A NOTE ON THE BOUNDEDNESS OF A SET OF SADDLE-POINTS*

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1. It is well-known that a saddle-point plays an important role in the theory of mathematical programming. The Kuhn-Tucker theorem of concave programming, as is frequently quoted, states that a maximun problem[1]:

Maximize $f(x)$ subject to $x \geqq 0$, $g(x) \geqq 0$.

is equivalent to a saddle-point problem:

Find a non-negative saddle-point of $\varphi(x,u)=f(x)+u.g(x)$.[2]

under some qualifications on $g(x)$. Among a number of constraint qualifications, the relations of which were extensively studied in [1], we are interested in the so-called (weak) Slater condition:

(S) *There exists an* $\hat{x} \geqq 0$ *such that* $g(\hat{x})>0$.

In the present note we shall show that, under strong version of the Slater condition, a set of saddle-points of concave-convex function is bounded.

2. Let $\varphi(x,u)$ be a real valued function defined for $x \geqq 0$ and $u \geqq 0$. By a saddle-point we mean the vector $(\bar{x},\bar{u})$ such that

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1) Throughout this paper, x denotes an n-vector whose i-th component is x_i and u denotes an m-vector whose j-th component is u_j. $g(x)$ is an m-vector valued function.

2) Let $f(x)$ be a utility function and $g(x)$ an excess supply function of resources, then a saddle-point of $\varphi(x,u)$ may be interpreted as an equilibrium point.

$\bar{x} \geqq 0, \ \bar{u} \geqq 0,$

$\varphi(x,\bar{u}) \leqq \varphi(\bar{x},\bar{u}) \leqq \varphi(x,u)$ for all $x \geqq 0, \ u \geqq 0.$

Let $\mathfrak{S}$ denote a set of saddle-points. $\varphi(x,u)$ is differentiable with respect to x and u. We shall assume that

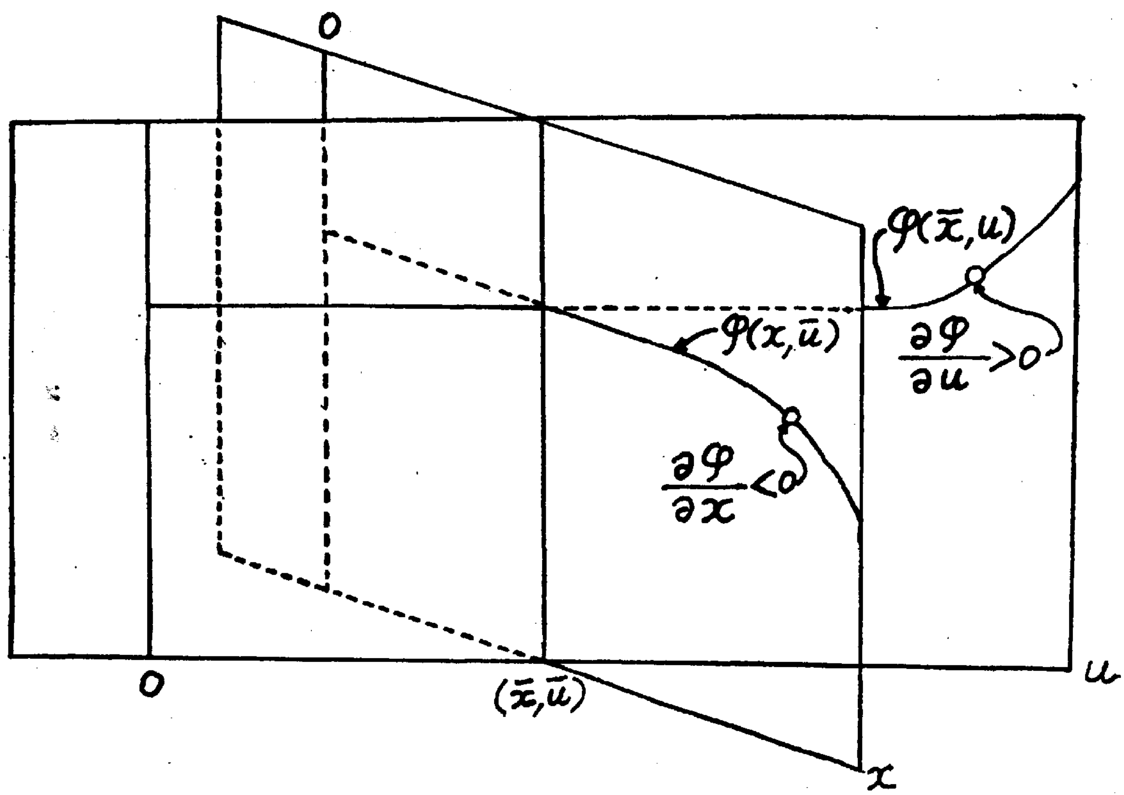
(G) *There exists an* $(\hat{x},\hat{u}) \geqq 0$ *such that*

$$\frac{\partial \varphi(\hat{x},\hat{u})}{\partial x_i} < 0, \quad i=1,\cdots,n,$$

$$\frac{\partial \varphi(\hat{x},\hat{u})}{\partial u_j} > 0, \quad j=1,\cdots m.$$

Now it is obvious that (G) implies (S) if $\varphi(x,u)=f(x)+u.g(x)$, so that (G) also makes the reduction of a maximum problem into a saddle-point problem possible. We call the condition (G) the strong Slater condition. We can prove

THEOREM : *Let* $\varphi(x,u)$ *be concave in* x *and convex in* u *and differentiable with respect to* x *and* u. *If the strong Slater condition* (G) *is satisfied, then* $\mathfrak{S}$ *is bounded.*



3. PROOF OF THEOREM. If $\mathfrak{S}$ is empty, then the theorem is trivially true. In what follows we shall assume $\mathfrak{S} \neq \phi$. For the differentiable concave-convex function, we have

$$\sum_{i=1}^{n} \frac{\partial\varphi(x,u)}{\partial x_i}(\bar{x}_i - x_i) - \sum_{j=1}^{m} \frac{\partial\varphi(x,u)}{\partial\varphi_i}(\bar{u}_j - u_j) \geqq 0 \text{ for all } x \geqq 0,\ u \geqq 0$$

where $(\bar{x}_1,\cdots,\bar{x}_n,\bar{u}_1,\cdots,\bar{u}_m)$ is any vector in $\mathfrak{S}$. Because $\mathfrak{S}$ is lower bounded by the definition, there exists, if $\mathfrak{S}$ is not bounded, some sequence $\{\bar{x}_{i'}^{(\nu)}\}$ or $\{\bar{u}_{j'}^{(\eta)}\}$

such that

$1 \leqq i' \leqq n,$

$\lim_{\nu\to\infty} \bar{x}_{i'}^{(\nu)} = +\infty,$

$\bar{x}_{i'}^{(\nu)}$ is i'-th component of some vector which belongs to $\mathfrak{S}$

or

$1 \leqq j' \leqq m,$

$\lim_{\eta\to\infty} u_{j'} = +\infty,$

$\bar{u}_{j'}^{(\eta)}$ is $n+j'$-th component of some vector which belongs to $\mathfrak{S}$.

Without loss of generality, we assume the existence of the sequence $\{\bar{x}_{i'}^{(\nu)}\}$.

By putting

$$F(\bar{x}_{i'}^{(\nu)}) = \frac{\partial\varphi(\hat{x},\hat{u})}{\partial x_{i'}}(\bar{x}_{i'}^{(\nu)} - \hat{x}_{i'})$$

we can infer that in view of (G)[3]

$$\lim_{\nu\to\infty} F(\bar{x}_{i'}^{(\nu)}) = -\infty$$

which leads to a contradiction. Q. E. D.

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3) If we assume only the first (second) inequality in the codnition (G) holds, then $\mathfrak{S}$ is bounded with respect to x-components (u-components).

4. It is easily shown that $\mathfrak{S}$ is closed if $\varphi(x,u)$ is continuous with respect to x and u. Hence it follows from the above theorem that *$\mathfrak{S}$ is compact provided that $\varphi(x,u)$ is differentiable concave-convex and the strong Slater condition (G) holds.*

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REFERENCE

〔1〕 Arrow, K.J., L.Hurwicz and H. Uzawa. "Constraint Qualifications in Maximization Problems," *Naval Research Logistics Quarterly*, Vol. 8 (1961), pp. 175—191.