Pointwise Multipliers From $BMOA^{\alpha}$ To $BMOA^{\beta}$

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Abstract

Let g be an analytic function on the open unit disk D in the complex plane C. We will study the following operator

$$I_g(f)(z) := \int_0^z f'(\zeta)g(\zeta)d\zeta \ , \ J_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta.$$

In this paper we study the operators I_g, J_g from $BMOA^{\alpha}$ to $BMOA^{\beta}$ (from D_{α} to D_{β}) ($\alpha \leq \beta$). And we study pointwise multipliers from $BMOA^{\alpha}$ to $BMOA^{\beta}$ (from D_{α} to D_{β}) ($\alpha \leq \beta$).

Key Words and Phrases : integration operator, Bloch space, Dirichlet spaces, $BMOA$, boundedness, multiplier.

§1. Introduction

Let $D = \{z \in C : |z| < 1\}$ denote the open unit disk in the complex plane C and let $\partial D = \{z \in C : |z| = 1\}$ denote the unit circle. For $1 \leq p < +\infty$, the Lebesgue space $L^p(D, dA)$ is defined to be the Banach space of Lebesgue measurable functions on the open unit disk D with

$$\| f \|_{L^p(dA)} := \left(\int_D |f(z)|^p dA(z) \right)^{\frac{1}{p}} < +\infty \ ,$$

where $dA(z)$ is the normalized area measure on D. The Bergman space $L_a^p(D)$ is defined to be the subspace of $L^p(D, dA)$ consisting of analytic functions. For $0 < p < +\infty$, the Hardy space H^p is defined to be the Banach space of analytic functions f on D with

$$\| f \|_p := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < +\infty \ .$$

For $z, w \in D$, let $\beta(z, w) := \frac{1}{2} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$, where $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$. For $0 < r < +\infty$

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and $z \in D$, let $D(z) = D(z,r) = \{w \in D : \ \beta(z,w) < r\}$ denote the Bergman disk. $|D(z,r)|$ denotes the normalized area of $D(z,r)$ and $|D(z,r)|$ is comparable to $(1-|z|^2)^2$.

The space of analytic functions on D of bounded mean oscillation , denoted by $BMOA$, consists of functions f in H^2 for which

$$\| f \|_{BMOA} := \sup_{z \in D} \| f \circ \varphi_z - f(z) \|_2 < +\infty \ .$$

Let $\alpha > 0$. Then α-Bloch space B^α is defined to be the space of analytic functions f on D such that

$$\| f \|_{B^\alpha} := \sup_{z \in D} (1-|z|^2)^\alpha \, |f'(z)| < +\infty \ .$$

And the little α-Bloch space, denoted by B_0^α, is the closed subspace of B^α consisting of functions f with $(1-|z|^2)^\alpha f'(z) \to 0$ $(|z| \to 1^-)$. Note that B^1, B_0^1 are the Bloch space B, the little Bloch space B_0, respectively.

The space $BMOA^\alpha$ is defined to be the space of analytic functions f on D such that

$$\| f \|^2_{BMOA^\alpha} := \sup_{a \in D} \int_D (1-|z|^2)^{2\alpha-2} \, |f'(z)|^2 \, (1-|\varphi_a(z)|^2) dA(z) < +\infty.$$

The space $BMOA_\alpha$ is defined to be the space of analytic functions f on D such that

$$\| f \|^2_{BMOA_\alpha} := \sup_{I \subset \partial D} \frac{|I|^{2\alpha-2}}{|I|} \int_{S(I)} |f'(z)|^2 \, (1-|z|^2) dA(z) < +\infty,$$

where I is any arc on the unit circle ∂D, $S(I) = \{ \ z \in D : \ |z| > 1-|I|, \ \frac{z}{|z|} \in I \ \}$, and $|I|$ is the normalized arc length on ∂D.

The space D_α is defined to be the space of analytic functions f on D such that

$$\| f \|^2_{D_\alpha} := \int_D (1-|z|^2)^\alpha \, |f'(z)|^2 \, dA(z) < +\infty.$$

Then note that $BMOA = BMOA^1 = BMOA_1$, $L_a^2 = D_2$ and $H^2 = D_1$.

Let X and Y be Banach spaces. Then a function f on D is a multiplier of X into Y if $fg \in Y$ for all g in X. In this case, we write $fX \subset Y$.

For g analytic on D, the operators I_g , J_g and M_g are defined on the above spaces by the following:

$$I_g(h)(z) := \int_0^z g(\zeta) h'(\zeta) d\zeta \ , \ J_g(f)(z) := \int_0^z f(\zeta) g'(\zeta) d\zeta \ , \ M_g(f)(z) := g(z)f(z).$$

In [P], Ch. Pommerenke showed that J_g is a bounded operator on Hardy space H^2 if and only if g is in $BMOA$, and this result was extended to other Hardy spaces H^p $1 \le p < +\infty$ in [AS1]. In [AS2], A.Aleman and A.G.Siskakis studied the operator J_g defined on weighted Bergman spaces.

In [Yo1], we proved the following result:

Theorem 1.1. The operator J_g is a bounded operator on B if and only if

$$\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|<+\infty,$$

and the operator J_g is a compact operator on B if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|=0.$$

And let $\alpha>1$. Then the operator J_g is a bounded operator on B^{α} if and only if $g\in B$. And the operator J_g is a compact operator on B^{α} if and only if $g\in B_0$.

In [Yo2], we also proved the following results :

Theorem 1.2. Let $\alpha\geq 1$ and g be analytic on D. Then the operator I_g is a bounded operator on B^{α} if and only if $g\in H^{\infty}$. And the operator I_g is a compact operator on B^{α} if and only if $g\equiv 0$.

Theorem 1.3. For g analytic on D, the following are equivalent :

(i) $gB\subset B$ $(gB_0\subset B_0)$;

(ii) Both I_g and J_g are bounded operators on B (or B_0) ;

(iii) $g\in H^{\infty}$, $\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|<+\infty.$

And let $\alpha>1$. The following are equivalent :

$(i)'$ $gB^{\alpha}\subset B^{\alpha}$ $(gB_0^{\alpha}\subset B_0^{\alpha}$) ;

$(ii)'$ I_g is a bounded operator on B^{α} (or B_0^{α}) ;

$(iii)'$ $g\in H^{\infty}$.

In Theorem 1.3, the equivalence of (i) and (iii), the equivalence of $(i)'$ and $(iii)'$ were proved in [Zhu3] and [Zhu4].

The space $BMOA^{\alpha}$ has been previous studied by R.Zhao in [Z1, p.51]. So $BMOA^{\alpha}$ is the same as $BMOA_2^{\alpha}$ in [Z1]; Pointwise multipliers on $BMOA$ have been characterized by D.Stegenga in [St] and J.M.Ortega and J.Farega in [OF]. Also , the boundedness of the operator J_g on $BMOA$ has been characterized by Siskakis and Zhao in [SZ].

In this paper we study the operators I_g, J_g from D_{α} to D_{β} (from $BMOA_{\alpha}$ to

$BMOA_\beta$) ($\alpha \le \beta$). And we also study the multipliers from D_α to D_β (from $BMOA_\alpha$ to $BMOA_\beta$) ($\alpha \le \beta$). And some of the techniques used to prove theorems were inspired by [OSZ] and [W]. Throughout this paper, C , K will denote positive constant whose value is not necessary the same at each occurrence.

§2. Multipliers from $BMOA$ to Bloch space

In this section, we study multipliers from $BMOA$ to Bloch space.

Theorem 2.1. For g analytic on D, the following are equivalent:

(i) $gBMOA \subset B$;

(ii) $I_g, J_g : BMOA \to B$ are bounded operators ;

(iii) $g \in H^\infty$, $\sup_{z\in D}(1-|z|^2)\left(\log \frac{1}{1-|z|^2}\right)|g'(z)| < +\infty$.

Proof. First, we prove that J_g : $BMOA \to B$ is bounded operator if and only if

$$\sup_{z\in D}(1-|z|^2)\left(\log \frac{1}{1-|z|^2}\right)|g'(z)| < +\infty.$$

Let $f \in BMOA$. Put $L := J_g f$. Then we see

$$(1-|z|^2)|L'(z)| = (1-|z|^2)|f(z)||g'(z)| = (1-|z|^2)\log\frac{1}{1-|z|^2}|g'(z)|\frac{|f(z)|}{\log\frac{1}{1-|z|^2}}.$$

Since $|f(z)| \le C \parallel f \parallel_{BMOA} \log \frac{1}{1-|z|^2}$ (see [SZ]), hence we have

$$\parallel J_g f \parallel_B = \sup_{z\in D}(1-|z|^2)|L'(z)| \quad \le \quad C\sup_{z\in D}(1-|z|^2)\log\frac{1}{1-|z|^2}|g'(z)| \parallel f \parallel_{BMOA} .$$

To prove the converse, suppose that J_g is bounded on $BMOA$. For $a \in D$, put $f_a(z) = \log \frac{1}{1-\overline{a}z}$. Then it is clear that $\{f_a\}$ is a bounded set in $BMOA$. For $z \in D(a,r)$, we have $\log \frac{1}{1-|a|^2} \le C\left|\log\frac{1}{1-\overline{a}z}\right|$. So by using the subharmonicity of $|g'(z)|$ and the fact that there is a constant $C_1 > 0$ (depending only on r) such that

$$\int_{D(a,r)} \frac{1}{(1-|z|^2)^2} dA(z) \le C_1 < \infty,$$

$$\begin{aligned}(1-|a|^2)^2\left(\log\frac{1}{1-|a|^2}\right)^2 |g'(a)|^2 &\le C'\left(\log\frac{1}{1-|a|^2}\right)^2\int_{D(a,r)}|g'(z)|^2 dA(z)\\ &\le CC'\int_{D(a,r)}\left|\log\frac{1}{1-\bar{a}z}\right|^2|g'(z)|^2 dA(z)\\ &\le CC'\sup_{z\in D(a,r)}(1-|z|^2)^2\left|\log\frac{1}{1-\bar{a}z}\right|^2|g'(z)|^2\int_{D(a,r)}\frac{1}{(1-|z|^2)^2}dA(z)\\ &\le CC'C_1\sup_{z\in D}(1-|z|^2)^2\left|\log\frac{1}{1-\bar{a}z}\right|^2|g'(z)|^2\\ &\le CC'C_1\sup_{a\in D}\| J_g f_a\|_B^2\\ &\le CC'C_1\| J_g\|^2\sup_{a\in D}\| f_a\|_{BMOA}^2<\infty.\end{aligned}$$

Next, we prove that $I_g : BMOA \to B$ is a bounded operator if and only if $g \in H^\infty$. Let $f \in BMOA$. Put $L := I_g f$. Then we see for some constant $C > 0$,

$$\begin{aligned}(1-|z|^2)|L'(z)| &= (1-|z|^2)|f'(z)||g(z)|\\ &\le \| g\|_\infty (1-|z|^2)|f'(z)|\\ &\le C\| g\|_\infty\| f\|_{BMOA}.\end{aligned}$$

Hence $\| I_g f\|_B \le C\| g\|_\infty \| f\|_{BMOA}$.

To prove the converse, suppose that I_g is bounded on $BMOA$. For $a \in D$, put $f_a(z) = \log\frac{1}{1-\bar{a}z}$. Then

$$\begin{aligned}|a|^2|g(a)|^2 &\le C\frac{|a|^2}{(1-|a|^2)^2}\int_{D(a,r)}|g(z)|^2 dA(z)\\ &\sim C\int_{D(a,r)}\left|\left(\log\frac{1}{1-\bar{a}z}\right)'\right|^2|g(z)|^2 dA(z)\\ &\sim C\int_{D(a,r)}\frac{dA(z)}{(1-|z|^2)^2}\sup_{z\in D(a,r)}(1-|z|^2)^2|f_a'(z)|^2|g(z)|^2\\ &\sim C\| I_g f_a\|_B^2 \le C\| I_g\|^2\| f_a\|_{BMOA}^2<\infty.\end{aligned}$$

Hence we see $\sup_{z\in D}|g(z)| < \infty$. Thus we see that the equivalence of (ii) and (iii) holds. So it suffices to show that $gBMOA \subset B$ implies $g \in H^\infty$. Put $k_a(z) := \log\frac{1}{1-\bar{a}z} - \log\frac{1}{1-|a|^2}$ $(a, z \in D)$. Since $gBMOA \subset B$ and $k_a(a) = 0$,

$$\begin{aligned}|a||g(a)| &= (1-|a|^2)\left|k_a(a)g'(a)+k_a'(a)g(a)\right|\\ &\le \sup_{z\in D}(1-|z|^2)\left|k_a(z)g'(z)+k_a'(z)g(z)\right| < +\infty.\end{aligned}$$

Hence we see that $g \in H^{\infty}$. $\square$

We also get the following results, but we omit to prove them because we can prove as the proof of the previous theorem. In the following theorem, the equivalence of (ii) and (v) was proved in [OSZ].

Theorem 2.2. Let $0 < \alpha < 1$ and $\alpha \leq \beta$. For g analytic on D, the following are equivalent:

(i) $gBMOA^{\alpha} \subset B^{\beta}$;

(ii) $gB^{\alpha} \subset B^{\beta}$;

(iii) $J_g : BMOA^{\alpha} \to B^{\beta}$ is a bounded operator ;

(iv) $J_g : B^{\alpha} \to B^{\beta}$ is a bounded operator ;

(v) $\sup_{z \in D} (1-|z|^2)^{\beta} |g'(z)| < +\infty$.

In the following theorem, the equivalence of (ii) and (v) was proved in [OSZ].

Theorem 2.3. Let $\alpha = 1$ and $\beta > 1$. For g analytic on D, the following are equivalent:

(i) $gBMOA^{\alpha} \subset B^{\beta}$;

(ii) $gB^{\alpha} \subset B^{\beta}$;

(iii) $J_g : BMOA^{\alpha} \to B^{\beta}$ is a bounded operator ;

(iv) $J_g : B^{\alpha} \to B^{\beta}$ is a bounded operator ;

(v) $\sup_{z \in D} (1-|z|^2)^{\beta} \left(\log \frac{1}{1-|z|^2} \right) |g'(z)| < +\infty$.

In the following theorem, the equivalence of (ii) and (vii) was proved in [OSZ].

Theorem 2.4. Let $\alpha > 1$ and $\alpha < \beta$. For g analytic on D, the following are equivalent:

(i) $gBMOA^{\alpha} \subset B^{\beta}$;

(ii) $gB^{\alpha} \subset B^{\beta}$;

(iii) $I_g : BMOA^{\alpha} \to B^{\beta}$ is a bounded operator ;

(iv) $J_g : BMOA^{\alpha} \to B^{\beta}$ is a bounded operator ;

(v) $I_g : B^\alpha \to B^\beta$ is a bounded operator ;

(vi) $J_g : B^\alpha \to B^\beta$ is a bounded operator ;

(vii) $\sup_{z\in D}(1-|z|^2)^{\beta-\alpha+1}|g'(z)| < +\infty$.

In the following theorem, the equivalence of (ii) and (v) was proved in [OSZ].

Theorem 2.5. Let $\alpha > 1$ and $\alpha = \beta$. For g analytic on D, the following are equivalent:

(i) $gBMOA^\alpha \subset B^\beta$;

(ii) $gB^\alpha \subset B^\beta$;

(iii) $I_g : BMOA^\alpha \to B^\beta$ is a bounded operator ;

(iv) $I_g : B^\alpha \to B^\beta$ is a bounded operator ;

(v) $g \in H^\infty$.

§3. Multipliers from $BMOA^\alpha$ to $BMOA^\beta$

In this section we study the operators I_g and J_g from $BMOA^\alpha$ to $BMOA^\beta$, and the operators I_g and J_g from $BMOA_\alpha$ to $BMOA_\alpha$.

Theorem 3.1. Let $\alpha \le \beta$. For g analytic on D, the operator $I_g : BMOA^\alpha \to BMOA^\beta$ is a bounded operator if and only if

$$\sup_{z\in D}(1-|z|^2)^{\beta-\alpha}|g(z)| < \infty.$$

Proof. If $\sup_{z\in D}(1-|z|^2)^{\beta-\alpha}|g(z)| < \infty$, it is trivial that $I_g : BMOA^\alpha \to BMOA^\beta$ is bounded. So we only need to prove the converse. Firstly, we prove the case $\alpha = 1$. Let $f_a(z) := \log\frac{1}{1-\bar{a}z}$. Then it is clear that $\{f_a\}$ is a bounded set in

$BMOA^1 = BMOA$. Since $1-|z|^2$ is comparable to $1-|a|^2$ and $|1-\overline{a}z|$ for $z \in D(a,r)$, we have

$$\begin{aligned}
&(1-|a|^2)^{2(\beta-1)}|a|^2|g(a)|^2 \\
\leq\ & C\frac{|a|^2}{(1-|a|^2)^2}\int_{D(a,r)}(1-|z|^2)^{2(\beta-1)}|g(z)|^2dA(z) \\
\sim\ & C\int_{D(a,r)}(1-|z|^2)^{2(\beta-1)}\left|\left(\log\frac{1}{1-\overline{a}z}\right)'\right|^2|g(z)|^2dA(z) \\
\sim\ & C\int_{D(a,r)}(1-|z|^2)^{2(\beta-1)}\left|\left(\log\frac{1}{1-\overline{a}z}\right)'\right|^2|g(z)|^2\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}dA(z) \\
=\ & C\int_{D(a,r)}(1-|z|^2)^{2(\beta-1)}\left|f_a'(z)\right|^2|g(z)|^2(1-|\varphi_a(z)|^2)dA(z) \\
\leq\ & C\int_{D}(1-|z|^2)^{2(\beta-1)}\left|f_a'(z)\right|^2|g(z)|^2(1-|\varphi_a(z)|^2)dA(z) \\
\leq\ & C\int_{D}(1-|z|^2)^{2(\beta-1)}\left|(I_gf_a)'(z)\right|^2(1-|\varphi_a(z)|^2)dA(z) \\
=\ & C\parallel I_gf_a\parallel^2_{BMOA^\beta}\leq C\parallel I_g\parallel^2\parallel f_a\parallel^2_{BMOA}<+\infty.
\end{aligned}$$

In the case $\alpha \neq 1$, by puting $f_a(z) := (1-\overline{a}z)^{1-\alpha}$, we can prove that as well. So we omit it. □

Theorem 3.2. Let $\alpha \leq \beta$ and $0<\alpha<1$. For g analytic on D, $J_g : BMOA^\alpha \to BMOA^\beta$ is a bounded operator if and only if

$$g \in BMOA^\beta.$$

Proof. Suppose that

$$g \in BMOA^\beta.$$

If $h \in BMOA^\alpha$, then

$$\begin{aligned}
(1-|a|^2)^{2\alpha}|h'(a)|^2 &\leq C(1-|a|^2)^{2\alpha}\frac{1}{(1-|a|^2)^2}\int_{D(a,r)}|h'(z)|^2dA(z) \\
&\leq C(1-|a|^2)^{2(\alpha-1)}\int_{D(a,r)}|h'(z)|^2(1-|\varphi_a(z)|^2)dA(z) \\
&\leq C\int_{D}(1-|z|^2)^{2(\alpha-1)}|h'(z)|^2(1-|\varphi_a(z)|^2)dA(z)
\end{aligned}$$

So we have

$$\parallel h\parallel^2_\infty\leq C\left(|h(0)|^2+\sup_{a\in D}(1-|a|^2)^{2\alpha}|h'(a)|^2\right)\leq C\left(|h(0)|^2+\parallel h\parallel^2_{BMOA^\alpha}\right).$$

Thus we see $BMOA^{\alpha} \subset H^{\infty}$. Hence we have

$$\begin{aligned} \| J_g f \|^2_{BMOA^{\beta}} &= \int_D (1-|z|^2)^{2(\beta-1)} |g'(z) f(z)|^2 (1-|\varphi_a(z)|^2) dA(z) \\ &\leq \| f \|^2_{\infty} \| g \|^2_{BMOA^{\beta}} \\ &\leq C \| f \|^2_{BMOA^{\alpha}} \| g \|^2_{BMOA^{\beta}} \end{aligned}$$

Hence we have that $J_g : BMOA^{\alpha} \to BMOA^{\beta}$ is a bounded operator.

It is trivial that the converse holds. In fact, since a non-zero constant c belongs to $BMOA^{\alpha}$, we have $J_g c \in BMOA^{\beta}$. Hence $g \in BMOA^{\beta}$. □

By using Theorem 3.1 and Theorem 3.2, we have the following corollary :

Corollary 3.3. Let $\alpha \leq \beta$ and $0 < \alpha < 1$. For g analytic on D, the following are equivalent:

(i) $gBMOA^{\alpha} \subset BMOA^{\beta}$;

(ii) $J_g : BMOA^{\alpha} \to BMOA^{\beta}$ is a bounded operator ;

(iii) $g \in BMOA^{\beta}$.

Proof. We only prove the case $\beta > \alpha$, because we can prove the case $\beta = \alpha$ as well. Then, since $\sup_{z\in D}(1-|z|^2)^{\beta-\alpha}|g(z)| \sim \sup_{z\in D}(1-|z|^2)^{\beta-\alpha+1}|g'(z)|$ and

$$\sup_{z\in D}(1-|z|^2)^{\beta-\alpha+1}|g'(z)| \leq \sup_{z\in D}(1-|z|^2)^{\beta}|g'(z)| \leq C \| g \|_{BMOA^{\beta}},$$

we see that the boundedness of J_g implies the boundedness of I_g because of Theorem 3.1 and Theorem 3.2. So it follows that (ii) implies (i). To prove that (i) implies (iii), suppose that $gBMOA^{\alpha} \subset BMOA^{\beta}$. Since a non-zero constant c belongs to $BMOA^{\alpha}$, we have $cg \in BMOA^{\beta}$. Thus we have $g \in BMOA^{\beta}$. □

Proposition 3.4. Let $1 < \alpha \leq \beta$. For g analytic on D, if $J_g : BMOA_{\alpha} \to BMOA_{\beta}$ is a bounded operator, then

$$g \in BMOA_{\beta-\alpha+1}.$$

And if $g \in BMOA^{\beta-\alpha+1}$, then $J_g : BMOA^{\alpha} \to BMOA^{\beta}$ is a bounded operator.

Proof. Let $1 < \alpha \leq \beta$. Suppose that $J_g : BMOA_{\alpha} \to BMOA_{\beta}$ is a bounded operator. For any arc $I \subset \partial D$, let $a = (1-|I|)\zeta$, where ζ is the center of I. Put

$f_a(z) = (1-\bar{a}z)^{1-\alpha}$, $(a \in D)$. Then $\{f_a\}$ is a bounded set in $BMOA$, and for any $z \in S(I)$, there is a constant $C>0$, such that $\frac{1}{C}|I|^{1-\alpha} \le |f_a(z)| \le C|I|^{1-\alpha}$. So we have

$$\begin{aligned}
&\frac{|I|^{2(\beta-\alpha)}}{|I|}\int_{S(I)}|g'(z)|^2(1-|z|^2)dA(z)\\
&\le \frac{1}{C}\frac{|I|^{2(\beta-1)}}{|I|}\int_{S(I)}|f_a(z)|^2|g'(z)|^2(1-|z|^2)dA(z)\\
&= \frac{1}{C}\frac{|I|^{2(\beta-1)}}{|I|}\int_{S(I)}|(J_gf_a)'(z)|^2(1-|z|^2)dA(z)\\
&\le \frac{1}{C}\parallel J_gf_a\parallel^2_{BMOA_\beta}\\
&\le \frac{1}{C}\parallel J_g\parallel^2\parallel f_a\parallel^2_{BMOA_\alpha}<\infty.
\end{aligned}$$

Thus we have $g \in BMOA_{\beta-\alpha+1}$.

Next, suppose that $g \in BMOA^{\beta-\alpha+1}$. Since $|f(z)| \le C(1-|z|^2)^{1-\alpha}\parallel f\parallel_{BMOA^\alpha}$ for all $f \in BMOA^\alpha$ for some constant $C>0$, we have

$$\begin{aligned}
\parallel J_gf\parallel^2_{BMOA^\beta} &= \sup_{a\in D}\int_D(1-|z|^2)^{2(\beta-1)}|f(z)|^2|g'(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&\le C^2\parallel f\parallel^2_{BMOA^\alpha}\int_D(1-|z|^2)^{2(\beta-\alpha)}|g'(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&= C^2\parallel f\parallel^2_{BMOA^\alpha}\parallel g\parallel^2_{BMOA^{\beta-\alpha+1}}.
\end{aligned}$$

Hence $J_g : BMOA^\alpha \to BMOA^\beta$ is bounded operator. In the case of $\alpha=1$, we can prove it by using a test function $f_a(z) = \log\frac{1}{1-\bar{a}z}$ for all $a \in D$ and the estimate $|f(z)| \le C\left(\log\frac{1}{1-|z|^2}\right)\parallel f\parallel_{BMOA}$ for all $f \in BMOA$ as well. So we omit it. □

Proposition 3.5. Let $1<\alpha\le\beta$. For g analytic on D, then $J_g : BMOA^\alpha \to BMOA^\beta$ is a bounded operator if and only if $g \in B^{\beta-\alpha+1}$.

Proof. Let $g \in B^{\beta-\alpha+1}$. Let $f \in BMOA^\alpha$. Then

$$\begin{aligned}
\parallel J_gf\parallel^2_{BMOA^\beta} &= \sup_{a\in D}\int_D(1-|z|^2)^{2(\beta-1)}|f(z)|^2|g'(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&\le \parallel g\parallel^2_{B^{\beta-\alpha+1}}\sup_{a\in D}\int_D(1-|z|^2)^{2(\alpha-2)}|f(z)|^2(1-|\varphi_a(z)|^2)dA(z).
\end{aligned}$$

Since $f \in BMOA^\alpha$ we get

$$\sup_{a\in D}\int_D(1-|z|^2)^{2(\alpha-1)}|f'(z)|^2(1-|\varphi_a(z)|^2)dA(z)<\infty.$$

Applying Proposition 2 in [Z2] to the antiderivative F of f we see that the above inequality is equivalent to

$$\sup_{a\in D}\int_D (1-|z|^2)^{2(\alpha-2)}|f(z)|^2(1-|\varphi_a(z)|^2)dA(z)<\infty.$$

Therefore $\| J_g f \|_{BMOA^{\beta}} < \infty$ and so $J_g f \in BMOA^{\beta}$. An application of the Closed Graph Theorem gives the boundedness of the operator J_g.

Conversely, let $J_g : BMOA^{\alpha} \to BMOA^{\beta}$ be bounded. It is easy to see that $\{f_a(z) = (1-\overline{a}z)^{1-\alpha}\}$ is a bounded set in $BMOA^{\alpha}$. So

$$\begin{aligned}
\infty &> \sup_{a\in D}\| J_g f_a \|^2_{BMOA^{\beta}}\\
&= \sup_{a\in D}\int_D (1-|z|^2)^{2(\beta-1)}|f_a(z)|^2|g'(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&\geq \sup_{a\in D}\int_{D(a,r)} \frac{(1-|z|^2)^{2(\beta-1)}}{|1-\overline{a}z|^{2(\alpha-1)}}|g'(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&\geq C\sup_{a\in D}(1-|a|^2)^{2(\beta-\alpha)}\int_{D(a,r)}|g'(z)|^2dA(z)\\
&\geq C\sup_{a\in D}(1-|a|^2)^{2(\beta-\alpha)}(1-|a|^2)^2|g'(a)|^2\\
&= C\| g \|^2_{B^{\beta-\alpha+1}}
\end{aligned}$$

Thus $g \in B^{\beta-\alpha+1}$. The proof is complete. □

Corollary 3.6. Let $1<\alpha<\beta$. For g analytic on D, if $g\in BMOA^{\beta-\alpha+1}$, then $gBMOA^{\alpha}\subset BMOA^{\beta}$.

Proof. Let $1<\alpha<\beta$. If $g\in BMOA^{\beta-\alpha+1}$, then $J_g : BMOA^{\alpha}\to BMOA^{\beta}$ is bounded operator. And if $g\in BMOA^{\beta-\alpha+1}$,

$$\sup_{z\in D}(1-|z|^2)^{\beta-\alpha+1}|g'(z)|<\infty.$$

Since $\sup_{z\in D}(1-|z|^2)^{\beta-\alpha+1}|g'(z)| \sim \sup_{z\in D}(1-|z|^2)^{\beta-\alpha}|g(z)|$, $I_g : BMOA^{\alpha}\to BMOA^{\beta}$ is bounded operator. Hence we have $gBMOA^{\alpha}\subset BMOA^{\beta}$. □

Together with Corollary 3.3, the following result gives a relative complete description of multipliers between $BMOA^{\alpha}$ and $BMOA^{\beta}$ (except for the case $\alpha=1$, $\beta\neq 1$).

Corollary 3.7. (i) Let $1 < \alpha < \beta$. Then g is a multiplier from $BMOA^\alpha$ into $BMOA^\beta$ if and only if $g \in B^{\beta-\alpha+1}$.
(ii) Let $\alpha > 1$. Then g is a multiplier from $BMOA^\alpha$ into itself if and only if $g \in H^\infty$.
(iii) Let $\alpha > \beta$ and g is a multiplier from $BMOA^\alpha$ into $BMOA^\beta$, then $g \equiv 0$.

Proof. Let $1 < \alpha < \beta$. Let $g \in B^{\beta-\alpha+1}$ as $\alpha < \beta$ and $g \in H^\infty$ as $\alpha = \beta$. Since $H^\infty \subset B$, by Theorem 3.1 and Theorem 3.2, both I_g and J_g are bounded operators from $BMOA^\alpha$ into $BMOA^\beta$. So g is a multiplier from $BMOA^\alpha$ into $BMOA^\beta$.

Conversely, suppose that g is a multiplier from $BMOA^\alpha$ into $BMOA^\beta$. As in the proof of Proposition 3.5, for any $a \in D$, $\{f_a(z) = (1-\overline{a}z)^{1-\alpha}\}$ is a bounded set in $BMOA^\alpha$. Thus $\{gf_a\}$ is a bounded set in $BMOA^\beta$. Thus by Proposition 2 of [Z2] we have

$$\begin{aligned}
\infty &> \sup_{a\in D}\int_D (1-|z|^2)^{2(\beta-2)}|f_a(z)|^2|g(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&\geq \sup_{a\in D}\int_{D(a,r)} \frac{(1-|z|^2)^{2(\beta-2)}}{|1-\overline{a}z|^{2(\alpha-1)}}|g(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&\geq C\sup_{a\in D}(1-|a|^2)^{2(\beta-\alpha-1)}\int_{D(a,r)}|g(z)|^2dA(z)\\
&\geq C\sup_{a\in D}(1-|a|^2)^{2(\beta-\alpha-1)}(1-|a|^2)^2|g(a)|^2\\
&= C\sup_{a\in D}(1-|a|^2)^{2(\beta-\alpha)}|g(a)|^2
\end{aligned}$$

which implies that $g \in B^{\beta-\alpha+1}$ as $\alpha < \beta$; $g \in H^\infty$ as $\alpha = \beta$ and $g \equiv 0$ as $\alpha > \beta$. The proof is complete. □

Theorem 3.8. *Let $\alpha > 0$, the operator $I_g : BMOA_\alpha \to BMOA_\alpha$ is bounded if and only if*

$$g \in H^\infty.$$

Proof. If $\sup_{z\in D}|g(z)| < \infty$, it is trivial that $I_g : BMOA_\alpha \to BMOA_\alpha$ is bounded. So we only need to prove the converse. Note that the quantity

$$\sup_{a\in D}(1-|a|^2)^{2\alpha-2}\int_D |f'(z)|^2(1-|\varphi_a(z)|^2)dA(z)$$

is comparable to the quantity

$$\| f \|^2_{BMOA_\alpha} := \sup_{I\subset\partial D}\frac{|I|^{2\alpha-2}}{|I|}\int_{S(I)}|f'(z)|^2(1-|z|^2)dA(z).$$

Suppose that I_g is bounded on $BMOA_\alpha$. In the case of $\alpha \neq 1$, consider the test function $h_a(z) := (1-\overline{a}z)^{1-\alpha}$ for all $a \in D$. Then it is clear that $\{h_a\}$ is a bounded set in $BMOA_\alpha$. For any $a \in D$,

$$\begin{aligned}
|a|^2|g(a)|^2 &\leq C\frac{|a|^2}{(1-|a|^2)^2}\int_{D(a,r)}|g(z)|^2dA(z)\\
&\sim C(1-|a|^2)^{2(\alpha-1)}\int_{D(a,r)}|h_a'(z)|^2|g(z)|^2dA(z)\\
&\sim C(1-|a|^2)^{2(\alpha-1)}\int_{D(a,r)}|h_a'(z)|^2|g(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&\leq C(1-|a|^2)^{2(\alpha-1)}\int_{D}|h_a'(z)|^2|g(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
&\leq C\parallel I_g h_a\parallel_{BMOA_\alpha}^2 \leq C\parallel I_g\parallel^2\parallel h_a\parallel_{BMOA_\alpha}^2 \leq 2^{2\alpha}C\parallel I_g\parallel^2 < \infty.
\end{aligned}$$

Hence we see $\sup_{z\in D}|g(z)| < \infty$.

In the case of $\alpha = 1$, we can prove it by using the test function $h_a(z) := \log\frac{1}{1-\overline{a}z}$ as well. So we omit to prove it. □

Theorem 3.9. *Let $\alpha > 1$, the operator $J_g : BMOA_\alpha \to BMOA_\alpha$ is bounded if and only if*

$$g \in BMOA.$$

Proof. Suppose that $g \in BMOA$. Then we have

$$\begin{aligned}
\parallel J_g f\parallel_{BMOA_\alpha}^2 &= \sup_{I\subset\partial D}\frac{|I|^{2\alpha-2}}{|I|}\int_{S(I)}|g'(z)|^2|f(z)|^2(1-|z|^2)dA(z)\\
&\leq \sup_{I\subset\partial D}\frac{2|I|^{2\alpha-2}}{|I|}\int_{S(I)}|g'(z)|^2|f(z)-f(u)|^2(1-|z|^2)dA(z)\\
&\quad+\sup_{I\subset\partial D}\frac{2|I|^{2\alpha-2}}{|I|}\int_{S(I)}|g'(z)|^2|f(u)|^2(1-|z|^2)dA(z)\\
&=: I_1 + I_2.
\end{aligned}$$

Since $g \in BMOA$, we see $g\circ\varphi_u \in BMOA$. Thus $d\mu_g = |(g\circ\varphi_u)'(z)|^2(1-|z|^2)dA(z)$ is a Carleson measure. So we have for $u = (1-|I|)\zeta$, ζ the center of I,

$$I_1 \leq C\parallel g\parallel_{BMOA}^2(1-|u|^2)^{2(\alpha-1)}\int_0^{2\pi}\left|f\circ\varphi_u(e^{i\theta})-f(u)\right|^2 d\theta.$$

$$\begin{aligned}
&(1-|u|^2)^{2(\alpha-1)}\int_0^{2\pi}\left|f\circ\varphi_u(e^{i\theta})-f(u)\right|^2 d\theta\\
&\leq \sup_{u\in D}(1-|u|^2)^{2(\alpha-1)}\parallel f\circ\varphi_u(z)-f(u)\parallel_{H^2}^2\\
&\sim \sup_{u\in D}(1-|u|^2)^{2(\alpha-1)}\left(\int_D|f'(z)|^2(1-|\varphi_u(z)|^2)dA(z)\right)\\
&\sim \parallel f\parallel_{BMOA_\alpha}^2.
\end{aligned}$$

Hence there exist some positive constant $K > 0$ such that

$$I_1 \leq K \parallel g \parallel_{BMOA}^2 \parallel f \parallel_{BMOA_\alpha}^2 .$$

On the other hand, since $BMOA_\alpha \subset B_\alpha$, we have that $\parallel f \parallel_{B_\alpha} \leq \parallel f \parallel_{BMOA_\alpha}$ and $|f(u)| \leq \dfrac{C}{(1-|u|)^{\alpha-1}} \parallel f \parallel_{B_\alpha}$. So we have
$|f(u)| \leq \dfrac{C}{(1-|u|)^{\alpha-1}} \parallel f \parallel_{BMOA_\alpha}$. Hence we see

$$I_2 \leq \sup_{I \subset \partial D} \frac{C|I|^{2\alpha-2}}{|I|} \frac{\parallel f \parallel_{BMOA_\alpha}^2}{(1-|u|)^{2(\alpha-1)}} \int_{S(I)} |g'(z)|^2 (1-|z|^2) dA(z).$$

Thus

$$I_2 \leq \parallel f \parallel_{BMOA_\alpha}^2 \parallel g \parallel_{BMOA}^2 .$$

Hence the operator J_g is bounded on $BMOA_\alpha$.

To prove the converse, suppose that for $a = (1-|I|)\zeta$, ζ the center of I. Then there exists a bounded set $\{f_a\}$ in $BMOA_\alpha$ such that $\dfrac{1}{|I|^{\alpha-1}} \leq C_1 |f_a(z)|$ for all $z \in S(I)$. Let $I \subset \partial D$ and $a = (1-|I|)\zeta$, ζ the center of I.Then we have

$$\begin{aligned}
&\frac{1}{|I|} \int_{S(I)} |g'(z)|^2 (1-|z|^2) dA(z) \\
&\leq \frac{C_1 |I|^{2(\alpha-1)}}{|I|} \int_{S(I)} |g'(z)|^2 |f_a(z)|^2 (1-|z|^2) dA(z) \\
&\leq \frac{C_1 |I|^{2(\alpha-1)}}{|I|} \int_{S(I)} \left|(J_g f_a(z))'\right|^2 (1-|z|^2) dA(z) \\
&\leq C_1 \parallel J_g f_a \parallel_{BMOA_\alpha}^2 \\
&\leq C_1 \parallel J_g \parallel^2 \parallel f_a \parallel_{BMOA_\alpha}^2 < +\infty.
\end{aligned}$$

Hence we have that $g \in BMOA$. □

By using Theorem 3.8 and Theorem 3.9, we have the following corollary :

Corollary 3.10. Let $\alpha > 1$. For g analytic on D, the following are equivalent:

(i) $gBMOA_\alpha \subset BMOA_\alpha$;

(ii) $I_g : BMOA_\alpha \to BMOA_\alpha$ is bounded operator ;

(iii) $g \in H^\infty$.

Proof. Since

$$\| g \|_{BMOA} \leq \sup_{z\in D} |g(z)|,$$

we see that the boundedness of I_g implies the boundedness of J_g because of Theorem 3.8 and Theorem 3.9. So it follows that (ii) implies (i). To prove that (i) implies (iii), suppose that $gBMOA_{\alpha} \subset BMOA_{\alpha}$. Put $k_a(z) := (1-\overline{a}z)^{1-\alpha} - (1-|a|^2)^{1-\alpha}$ $(a, z \in D)$. Since $gBMOA_{\alpha} \subset BMOA_{\alpha}$ and $k_a(a) = 0$,

$$\begin{aligned} |a||g(a)| &= (1-|a|^2)^{\alpha} \left|k_a(a)g'(a) + k_a'(a)g(a)\right| \\ &\leq \sup_{z\in D}(1-|z|^2)^{\alpha} \left|k_a(z)g'(z) + k_a'(z)g(z)\right| \\ &= \| k_a g \|_{B^{\alpha}} \leq \| k_a g \|_{BMOA_{\alpha}} < +\infty. \end{aligned}$$

Hence we see that $g \in H^{\infty}$. □

§4. Multipliers from D_{α} to D_{β}

As $\alpha > 1$, it is known that the space D_{α} is exactly the weighted Bergman space $L_a^{2,\alpha-2}$ which contains analytic functions f on D satisfying

$$\int_D |f(z)|^2(1-|z|^2)^{\alpha-2}dA(z) < \infty$$

(see, for example, [HKZ, p12, Proposition 1.11]). But pointwise multipliers between weighted Bergman spaces have been completely characterized recently by R.Zhao in [Z2, Theorem 1]. In this section we study the operators I_g and J_g from D_{α} to D_{β}.

Theorem 4.1. Let $\alpha \leq \beta$. For g analytic on D, the operator $I_g : D_{\alpha} \to D_{\beta}$ is bounded if and only if

$$\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)}|g(z)| < \infty.$$

Proof. Suppose that

$$\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)}|g(z)| < +\infty.$$

Let $f \in D_\alpha$. Then we have

$$\begin{aligned}\| I_g f \|_{D_\beta}^2 &= \int_D |g(z)f'(z)|^2(1-|z|^2)^\beta dA(z) \\ &= \int_D \left(|g(z)|(1-|z|^2)^{\frac{1}{2}\beta-\frac{1}{2}\alpha}\right)^2 |f(z)|^2(1-|z|^2)^\alpha dA(z) \\ &\le \left(\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}\beta-\frac{1}{2}\alpha}|g(z)|\right)^2 \int_D |f(z)|^2(1-|z|^2)^\alpha dA(z) \\ &\le \left(\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}\beta-\frac{1}{2}\alpha}|g(z)|\right)^2 \| f \|_{D_\alpha}^2\end{aligned}$$

To prove the converse, for $a \in D$, $m > 1-\frac{1}{2}\alpha$, let $g_a(z) = (1-|a|^2)^{m+\frac{1}{2}\alpha-1}\dfrac{z-a}{(1-\overline{a}z)^{m+\alpha}}$. Then $g_a'(a) = (1-|a|^2)^{-\frac{1}{2}\alpha-1}$. Since $g(a)g_a'(a) = (1-|a|^2)^{-\frac{1}{2}\alpha-1}g(a)$, we have

$$\begin{aligned}(1-|a|^2)^{-(\alpha+2)}|g(a)|^2 &= |g(a)g_a'(a)|^2 \\ &\le C\frac{1}{(1-|a|^2)^2}\int_{D(a,r)} |g(z)g_a'(z)|^2 dA(z) \\ &\le C\frac{1}{(1-|a|^2)^{2+\beta}}\int_{D(a,r)} |(I_g g_a)'(z)|^2(1-|z|^2)^\beta dA(z) \\ &\le C\frac{1}{(1-|a|^2)^{2+\beta}} \| I_g g_a \|_{D_\beta}^2 \le C\frac{1}{(1-|a|^2)^{2+\beta}} \| I_g \|^2 \| g_a \|_{D_\alpha}^2\end{aligned}$$

Hence

$$\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)}|g(z)| < \infty. \qquad \square$$

Lemma A.([AS2, Lemma 2]) Let $\alpha > 1$. *There is a constant* C_1 *such that*

$$\int_D |f(z)|^2(1-|z|^2)^{\alpha-2}dm(z) \le C_1\int_D |f'(z)|^2(1-|z|^2)^\alpha dm(z),$$

for all analytic functions f *on* D.

Proof. This follows from Lemma 2 of [AS2]. $\square$

Theorem 4.2. Let $1 < \alpha \le \beta$. For g analytic on D, the operator $J_g : D_\alpha \to D_\beta$ is a bounded operator if and only if

$$\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)+1}|g'(z)| < \infty.$$

Proof. Let $f \in D_\alpha$. Then we see by Lemma A,

$$\begin{aligned}
\| J_g f \|_{D_\beta}^2 &= \int_D |g'(z)f(z)|^2(1-|z|^2)^\beta dA(z) \\
&= \int_D \left(|g'(z)|(1-|z|^2)^{\frac{1}{2}\beta-\frac{1}{2}\alpha+1}\right)^2 |f(z)|^2(1-|z|^2)^{\alpha-2}dA(z) \\
&\le \left(\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}\beta-\frac{1}{2}\alpha+1}|g'(z)|\right)\int_D |f(z)|^2(1-|z|^2)^{\alpha-2}dA(z) \\
&\le \| g \|^2_{B^{\frac{1}{2}(\beta-\alpha)+1}} \int_D |f'(z)|^2(1-|z|^2)^\alpha dA(z) = \| g \|^2_{B^{\frac{1}{2}(\beta-\alpha)+1}} \| f \|^2_{D_\alpha}
\end{aligned}$$

To prove the converse, put $g_a(z) = (1-|a|^2)^{\frac{1}{2}\alpha}\dfrac{1}{(1-\bar{a}z)^\alpha}$. Then we see that $g_a(z) \in D_\alpha$. Since $(1-|z|^2)$ is comparable to $(1-|a|^2)$ for all $z \in D(a,r)$, we have

$$\begin{aligned}
|g'(a)|^2(1-|a|^2)^{\beta-\alpha+2} &\le C\int_{D(a,r)} |g'(z)|^2(1-|z|^2)^{\beta-\alpha}dA(z) \\
&= C\int_{D(a,r)} |g'(z)|^2\frac{(1-|a|^2)^\alpha}{|1-\bar{a}z|^{2\alpha}}\frac{|1-\bar{a}z|^{2\alpha}}{(1-|a|^2)^\alpha}(1-|z|^2)^{\beta-\alpha}dA(z) \\
&\sim C\int_{D(a,r)} |g'(z)|^2|g_a(z)|^2(1-|z|^2)^\beta dA(z) \\
&\le C\int_D |(J_g g_a)'(z)|^2(1-|z|^2)^\beta dA(z) = C \| J_g \|^2 \| g_a \|^2_{D_\alpha} < \infty
\end{aligned}$$

Hence we have

$$\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)+1}|g'(z)| < +\infty. \qquad \square$$

Corollary 4.3. Let $1 < \alpha < \beta$. For g analytic on D, the following are equivalent:

(*i*) $gD_\alpha \subset D_\beta$;

(*ii*) $I_g : D_\alpha \to D_\beta$ is a bounded operator ;

(*iii*) $J_g : D_\alpha \to D_\beta$ is a bounded operator ;

(*iv*) $\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)+1}|g'(z)| < +\infty$.

Proof. Since the equivalence of $\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)}|g(z)| < \infty$ and $\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)+1}|g'(z)| < \infty$ follows from the result of [Zhu3], the equivalence of (ii),(iii),(iv) follows from Theorem 4.1 and Theorem 4.2. It is trivial that (iv) implies (i). In fact, supposing (iv), by using the equivalence of (ii),(iii),(iv), for all $f \in D_\alpha$,

$$\begin{aligned}
&\int_D (1-|z|^2)^\beta|f'(z)g(z)+f(z)g'(z)|^2dA(z) \\
&\le \| J_g f \|^2_{D_\beta} + \| I_g f \|^2_{D_\beta} \le \left(\| J_g \|^2\| + \| I_g \|^2\right) \| f \|^2_{D_\beta} .
\end{aligned}$$

So we have $gD_\alpha \subset D_\beta$.

To prove the converse, suppose that $gD_\alpha \subset D_\beta$. Then for $a \in D$, $m > 1 - \frac{1}{2}\alpha$, let $g_a(z) = (1-|a|^2)^{m+\frac{1}{2}\alpha-1}\dfrac{z-a}{(1-\overline{a}z)^{m+\alpha}}$. Then $g_a'(a) = (1-|a|^2)^{-\frac{1}{2}\alpha-1}$ and $g_a(a) = 0$. Since $g(a)g_a'(a) = (1-|a|^2)^{-\frac{1}{2}\alpha-1}g(a)$ and $g'(a)g_a(a) = 0$, we have

$$\begin{aligned}(1-|a|^2)^{-(\alpha+2)}|g(a)|^2 &= |g(a)g_a'(a) + g'(a)g_a(a)|^2 \\ &\leq C\frac{1}{(1-|a|^2)^2}\int_{D(a,r)}|g(z)g_a'(z)+g'(z)g_a(z)|^2 dA(z) \\ &\leq C\frac{1}{(1-|a|^2)^{2+\beta}}\int_{D(a,r)}|(gg_a)'(z)|^2(1-|z|^2)^\beta dA(z) \\ &\leq C\frac{1}{(1-|a|^2)^{2+\beta}}\parallel M_g g_a \parallel_{D_\beta}^2 \leq C\frac{1}{(1-|a|^2)^{2+\beta}}\parallel M_g \parallel^2 \parallel g_a \parallel_{D_\alpha}^2\end{aligned}$$

Hence

$$\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}(\beta-\alpha)}|g(z)| < \infty. \qquad \square$$

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