Generalized Rybczynski's Theorem: Its analytical expression*

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1. The Rybczynski's theorem as generalized by Guha [2] and Amano [1] are usually presented in geometrical forms. Even the original theorem itself is seldom expressed in an analytical form.[1] Geometrical 'proof' of the original and the generalized versions of the theorem is quite sufficient for qualitative discussions. Nevertheless, an analytical expression of it is rather simple and it can be readily employed for further studies of related subjects.

2. Let the production functions be represented by

$$X=F^1(K_1, L_1)$$

and

$$Y=F^2(K_2, L_2),$$

where X, Y, K_1, K_2, L_1, L_2 are, respectively, quantities of products, quantities of a factor (which we call capital), and quantities of another factor (which we call labor), employed in the two sectors.

From the constant-return-to-scale assumption on the production functions, we have

$$X=L_1f_1(K_1/L_1)$$

$$Y=L_2f_2(K_2/L_2)$$

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1) Except Takayama [6], chapter II, section 1.

where $f_1(K_1/L_1)\equiv F^1(K_1/L_1,\ 1)$ and $f_2(K_2/L_2)\equiv F^2(K_2/L_2,\ 1)$.

3. Under competitive conditions, the price p of the product of the first sector in terms of the product of the second sector must satisfy the relation

(1) $p=f_2'(k_2)/f_1'(k_1)$

where $k_1\equiv K_1/L_1$ and $k_2\equiv K_2/L_2$ are capital-intensities of the two sectors. If we introduce the wage-rental ratio $\omega\equiv w/r$, then the following must hold[2]

(2) $\omega=\dfrac{f_1}{f_1'}-k_1=\dfrac{f_2}{f_2'}-k_2$

From (1), we have[3]

$$dp=\frac{1}{(f_1')^2}(f_1'\cdot f_2''\cdot dk_2-f_2'\cdot f_1''\cdot dk_1)$$

and from (2), we have

(3) $d\omega=\dfrac{f_1\cdot f_1''}{(f_1')^2}dk_1=\dfrac{f_2\cdot f_2''}{(f_2')^2}dk_2$

Hence, for $dp=0$ to hold, we must have either

(4a) $dk_1=dk_2=0$

or

(4b) $dk_2=\dfrac{f_2'\cdot f_1''}{f_1'\cdot f_2''}dk_1\neq 0$

But (4b) and the second relation of (3) jointly imply

(5) $f_1/f_2=f_1'/f_2'$

But $f_1(k_1)=(\omega+k_1)f_1'(k_1)$ and $f_2(k_2)=(\omega+k_2)f_2'(k_2)$ from relations (2). Therefore equation (5) implies

2) In the following, $f_1(k_1)$ and $f_1'(k_2)$ and the like are denoted by f_1 and f_1' etc.

Relation (2) is straightfoward from

$$r=\mathrm{MP}_k^2=p\cdot\mathrm{MP}_k^1=f_2'=p\cdot f_1'$$

and

$$w=\mathrm{MP}_l^2=p\cdot\mathrm{MP}_l^1=f_2-k_2\cdot f_2'=p\cdot(f_1-k_1\cdot f_1')$$

3) Of course, it is assumed that $f_1'>0$, $f_2'>0$, $f_1''<0$, $f_2''<0$ and that $C>0$, $I>0$, $k_1>0$, $k_2>0$, thus excluding the case of complete specialization.

(6) $k_1 = k_2$

On the other hand, for $d\omega = 0$ to hold, we must have from (3) that $dk_1 = dk_2 = 0$.

Summarizing,[4)]

Lemma 1. Suppose $k_1 \neq k_2$. Then, $dp=0$ if and only if $d\omega=0$, and $d\omega=0$ if and only if $dk_1=dk_2=0$.

4. From the assumed property of linear homogeneity of production functions, we also have

$$(7)\quad \begin{aligned} dX &= f_1' \cdot dK_1 + (f_1 - k_1 f_1') dL_1 = X \cdot \frac{dL_1}{L_1} + \left(\frac{dK_1}{K_1} - \frac{dL_1}{L_1}\right) \cdot K_1 \cdot f_1' \\ dY &= f_2' \cdot dK_2 + (f_2 - k_2 f_2') dL_2 = Y \cdot \frac{dL_2}{L_2} + \left(\frac{dK_2}{K_2} - \frac{dL_2}{L_2}\right) \cdot K_2 \cdot f_2' \end{aligned}$$

From the Lemma 1, we must have $dk_1 = dk_2 = 0$ for $dp=0$.

But since

$$dk_1 = k_1 \cdot \left(\frac{dK_1}{K_1} - \frac{dL_1}{L_1}\right)$$

and

$$dk_2 = k_2 \cdot \left(\frac{dK_2}{K_2} - \frac{dL_2}{L_2}\right),$$

it is necessary for $dp=0$ that

$$\frac{dK_1}{K_1} - \frac{dL_1}{L_1} = 0 \text{ and } \frac{dK_2}{K_2} - \frac{dL_2}{L_2} = 0.$$

Hence,

Lemma 2. Suppose $k_1 \neq k_2$. If price p is kept constant, then proportional changes of the outputs of the two sectors are equal to the proportional changes of labor inputs in each sector, namely,

$$(8)\quad \begin{aligned} dX &= X \cdot \frac{dL_1}{L_1} \\ dY &= Y \cdot \frac{dL_2}{L_2} \end{aligned}$$

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4) As a matter of fact, this property is so obvious that majority of the expositions of the Rybczynski's theorem do not provide much space for the discussion of this kind.

5. Define factor-endowment ratio k by $k \equiv K/L$ and proportions l_1 and l_2 by $l_1 \equiv L_1/L$ and $l_2 \equiv L_2/L$ where K and L are the total quantities of factors of production.

Obviously,

(9) $k = l_1 k_1 + l_2 k_2$

and $l_1 + l_2 = 1$. From (9) we obtain, if $dk_1 = dk_2 = 0$, that

(10a) $$dk = (k_1 - k_2) \cdot l_1 \cdot \left(\frac{dL_1}{L_1} - \frac{dL}{L} \right) = (k_2 - k_1) \cdot l_2 \cdot \left(\frac{dL_2}{L_2} - \frac{dL}{L} \right)$$

On the other hand, from the definition of k, we have

(10b) $$dk = k \cdot \left(\frac{dK}{K} - \frac{dL}{L} \right)$$

From (10a) and (10b), we obtain, provided $k_1 \neq k_2$,

(11) $$\frac{dL_1}{L_1} = \frac{1}{(k_1 - k_2) \cdot l_1} \cdot \left(k \cdot \frac{dK}{K} - k_2 \cdot \frac{dL}{L} \right)$$
$$\frac{dL_2}{L_2} = \frac{1}{(k_2 - k_1) \cdot l_2} \cdot \left(k \cdot \frac{dK}{K} - k_1 \cdot \frac{dL}{L} \right)$$

Substituting these relations into equations (8), our fundamental relations are obtained :

(12) $$dX = \frac{X}{(k_1 - k_2) \cdot l_1} \cdot \left(k \cdot \frac{dK}{K} - k_2 \cdot \frac{dL}{L} \right)$$
$$dY = \frac{Y}{(k_2 - k_1) \cdot l_2} \cdot \left(k \cdot \frac{dK}{K} - k_1 \cdot \frac{dL}{L} \right)$$

6. Original Rybczynski's Theorem. First, consider special cases where one of the factors increases while the other remains stationary.

Case (α). $L = constant.$

Here $dL = 0$ and we have

$$\frac{dX}{dK} = \frac{1}{(k_1 - k_2) \cdot l_1} \cdot \frac{X}{L} \gtrless 0$$
$$\frac{dY}{dK} = \frac{1}{(k_2 - k_1) \cdot l_2} \cdot \frac{Y}{L} \lessgtr 0$$
according as $k_1 \gtrless k_2$

Case (β). $K = constant.$

Here $dK=0$ and we have

$$\frac{dX}{dL}=-\frac{k_2}{(k_1-k_2)\cdot l_1}\cdot\frac{X}{L}\lesseqgtr 0 \qquad \text{according as } k_1\gtrless k_2$$
$$\frac{dY}{dL}=-\frac{k_1}{(k_2-k_1)\cdot l_2}\cdot\frac{Y}{L}\gtreqless 0$$

Thus summarizing,

	$L=constant$	$K=constant$
$k_1>k_2$:	$\frac{dX}{dK}>0,\ \frac{dY}{dK}<0$;	$\frac{dX}{dL}<0,\ \frac{dY}{dL}>0.$
$k_1<k_2$:	$\frac{dX}{dK}<0,\ \frac{dY}{dK}>0$;	$\frac{dX}{dL}>0,\ \frac{dY}{dL}<0.$

7. Generalized Rybczynski's Theorem. Now consider the general case where the quantities of both factors change. Since $dL\neq 0$, we have, from our fundamental equations (12),

$$(12')\qquad \begin{aligned} dX&=\frac{X}{(k_1-k_2)\cdot l_1}\cdot\left(\frac{dK}{dL}-k_2\right)\cdot\frac{dL}{L}\\ dY&=\frac{Y}{(k_2-k_1)\cdot l_2}\cdot\left(\frac{dK}{dL}-k_1\right)\cdot\frac{dL}{L}\end{aligned}$$

For virtually all practical purposes, we may assume that available total quantity of labor increases rather than decreases, i. e., *we may assume* $dL>0$. Case of decreasing labor $dL<0$ may similarly be analyzed.

Under the assumed condition of $dL>0$, two cases can be distinguished according to the relative capital-intensities.

Case (i). $k_1>k>k_2$.

$$dX\gtreqless 0 \quad \text{according as} \quad \frac{dK}{dL}\gtreqless k_2$$

$$dY\lesseqgtr 0 \quad \text{according as} \quad \frac{dK}{dL}\gtreqless k_1$$

Case (ii). $k_1<k<k_2$.

$$dX\lesseqgtr 0 \quad \text{according as} \quad \frac{dK}{dL}\gtreqless k_2$$

$$dY\gtreqless 0 \quad \text{according as} \quad \frac{dK}{dL}\gtreqless k_1$$

Summarizing,

	$dL>0$, $dK/dL<k_2$,	$k_2<dK/dL<k_1$,	$k_1<dK/dL$
$k_1>k_2$:	$dX<0$	$dX>0$	$dX>0$
	$dY>0$	$dY>0$	$dY<0$

and

	$dL<0$, $dK/dL<k_1$,	$k_1<dK/dL<k_2$,	$k_2<dK/dL$
$k_1<k_2$:	$dX>0$	$dX>0$	$dX<0$
	$dY<0$	$dY>0$	$dY>0$

The case of labor decrease can be similarly analyzed by using equations (12′).

8. For some purposes, it is necessary to employ the theorem on per-capita basis. Define per-capita outputs x and y of the products by[5)]

$$x=X/L \quad \text{and} \quad y=Y/L$$

Then,

$$dx=x\cdot\left(\frac{dX}{X}-\frac{dL}{L}\right)$$

and

$$dy=y\cdot\left(\frac{dY}{Y}-\frac{dL}{L}\right)$$

Substituting (8) into these relations, we have

$$dx=x\cdot\left(\frac{dL_1}{L_1}-\frac{dL}{L}\right)$$

$$dy=y\cdot\left(\frac{dL_2}{L_2}-\frac{dL}{L}\right)$$

Then, by using relations (11),

$$\text{(13)} \quad \begin{aligned} \frac{dx}{x}&=\frac{k}{(k_1-k_2)\cdot l_1}\cdot\left(\frac{dK}{K}-\frac{dL}{L}\right)=\frac{1}{(k_1-k_2)l_1}dk \\ \frac{dy}{y}&=\frac{k}{(k_2-k_1)\cdot l_2}\cdot\left(\frac{dK}{K}-\frac{dL}{L}\right)=\frac{1}{(k_2-k_1)l_2}dk \end{aligned}$$

5) x and y are per-capita outputs of the products, and should not be confused with the labor-productivities of each sector, X/L_1 and Y/L_2 which are related to per-capita outputs by identities $X/L_1=x/l_1$ and $Y/L_2=y/l_2$.

The second equalities of the above equations are due to the relation (10b). Rewriting (13),

$$(13') \quad \begin{aligned} \frac{dx}{dk} &= \frac{x}{(k_1-k_2)\cdot l_1} \gtrless 0 \\ \frac{dy}{dk} &= \frac{y}{(k_2-k_1)\cdot l_2} \lessgtr 0 \end{aligned} \quad \text{according as } k_1 > k_2$$

Thus,

$$(14) \quad sign\frac{dx}{dk} \neq sign\frac{dy}{dk}$$

and signs of dx and dy depend, besides relative capital-intensities, entirely on the sign of dk, which in turn depends on the relative growth rates of capital and labor, i.e., dK/K and dL/L.

9. A concluding remark. Qualitative conclusions summarized at the end of Sections 6 and 7 are familiar. As a matter of fact, they can be derived more easily by a geometric device of Bowley-Edgeworth diagram used in [1], [2] and [5]. Nevertheless, it may be claimed that our fundamental relations (12) and (12′) provide better quantitative estimates of the degree of 'biasedness' in production in the process of economic growth (Johnson [3]), and (12) or (12′) may easily be expressed in terms of elasticities and growth rates.

References

[1] A. Amano, "Factor Endowment and Relative Prices: A Generalization of Rybczynski's Theorem," *Economica*, vol. 30, no. 120, Nov. 1963, pp.413–414.

[2] A. Guha, "Factor and Commodity Prices in an Expanding Economy," *Quarterly Journal of Economics*, vol. LXXVII, no. 1, Feb. 1963, pp. 149–158.

[3] H. G. Johnson, "Economic Development and International Trade," *Money, Trade and Economic Growth* (London, 1962), pp. 75–103.

[4] K. Lancaster, "The Heckscher–Ohlin Trade Model: A Geometric Treatment," *Economica*, vol. 24, no. 93, Feb. 1957, pp. 19–39.

[5] T.M. Rybczynski, "Factor Endowment and Relative Commodity Prices," *Economica*, vol. 22, no. 88, Nov. 1955, pp. 336–341.

[6] A. Takayama, *Kokusai Keizaigaku* (Tokyo, 1963).