Greedy Approximation for the Source Location Problem with Vertex-Connectivity Requirements in Undirected Graphs ⋆

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Abstract

Let $G = (V, E)$ be a simple undirected graph with a set V of vertices and a set E of edges. Each vertex $v \in V$ has a demand $d(v) \in Z_+$, and a cost $c(v) \in R_+$, where Z_+ and R_+ denote the set of nonnegative integers and the set of nonnegative reals, respectively. The source location problem with vertex-connectivity requirements in a given graph G asks to find a set S of vertices minimizing $\sum_{v \in S} c(v)$ such that there are at least $d(v)$ pairwise vertex-disjoint paths from S to v for each vertex $v \in V - S$. It is known that the problem is not approximable within a ratio of $O(\ln \sum_{v \in V} d(v))$, unless NP has an $O(N^{\log \log N})$-time deterministic algorithm. Also, it is known that even if every vertex has a uniform cost and $d^* = 4$ holds, then the problem is NP-hard, where $d^* = \max\{d(v) \mid v \in V\}$.

In this paper, we consider the problem in the case where every vertex has uniform cost. We propose a simple greedy algorithm for providing a $\max\{d^*, 2d^* - 6\}$-approximate solution to the problem in $O(\min\{d^*, \sqrt{|V|}\}d^*|V|^2)$ time, while we also show that there exists an instance for which it provides no better than a $(d^* - 1)$-approximate solution. Especially, in the case of $d^* \leq 4$, we give a tight analysis to show that it achieves an approximation ratio of 3. We also show the APX-hardness of the problem even restricted to $d^* \leq 4$.

Key words: graph algorithm, greedy algorithm, undirected graph, location problem, vertex-connectivity

1 Introduction

Problems of selecting the best location of facilities in a given network to satisfy a certain property are called *location problems* [12]. Recently, the location problems with requirements measured by a network-connectivity have been studied extensively [2,3,5,7,6,9–11,15–18].

Connectivity and/or flow-amount are very important factors in applications to control and design of multimedia networks. In a multimedia network, a set S of some specified network nodes, such as the so-called mirror servers, may have functions of offering the same services for users. A user at a node v can use the service by communicating with at least one node $s \in S$ through a path between s and v. The flow-amount (which is the capacity of paths between S and v) affects the maximum data amount that can be transmitted from S to a user at a node v. Also, the edge-connectivity or the vertex-connectivity between S and v measures the robustness of the service against network failures. The concept of such connectivity and/or flow-amount between a node and a set of specified nodes was given by H. Ito [8], considering design of a reliable telephone network with plural switching apparatuses.

Given a graph, the problem of finding the best location of such a set S of vertices, called *sources*, under connectivity and/or flow-amount requirements from each vertex to S is called the *source location problem*, which is formulated as follows:

Problem 1 (*Source location problem with meausre* ψ)
Input : A graph $G = (V, E)$ with a set V of vertices and a set E of edges with nonnegative real capacities, a cost function $c : V \to R_+$ (where R_+ denotes the set of nonnegative reals), and a demand function $d : V \to R_+$.
Output : A vertex set $S \subseteq V$ such that $\psi(S, v) \geq d(v)$ holds for every vertex $v \in V - S$ and $\sum_{v \in S} c(v)$ is minimized, where $\psi(S, v)$ is a measure based on the edge-connectivity, vertex-connectivity or flow-amount between S and a vertex v in the input graph G.

For such measures $\psi(S, v)$, one may consider the minimum capacity $\lambda(S, v)$ of an edge cut $C \subseteq E$ that separates v from S, the minimum size $\kappa(S, v)$ of a vertex cut $C \subseteq V - S - v$ that separates S and v, or the maximum number

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$\hat{\kappa}(S,v)$ of paths between S and v such that no pair of paths has a common vertex in $V - v$.

Here let us review the developments in the source location problems in undirected graphs. The problem with $\psi = \lambda$ was first considered by Tamura *et al.* [17]. They showed that the problem with uniform costs and uniform demands can be solved in polynomial time. Also, Tamura *et al.* [18] showed that the case of uniform costs and general demands is solvable in polynomial time, while the fastest known algorithm for it achieves complexity $O(mM(n,m))$ due to Arata et al. [2], where $n = |V|$, $m = |\{\{u,v\} \mid u,v \in V\}|$, and $M(n,m)$ denotes the time for max-flow computation in the graph with n vertices and m edges. In general, Sakashita *et al.* [16] showed that the problem is strongly NP-hard. It is also known that when a given graph is a tree, the problem is weakly NP-hard [2] and there exists a pseudo-polynomial time algorithm for it [11,16].

For $\psi = \kappa$, Ito *et al.* [9] investigated the problem with uniform costs and uniform demands $d(v) = k$, presented a polynomial time algorithm in the case of $k \leq 2$, and showed the NP-hardness of the problem in the case of $k \geq 3$. They also showed that in the case of $k \leq 2$, even if a measure $\lambda(S,v) \geq \ell$ is added, then the problem is still solvable in polynomial time.

For $\psi = \hat{\kappa}$, Nagamochi *et al.* [15] showed that the problem with uniform demands $d(v) = k$ can be solved in $O(\min\{k, \sqrt{n}\}kn^2)$ time. In [7], Ishii et al. considered the problem with uniform costs and general demands, and showed that it can be solved in linear time in the case of $d^* \leq 3$, while it is NP-hard even restricted to $d^* = 4$, where $d^* = \max\{d(v)|v \in V\}$. They also showed that if $d^* \leq 3$, then even in the case of general costs, it is also solvable in polynomial time[6].

Also for directed graphs, many variants of problems have been investigated (see [3,5,10] for $\psi = \lambda$, [15] for $\psi = \hat{\kappa}$, and [14] for a survey).

Recently, Sakashita et al.[16] showed that no problems of the above three types of connectivity requirements in undirected/directed graphs are approximable within the ratio of $O(\ln \sum_{v \in V} d(v))$, unless NP has an $O(N^{\log \log N})$-time deterministic algorithm. They also gave $(1 + \ln \sum_{v \in V} d(v))$-approximation algorithms for all such problems if the capacity and demand functions are integral.

In this paper, we focus on the problem with $\psi = \hat{\kappa}$ in undirected graphs. As shown in [16], in general, it is unlikely that it is approximable within a ratio of $O(\ln \sum_{v \in V} d(v))$. Moreover, it was shown in [7] that even if the cost function is uniform and d^* is bounded from above by a constant, the problem is NP-hard. In this paper, after describing some definitions and preliminaries in Section 2, we show in Section 3 that if the cost function is uniform, then a simple greedy algorithm provides a $\max\{d^*, 2d^* - 6\}$-approximate solution in

$O(\min\{d^*, \sqrt{n}\}d^*n^2)$ time; the approximation ratio is constant if d^* is bounded from above by a constant. Especially, in Section 4, in the case of $d^* \leq 4$, we give a tight analysis to show that it achieves an approximation ratio of 3. We also show that the problem is APX-hard even restricted to uniform costs and $d^* \leq 4$.

Before closing this section, we summarize our method. First, we start with the source set $S = V$. Then, we pick vertices v, one by one, in nondecreasing order of their demands; only when $S - \{v\}$ remains feasible, then update $S := S - \{v\}$. It was shown in [2] that for the problem with $\psi = \lambda$ and uniform costs in undirected graphs, this algorithm provides an optimal solution. In our problem, this method may not achieve an optimal, but an approximation ratio of $\max\{d^*, 2d^* - 6\}$.

2 Main Theorems

Let $G = (V, E)$ be a simple undirected graph with a set V of *vertices* and a set E of *edges*, where we denote $|V|$ by n and $|E|$ by m. A singleton set $\{x\}$ may be simply written as x, and "$\subset$" implies proper inclusion while "$\subseteq$" means "$\subset$" or "$=$". The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex subset $V' \subseteq V$, $G[V']$ means the subgraph induced by V'. For a vertex set $X \subseteq V$, $N_G(X)$ is defined as the set of all vertices in $V - X$ which are adjacent to some of vertices in X. Moreover, let $N_G(\emptyset) = \emptyset$. For a vertex set $Y \subseteq V$ and a family $\mathcal{X}$ of vertex sets, Y *covers* $\mathcal{X}$ if each $X \in \mathcal{X}$ satisfies $X \cap Y \neq \emptyset$. For a family $\mathcal{X}$ of vertex sets in V, the *frequency* of a vertex v (with respect to $\mathcal{X}$) is defined as the number of sets of $\mathcal{X}$ which includes v, and let $f(V, \mathcal{X})$ denote the maximum frequency with respect to $\mathcal{X}$ of a vertex in V.

For a vertex $v \in V$ and a vertex set $X \subseteq V - \{v\}$ in G, we denote by $\hat{\kappa}_G(X, v)$ the maximum number of paths from v to X such that no pair of paths has a common vertex in $V - v$. For a vertex $v \in V$ and a vertex set $X \subseteq V$ with $v \in X$, let $\hat{\kappa}_G(X, v) = \infty$. By Menger's theorem, the following lemma holds.

Lemma 2 *For a vertex $v \in V$ and a vertex set $X \subseteq V - \{v\}$, $\hat{\kappa}_G(X, v) \geq k$ holds if and only if $|N_G(W)| \geq k$ holds for every vertex set $W \subseteq V - X$ with $v \in W$.*

In this paper, each vertex $v \in V$ in $G = (V, E)$ has a nonnegative integer *demand* $d(v)$. Let $d^* = \max\{d(v) \mid v \in V\}$. A vertex set $S \subseteq V$ is called a *source set* if it satisfies

$$\hat{\kappa}_G(S, v) \geq d(v) \text{ for all vertices } v \in V - S, \tag{1}$$

and we call each vertex $v \in S$ a *source*. In this paper, we consider the following source location problem with local vertex-connectivity requirements in an undirected graph (shortly, LVSLP or d^*LVSLP).

Problem 3 (*LVSLP or d^*LVSLP*)
Input : *An undirected graph $G = (V, E)$ and a demand function $d : V \to Z_+$ (where Z_+ denotes the set of nonnegative integers).*
Output : *A source set $S \subseteq V$ with the minimum cardinality.*

The main results of this paper are described as follows.

Theorem 4 *Given an undirected graph $G = (V,\ E)$ and a demand function $d : V \to Z^+$, LVSLP is $\max\{d^*, 2d^*-6\}$-approximable in $O(\min\{d^*, \sqrt{n}\}d^*n^2)$ time.*

Theorem 5 *A 3-approximate solution to 4LVSLP can be found in $O(n^2)$ time, while 4LVSLP is APX-hard.*

In the subsequent sections, we will prove these theorems constructively by giving an approximation algorithm for LVSLP. Also, we will show that there exists an instance for which the proposed algorithm provides no better than a $(d^* - 1)$-approximate solution.

In the rest of this section, we introduce several properties for LVSLP, which will be used in the subsequent sections. For a vertex set $X \subseteq V$, $d(X)$ denotes the maximum demand among all vertices in X, i.e., $d(X) = \max_{v \in X} d(v)$ (note that we define $\max_{\emptyset} = 0$). A vertex subset $W \subseteq V$ with $d(W) > |N_G(W)|$ is called a *deficient set*. We have the following property by Lemma 2.

Lemma 6 *A vertex set $S \subseteq V$ is a source set if and only if S satisfies $W \cap S \neq \emptyset$ for every deficient set W.*

A deficient set W is *minimal* if no proper subset of W is deficient. For a vertex $v \in V$, we say that a deficient set $W \subseteq V$ with $v \in W$ is a *minimal deficient set with respect to* v, if W is minimal deficient and $d(v) > |N_G(W)|$. A minimal deficient set has the following properties.

Lemma 7 [7] *Every minimal deficient set W with respect to $v \in W$ induces a connected graph.*

Lemma 8 *Let W be a minimal deficient set with respect to $v \in W$. If there is a set X with $v \notin X$, $|N_G(X) \cap W| = 1$, and $X \cap N_G(W) \neq \emptyset$, then $N_G(X) \cap W = \{v\}$.*

PROOF. Assume by contradiction that $v \in (W - X) - N_G(X)$. Now we have

$N_G((W-X)-N_G(X)) \subseteq (N_G(W)-X) \cup (W \cap N_G(X))$. Hence, it follows from $|N_G(W) \cap X| \geq 1$ and $|W \cap N_G(X)| = 1$ that $|N_G((W-X)-N_G(X))| \leq |N_G(W)-X| + |W \cap N_G(X)| \leq |N_G(W)| - |N_G(W) \cap X| + 1 \leq |N_G(W)| < d(v)$ and $(W-X)-N_G(X)$ is also a deficient set, contradicting the minimality of W. $\square$

For two vertex sets X and Y, we say that X and Y *intersect* each other, if none of $X \cap Y$, $X-Y$, and $Y-X$ is empty. For two vertex sets X and Y, the following holds.

$$|N_G(X)| + |N_G(Y)| \geq |N_G(X \cap Y)| + |N_G(X \cup Y)|. \quad (2)$$
$$|N_G(X)| + |N_G(Y)| \geq |N_G((X-Y)-N_G(Y))| + |N_G((Y-X)-N_G(X))|. \quad (3)$$

Lemma 9 *Let W_i, $i=1,2$ be minimal deficient sets with respect to $w_i \in W_i$. If W_1 and W_2 intersect each other, $w_1 \in W_1 - W_2$, and $w_2 \in W_2 - W_1$, then $w_1 \in N_G(W_2)$ or $w_2 \in N_G(W_1)$ hold.*

PROOF. Assume by contradiction that $\{w_1, w_2\} \cap (N_G(W_1) \cup N_G(W_2)) = \emptyset$. By $w_1 \in (W_1 - W_2) - N_G(W_2)$ and $w_2 \in (W_2 - W_1) - N_G(W_1)$, we have $(W_1 - W_2) - N_G(W_2) \neq \emptyset \neq (W_2 - W_1) - N_G(W_2)$. Now we have $|N_G(W_1)| < d(w_1)$ and $|N_G(W_2)| < d(w_2)$, since W_1 and W_2 are both deficient sets. It follows from (3) that we have $d(w_1) > |N_G((W_1 - W_2) - N_G(W_2))|$ or $d(w_2) > |N_G((W_2 - W_1) - N_G(W_1))|$ (say, $d(w_1) > |N_G((W_1 - W_2) - N_G(W_2))|$). Then $(W_1 - W_2) - N_G(W_2)$ is also deficient, which contradicts the minimality of W_1. Hence, it follows that $\{w_1, w_2\} \cap (N_G(W_1) \cup N_G(W_2)) \neq \emptyset$. $\square$

3 Greedy Algorithm

For a given graph $G=(V,E)$ and a demand function $d: V \to Z_+$, let $opt(G,d)$ denote the optimal value to LVSLP. In this section, we give a simple greedy algorithm, named GREEDY_LVSLP, for finding a $\max\{d^*, 2d^*-6\}$-approximate solution S to LVSLP in $O(\min\{d^*, \sqrt{n}\} d^* \, n^2)$ time. Below, assume that the given graph G is connected, since if G is disconnected, then we can consider the problem for each connected component separately.

The algorithm GREEDY_LVSLP is a greedy method to find a minimal feasible solution S_0. We start with the source set $S_0 = V$, and pick vertices $v \in V$, one by one, in nondecreasing order of their demands. Only when $S_0 - \{v\}$ remains to be a source set, we update $S_0 := S_0 - \{v\}$.

Algorithm 1 Algorithm GREEDY_LVSLP

Require: An undirected connected graph $G = (V, E)$ and a demand function $d : V \to Z_+$.

Ensure: A source set S such that $|S| \le \max\{d^*, 2d^* - 6\} opt(G, d)$.

1: Order vertices of V such that $d(v_1) \le \cdots \le d(v_n)$.
2: Initialize $j := 1$ and $S_0 := V$.
3: **for** $j = 1$ to n **do**
4:     **if** $S_0 - \{v_j\}$ is a source set **then**
5:         $S_0 \leftarrow S_0 - \{v_j\}$.
6:     **end if**
7: **end for**
8: Output S_0 as a solution.

Let $S_0 = \{s_1, s_2, \ldots, s_p\}$ be a source set obtained by the algorithm. Here we observe the following property, which will be used for proving the approximation results.

Lemma 10 *For each $s \in S_0$, there is a deficient set W satisfying the following conditions (i)–(iii):*

(i) $W \cap S_0 = \{s\}$.
(ii) *W is minimal with respect to s.*
(iii) $d(W) = d(s)$.

PROOF. From the construction, when the vertex s is picked in lines 4–6, $S'_0 - \{s\}$ does not satisfy (1) for the current source set S'_0. Before deleting s from S'_0, S'_0 is feasible and hence by Lemma 6, every deficient set contains a source in S'_0. On the other hand, $S'_0 - \{s\}$ is infeasible. Again by Lemma 6, there is a deficient set W with $W \cap S'_0 = \{s\}$ such that $W - \{s\}$ is not deficient. Moreover, since all vertices in $W - \{s\}$ have been already deleted, we can observe that $d(s) = \max\{d(v) \mid v \in W\} = d(W)$ holds by the sorting in line 1, and that $d(s) > |N_G(W)|$. It follows that there is a minimal deficient set W with respect to s satisfying $W \cap S'_0 = \{s\}$ and $d(W) = d(s)$. Moreover, by $S_0 \subseteq S'_0$, we have $W \cap S_0 = \{s\}$. $\square$

By this lemma and observations in its proof, we can see that for s and S'_0 defined in the proof, if there is no deficient set W with respect to s such that $W \cap S'_0 = \{s\}$ (i.e., $\hat{\kappa}_G(S'_0 - \{s\}, s) \ge d(s)$), then $S'_0 - \{s\}$ is a source set. Hence, we also have the following lemma.

Lemma 11 *In lines 4–6 of the algorithm GREEDY_LVSLP, a vertex set $S_0 - \{v_j\}$ is a source set if and only if $\hat{\kappa}_G(S_0 - \{v_j\}, v_j) \ge d(v_j)$.*

Let $\mathcal{W}_0 = \{W_1, W_2, \ldots, W_p\}$ be a family of deficient sets such that W_i satifies (i)–(iii) of Lemma 10 for $s_i \in S_0$. Here we observe that S_0 is $\max\{d^*, 2d^*-6\}$-approximate.

Lemma 12 $|S_0| \leq \max\{d^*, 2d^* - 6\}opt(G, d)$.

PROOF. If $d^* = 1$, then $|S_0| = 1$ clearly holds; S_0 is optimal. Consider the case where $d^* \geq 2$. Let S be an arbitrary source set. From the definition of $f(V, \mathcal{W}_0)$, we can observe that S can cover at most $|S|f(V, \mathcal{W}_0)$ sets in $\mathcal{W}_0$. On the other hand, Lemma 6 indicates that we have $S \cap W \neq \emptyset$ for every $W \in \mathcal{W}_0$. Therefore, $|S|f(V, \mathcal{W}_0) \geq |\mathcal{W}_0|$ must hold. It follows that we have $opt(G, d) \geq |\mathcal{W}_0|/f(V, \mathcal{W}_0) = |S_0|/f(V, \mathcal{W}_0)$. We will prove this lemma by showing that if $f(V, \mathcal{W}_0) \geq d^* + 1$, then $f(V, \mathcal{W}_0) \leq 2d^* - 5$ and that when$f(V, \mathcal{W}_0) = 2d^* - 5 \geq d^* + 1$, we have $opt(G, d) < 2d^* - 5$.

Assume that there is a family $\mathcal{W}' \subseteq \mathcal{W}_0$ of deficient sets with $|\mathcal{W}'| = \ell$, $\ell \geq d^* + 1$, and $\bigcap_{W \in \mathcal{W}'} W \neq \emptyset$. We first claim that for each $W \in \mathcal{W}'$, the number of sets $W_i \in \mathcal{W}'$ with $s_i \in N_G(W)$ is at most $d^* - 3$. From $|N_G(W)| \leq d^* - 1$, $\ell \geq d^* + 1$, and Lemma 10(i), there exists a set $W_j \in \mathcal{W}'$ with $s_j \notin N_G(W)$ (notice that $|N_G(W)| \leq d^* - 1$ holds for every $W \in \mathcal{W}_0$ since W is deficient and $d^* = \max\{d(v) \mid v \in V\}$). Again by Lemma 10(i), if this claim would not hold, then such W_j would satisfy $|W_j \cap N_G(W)| \leq 1$. Then $|W_j \cap N_G(W)| = 0$ would imply that $W_j - W$ and $W_j \cap W$ are disconnected, which contradicts Lemma 7, and $|W_j \cap N_G(W)| = 1$ would indicate that the vertex v with $W_j \cap N_G(W) = \{v\}$ satisfies $v = s_j$ by Lemmas 8 and 10(ii) (note that $W \cap N_G(W_j) \neq \emptyset$ holds by Lemma 7).

Consider the directed graph $H = (V_1, E_1)$ such that each vertex $v_i \in V_1$ corresponds to a set in $W_i \in \mathcal{W}'$, and that a directed edge (v_i, v_j) belongs to E_1 if and only if $s_j \in N_G(W_i)$. From the above claim, the outdegree of each vertex in V_1 is at most $d^* - 3$. On the other hand, $|E_1| \geq \ell(\ell - 1)/2$ holds, since Lemmas 9 and 10 imply that for every two sets $W_i, W_j \in \mathcal{W}'$, we have $s_i \in N_G(W_j)$ or $s_j \in N_G(W_i)$. It follows that $(d^* - 3)\ell \geq |E_1| \geq \ell(\ell - 1)/2$; $\ell \leq 2d^* - 5$.

Finally, we consider a special case of $\ell = 2d^* - 5$. Then, by the above inequality, the outdegree of each vertex in V_1 is exactly $d^* - 3$. Now notice that every $W_i \in \mathcal{W}'$ satisfies $|N_G(W_i)| \leq d^* - 1$ and $|N_G(W_i) \cap W_j| \geq 2$ for each $W_j \in \mathcal{W}' - \{W_i\}$ with $s_j \notin N_G(W_i)$ as observed above. It follows that we have $|N_G(W_i)| = d^* - 1$ and $N_G(W_i) \subseteq \bigcup_{W \in \mathcal{W}'} W$ for each $W_i \in \mathcal{W}'$; $N_G(\bigcup_{W \in \mathcal{W}'} W) = \emptyset$, $V = \bigcup_{W \in \mathcal{W}'} W$, $\mathcal{W}_0 = \mathcal{W}'$, and each $s_i \in S_0$ satisfies $d(s_i) = d^*$ (note that G is connected and that by $V = \bigcup_{W \in \mathcal{W}'} W$ and Lemma 10(i), any set in $\mathcal{W}_0 - \mathcal{W}'$ cannot exist). Observe that $opt(G, d) \geq 2$ since if $\{v\}$ would be an optimal solution for some $v \in V$, then $V - \{v\}$ would be a deficient set with respect to some $s \in S_0 - \{v\}$ and hence $\{v\}$ would be infeasible (note that

$|N_G(V - \{v\})| = 1 < d^*$ and $|S_0| \geq d^* + 1 > 1$). It follows that if $\ell = 2d^* - 5$, $|S_0| = \ell \leq (2d^* - 5)opt(G, d)/2$. $\square$

Finally, we show that the algorithm GREEDY_LVSLP can be implemented to run in $O(\min\{d^*, \sqrt{n}\}d^*n^2)$ time. From Lemma 11, we can observe that the procedure in lines 4–6 can be done in $O(\min\{d^*, \sqrt{n}\}m)$ time by using the network flow computation [4]. Since the procedure in lines 4–6 is executed at most n times, it follows that the total complexity is $O(\min\{d^*, \sqrt{n}\}mn)$.

Moreover, it was shown in [13] that for any graph H and any integer k, a sparse subgraph H_k of H with $O(d^*n)$ edges satisfying the following (i) and (ii) can be obtained in $O(|E(H)|)$ time. (i) The local vertex-connectivity less than k in H is preserved also in H_k. (ii) The local vertex-connectivity at least k in H is at least k also in H_k. Notice that since d^* is the maximum demand, what we need to concern is the connectivity less than d^*. Hence, by computing such a sparse subgraph G_{d^*} of G with $O(d^*n)$ edges and applying the algorithm GREEDY_LVSLP to this G_{d^*}, we can reduce the above complexity to $O(m + \min\{d^*, \sqrt{n}\}d^*n^2)$.

Summarizing the arguments given so far, Theorem 4 is now established.

4 The case of $d^* \leq 4$

In this section, we consider 4LVSLP. Let S_0 and $\mathcal{W}_0$ be a set of vertices obtained by algorithm GREEDY_LVSLP and the family of deficient sets corresponding to S_0, respectively, as defined in the previous section. Here we show that S_0 is 3-approximate and that this analysis is tight for the algorithm. We also show that 4LVSLP is APX-hard.

Assume that $d^* = 4$, since the case of $d^* \leq 3$ is solvable in polynomial time, as shown in [7]. Also assume that $|S_0| \geq 4$, since $|S_0| \leq 3$ implies that S_0 is 3-approximate. If the frequency of each vertex with respect to $\mathcal{W}_0$ is at most three, then S_0 is 3-approximate as observed in the proof of Lemma 12. However, there exists an instance which has a vertex with frequency four. We first start with characterizing such cases through the following preparatory lemmas.

Lemma 13 *Let W_i and W_j denote deficient sets in $\mathcal{W}_0$ with $W_i \cap W_j \neq \emptyset$.*

(i) $|N_G(W_i \cup W_j)| \geq 1$.
(ii) $W_i \cap N_G(W_j) \neq \emptyset \neq W_j \cap N_G(W_i)$ *holds;* $|N_G(W_i \cap W_j)| \geq 2$.
(iii) *If* $|N_G(W_i \cap W_j)| = 2$, *then no set* $W \in \mathcal{W}_0 - \{W_i, W_j\}$ *satisfies* $W \cap W_i \cap W_j \neq \emptyset$.

(iv) *If* $|N_G(W_i \cup W_j)| = 1$, *then at most one set* $W \in \mathcal{W}_0 - \{W_i, W_j\}$ *satisfies* $W \cap W_i \cap W_j \neq \emptyset$.

(v) *If* $|N_G(W_i \cup W_j)| = 2$, *then for every* $W \in \mathcal{W}_0 - \{W_i, W_j\}$ *with* $W_i \cap W_j \cap W \neq \emptyset$, *we have* $N_G(W_i \cup W_j) \cap W \cap S_0 \neq \emptyset$.

PROOF. (i) Lemma 10(i) implies that $S_0 - \{s_i, s_j\} \subseteq V - (W_i \cup W_j)$ (note that $S_0 - \{s_i, s_j\} \neq \emptyset$ by $|S_0| \geq 4$). Hence, $N_G(W_i \cup W_j) = \emptyset$ would contradict the connectedness of G.

(ii) This follows from Lemma 7.

(iii) By (ii), we have $|W_i \cap N_G(W_j)| = |W_j \cap N_G(W_i)| = 1$. It follows from Lemma 8 that $W_i \cap N_G(W_j) = \{s_i\}$ and $W_j \cap N_G(W_i) = \{s_j\}$; $N_G(W_i \cap W_j) = \{s_i, s_j\}$. Lemma 10(i) indicates that any set $W \in \mathcal{W}_0 - \{W_i, W_j\}$ satisfies $W \cap N_G(W_i \cap W_j) = \emptyset$ and $W - (W_i \cup W_j) \neq \emptyset$. Hence, we can observe that no set $W \in \mathcal{W}_0 - \{W_i, W_j\}$ satisfies $W \cap W_i \cap W_j \neq \emptyset$, since if such a set W would exist, then $G[W]$ would be disconnected, contradicting Lemma 7.

(iv) Let $\{v\} = N_G(W_i \cup W_j)$. Assume that there is a set $W_\ell \in \mathcal{W}_0 - \{W_i, W_j\}$ with $W_i \cap W_j \cap W_\ell \neq \emptyset$. Then by applying Lemma 8 as $X = W_i \cup W_j$ and $W = W_\ell$, we have $v = s_\ell$. Hence, from Lemmas 7 and 10, we can observe that for any set $W \in \mathcal{W}_0 - \{W_i, W_j, W_\ell\}$, we have $W \cap W_i \cap W_j = \emptyset$.

(v) Let $W_\ell \in \mathcal{W}_0 - \{W_i, W_j\}$ be a set with $W_i \cap W_j \cap W_\ell \neq \emptyset$. If $|N_G(W_i \cup W_j) \cap W_\ell| = 1$, then $N_G(W_i \cup W_j) \cap W_\ell = \{s_\ell\}$ holds by Lemma 8. Consider the case where $|N_G(W_i \cup W_j) \cap W_\ell| = 2$; $N_G(W_i \cup W_j) \subseteq W_\ell$. Assume by contradiction that $s_\ell \in W_\ell - (W_i \cup W_j \cup N_G(W_i \cup W_j))$ holds. Lemma 9 indicates that $\{s_i, s_j\} \subseteq N_G(W_\ell)$. From the connectedness of G, $|S_0| \geq 4$, and $N_G(W_i \cup W_j) \subseteq W_\ell$, we can observe that $N_G(W_\ell) - (W_i \cup W_j) \neq \emptyset$. It follows from$|N_G(W_\ell)| \leq d^* - 1 = 3$ that $|N_G(W_\ell)| = 3$ and $d(s_\ell) = 4$; $N_G(W_\ell) \cap (W_i \cup W_j) = \{s_i, s_j\}$ and $|N_G(W_\ell) - (W_i \cup W_j)| = 1$. Moreover, $N_G(W_\ell - (W_i \cup W_j \cup N_G(W_i \cup W_j))) \subseteq N_G(W_i \cup W_j) \cup (N_G(W_\ell) - (W_i \cup W_j))$ and hence $|N_G(W_\ell - (W_i \cup W_j \cup N_G(W_i \cup W_j)))| \leq |N_G(W_i \cup W_j)| + |N_G(W_\ell) - (W_i \cup W_j)| \leq 3$, contradicting the minimality of W_ℓ. $\square$

Lemma 14 $f(V, \mathcal{W}_0) \leq 4$ *holds. In particular, for a vertex* $v \in V$ *whose frequency is four, the four distinct sets* $W_i \in \mathcal{W}_0$, $i = 1, 2, 3, 4$ *with* $v \in W_i$ *satisfy the following* (4)*:*

For some two sets W_1, W_2, $d(s_1) = 4$ *and* $d(s_2) \geq 3$ *hold and any set in* $\mathcal{W}_0 - \{W_1, W_2, W_3, W_4\}$ *is disjoint with* $W_1 \cup W_2$. (4)

PROOF. Let W_1 and W_2 denote deficient sets in $\mathcal{W}_0$ with $W_1 \cap W_2 \neq \emptyset$.

We observe how many sets in $\mathcal{W}_0 - \{W_1, W_2\}$ can intersect with $W_1 \cap W_2$. From Lemma 13(i)(ii), we have $|N_G(W_1 \cup W_2)| \geq 1$ and $|N_G(W_1 \cap W_2)| \geq 2$. Moreover, Lemma 13(iii) says that if $|N_G(W_1 \cap W_2)| = 2$, then every set $W \in \mathcal{W}_0 - \{W_1, W_2\}$ is disjoint with $W_1 \cap W_2$.

Consider the case where $|N_G(W_1 \cap W_2)| \geq 3$. By (2) and $|N_G(W)| \leq d^* - 1 \leq 3$ for each $W \in \mathcal{W}_0$, we have $|N_G(W_1 \cup W_2)| \leq 3$. In particular, if $|N_G(W_1 \cup W_2)| = 3$ (resp. $|N_G(W_1 \cup W_2)| = 2$), then we have $|N_G(W_1)| = |N_G(W_2)| = |N_G(W_1 \cap W_2)| = 3$ (resp. $|N_G(W_1)| = 3$ and $|N_G(W_2)| \geq 2$ without loss of generality). There are the following three possible cases (I) $|N_G(W_1 \cup W_2)| = 1$, (II) $|N_G(W_1 \cup W_2)| = 2$, $|N_G(W_1)| = 3$, and $|N_G(W_2)| \geq 2$, and (III) $|N_G(W_1 \cup W_2)| = 3$ and $|N_G(W_1)| = |N_G(W_2)| = |N_G(W_1 \cap W_2)| = 3$.

(I) Lemma 13(iv) implies that the frequency of each vertex in $W_1 \cap W_2$ is at most three.

(II) Assume that there are two distinct sets $W_3, W_4 \in \mathcal{W}_0 - \{W_1, W_2\}$ such that $W_1 \cap W_2 \cap W_3 \cap W_4 \neq \emptyset$. Lemma 13(v) implies that $N_G(W_1 \cup W_2) = \{s_3, s_4\}$. Hence, any other set $W \in \mathcal{W}_0$ cannot intersect with $W_1 \cup W_2$ by $W \cap \{s_3, s_4\} = \emptyset$ and the connectedness of $G[W]$. Therefore, we can observe that the frequency of each vertex in $W_1 \cap W_2$ is at most four and that if $W_1 \cap W_2 \cap W_3 \cap W_4 \neq \emptyset$ holds, then (4) holds.

(III) Assume that there is a set $W_3 \in \mathcal{W}_0 - \{W_1, W_2\}$ with $W_1 \cap W_2 \cap W_3 \neq \emptyset$. We also assume that $|N_G(W_3 \cup W_1)| = |N_G(W_2 \cup W_3)| = 3$ and hence $|N_G(W_3)| = 3$, since otherwise we can apply the above arguments. Note that $d(s_1) = d(s_2) = d(s_3) = 4$. Then we have the following claim, which proves this lemma. $\square$

Claim 15 *Every set in $\mathcal{W}_0 - \{W_1, W_2, W_3\}$ is disjoint with $W_1 \cap W_2 \cap W_3$.*

PROOF. We have $|N_G(W_3) \cap (W_1 \cup W_2)| \geq 2$, since otherwise if $|N_G(W_3) \cap (W_1 \cup W_2)| = 1$ would hold, then Lemma 8 would indicate that $N_G(W_3) \cap (W_1 \cup W_2) = \{s_1\} = \{s_2\}$, a contradiction.

Now by $|N_G(W_1 \cap W_2)| = 3$, there are the following two possible cases: (III-1) $|N_G(W_1) \cap W_2| = |N_G(W_2) \cap W_1| = |N_G(W_1 \cap W_2) - (W_1 \cup W_2)| = 1$, (III-2) $|N_G(W_1) \cap W_2| = 2$ and $|N_G(W_2) \cap W_1| = 1$ without loss of generality. In both cases, we have $N_G(W_2) \cap W_1 = \{s_1\}$.

(III-1) By Lemma 8, we have $N_G(W_1) \cap W_2 = \{s_2\}$. Let $\{v_{12}\} = N_G(W_1 \cap W_2) - (W_1 \cup W_2)$. By the connectedness of $G[W_3]$ and $\{s_1, s_2\} \cap W_3 = \emptyset$, we have $v_{12} \in W_3$. If $v_{12} = s_3$, then $N_G(W_1 \cap W_2) = \{s_1, s_2, s_3\}$ holds and hence every set in $\mathcal{W}_0 - \{W_1, W_2, W_3\}$ is disjoint with $W_1 \cap W_2$. Consider the case where $v_{12} \neq s_3$. Note that from the connectedness of $G[W_1]$, $N_G(W_3) \cap ((W_1 \cap W_2) \cup \{s_1\}) \neq \emptyset$

holds. Then $(W_1 \cap W_2) \cup \{s_1\}$ has a neighbour in W_3 other than v_{12}, since if $N_G((W_1 \cap W_2) \cup \{s_1\}) \cap W_3 = \{v_{12}\}$ would hold, then by applying Lemma 8 as $X = (W_1 \cap W_2) \cup \{s_1\}$ and $W = W_3$, $v_{12} = s_3$ would hold. It follows that $s_1 \in N_G(W_3)$. Similarly, $s_2 \in N_G(W_3)$ holds. On the other hand, we have $N_G(W_2) - \{s_1\} \subseteq W_3$, since otherwise $|N_G(W_2) \cap W_3| = |\{v_{12}\}| = 1$ would hold and Lemma 8 would imply that $v_{12} = s_3$. It follows that $N_G(W_2) \subseteq W_3 \cup N_G(W_3)$ and $|N_G(W_3) - W_2| \leq 2$. Therefore, we have $|N_G(W_2 \cup W_3)| \leq 2$, contradicting the assumption.

(III-2) Let $\{v_1, v_2\} = N_G(W_1) \cap W_2$, $\{v_3\} = N_G(W_1) - \{v_1, v_2\}$, and $\{v_4, v_5\} = N_G(W_2) - \{s_1\}$. By the connectedness of $G[W_3]$, we have $\{v_1, v_2\} \cap W_3 \neq \emptyset$ (say, $v_1 \in W_3$) and $\{v_4, v_5\} \cap W_3 \neq \emptyset$ (say, $v_4 \in W_3$). Then we claim that $(W_1 \cap W_2) \cup \{s_1\}$ has a neighbour in W_3 other than v_1, since if $|N_G((W_1 \cap W_2) \cup \{s_1\}) \cap W_3| = 1$, then by applying Lemma 8 as $X = (W_1 \cap W_2) \cup \{s_1\}$ and $W = W_3$, $s_3 = v_1 \in W_2$ would hold, contradicting Lemma 10(i) (note that $N_G(W_3) \cap ((W_1 \cap W_2) \cup \{s_1\}) \neq \emptyset$ by the connectedness of $G[W_1]$).

Here we claim that $s_1 \in N_G(W_3)$. If $v_2 \notin W_3$, then $(N_G((W_1 \cap W_2) \cup \{s_1\}) \cap W_3) - \{v_1\}$ is included in $((W_1 - W_2) - \{s_1\}) \cup \{v_3\}$; $s_1 \in N_G(W_3)$. If $v_2 \in W_3$, then $|N_G(W_3) \cap W_1| \geq 2$ cannot hold, since otherwise $|N_G(W_1 \cup W_3)| \leq 2$ would hold. It follows from Lemma 8 that $v_2 \in W_3$ indicates $N_G(W_3) \cap W_1 = \{s_1\}$.

We next claim that $N_G(W_3) - (W_2 \cup \{v_5, s_1\}) \neq \emptyset$ and $N_G(W_3) \cap (W_2 \cup \{v_5\}) = \{s_2\}$. If $N_G(W_3) \subseteq W_2 \cup \{v_5, s_1\}$, then it follows that $N_G(W_3) \subseteq W_2 \cup N_G(W_2)$ and $|N_G(W_2 \cup W_3)| \leq 2$, a contradiction. Note that $|N_G(W_3) - \{s_1\}| \leq 2$ and $W_2 \cap N_G(W_3) \neq \emptyset$. Hence, we have $|W_2 \cap N_G(W_3)| = 1$, $v_5 \notin N_G(W_3)$, and $N_G(W_3) - (W_2 \cup \{v_5, s_1\}) \neq \emptyset$. Moreover, by applying Lemma 8 as $X = W_3$ and $W = W_2$, we have $N_G(W_3) \cap W_2 = \{s_2\}$.

On the other hand, $v_5 \notin W_3$, since otherwise we would have $N_G(W_2) \subseteq W_3 \cup N_G(W_3)$ and $|N_G(W_2 \cup W_3)| \leq 2$, a contradiction. By Lemma 8 and $W_3 \cap N_G(W_2) = \{v_4\}$, we have $v_4 = s_3$. It follows that $N_G(W_2 \cap W_3) = \{s_1, s_2, s_3\}$, and every set $W \in \mathcal{W}_0 - \{W_1, W_2, W_3\}$ is disjoint with $W_1 \cap W_2 \cap W_3$. $\square$

Lemma 16 *Let W_i, W_j be two minimal deficient sets with respect to v_i and v_j, respectively, such that $W_i \cap W_j \neq \emptyset$, $|N_G(W_i \cup W_j)| \leq 2$, $d(v_i) = 4$, and $\{v_i, v_j\} \cap (W_i \cap W_j) = \emptyset$. Then for any feasible solution S to 4LVSLP, we have $|S \cap (W_i \cup W_j)| \geq 2$.*

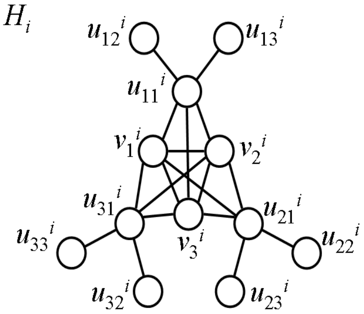
PROOF. By Lemma 6, $S \cap (W_i \cup W_j) \neq \emptyset$ holds; let $s \in S \cap (W_i \cup W_j)$. Now we have $|N_G(W_i \cup W_j - \{s\})| \leq |N_G(W_i \cup W_j)| + 1 \leq 3 < d(v_i)$. Hence, if $s \neq v_i$, then again by Lemma 6, we have $S \cap (W_i \cup W_j - \{s\}) \neq \emptyset$ and $|S \cap (W_i \cup W_j)| \geq 2$. If $s = v_i$, then $s = v_i \notin W_j$ holds and hence by Lemma 6 we have $(S - s) \cap W_j \neq \emptyset$. Also in this case, $|S \cap (W_i \cup W_j)| \geq 2$. $\square$

Lemma 17 *S_0 is 3-approximate.*

PROOF. Let S^* denote an optimal solution. Since S^* is feasible, we have $W \cap S^* \neq \emptyset$ for every $W \in \mathcal{W}_0$. Consider a mapping $g : \mathcal{W}_0 \to S^*$ such that for each set $W \in \mathcal{W}_0$, $g(W) = s^*$ holds for some source $s^* \in S^*$ with $s^* \in W$. If $|\{W \in \mathcal{W}_0 \mid g(W) = s^*\}| \leq 3$ holds for each source $s^* \in S^*$, then we have $|\mathcal{W}_0| \leq 3|S^*|$, from which $|S_0| = |\mathcal{W}_0| \leq 3|S^*|$. We claim that there is such a mapping.

Assume that for a mapping g, there is a source $s_1^* \in S^*$ which at least four sets in $\mathcal{W}_0$ is mapped to. By Lemma 14, $f(V, \mathcal{W}_0) \leq 4$ holds, and hence the number of sets in $\mathcal{W}_0$ mapped to s_1^* is exactly four. Moreover, the four sets W_1, W_2, W_3, W_4 in $\mathcal{W}_0$ with $g(W_i) = s_1^*$, $i = 1, 2, 3, 4$ satisfy (4); $|N_G(W_1 \cup W_2)| = 2$, $d(s_1) = 4$, and $W \cap (W_1 \cup W_2) = \emptyset$ for each $W \in \mathcal{W}_0 - \{W_1, W_2, W_3, W_4\}$ (notice that in this case, $|N_G(W_1 \cup W_2)| = 2$ holds by the proof of Lemma 14).

Now Lemma 16 implies that $W_1 \cup W_2$ includes a source $s_2^* \in S^* - \{s_1^*\}$. Notice that no set in $\mathcal{W}_0$ is mapped to s_2^* in g because every set $W \in \mathcal{W}_0 - \{W_1, W_2, W_3, W_4\}$ satisfies $s_2^* \notin W$ and each of W_i, $i = 1, 2, 3, 4$ has been mapped to s_1^*. So, we can decrease the number of sets in $\mathcal{W}_0$ mapped to s_1^* by one, by remapping one of two sets W_1 and W_2 including s_2^* to s_2^*. Consequently, by repeating this arguments, we can obtain a mapping with the required property. □



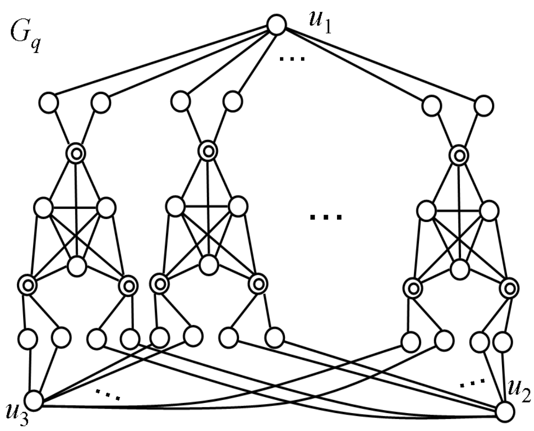


Fig. 1. Illustration of a tight example G for the algorithm GREEDY_LVSLP in 4LVSLP. (a) shows the graph H_i which is a subgraph of G in (b). In G, each vertex v with $d(v) = 4$ is drawn as double circles.

We now give a tight example for the algorithm GREEDY_LVSLP. Let $H_i = (V_i, E_i)$ be the graph where $V_i = \bigcup_{j=1}^{3}\{v_j^i, u_{j1}^i, u_{j2}^i, u_{j3}^i\}$ and $E_i = (\bigcup_{j,\ell}(v_j^i, v_\ell^i)) \cup (\bigcup_{j=1}^{3}\{(u_{j1}^i, u_{j2}^i), (u_{j1}^i,\ u_{j3}^i), (u_{j1}^i, v_1^i), (u_{j1}^i, v_2^i), (u_{j1}^i, v_3^i)\})$ (see Fig. 1(a)). Let $G_q = (V, E)$ be the graph where $V = \{u_1, u_2, u_3\} \cup (\bigcup_{i=1}^{q} V_i)$, $q \geq 4$ and $E = \bigcup_{i=1}^{q}(E_i \cup (\bigcup_{j=1}^{3}\{(u_j, u_{j2}^i), (u_j, u_{j3}^i)\}))$, $d(u_{j1}^i) = 4$ for each $i \in \{1, 2, \ldots, q\}$ and $j \in \{1, 2, 3\}$, and $d(v) = 0$ for all other vertices (see Fig. 1(b)). For G_q and d, the algorithm GREEDY_LVSLP returns a source set $S_0 = \bigcup_{i=1}^{q}\{u_{11}^i, u_{21}^i, u_{31}^i\}$ and $\mathcal{W}_0 = \bigcup_{i=1}^{q}\{\{u_{11}^i, u_{12}^i, u_{13}^i, v_1^i, v_2^i, v_3^i\}, \{u_{21}^i, u_{22}^i, u_{23}^i, v_1^i, v_2^i, v_3^i\}, \{u_{31}^i, u_{32}^i, u_{33}^i, v_1^i,\ v_2^i,\ v_3^i\}\}$. On the other hand, $\{v_1^1, v_1^2, \ldots, v_1^q\}$ is an optimal solution. This example shows that our analysis of the algorithm is tight. Here we remark that in a similar way, we can construct an instance in which GREEDY_LVSLP returns a solution S with $|S| = (d^* - 1)opt(G, d)$ for a general d. Namely, we have the following lemma.

Lemma 18 *For d^*LVSLP, there exists a graph for which the algorithm GREEDY_LVSLP provides no better than a $(d^* - 1)$-approximate solution. In particular, for 4LVSLP, such a graph is a tight example. A graph in Fig. 1* (b) *is one of such examples.*

Finally, we show that the problem is APX-hard. In [7], it was shown that 4LVSLP is NP-hard by a reduction from the minimum vertex cover problem restricted to 3-regular graphs:

Vertex-cover problem in a 3-regular graph (VC3R)

INSTANCE: $(G = (V, E), k)$: A 3-regular graph $G = (V, E)$ and an integer k.

QUESTION: Is there a vertex cover X with $|X| \leq k$ in G?

where a set $V' \subseteq V$ of vertices is called a *vertex cover* if every edge $e = (u, v) \in E$ satisfies $\{u, v\} \cap V' \neq \emptyset$, and a graph is called *k-regular* if the degree of every vertex is exactly k. As shown in [1], the minimum vertex cover problem is APX-hard, even restricted to 3-regular graphs. We can prove the APX-hardness of 4LVSLP by using the same reduction as [7].

Lemma 19 *4LVSLP is APX-hard.*

PROOF. We start with reviewing a reduction from the minimum vertex cover problem in a 3-regular graph to 4LVSLP, which was shown in [7].

Take an instance $I_{VC3R} = (G_1 = (V_1, E_1), k)$ of VC3R, where $n_1 = |V_1|$ and $m_1 = |E_1|$. Let $G_2 = (V_2, E_2)$ be the graph obtained from G_1 by replacing each edge $e = (v_i, v_j) \in E_1$ with three edges $(v_i, v_{i,j})$, $(v_{i,j}, v_{j,i})$, and $(v_{j,i}, v_j)$ introducing two new vertices $v_{i,j}$ and $v_{j,i}$; $V_2 = V_1 \cup V_{2,E}$ and

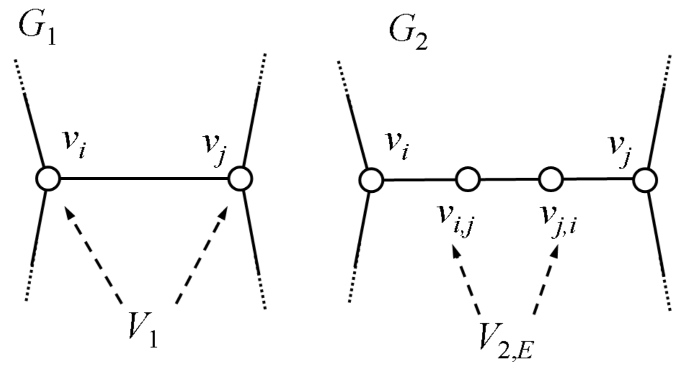


Fig. 2. Illustration of an edge (v_i, v_j) in G_1 and the corresponding edges in G_2 in the proof of Lemma 19.

$E_2 = \cup_{(v_i,v_j)\in E_1}\{(v_i, v_{i,j}), (v_{i,j}, v_{j,i}), (v_{j,i}, v_j)\}$, where $V_{2,E} = \bigcup_{(v_i,v_j)\in E_1}\{v_{i,j}, v_{j,i}\}$ (see Fig. 2). From G_2, we construct an instance $I_{LVSLP} = (G_3 = (V_3, E_3), d)$ of 4LVSLP as follows.

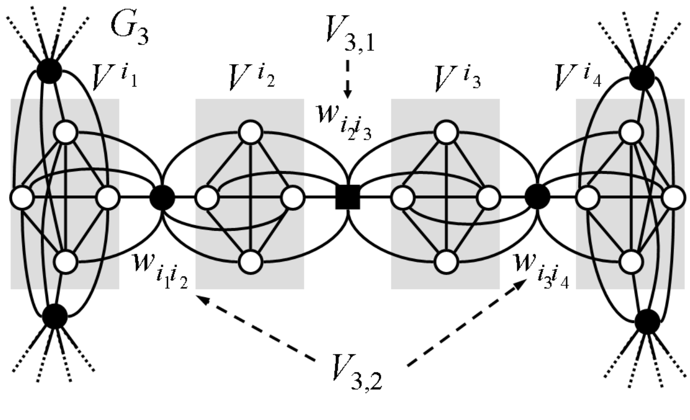


Fig. 3. Illustration of a subgraph of G_3 in the proof of Lemma 19 constructed from G_2, where $\{w_{i_2}, w_{i_3}\} \subseteq V_{2,E}$, $\{w_{i_1}, w_{i_4}\} \subseteq V_2 - V_{2,E}$, and $\{(w_{i_1}, w_{i_2}), (w_{i_2}, w_{i_3}), (w_{i_3}, w_{i_4})\} \subseteq E_2$. Each vertex in $V_{3,1}$ and $V_{3,2}$ is drawn as a black square and a black circle, respectively.

For each $w_i \in V_2$, we construct the complete graph (V^i, E^i) with $|V^i| = 4$. For each $e = (w_i, w_j) \in E_2$, we construct one vertex w_{ij}. Let $V_{3,1} = \{w_{ij} \mid (w_i, w_j) \in E_2,\ i < j,\ \{w_i, w_j\} \subseteq V_{2,E}\}$ and $V_{3,2} = \{w_{ij} \mid (w_i, w_j) \in E_2,\ w_i \in V_{2,E},\ w_j \in V_2 - V_{2,E}\}$. We construct G_3 from G_2 by replacing each vertex $w_i \in V_2$ by (V^i, E^i) and each edge $e = (w_j, w_\ell) \in E_2$ by the vertex $w_{j\ell}$, and adding edges connecting $w_{j\ell}$ and $V^j \cup V^\ell$ for each edge $e = (w_j, w_\ell) \in E_2$; let

$V_3 = (\cup_{w_i \in V_2} V^i) \cup V_{3,1} \cup V_{3,2}$ and $E_3 = (\cup_{w_i \in V_2} E^i) \cup$ $(\cup_{w_{ij} \in V_{3,1} \cup V_{3,2}} \{(w_{ij}, u) | u \in V^i \cup V^j\})$ (see Fig. 3). Let $d(x) = 3$ for each vertex $x \in V_{3,1}$ and $d(x) = 4$ for each vertex $x \in V_{3,2}$ and $d(x) = 0$ otherwise. Clearly, G_3 can be constructed in polynomial time in n_1 and m_1.

In [7], the following properties were shown.

Claim 20 (i) *Let X_1 be a vertex cover in G_1. Then, $X_2 = X_1 \cup \{v_{i,j} \in V_{2,E} \mid (v_i, v_j) \in E_1, v_i \notin X_1\} \cup \{v_{i,j} \in V_{2,E} \mid (v_i, v_j) \in E_1, i < j, \{v_i, v_j\} \subseteq X_1\}$ is a vertex cover in G_2. Moreover, the vertex set obtained by choosing exactly one vertex in V^i for each $w_i \in X_2$ is a source set in G_3.*

(ii) *Let S be a source set in G_3. Let S' be the vertex set obtained from S by replacing each $w_{ij} \in (V_{3,1} \cup V_{3,2}) \cap S$ with some $w' \in V^i \cup V^j$, and $X_2 = \{w_i \in V_2 \mid V^i \cap S' \neq \emptyset\}$ be the vertex set in G_2. Let X_2' be the vertex set obtained from X_2 by replacing each $v_{i,j}$ with $\{v_{i,j}, v_{j,i}\} \subseteq V_{2,E} \cap X_2$ with v_i. Then $X_2' \cap V_1$ is a vertex cover in G_1.*

By this claim, we observe that G_1 has a vertex cover with cardinality at most k if and only if G_3 has a source set with cardinality at most $k + m_1$; $opt(G_3, d) = opt_{VC}(G_1) + m_1$ holds, where $opt_{VC}(G)$ denotes the minimum size $|X|$ of a vertex cover X in G. Now since G_1 is 3-regular, we have $m_1 \leq 3opt_{VC}(G_1)$. It follows that $opt(G_3, d) = opt_{VC}(G_1) + m_1 \leq 4opt_{VC}(G_1)$.

Let S be an arbitrary source set in G_3, and X be a vertex cover in G_1 obtained from S according to Claim 20(ii). Note that $|X| \leq |S| - m_1$. Then, we have

$$\frac{|X| - opt_{VC}(G_1)}{opt_{VC}(G_1)} \leq 4\frac{(|S| - m_1) - (opt(G_3, d) - m_1)}{opt(G_3, d)} = 4\frac{|S| - opt(G_3, d)}{opt(G_3, d)}.$$

Therefore, if we would have a polynomial-time approximation scheme for 4LVSLP, then we would have a polynomial-time approximation scheme for VC3R. □

5 Concluding Remarks

In this paper, given an undirected graph $G = (V, E)$ and a demand function $d : V \to Z_+$, we have considered the problem of finding a set $S \subseteq V$ with the minimum cardinality such that for every vertex v, there exist $d(v)$ paths between every vertex $v \in V - S$ and S such that no pair of paths has a common vertex in $V - v$. We have shown that a simple greedy algorithm finds a $\max\{d^*, 2d^* - 6\}$-approximate solution to the problem in $O(\min\{d^*, \sqrt{n}\}d^* n^2)$ time. Especially, restricted to $d^* \leq 4$, we have given a tight analysis to show that it achieves an approximation ratio of 3, while the problem is APX-hard. However, it is still open whether the problem is approximable within a constant which is independent of d^*.

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