A COMMENT ON DEBREU'S LEMMA*

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1.

G. Debreu has stated in [1] a theorem of utility representation by a continuous real-valued function. But we find some defects in his proof of Lemma I of the paper. In this note, we first present a counter-example to his proof[1)] and then we give another proof of his Lemma I. Finally some set-theoretical and topological aspects of the utility representation problem is considered.

We shall use same notations and definitions as in (1).

2.

His Lemma I is as follows:

Lemma I. *Let X be a completely ordered set whose quotient A is countable. There exists on X a real, order-preserving function, continuous in any natural topology.*

He states that the gaps of unwanted types can be eliminated by means of a non-decreasing step function $\theta(\alpha)$ and that the new function $\phi^*(a)=\psi(a)-\theta[\psi(a)]$ is still order-preserving. But this is not generally true.

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1) The author owes this example to Professor Takashi Ito of Hokkaido University, Department of Mathematics.

Let A be a set of rational numbers in $(0, 1)$. Since A is countable we can arrange the elements of A as $A=\{r_1, r_2, \cdots\cdots\}$.

Now we define $\psi(a)=\sum_{r_i\leqq a}\frac{1}{2_i}$. This function is easily seen to be order-preserving, but at each point r_i, ψ has a gap of unwanted type, i. e. of $(2\text{-}1')$ type, and its corresponding gap is equal to $1/2^i$.

Then $\theta[\psi(a)]=\psi(a)$ and $\phi^*(a)=0$ for all $a\epsilon A$ and ϕ^* does not preserve the order $\leqq$ at all.

3.

We shall now show that the Lemma I is valid if we specify the construction of ψ from the beginning.

Let $A=\{a_1, a_2, \cdots\cdots\}$. We define $\psi(a)$ by induction. First, we define $\psi(a_1)=1/2$. Next, suppose $\psi(a_1), \cdots, \psi(a_n)$ have been defined, then there are following three cases

(a) $a_{n+1}>a_i$ for all $i=1, \cdots\cdots, n$.

Let $\max_{1\leqq i\leqq n} a_i=a$. We define $\psi(a_{n+1})=\frac{1}{2}(1+\psi(a))$.

(b) $a_{n+1}<a_i$ for all $i=1, \cdots\cdots$, n.

Let $\min_{1\leqq i\leqq n} a_i=a$. We define $\psi(a_{n+1})=\frac{1}{2}\psi(a)$.

(c) $a_i<a_{n+1}<a_j$ for some $i, j=1, \cdots\cdots, n$ and $a_k\leqq a_i$ or $a_k\geqq a_j$ for all $k=1, \cdots\cdots, n$. We define $\psi(a_{n+1})=\frac{1}{2}(\psi(a_i)+\psi(a_j))$.

Then $\psi(a)$ has no gaps of type $(1\text{-}2')$ or $(2\text{-}1')$.[2]

Proof. Let $\alpha'\bar{\epsilon}\psi(A)$ and $R_{\alpha'}=\{\alpha\epsilon\psi(A)/\alpha<\alpha'\}\neq\phi$ have no maximum element. If $R^{\alpha'}=\{\alpha\epsilon\psi(A)/\alpha>\alpha'\}$ has minimum element, this will lead to a contradiction.

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2) Though G. Debreu states that $(2-2')$ type is unwanted, this is not unwanted at this stage. The truely unwanted types are only types $(1-2')$ and $(2-1')$.

For suppose the minimum element of $R^{\alpha'}$ be $\psi(a_{i_1})$. Since there is no maximum in $R_{\alpha'} \neq \phi$, there exists an element $a_{i_2} \in A$ such that $i_2 > i_1$ and $\psi(a_{i_2}) < \alpha'$.

Similarly, the set $\{a_i \in A / a_i > a_{i_2},\ i > i_2\}$ is not empty. Let a_{i_3} be the element of this set which has the smallest suffix. Then we have $\psi(a_{i_3}) = \frac{1}{2}(\psi(a_{i_2}) + \psi(a_{i_1}))$.

If We define $a_{i_4}, a_{i_5}, \cdots\cdots$ analogously, we have $\psi(a_{i_\nu}) \to \psi(a_{i_1})$ as $\nu \to \infty$ and $\lim_{\nu\to\infty} \psi(a_{i_\nu}) > \alpha'$ which is a contradiction.

The proof for the case of the type (1-2′) is similar and is omitted.

4.

G. Debreu has shown in Theorem I in [1] that there exists a continuous mapping from X into real space if X is completely ordered, separable, connected and if X_x and X^x is closed for all $x \in X$.

In this case $\phi(X)$ is an interval since the image of connected set is connected and the connected set in real line is an interval.

Moreover $X/\sim$ becomes similar[3] to the interval, hence the types of order of $X/\sim$ is either λ, or $1+\lambda$, or $\lambda+1$, or $1+\lambda+1$.

But even if $\phi(X)$ is an interval, X is not connected. This can be seen by inducing discrete topology on X.

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3) Completely ordered asymmetric sets $(X, \leqq)$ and $(Y, \leqq^*)$ is called similar if there is a one-to-one correspondence f between X and Y and $x \leqq x'$ if and only if $f(x) \leqq^* f(x')$.

〔1〕 G. Debreu, Representation of a preference ordering by a numerical function, in *Decision Processes* edited by R. M. Thrall, C. H. Coombs and R. L. Davis (John Wiley & Sons, N. Y., 1954).