0.6613
l	0.6656	0.7306	1.0000	0.5877	0.5512	0.4482	−0.4256	0.9953
$y/l, y'$	0.9950	0.9818	0.5877	1.0000	−0.3512	0.9867	0.4818	0.5068
λ_1	−0.2557	−0.1670	0.5512	−0.3512	1.0000	−0.4988	−0.9893	0.6292
w	0.9655	0.9378	0.4482	0.9867	−0.4988	1.0000	0.6179	0.3598
wl/y	0.3917	0.3067	−0.4256	0.4818	−0.9893	0.6179	1.0000	−0.5110
θ	0.5905	0.6613	0.9953	0.5068	0.6292	0.3598	−0.5110	1.0000

2. Demand cycle case

(a) 1 year cycle

	r	y, l	$y/l, y'$	λ_1	w	wl/y	θ
r	1.0000	−0.9207	0.9207	−0.3067	−0.8161	−0.8286	−0.8328
y, l	−0.9207	1.0000	−1.0000	0.06592	0.6928	0.7113	0.7177
$y/l, y'$	0.9207	−1.0000	1.0000	−0.06592	−0.6928	−0.7113	−0.7177
λ_1	−0.3067	0.06592	−0.06592	1.0000	0.7652	0.7483	0.7421
w	−0.8161	0.6928	−0.6928	0.7652	1.0000	0.9997	0.9994
wl/y	−0.8286	0.7113	−0.7113	0.7483	0.9997	1.0000	1.0000
θ	−0.8328	0.7177	−0.7177	0.7421	0.9994	1.0000	1.0000

(b) 5 year cycle

	r	y, l	$y/l, y'$	λ_1	w	wl/y	θ
r	1.0000	−0.8232	0.8232	0.1972	−0.8982	−0.9277	−0.7185
y, l	−0.8232	1.0000	−1.0000	0.3137	0.6350	0.7129	0.9816
$y/l, y'$	0.8232	−1.0000	1.0000	−0.3137	−0.6350	−0.7129	−0.9816
λ_1	0.1972	0.3137	−0.3137	1.0000	−0.5338	−0.4418	0.4887
w	−0.8982	0.6350	−0.6350	−0.5338	1.0000	0.9944	0.4763
wl/y	−0.9277	0.7129	−0.7129	−0.4418	0.9944	1.0000	0.5662
θ	−0.7185	0.9816	−0.9816	0.4887	0.4763	0.5662	1.0000

(c) 10 year cycle

	r	y, l	$y/l, y'$	λ_1	w	wl/y	θ
r	1.0000	−0.5390	0.5390	0.3586	−0.7638	−0.8831	−0.4579
y, l	−0.5390	1.0000	−1.0000	0.5512	−0.04928	0.1998	0.9950
$y/l, y'$	0.5390	−1.0000	1.0000	−0.5512	0.04928	−0.1998	−0.9950
λ_1	0.3586	0.5512	−0.5512	1.0000	−0.8598	−0.7066	0.6288
w	−0.7638	−0.04928	0.04928	−0.8598	1.0000	0.9688	−0.1463
wl/y	−0.8831	0.1998	−0.1998	−0.7066	0.9688	1.0000	0.1034
θ	−0.4579	0.9950	−0.9950	0.6288	−0.1463	0.1034	1.0000

D.4. *Peak shift matrices*

(Units: year)

1. Productivity cycle case

(a) 1 year cycle

	α, y, y'	y/l	λ_1	w	wl/y	θ
α, y, y'	0.00	0.00	0.82	0.95	0.59	0.93
y/l	0.00	0.00	0.82	0.96	0.59	0.93
λ_1	0.18	0.18	0.00	0.14	0.77	0.12
w	0.05	0.04	0.86	0.00	0.64	0.98
wl/y	0.41	0.41	0.23	0.36	0.00	0.34
θ	0.07	0.07	0.88	0.02	0.66	0.00

(b) 5 year cycle

	α, y, y'	y/l	λ_1	w	wl/y	θ
α, y, y'	0.00	0.05	3.80	0.10	1.10	4.60
y/l	5.00	0.00	3.70	0.05	1.10	4.60
λ_1	1.30	1.30	0.00	1.40	2.40	0.85
w	4.90	5.00	3.70	0.00	1.00	4.50
wl/y	3.90	4.00	2.70	4.00	0.00	3.50
θ	0.40	0.45	4.20	0.50	1.50	0.00

(c) 10 year cycle

	α, y, y'	y/l	λ_1	w	wl/y	θ
α, y, y'	0.00	0.30	7.20	0.60	2.00	8.60
y/l	9.70	0.00	6.90	0.30	1.70	8.30
λ_1	2.80	3.10	0.00	3.30	4.80	1.40
w	9.40	9.70	6.70	0.00	1.40	8.10
wl/y	8.00	8.30	5.20	8.60	0.00	6.60
θ	1.40	1.70	8.60	1.90	3.40	0.00

2. Demand cycle case

(a) 1 year cycle

	β, r	y	y/l	y'	λ_1	w	wl/y	θ
β, r	0.00	0.54	0.04	0.04	0.30	0.41	0.42	0.42
y	0.46	0.00	0.50	0.50	0.76	0.87	0.88	0.88
y/l	0.96	0.50	0.00	0.00	0.26	0.37	0.38	0.38
y'	0.96	0.50	0.00	0.00	0.26	0.37	0.38	0.38
λ_1	0.70	0.24	0.74	0.74	0.00	0.11	0.12	0.12
w	0.59	0.13	0.63	0.63	0.89	0.00	0.00	0.01
wl/y	0.58	0.12	0.62	0.62	0.88	0.00	0.00	0.00
θ	0.58	0.12	0.62	0.62	0.88	0.99	0.00	0.00

(b) 5 year cycle

	β, r	y	y/l	y'	λ_1	w	wl/y	θ
β, r	0.00	2.10	4.60	4.60	1.10	2.80	2.70	2.00
y	2.90	0.00	2.50	2.50	4.00	0.70	0.60	4.90
y/l	0.40	2.50	0.00	0.00	1.50	3.20	3.10	2.40
y'	0.40	2.50	0.00	0.00	1.50	3.20	3.10	2.40
λ_1	3.90	1.00	3.50	3.50	0.00	1.70	1.60	0.85
w	2.20	4.30	1.80	1.80	3.30	0.00	4.90	4.20
wl/y	2.30	4.40	1.90	1.90	3.40	0.10	0.00	4.30
θ	3.10	0.15	2.70	2.70	4.20	0.85	0.75	0.00

(c) 10 year cycle

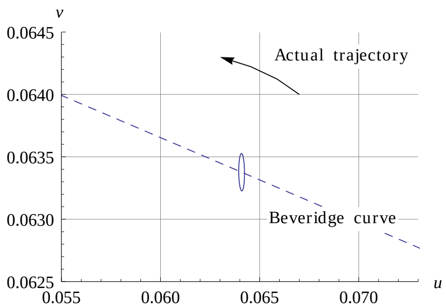
	β, r	y	y/l	y'	λ_1	w	wl/y	θ
β, r	0.00	3.50	8.50	8.50	1.90	6.00	5.60	3.30
y	6.50	0.00	5.00	5.00	8.40	2.60	2.20	9.80
y/l	1.50	5.00	0.00	0.00	3.40	7.60	7.20	4.80
y'	1.50	5.00	0.00	0.00	3.40	7.60	7.20	4.80
λ_1	8.10	1.60	6.60	6.60	0.00	4.10	3.80	1.40
w	4.00	7.40	2.40	2.40	5.90	0.00	9.60	7.30
wl/y	4.40	7.80	2.80	2.80	6.20	0.40	0.00	7.70
θ	6.70	0.20	5.20	5.20	8.60	2.70	2.30	0.00

The tables show phase shifts which is defined to be the argmax of the cross-correlation function, i.e. $\arg\max_{s\in[0,\tau)} \int_0^\tau a(t)\, b(t+s)\, \mathrm{d}t$ where a is a variable from rows, b from columns and τ the length of cycles. The number in the cells shows the forward shift of the column variable compared to the row variable in year units. It can be approximated by $\arg\max_s \mathrm{Corr}_t\big(a(t),\ b(t+s)\big)$.

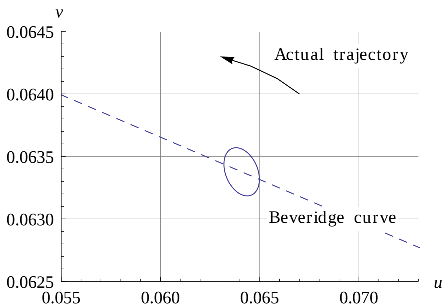
D.5. *uv dynamics*

1. Productivity cycle case

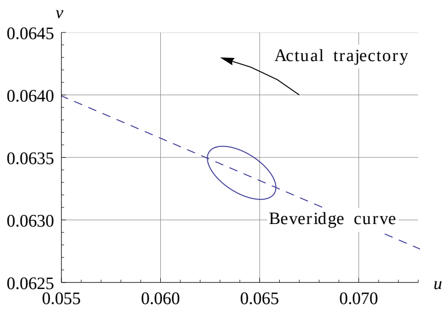
(a) 1 year cycle



(b) 5 year cycle

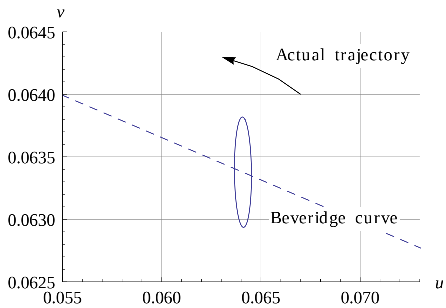


(c) 10 year cycle

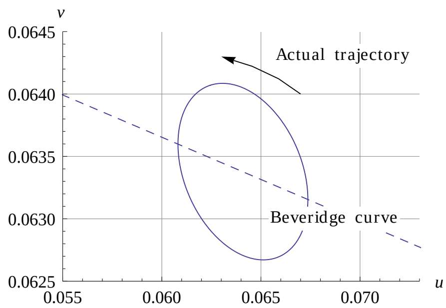


2. Demand cycle case

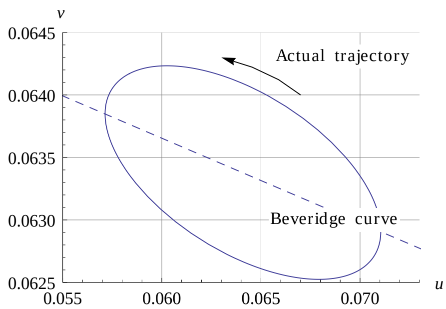
(a) 1 year cycle



(b) 5 year cycle



(c) 10 year cycle



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