Minimum Augmentation of Edge-Connectivity between Vertices and Sets of Vertices in Undirected Graphs*

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Abstract

Given an undirected multigraph $G = (V, E)$, a family $\mathcal{W}$ of areas $W \subseteq V$, and a target connectivity $k \geq 1$, we consider the problem of augmenting G by the smallest number of new edges so that the resulting graph has at least k edge-disjoint paths between v and W for every pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$. So far this problem was shown to be NP-complete in the case of $k = 1$ and polynomially solvable in the case of $k = 2$. In this paper, we show that the problem for $k \geq 3$ can be solved in $O(m+n(k^3+n^2)(p+kn+n\log n)\log k+pkn^3\log(n/k))$ time, where $n = |V|$, $m = |\{\{u, v\}|(u, v) \in E\}|$, and $p = |\mathcal{W}|$.

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1 Introduction

In a communication network, graph connectivity is a fundamental measure of its robustness. An undirected graph $G = (V, E)$ is k-edge-connected if the deletion of any $k-1$ or fewer edges leaves a connected graph; equivalently, there exist at least k pairwise edge-disjoint paths between every two vertices. The connectivity augmentation problem asks to add to a given graph the smallest number of new edges such that the connectivity of the graph increases up to a specified value k. The problem has important applications such as the network design problem [8], the rigidity problem in grid frameworks [3], the data security problem [14], the rectangular dual graph problem in floor-planning [21], and the graph drawing problem [13], and many efficient algorithms have been developed so far.

Most of all those researches have dealt with connectivity between two vertices in a graph. However, in many real-wold networks, the connectivity between every two vertices is not necessarily required. For example, in a multimedia network, some vertices of the network may have functions of offering several types of services for users. For a set W of vertices offering certain service i, a user at a vertex v can use service i by communicating with one vertex $w \in W$ through a path between w and v. In such networks, it is desirable that the network has some pairwise disjoint paths from the vertex v to *at least one* of vertices in W. This means that the measure of reliability is the connectivity between a vertex and a set of vertices rather than that between two vertices. From this point of view, H. Ito et al. considered the node to area connectivity (NA-connectivity, for short) as a concept that represents the connectivity between vertices and sets of vertices (areas) in a graph [9, 10, 12]. As related problems, the problem of locating a set W of vertices offering service with requirements measured by connectivity has been also studied [1, 11, 12, 19].

In this paper, given a graph $G = (V, E)$ with a family $\mathcal{W}$ of areas $W \subseteq V$, and a positive integer k, we consider the problem of asking to augment G by adding the smallest number of new edges so that the resulting graph has at least k pairwise edge-disjoint paths between v and W for every pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$. We call this problem *k-NA-edge-connectivity augmentation problem* (for short, *k-NA-ECAP*). Figure 1 gives an instance of 3-NA-ECAP. In the graph G in (i), some pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$ (say, v_8 and W_3) cannot have three edge-disjoint paths between them, and 3-NA-ECAP asks to add the minimum number of new edges to G to construct a graph like (ii) in which there are at least three edge-disjoint paths between every pair of $v \in V$ and $W \in \mathcal{W}$. H. Miwa et al. [15] showed that 1-NA-ECAP is NP-hard by a reduction from SET SPLITTING.

Lemma 1.1 [15] *1-NA-ECAP is NP-hard.*

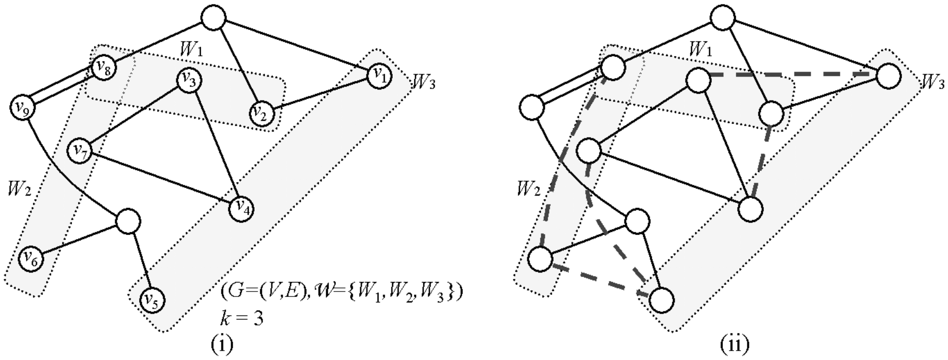

Figure 1: Illustration of an instance of 3-NA-ECAP. (i) An initial graph $G = (V, E)$ with a family $\mathcal{W} = \{W_1, W_2, W_3\}$ of areas. (ii) A 3-NA-edge-connected graph obtained from G by adding a set of edges drawn as broken lines; there are at least three edge-disjoint paths between every pair of a vertex $v \in V$ and an area $W \in \mathcal{W}$.

Proof sketch: SET SPLITTING is reduced to 1-NA-ECAP:

SET SPLITTING [4]

Instance: A family $\mathcal{C}$ of subsets of a finite ground set S.

Objective: Find a bipartition $\{S_1, S_2\}$ of S such that no subset in C is entirely included in either S_1 or S_2.

Given an instance of SET SPLITTING, we construct an instance of 1-NA-ECAP as follows: Let $G = (S, \emptyset)$ and $\mathcal{W} = \mathcal{C}$. Then we can prove that G can be augmented by adding at most $|S| - 2$ edges if and only if S has a feasible bipartition. Indeed, if there is a feasible bipartition $\{S_1, S_2\}$ of S to SET SPLITTING, then the union of an edge set spanning S_1 and that spanning S_2 is a solution to 1-NA-ECAP in G. If there is a solution E' with cardinality at most $|S| - 2$ to 1-NA-ECAP, then the augmented graph (S, E') is disconnected and each component in (S, E') does not include any subset in $\mathcal{C}$, which implies that a feasible bipartition of S exists. $\square$

They also showed that 2-NA-ECAP can be solved in polynomial time. However, it was still open whether the problem in the case of $k \geq 3$ is polynomially solvable or not.

Notice that if some area $W \in \mathcal{W}$ satisfies $|W| = 1$, then k-NA-ECAP is equivalent to the classical *k-edge-connectivity augmentation problem* (for short, *k-ECAP*) which augments the edge-connectivity of a given graph. Indeed, a graph has pairwise k edge-disjoint paths between each pair of a vertex and an area for any given family of areas if it is k-edge-connected, and a graph is k-edge-connected if it has k pairwise edge-disjoint paths between one fixed vertex and any other vertex. It was shown that k-ECAP is polynomially solvable by T. Watanabe et al. [22] and A. Frank[5]. Many algorithms for k-ECAP have been studied [5, 6, 7, 18, 22]. In [5], A. Frank also solved a more general problem such as local edge-connectivity augmentation problems and minimum node-cost and degree-constrained k-ECAP.

Mainly, there are two kinds of algorithms for k-ECAP; one is to augment the connectivity up to the target value k, one by one, by using the structure of an original graph [7, 22], and the other one is to add a new vertex s and the minimum number of new edges between s and G to construct a k-edge-connected graph G' and convert G' into a k-edge-connected graph eliminating s by the so-called "edge-splitting" operation [5, 18]. The algorithm by H. Miwa et al. [15] is based on the former one. In this paper, by following the latter approach, we establish a min-max formula to the k-NA-ECAP with $k \geq 3$, and show that the problem can be solved in $O(m + n(k^3 + n^2)(p + kn + n \log n) \log k + pkn^3 \log(n/k))$ time, where $n = |V|$, $m = |\{\{u, v\}|(u, v) \in E\}|$, and $p = |\mathcal{W}|$.

The paper is organized as follows. In Section 2, we define k-NA-ECAP, after introducing some basic notations. In Section 3, we derive lower bounds on the optimal value $opt_k(G, \mathcal{W})$ to k-NA-ECAP, and state our main result that a min-max formula to the k-NA-ECAP with $k \geq 3$ is established and that k-NA-ECAP is polynomially solvable for $k \geq 3$. In Section 4, we show several properties about the edge-splitting operation, which is a key operation for augmenting graph connectivities. Based on these, we give an algorithm, called NAEC-AUG, for finding a solution E' with $|E'| = opt_k(G, \mathcal{W})$ in Section 5. We prove the correctness of algorithm NAEC-AUG in Section 6. In Section 7, we give concluding remarks.

2 Problem Definition

Let $G = (V, E)$ stand for an undirected graph with a set V of *vertices* and a set E of *edges*. An edge with end vertices u and v is denoted by (u, v). We denote $|V|$ by n and $|\{\{u, v\}|(u, v) \in E\}|$ by m. A singleton set $\{x\}$ may be simply written as x, and " $\subset$ " implies proper inclusion while " $\subseteq$ " means " $\subset$ " or " $=$ ". In $G = (V, E)$, its vertex set V and edge set E may be denoted by $V(G)$ and $E(G)$, respectively. For a subset $V' \subseteq V$ in G, $G[V']$ denotes the subgraph induced by V'. For an edge set E' with $E' \cap E = \emptyset$, we denote the augmented graph $(V, E \cup E')$ by $G + E'$. For an edge set E', we denote by $V[E']$ the set of all end vertices of edges in E'.

An *area graph* is defined as a graph $G = (V, E)$ with a family $\mathcal{W}$ of vertex subsets $W \subseteq V$ which are called *areas* (see Figure 1). We denote an area graph G with $\mathcal{W}$ by $(G, \mathcal{W})$. In the sequel, we may denote $(G, \mathcal{W})$ simply by G if no confusion arises. For two disjoint subsets X, $Y \subset V$ of vertices, we denote by $E_G(X, Y)$ the set of edges $e = (x, y)$ such that $x \in X$ and $y \in Y$, and also denote $|E_G(X, Y)|$ by $d_G(X, Y)$. In particular, $E_G(u, v)$ is the set of edges with end vertices u and v. A *cut* is defined as a subset X of V with $\emptyset \neq X \neq V$, and the *size* of a cut X is defined by $d_G(X, V - X)$, which may also be written as $d_G(X)$. Moreover, we define $d(\emptyset) = 0$. For two cuts $X, Y \subseteq V$ in a graph $G = (V, E)$, we say that X and Y *cross* each other in G or X crosses with Y if none of $X \cap Y$, $X - Y$, $Y - X$, and $V - (X \cup Y)$ is empty. For a graph $G = (V, E)$, every two cuts $X, Y \subset V$ satisfy the following equalities:

$$d_G(X) + d_G(Y) = d_G(X - Y) + d_G(Y - X) + 2d_G(X \cap Y, V - (X \cup Y)). \quad (2.1)$$

$$d_G(X) + d_G(Y) = d_G(X \cup Y) + d_G(X \cap Y) + 2d_G(X - Y, Y - X). \quad (2.2)$$

For two cuts $X, Y \subset V$ with $X \cap Y = \emptyset$ in G, we denote by $\lambda_G(X, Y)$ the minimum size of cuts which separate X and Y, i.e., $\lambda_G(X, Y) = \min\{d_G(S) \mid S \supseteq X, S \subseteq V - Y\}$. For two cuts $X, Y \subset V$ with $X \cap Y \neq \emptyset$ in G, we define $\lambda_G(X, Y) = \infty$. The *edge-connectivity* of G, denoted by $\lambda(G)$, is defined as $\min_{X \subset V, Y \subset V} \lambda_G(X, Y)$. For a vertex $x \in V$ and a set $W \subseteq V$ of vertices, the *node-to-area edge-connectivity* (*NA-edge-connectivity*, for short) between x and W is defined as $\lambda_G(x, W)$. Note that $\lambda_G(x, W) = \infty$ holds for $x \in W$. For an area graph $(G, \mathcal{W})$, the *NA-edge-connectivity* of G, denoted by $\lambda(G, \mathcal{W})$, is defined as $\min_{x \in V, W \in \mathcal{W}} \lambda_G(x, W)$. Note that the area graph $(G, \mathcal{W})$ in Figure 1(i) satisfies $\lambda(G, \mathcal{W}) = 1$. If $\lambda(G, \mathcal{W}) \geq k$ holds, then we say that $(G, \mathcal{W})$ is *k-NA-edge-connected*.

In this paper, we consider the following problem, called k-NA-ECAP.

Problem 2.1 (k-NA-edge-connectivity augmentation problem, k-NA-ECAP)
Input: *An area graph* $(G = (V, E), \mathcal{W})$ *and a positive integer* k.
Output: *A set* E^* *of new edges with* $\lambda(G + E^*, \mathcal{W}) \geq k$ *such that* $|E^*|$ *is the minimum.* □

3 Lower Bounds on the Optimal Value

For an area graph $(G, \mathcal{W})$ and a fixed integer k, let $opt_k(G, \mathcal{W})$ denote the optimal value to k-NA-ECAP in G, i.e., the minimum size $|E^*|$ of a set E^* of new edges such that $G + E^*$ is k-NA-edge-connected. In this section, we derive a lower bound on $opt_k(G, \mathcal{W})$ to k-NA-ECAP with $(G, \mathcal{W})$.

A family $\mathcal{X} = \{X_1, \ldots, X_t\}$ of cuts in G is called a *partition of* V, if every two cuts $X_i, X_j \in \mathcal{X}$ satisfy $X_i \cap X_j = \emptyset$ and $\cup_{X \in \mathcal{X}} X_i = V$ holds. For a subset $X \subseteq V$ of vertices, a partition of X is called a *subpartition of* V. In an area graph $(G, \mathcal{W})$, a cut X is called *type* (*A*) if $X \cap W = \emptyset$ holds for some area $W \in \mathcal{W}$, and a cut X is called *type* (*B*) if $X \supseteq W$ holds for some area $W \in \mathcal{W}$ (note that a cut X of type (B) satisfies $X \neq V$ by the definition of a cut). Note that X is of type (A) if and only if $V - X$ is of type (B), and that each cut can be of both types.

Let $p(X) = \max\{0, k - d_G(X)\}$ for each cut X of type (A) or (B) and $p(X) = 0$ for all other subsets $X \subseteq V$. We easily see the following property.

Lemma 3.1 *An area graph* $(G, \mathcal{W})$ *satisfies* $\lambda(G, \mathcal{W}) \geq k$ *if and only if* $d_G(X) \geq k$ *holds for every cut* $X \subset V$ *of type* (*A*) *or* (*B*); $p(X) = 0$ *for all cuts* X. □

For each cut X of type (A) or (B) with $d_G(X) < k$, it is necessary to add at least $k - d_G(X)$ edges between X and $V - X$. Thus if X is of type (A) (resp., type (B)), then the NA-edge-connectivity between a vertex in X (resp., $V - X$) and an area $W \in \mathcal{W}$ with $W \cap X = \emptyset$ (resp., $W \subseteq X$) need be augmented to at least k.

Let

$$\alpha_k(G, \mathcal{W}) = \max_{\mathcal{X}} \Big\{ \sum_{X \in \mathcal{X}} p(X) \Big\}, \tag{3.1}$$

where the maximization is taken over all subpartitions of V. Then any feasible solution to k-NA-ECAP with $(G, \mathcal{W})$ must contain an edge which joins two vertices from a cut X of type (A) or (B) and the cut $V - X$. Thus we have the following lemma.

Lemma 3.2 *$opt_k(G, \mathcal{W}) \geq \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ holds.* □

The area graph $(G, \mathcal{W})$ in Figure 1(i) satisfies $\alpha_3(G, \mathcal{W}) = 10$. We have $\sum_{X \in \mathcal{X}} p(X) = \sum_{X \in \mathcal{X}} (3 - d_G(X)) = 10$ for the subpartition $\mathcal{X} = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_7\}, \{v_8, v_9\}\}$ of V.

In the classical k-ECAP, which is a special case of k-NA-ECAP, it is known that this type of lower bound based on subpartitions of V is equal to the optimal value [5, 22]. However, in k-NA-ECAP, there are area graphs $(G, \mathcal{W})$ with $opt_k(G, \mathcal{W}) > \lceil \alpha_k(G, \mathcal{W})/2 \rceil$. Figure 2 gives an instance for $k = 2$. Each cut $\{v_i\}$ is of type (A) satisfies $p(v_i) = k - d_G(v_i) = 1$ for $i = 1, 2, 3, 4$. Thus $\lceil \alpha_2(G, \mathcal{W})/2 \rceil = 2$. In order to make $(G, \mathcal{W})$ 2-NA-edge-connected by adding two new edges, we must add $e = (v_1, v_2)$ and $e' = (v_3, v_4)$ without loss of generality. $G + \{e, e'\}$ is not 2-NA-edge-connected by $\lambda_{G+\{e,e'\}}(v_3, W_1) = 1$. We will show that all such instances can be

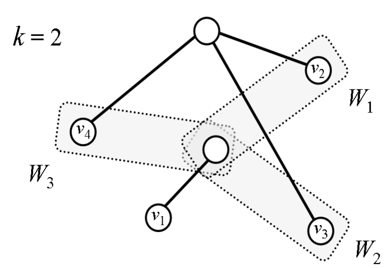

Figure 2: Illustration of an area graph $(G, \mathcal{W})$ with $opt_2(G, \mathcal{W}) = \lceil \frac{\alpha_2(G, \mathcal{W})}{2} \rceil + 1$.

completely characterized as follows.

Definition 3.3 *We say that an area graph $(G, \mathcal{W})$ has property* (P) *if $\alpha_k(G, \mathcal{W})$ is even and there is a subpartition $\mathcal{X} = \{X_1, \ldots, X_q\}$ of V with $\sum_{X \in \mathcal{X}} p(X) = \alpha_k(G, \mathcal{W})$ satisfying the following conditions* (P1)–(P3)*:*

(P1) *Each cut $X_i \in \mathcal{X}$ is of type* (A).

(P2) *The cut X_1 satisfies $d_G(X_1) = k - 1$ and $X_1 \subset C_1$ for some component C_1 of G with $X_i \cap C_1 = \emptyset$ for each $i = 2, \ldots, q$.*

(P3) *For each* $i = 2, \ldots, q$, *there is a cut* Y_i *of type* (B) *with* $X_i \cup X_1 \subseteq Y_i$ *and* $\sum_{X \in \mathcal{X}, X \subset Y_i} (k - d_G(X)) \leq (k+1) - d_G(Y_i)$ *such that every cut* $X \in \mathcal{X}$ *satisfies* $X \subset Y_i$ *or* $X \cap Y_i = \emptyset$. □

Note that $(G, \mathcal{W})$ in Figure 2 has property (P) because $\alpha_2(G, \mathcal{W}) = 4$ holds and the subpartition $\mathcal{X}$ of V consisting of $X_i = \{v_i\}$, $i = 1, 2, 3, 4$ satisfies $Y_2 = C_1 \cup \{v_2\}$, $Y_3 = C_1 \cup \{v_3\}$, and $Y_4 = C_1 \cup \{v_4\}$ for the component C_1 of G containing v_4.

Lemma 3.4 *If* $(G, \mathcal{W})$ *has property* (P), *then* $opt_k(G, \mathcal{W}) \geq \lceil \alpha_k(G, \mathcal{W})/2 \rceil + 1$ *holds.*

Proof: Assume by contradiction that there is an edge set E^* with $\lambda(G + E^*, \mathcal{W}) \geq k$ and $|E^*| = \alpha_k(G, \mathcal{W})/2$ (note that $\alpha_k(G, \mathcal{W})$ is even). Let $\mathcal{X} = \{X_1, \ldots, X_q\}$ denote a subpartition of V satisfying $\sum_{X \in \mathcal{X}} p(X) = \alpha_k(G, \mathcal{W})$ and the above (P1)–(P3). Since $|E^*| = \alpha_k(G, \mathcal{W})/2$ holds, each cut $X \in \mathcal{X}$ satisfies $d_{G+E^*}(X) = k$, and hence $d_{G'}(X) = k - d_G(X)$, where $G' = (V, E^*)$. Therefore, any edge $(x_i, x_j) \in E^*$ satisfies $x_i \in X_i$ and $x_j \in X_j$ for some two cuts $X_i, X_j \in \mathcal{X}$ with $X_i \neq X_j$. From this, there exists a cut $X_\ell \in \mathcal{X}$ with $\ell \neq 1$ and $E_{G'}(X_\ell, X_1) \neq \emptyset$. Since $(G, \mathcal{W})$ satisfies property (P), there is a cut Y_ℓ which satisfies (P3), and hence $\sum_{v \in Y_\ell} d_{G'}(v) \leq (k+1) - d_G(Y_\ell)$. Since $G'[Y_\ell]$ contains one edge in $E_{G'}(X_\ell, X_1)$, we have $d_{G'}(Y_\ell) \leq (k-1) - d_G(Y_\ell)$, which implies that $G + E^*$ is not k-NA-edge-connected, a contradiction. □

In this paper, we prove that k-NA-ECAP enjoys the following min-max theorem and is polynomially solvable.

Theorem 3.5 *For* k*-NA-ECAP with* $k \geq 3$, $opt_k(G, \mathcal{W}) = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ *holds if* $(G, \mathcal{W})$ *does not have property (P), and* $opt_k(G, \mathcal{W}) = \lceil \alpha_k(G, \mathcal{W})/2 \rceil + 1$ *holds otherwise. Moreover, a solution* E^* *with* $|E^*| = opt_k(G, \mathcal{W})$ *can be obtained in* $O(m + n(n^2 + k^3)(p + kn + n \log n) \log k)$ *time.* □

4 Extensions and Edge-Splittings

4.1 Extensions

As mentioned in Section 1, we adapt the so-called "edge-splitting" method for solving k-NA-ECAP. In the edge-splitting method, after creating a new vertex s outside of G and adding new edges between s and G, we find an appropriate edge set to be added to G by splitting off a pair of edges incident to s in the extended graph. Given an area graph $(G, \mathcal{W})$, a graph $H = (V \cup \{s\}, E \cup F)$ obtained from $(G, \mathcal{W})$ by adding a new vertex s and a set F of new edges connecting s and V is called a *k-extension of* $(G, \mathcal{W})$ if

$$d_H(X) \geq k \text{ holds for each cut } X \subset V \text{ of type (A) or type (B).} \tag{4.1}$$

In particular, a k-extension $H = (V \cup \{s\}, E \cup F)$ of $(G, \mathcal{W})$ is called *critical* if $(V \cup \{s\}, E \cup F')$ violates (4.1) for any $F' \subset F$. We here show that a critical k-extension $H = (V \cup \{s\}, E \cup F)$ of $(G, \mathcal{W})$ satisfies $|F| = \alpha_k(G)$.

Theorem 4.1 *Let $(G = (V, E), \mathcal{W})$ be an area graph, and k be a nonnegative integer. A critical k-extension $H = (V \cup \{s\}, E \cup F)$ of $(G, \mathcal{W})$ satisfies $|F| = \alpha_k(G, \mathcal{W})$.* □

In $(G, \mathcal{W})$, the following properties hold:

$$\text{If a cut } X \subset V \text{ is of type } (A)\text{, then some area } W \text{ satisfies } W \cap X = \emptyset \text{ and hence every cut } X' \subseteq X \text{ is also of type } (A). \tag{4.2}$$

$$\text{If a cut } X \subset V \text{ is of type } (B)\text{, then some area } W \text{ satisfies } W \subseteq X \text{ and hence every cut } X' \supseteq X \text{ with } X' \neq V \text{ is also of type } (B). \tag{4.3}$$

Let V be a finite ground set and let $q : 2^V \to Z^+$ be an integer-valued function with $q(\emptyset) = 0$, where Z^+ denotes the set of nonnegative integers. A set function q is called *skew-supermodular* if $q(X) + q(Y) \leq q(X \cap Y) + q(X \cup Y)$ or $q(X) + q(Y) \leq q(X - Y) + q(Y - X)$ hold for every two subsets X and Y of V with $q(X) > 0, q(Y) > 0$. A set function q is called *symmetric* if $q(X) = q(V - X)$ holds for all $X \subseteq V$. In [5, Section 7], [20, Lemma 5.1], it was shown that given a symmetric skew-supermodular integer-valued function $q : 2^V \to Z^+$, a function $z : V \to Z^+$ such that $\sum_{v \in V} z(v)$ is the minimum and $\sum_{v \in X} z(v) \geq q(X)$ holds for every $X \subseteq V$ can be found by a greedy algorithm. Now it is not difficult to see from (2.1), (2.2), (4.2), and (4.3) that p is a symmetric skew-supermodular integer-valued function. Note that $H = (V \cup \{s\}, E \cup F)$ is a k-extension of $(G = (V, E), \mathcal{W})$ if and only if the function $z : V \to Z^+$ with $z(v) = d_H(s, v)$ satisfies $z(X) \geq p(X)$ for every $X \subseteq V$. This observation proves Theorem 4.1.

Remarks: In [2], A. Benczúr and A. Frank considered a problem of covering a symmetric crossing supermodular function q by the minimum number of edges, which is a generalization of k-ECAP, where a set function $q : 2^V \to Z^+$ is called *crossing supermodular* if $q(X) + q(Y) \leq q(X \cap Y) + q(X \cup Y)$ holds for every two crossing subsets X and Y of V with $q(X) > 0, q(Y) > 0$. They gave a polynomial time algorithm for solving the problem, under the assumption that a polynomial oracle for $\min\{\sum_{v \in X} z(v) + d_{(V,E')}(X) - q(X) \mid X \subset V\}$ is given, where $z : V \to Z^+$ denotes a function on V and E' denotes a set of edges. On the other hand, the function p in k-NA-ECAP is not crossing supermodular. For example, in the graph in Figure 2, two cuts $X = C_1 \cup \{v_3\}$ and $X' = C_1 \cup \{v_4\}$ satisfy $p(X) = p(X') = 1$ and $p(X \cap X') = p(X \cup X') = 0$, where C_1 denotes the component containing v_1 (note that $X \cap X' = C_1$ is neither of type (A) nor of type (B)). □

4.2 Edge-splitting theorems

For a graph $H = (V \cup \{s\}, E)$ and a designated vertex $s \notin V$, an operation called *edge-splitting* (*at* s) is defined as deleting two edges $(s,u),(s,v) \in E$ and adding one new edge (u,v). That is, the graph $H' = (V \cup \{s\}, (E - \{(s,u),(s,v)\}) \cup \{(u,v)\})$ is obtained from such edge-splitting operation. Then we say that H' is obtained from H by *splitting* a pair of edges (s,u) and (s,v) (or by splitting (s,u) and (s,v)). A sequence of splittings is *complete* if the resulting graph H' does not have any neighbor of s. The edge-splitting operation is known to be a useful tool for solving connectivity augmentation problems [5].

Given a k-extension $H = (V \cup \{s\}, E \cup F)$ of $(G, \mathcal{W})$, a pair $\{(s,u),(s,v)\}$ is called *admissible* if the graph H' obtained from H by splitting (s,u) and (s,v) is also a k-extension of $H' - s = G + \{(u,v)\}$; H' satisfies (4.1). Notice that given an area graph $(G, \mathcal{W})$, if there is a complete admissible splitting at s in its critical k-extension $H = (V \cup \{s\}, E \cup F)$, then the set E' of split edges is an optimal solution of k-NA-ECAP to $(G, \mathcal{W})$. Indeed, every cut $X \subset V$ of type (A) or (B) satisfies $d_{G+E'}(X) \geq k$, implying $\lambda(G + E', \mathcal{W}) \geq k$, since $H' = (V \cup \{s\}, E \cup E')$ satisfies (4.1) and $d_{H'}(s, V) = 0$, and Theorem 4.1 implies that we have $|E'| = |F|/2 = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$, which is a lower bound on $opt_k(G, \mathcal{W})$. However, as indicated by the graph in Figure 2, not all k-extensions of $(G, \mathcal{W})$ have a complete admissible splitting at s.

In the sequel, we consider situations where a k-extension H of $(G, \mathcal{W})$ has no admissible pair of edges. In [20], Z. Nutov gave some splitting off theorem under a more general setting in such a sense that it can be applied to any skew-supermodular set function p. However, we here show the splitting theorems and lemmas which are specified to k-NA-ECAP without applying Nutov's theorem, because we need to prove more than it for achieving a polynomial time algorithm to k-NA-ECAP, and because of the completeness of the paper. The following theorem is a key theorem for our algorithm.

Theorem 4.2 *Let $(G = (V, E), \mathcal{W})$ be an area graph and $H = (V \cup \{s\}, E \cup F)$ be a k-extension of $(G, \mathcal{W})$ with $k \geq 2$ and an even $d_H(s)$. If no pair of two edge in F is admissible, then we have $d_H(s) = 4$ and G has two components C_1 and C_2 with $d_H(s, C_1) = 3$ and $d_H(s, C_2) = 1$. Moreover, in the graph $H + e^*$ obtained by adding one arbitrary new edge e^* to $E_G(C_1, C_2)$, there is a complete admissible splitting at s.* □

Before giving a proof of this theorem, we show several preparatory lemmas. For a graph $G = (V, E)$, every three cuts X, Y, and Z satisfy the following inequality.

$$\begin{aligned} d_G(X) + d_G(Y) + d_G(Z) \geq\ & d_G(X - Y - Z) + d_G(Y - X - Z) \\ & + d_G(Z - X - Y) + d_G(X \cap Y \cap Z) \\ & + 2d_G(X \cap Y \cap Z, V - (X \cup Y \cup Z)). \end{aligned} \quad (4.4)$$

For a k-extension $H = (V \cup \{s\}, E \cup F)$ of $(G, \mathcal{W})$, a pair $\{(s,u),(s,v)\} \subseteq F$ of two edges is not admissible if there is a cut $Y \subset V$ of type (A) or (B) with $\{u,v\} \subseteq Y$ and $d_H(Y) \le k+1$. Such cut Y is called a *dangerous cut*. Conversely, a pair $\{(s,u),(s,v)\}$ is not admissible only if there is a dangerous cut $Y \subset V$ with $\{u,v\} \subseteq Y$. We give the following two lemmas on the properties of dangerous cuts.

Lemma 4.3 *Let $H = (V \cup \{s\}, E \cup F)$ be a k-extension of $(G = (V,E), \mathcal{W})$ and $Y \subset V$ be a dangerous cut. Then we have $d_H(s, V-Y) \ge d_H(s,Y) - 1 > 0$.*

PROOF: Since Y is a dangerous cut, we have $d_H(Y) = d_H(s,Y) + d_H(Y, V-Y) \le k+1$. Moreover, Y is of type (A) or (B), and hence so is $V-Y$, which implies $d_H(V-Y) = d_H(s, V-Y) + d_H(Y, V-Y) \ge k$ by (4.1). Hence we have $d_H(s, V-Y) \ge k - d_H(Y, V-Y) \ge d_H(s,Y) - 1$. From the definition of dangerous cuts, $d_H(s,Y) \ge 2$ holds. □

Lemma 4.4 *Let $H = (V \cup \{s\}, E \cup F)$ be a k-extension of $(G = (V,E), \mathcal{W})$ with an even $d_H(s)$. Assume that there are two dangerous cuts $Y_1, Y_2 \subset V$ with $d_H(s, Y_1 - Y_2) > 0$, $d_H(s, Y_2 - Y_1) > 0$, and $d_H(s, Y_1 \cap Y_2) > 0$. Then we have $d_H(s, V - Y_1 - Y_2) > 0$.*

PROOF: Assume $d_H(s, Y_1 - Y_2) \ge d_H(s, Y_2 - Y_1)$ without loss of generality. By Lemma 4.3, we have $d_H(s, Y_2 - Y_1) + d_H(s, V - Y_1 - Y_2) = d_H(s, V - Y_1) \ge d_H(s, Y_1) - 1 = d_H(s, Y_1 - Y_2) + d_H(s, Y_1 \cap Y_2) - 1 \ge d_H(s, Y_2 - Y_1) + d_H(s, Y_1 \cap Y_2) - 1$. Hence we have $d_H(s, V - Y_1 - Y_2) = 0$ would imply that the above inequalities hold by equality since $d_H(s, Y_1 \cap Y_2) \ge 1$. This means $d_H(Y_1 - Y_2) = d_H(Y_2 - Y_1)$, which implies $d_H(s) = 2d_H(Y_1 - Y_2) + 1$, contradicting that $d_H(s)$ is even. □

We show the following properties for cuts Y of type (A) or (B) with $d_H(Y) \le k+1$ (note that Y is not necessarily dangerous).

Lemma 4.5 *Let $H = (V \cup \{s\}, E \cup F)$ be a k-extension of $(G = (V,E), \mathcal{W})$ with $k \ge 2$. For every cut $Y \subset V$ of type (A) with $d_H(Y) \le k+1$, $\lambda(G[Y]) \ge k - \lfloor \frac{d_H(Y)}{2} \rfloor$ (≥ 1) holds.*

PROOF: By (4.2), for any partition $\{Y_1, Y_2\}$ of Y, Y_i is also of type (A). Hence by (4.1), we have $d_H(Y_i) \ge k$ for $i = 1, 2$, and $d_H(Y_1, Y_2) = \lfloor \frac{1}{2}(d_H(Y_1) + d_H(Y_2)) \rfloor - \lfloor \frac{d_H(Y)}{2} \rfloor \ge k - \lfloor \frac{d_H(Y)}{2} \rfloor \ge 1$. □

Lemma 4.6 *Let $H = (V \cup \{s\}, E \cup F)$ be a k-extension of $(G = (V,E), \mathcal{W})$, and Y_1 and Y_2 be two cuts with $d_H(Y_1) \le k+1$, $d_H(Y_2) \le k+1$, and $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) > 0$ such that both Y_1 and Y_2 are of type (A) or of type (B). If Y_1 and Y_2 cross each other in H, then both of the cuts $Y_1 - Y_2$ and $Y_2 - Y_1$ are of type (A) and we have $d_H(Y_1) = d_H(Y_2) = k+1$, $d_H(Y_1 - Y_2) = d_H(Y_2 - Y_1) = k$, and $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) = 1$.*

PROOF: If both of Y_1 and Y_2 are of type (A) (resp., type (B)), then both of the cuts $Y_1 - Y_2$ and $Y_2 - Y_1$ are of type (A) by (4.2) (resp., by $(Y_1 - Y_2) \cap W_2 = \emptyset = (Y_2 - Y_1) \cap W_1$, where W_i denotes an area with $W_i \subseteq Y_i$). Hence, $d_H(Y_1 - Y_2) \geq k$ and $d_H(Y_2 - Y_1) \geq k$ hold by (4.1). By $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) > 0$, $d_H(Y_1 - Y_2) \geq k$, $d_H(Y_2 - Y_1) \geq k$, and (2.1), we have $2(k+1) \geq d_H(Y_1) + d_H(Y_2) = d_H(Y_1 - Y_2) + d_H(Y_2 - Y_1) + 2d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) \geq k + k + 2$. This proves the lemma. □

We first show Theorem 4.2 in a special case where $d_H(s, C) \leq 1$ holds for every component C in G. In this case, we can see that there is a complete admissible splitting at s.

Lemma 4.7 *Let G, H, and k satisfy the assumption of Theorem 4.2. If $d_H(s, C) \leq 1$ holds for each component in G, then there is a complete admissible splitting at s.*

PROOF: We prove the lemma by showing that every pair of two edges in F is admissible. Let $\mathcal{C} = \{C_1, \ldots, C_q\}$ denote the family of all components with $d_H(s, C_i) = 1$, $C_i \subseteq V$, where $\{(s, v_i)\} = E_H(s, C_i)$. Then every area $W \in \mathcal{W}$ satisfies $W \cap C_i \neq \emptyset$ for any $C_i \in \mathcal{C}$ since a component $C \in \mathcal{C}$ with $W \cap C = \emptyset$ is of type (A) and satisfies $d_H(C_i) = d_H(s, C_i) \geq k$ by (4.1), contradicting $k \geq 2$.

Assume by contradiction that some pair $\{(s, v_1), (s, v_2)\}$ is not admissible. Then there is a dangerous cut $Y \subset V$ with $\{v_1, v_2\} \subseteq Y$. Lemma 4.5 implies that Y is of type (B) since two vertices v_1 and v_2 are contained in distinct components in G and hence $G[Y]$ is not connected. Let $W_1 \in \mathcal{W}$ be an area with $W_1 \subseteq Y$. Lemma 4.3 says that there is an edge $(s, v_3) \in E_H(s, V - Y)$. By $(C_3 - Y) \cap W_1 = \emptyset$ and (4.1), we have $d_H(C_3 - Y) \geq k$. Hence $d_G(C_3 - Y) = d_G(C_3 - Y, C_3 \cap Y) \geq k - 1$ by $C_3 \cap W_1 \neq \emptyset$ and $(s, v_3) \in E_H(s, V - Y)$. Hence we have $d_H(Y) \geq d_H(s, Y) + d_G(C_3 - Y, C_3 \cap Y) \geq k + 1$ by $d_H(s, Y) \geq 2$. By $d_H(Y) \leq k + 1$, we have $d_H(Y) = k + 1$, $d_H(s, Y) = 2$, and $d_G(Y) = d_G(C_3 - Y, C_3 \cap Y)$. This means that the number of components $C \in \mathcal{C}$ with $C \subseteq Y$ is two and the number of components $C \in \mathcal{C}$ with $C - Y \neq \emptyset$ is one. There is no other edge $(s, v_j) \in E_H(s, V - Y)$, $j \neq 3$ because if $C_j - Y \neq \emptyset$ holds for another component $C_j \in \mathcal{C}$, then $d_G(C_j \cap Y, C_j - Y) > 0$ holds by $C_j \cap W_1 \neq \emptyset$, from which $d_H(Y) > k + 1$ would hold. Hence $|\mathcal{C}| = 3$ holds. It contradicts that $d_H(s)$ is even. □

The following theorem shows a property of a k-extension of $(G, \mathcal{W})$ which has no edge incident to s admissible with a fixed edge (s, u).

Theorem 4.8 *Let $H = (V \cup \{s\}, E \cup F)$ be a k-extension of $(G = (V, E), \mathcal{W})$ with an even $d_H(s)$ and $k \geq 2$, and $(s, u) \in F$ be an edge. If $\{(s, u), e\}$ is not admissible for any edge $e \in F$, then there are three dangerous cuts Y_1, Y_2, and Y_3 with $u \in Y_1 \cap Y_2 \cap Y_3$, $d_H(Y_1) = d_H(Y_2) = d_H(Y_3) = k + 1$, $d_H(s, Y_1 - Y_2 - Y_3) > 0$, $d_H(s, Y_2 - Y_1 - Y_3) > 0$, and $d_H(s, Y_3 - Y_1 - Y_2) > 0$ such that one of the following (i) and (ii) holds.*

(i) Each Y_i is of type (B). We have $d_G(Y_1 \cap Y_2 \cap Y_3) = 0$, $d_H(s, Y_1 \cap Y_2 \cap Y_3) = 1$, and $d_H(Y_1 - Y_2 - Y_3) = d_H(Y_2 - Y_1 - Y_3) = d_H(Y_3 - Y_1 - Y_2) = k$.
(ii) Both of Y_1 and Y_2 are of type (A) and Y_3 is of type (B) (without loss of generality). We have $d_G(Y_3 - Y_1 - Y_2) = 0$, $d_H(s, Y_3 - Y_1 - Y_2) = 1$, and $d_H(Y_1 - Y_2 - Y_3) = d_H(Y_2 - Y_1 - Y_3) = d_H(Y_1 \cap Y_2 \cap Y_3) = k$. Moreover, $V = Y_1 \cup Y_2 \cup Y_3$ holds.

PROOF: Since $\{(s,u),(s,v_i)\}$ is non-admissible for any edge $(s,v_i) \in F - \{(s,u)\}$, there is a dangerous cut containing u and v_i in H. For each edge $(s,v_i) \in F - \{(s,u)\}$, let Y_i denote a dangerous cut containing two vertices u and v_i. Let $\mathcal{Y} = \{Y_i \mid (s,v_i) \in F - \{(s,u)\}\}$. Lemma 4.3 implies $|\mathcal{Y}| \geq 2$. We first claim that for any two cuts Y_1, Y_2 in $\mathcal{Y}$,

$$d_H(s, V - Y_1 - Y_2) > 0 \text{ holds.} \tag{4.5}$$

By Lemma 4.4 and $d_H(s, Y_1 \cap Y_2) > 0$, (4.5) holds if $d_H(s, Y_1 - Y_2) > 0$ and $d_H(s, Y_2 - Y_1) > 0$ hold. On the other hand, assume $d_H(s, Y_1 - Y_2) = 0$ without loss of generality. Then we have $d_H(s, Y_1 \cup Y_2) = d_H(Y_2)$. By Lemma 4.3 and $E_H(s, Y_2) \supseteq E_H(s, Y_1)$, we have $d_H(s, V - Y_1 - Y_2) > 0$.

By (4.5), there are an edge $(s, v_3) \in E_H(s, V - Y_1 - Y_2)$ and the corresponding dangerous cut $Y_3 \in \mathcal{Y}$ with $\{u, v_3\} \subseteq Y_3$. Then we can see the following claim.

Claim 4.9 *For three cuts Y_1, Y_2, $Y_3 \in \mathcal{Y}$ such that $d_H(s, Y_3 - Y_1 - Y_2) > 0$ and $d_H(s, Y_1 \cup Y_2) \geq \max\{d_H(s, Y_1 \cup Y_3), d_H(s, Y_2 \cup Y_3)\}$, we have*

$$d_H(s, Y_1 - Y_2 - Y_3) > 0, d_H(s, Y_2 - Y_1 - Y_3) > 0, \text{ and } d_H(s, Y_3 - Y_1 - Y_2) > 0. \tag{4.6}$$

□

Hence by choosing $Y_1, Y_2 \in \mathcal{Y}$ such that $d_H(s, Y_1 \cup Y_2)$ is the maximum, Y_1, Y_2, and Y_3 satisfy (4.6).

Then there are the following four possible cases.

(Case-1) Each cut Y_i, $i = 1, 2, 3$ is of type (A).
(Case-2) Each cut Y_i, $i = 1, 2, 3$ is of type (B).
(Case-3) Y_1 is of type (A) and both of Y_2 and Y_3 are of type (B).
(Case-4) Both of Y_1 and Y_2 are of type (A) and Y_3 is of type (B).

Let W_i denote an area in $\mathcal{W}$ with $W_i \subseteq Y_i$ if Y_i is of type (B). Note that in each case, every Y_i satisfies $d_H(Y_i) \leq k + 1$ for $i = 1, 2, 3$, and $d_H(Y_1 \cap Y_2 \cap Y_3, V \cup \{s\} - (Y_1 \cup Y_2 \cup Y_3)) \geq d_H(s, Y_1 \cap Y_2 \cap Y_3) \geq d_H(s, u) > 0$ holds. By (4.6), we have $Y_1 - Y_2 - Y_3 \neq \emptyset$, $Y_2 - Y_3 - Y_1 \neq \emptyset$, and $Y_3 - Y_1 - Y_2 \neq \emptyset$.

(Case-1) By (4.1) and (4.2), we have $d_H(Y_1 - Y_2 - Y_3) \geq k$, $d_H(Y_2 - Y_3 - Y_1) \geq k$, $d_H(Y_3 - Y_1 - Y_2) \geq k$, and $d_H(Y_1 \cap Y_2 \cap Y_3) \geq k$. By (4.4), we have $3(k+1) \geq \sum_{i=1}^{3} d_H(Y_i) \geq d_H(Y_1 - Y_2 - Y_3) +$

$d_H(Y_2-Y_3-Y_1)$ $+d_H(Y_3-Y_1-Y_2)+d_H(Y_1\cap Y_2\cap Y_3)$ $+2d_H(Y_1\cap Y_2\cap Y_3, V\cup\{s\}-(Y_1\cup Y_2\cup Y_3))$ $\geq 4k+2$, contradicting $k\geq 2$. This case cannot occur.

(Case-2) By $(Y_1-Y_2-Y_3)\cap W_2=\emptyset$, the cut $Y_1-Y_2-Y_3$ is of type (A) and hence satisfies $d_H(Y_1-Y_2-Y_3)\geq k$ by (4.1). Similarly, we have $d_H(Y_2-Y_3-Y_1)\geq k$ and $d_H(Y_3-Y_1-Y_2)\geq k$. We have $d_H(Y_1\cap Y_2\cap Y_3)\geq d_H(s, Y_1\cap Y_2\cap Y_3)\geq 1$. By (4.4), we have $3(k+1)\geq\sum_{i=1}^{3} d_H(Y_i)\geq$ $d_H(Y_1-Y_2-Y_3)+d_H(Y_2-Y_3-Y_1)$ $+d_H(Y_3-Y_1-Y_2)+d_H(Y_1\cap Y_2\cap Y_3)$ $+2d_H(Y_1\cap Y_2\cap Y_3, V\cup\{s\}-(Y_1\cup Y_2\cup Y_3))\geq 3k+3$. This implies that every inequality turns out to be an equality. The cuts Y_1, Y_2, and Y_3 satisfy the statement (i) of this theorem.

(Case-3) By (4.1) and (4.2), we have $d_H(Y_1-Y_2-Y_3)\geq k$ and $d_H(Y_1\cap Y_2\cap Y_3)\geq k$. By $(Y_2-Y_3-Y_1)\cap W_3=\emptyset$ and (4.1), we have $d_H(Y_2-Y_3-Y_1)\geq k$. Similarly, $d_H(Y_3-Y_1-Y_2)\geq k$ holds. Similarly to Case-1, by $k\geq 2$ and (4.4), we see that this case cannot occur.

(Case-4) By (4.1) and (4.2), we have $d_H(Y_1-Y_2-Y_3)\geq k$, $d_H(Y_2-Y_3-Y_1)\geq k$, and $d_H(Y_1\cap Y_2\cap Y_3)\geq k$. By (4.6), we have $d_H(Y_3-Y_1-Y_2)\geq d_H(s, Y_3-Y_1-Y_2)\geq 1$. Similarly to Case-2, we see by (4.4) that every inequality turns out to be an equality. Hence the cuts Y_1, Y_2, and Y_3 satisfy the statement (ii) of this theorem, except $V=Y_1\cup Y_2\cup Y_3$.

Let Y_1, Y_2, Y_3 be three cuts satisfying Case-4 such that $d_H(s, Y_1\cup Y_2\cup Y_3)$ is the maximum, and W_3 be an area in $\mathcal{W}$ with $W_3\subseteq Y_3$. $d_H(s, Y_3-Y_1-Y_2)=1$ means that $d_H(s, Y_1\cup Y_2)$ is the maximum. We now have $d_G(Y_3-Y_1-Y_2)=0$.

Assume by contradiction that $V-Y_1-Y_2-Y_3\neq\emptyset$ holds. We show the following claim.

Claim 4.10 *$E_H(s, V-(Y_1\cup Y_2\cup Y_3))=\emptyset$ holds.*

PROOF: Assume by contradiction that there is an edge $(s, v_4)\in E_H(s, V-(Y_1\cup Y_2\cup Y_3))$. Let Y_4 be the corresponding dangerous cut in $\mathcal{Y}$ with $\{u, v_4\}\subseteq Y_4$. Assume that Y_4 is of type (A). Then by $d_G(Y_3-Y_1-Y_2)=0$ and Lemma 4.5, we have $Y_4\cap(Y_3-Y_1-Y_2)=\emptyset$. Hence from the maximality of $d_H(s, Y_1\cup Y_2)$, the cuts Y_1, Y_2, and Y_4 satisfy Case-1, which cannot occur. Assume that Y_4 is of type (B) and the cuts Y_1, Y_2, and Y_4 satisfy Case-4. Then we have $d_G(Y_4-Y_1-Y_2)=0$ and $d_H(Y_4-Y_1-Y_2)=1$, implying that $Z=Y_4-Y_1-Y_2-(Y_3-Y_1-Y_2)$ satisfies $Z\cap W_3=\emptyset$ but $d_H(Z)=1$, a contradiction to (4.1). Assume that Y_4 is of type (B) and the cuts Y_1, Y_2, and Y_4 do not satisfy Case-4. Then we have $d_H(s, Y_1-Y_2-Y_4)=0$ without loss of generality. The cuts Y_2, Y_3, and Y_4 do not satisfy Case-3 from the above arguments, and hence $d_H(s, Y_3-Y_2-Y_4)=0$ or $d_H(s, Y_2-Y_3-Y_4)=0$ hold. This means $d_H(s, Y_2\cup Y_4)>d_H(s, Y_1\cup Y_2\cup Y_3)$ or $d_H(s, Y_3\cup Y_4)>d_H(s, Y_1\cup Y_2\cup Y_3)$. This and Lemma 4.4 imply that there are three dangerous cuts Y_1', Y_2', and Y_3' satisfying Case-4 with $d_H(s, Y_1'\cup Y_2'\cup Y_3')>d_H(s, Y_1\cup Y_2\cup Y_3)$, contradicting the maximality of $d_H(s, Y_1\cup Y_2\cup Y_3)$ (note that such cuts Y_1', Y_2', and Y_3' do not satisfy Case-2 since otherwise the component C' of G with $u\in C'$ satisfies $d_H(s, C')=d_H(s, u)=1$, contradicting that the cut Y_1 satisfies $u\in Y_1$ and $d_H(s, Y_1)\geq 2$ and $G[Y_1]$ is connected by Lemma 4.5). Therefore we have $E_H(s, V-(Y_1\cup Y_2\cup Y_3))=\emptyset$. □

By $V - Y_1 - Y_2 - Y_3 \neq \emptyset$ and $d_G(Y_3 - Y_1 - Y_2) = 0$, we have $d_H(V - Y_1 - Y_2 - Y_3) = d_G(V - Y_1 - Y_2 - Y_3) = d_G(Y_1 \cup Y_2)$. By $(V - Y_1 - Y_2 - Y_3) \cap W_3 = \emptyset$ and (4.1), $d_H(V - Y_1 - Y_2 - Y_3) \geq k$ holds, from which $d_G(Y_1 \cup Y_2) \geq k$ holds. By $d_H(s, Y_1 - Y_2) > 0$, $d_H(s, Y_2 - Y_1) > 0$, and $d_H(s, Y_1 \cap Y_2) > 0$, we have $d_H(s, Y_1 \cup Y_2) \geq 3$, from which $d_H(Y_1 \cup Y_2) \geq k + 3$ holds. (4.1) and (4.2) imply that $d_H(Y_1 \cap Y_2) \geq k$ holds. By (2.2), we have $2(k + 1) \geq d_H(Y_1) + d_H(Y_2) \geq d_H(Y_1 \cap Y_2) + d_H(Y_1 \cup Y_2) \geq k + k + 3$, a contradiction. Therefore $V = Y_1 \cup Y_2 \cup Y_3$ holds. $\square$

PROOF OF THEOREM 4.2: Assume that H has no admissible pair of two edges in F. Let C_1 denote a component in G with $d_H(s, C_1) \geq 2$ and $(s, u_0) \in E_H(s, C_1)$. Then Theorem 4.8 says that there are three dangerous cuts Y_1, Y_2, and Y_3 satisfying (i) or (ii) in Theorem 4.8 for $u = u_0$. If they satisfy (i), then $d_H(s, Y_1 \cap Y_2 \cap Y_3) = d_H(s, u_0) = 1$ and $d_G(Y_1 \cap Y_2 \cap Y_3) = 0$ hold, which implies that $C_1 \subseteq Y_1 \cap Y_2 \cap Y_3$ holds, contradicting $d_H(s, C_1) \geq 2$. Hence such three dangerous cuts Y_1, Y_2, and Y_3 satisfy (ii) in Theorem 4.8. Let Y_1, Y_2 be of type (A) and Y_3 be of type (B). Since Lemma 4.5 says that both of $G[Y_1]$ and $G[Y_2]$ are connected, $G[Y_1 \cup Y_2]$ is also connected. By Theorem 4.8, $d_H(s, Y_3 - Y_1 - Y_2) = 1$ and $d_G(Y_3 - Y_1 - Y_2) = 0$ hold. Hence we have $C_1 = Y_1 \cup Y_2$. Let (s, u_3) be the edge with $\{(s, u_3)\} = E_H(s, Y_3 - Y_1 - Y_2)$ and C_2 be the component in G with $u_3 \in C_2$.

We show that $d_H(s) = 4$ holds. Now Y_1 and Y_2 cross each other in H. Lemma 4.6 says that

$$d_H(Y_1 - Y_2) = d_H(Y_2 - Y_1) = k, \tag{4.7}$$

$d_H(s, Y_1 \cap Y_2) = 1$, and $E_H(s, Y_1 \cap Y_2) = \{(s, u_0)\}$ hold. Hence it suffices to show that $d_H(s, Y_1 - Y_2) = d_H(s, Y_2 - Y_1) = 1$ holds.

We here show that $d_H(s, Y_1 - Y_2) = 1$ holds ($d_H(s, Y_2 - Y_1) = 1$ can be proved similarly). Assume by contradiction that $d_H(s, Y_1 - Y_2) \geq 2$ holds. For an edge $(s, u_1) \in E_H(s, Y_1 - Y_2)$, no pair $\{(s, u_1), e\}$ with $e \in F - \{(s, u_1)\}$ is admissible in H. By Theorem 4.8, H has three dangerous cuts Z_1, Z_2, and Z_3 with $u_1 \in Z_1 \cap Z_2 \cap Z_3$ and satisfying (i) or (ii) in Theorem 4.8 for $u = u_1$. Since $d_H(s, C_1) \geq 2$ holds, the cuts Z_1, Z_2, and Z_3 satisfy (ii). Let Z_1, Z_2 be of type (A) and Z_3 be of type (B). We have

$$V = Z_1 \cup Z_2 \cup Z_3, \ d_H(s, Z_3 - Z_1 - Z_2) = 1. \tag{4.8}$$

Then two cuts Z_1 and $Y_1 - Y_2$ do not cross each other in H, since otherwise Lemma 4.6 says $d_H(Y_1 - Y_2) = k + 1$, contradicting (4.7) (note that both of Z_1 and $Y_1 - Y_2$ are of type (A) and we have $d_H(Z_1) \leq k + 1$, $d_H(Y_1 - Y_2) = k$, and $(s, u_1) \in E_H(s, (Y_1 - Y_2) \cap Z_1)$). Similarly, two cuts Z_2 and $Y_1 - Y_2$ do not cross each other in H. By these properties, $d_H(s, Z_1 - Z_2) > 0$, and $d_H(s, Z_2 - Z_1) > 0$, there are two possible cases of (a) $Y_1 - Y_2 \subseteq Z_1 \cap Z_2$, and (b) $Y_1 - Y_2 \supseteq Z_1 \cup Z_2$. In the case of (a), by applying Lemma 4.6 to Z_1 and Z_2, we have $d_H(s, Z_1 \cap Z_2) = 1$, contradicting

$d_H(s, Z_1 \cap Z_2) \geq d_H(s, Y_1 - Y_2) \geq 2$. In the case of (b), we have $Z_3 \supseteq V - (Y_1 - Y_2)$ by (4.8). From this and $d_H(s, Y_2) \geq 2$, $d_H(s, Z_3 - Z_1 - Z_2) \geq 2$ holds, contradicting (4.8).

We finally show that after adding one edge e^* to $E_H(C_1, C_2)$, there is a complete splitting at s while preserving (4.1) in $H + e^*$. In $H' = H + e^*$, all neighbors of s are contained in the single component $C_1 \cup C_2$ in $H'[V]$. This implies that for any edge $(s, u) \in F$, there is another edge $(s, v) \in F - \{(s, u)\}$ such that $\{(s, u), (s, v)\}$ is admissible, since otherwise Theorem 4.8 (i) or (ii) hold, that is, there is a component C' in $H'[V]$ with $d_H(s, C') = 1$, contradicting that all neighbors of s in H' are contained in the single component in $H'[V]$. Since all neighbors of s remain to be contained in the component $C_1 \cup C_2$ after any splitting at s, we see that there is a complete splitting at s while preserving (4.1) in $H + e^*$. □

Before closing this section, we show that if at least one area is included in a component, then there is a complete admissible splitting at s in H, and, $opt_k(G, \mathcal{W}) = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ holds.

Theorem 4.11 *For the k-NA-ECAP with $k \geq 2$, $opt_k(G, \mathcal{W}) = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ holds if at least one area is included in a component of G.*

PROOF: Let $H = (V \cup \{s\}, E \cup F)$ be a critical k-extension of $(G, \mathcal{W})$. If $|F|$ is odd, then we add one extra edge conncecting s and a vertex in V to F and redenote the resulting graph by H; $|F|/2 = \lceil \alpha_k(G)/2 \rceil$ holds by Theorem 4.1. It suffices to show that in H, there is an edge $e' \in F - \{e\}$ such that $\{e, e'\}$ is admissible in H for any edge $e \in F$. Let $\mathcal{C}$ be the family of all components in G. Since some area $W \in \mathcal{W}$ satisfies $W \subseteq C$, then each $C \in \mathcal{C}$ is of type (A) or (B) and hence satisfies $d_H(C) = d_H(s, C) \geq k \geq 2$ by (4.1) and $k \geq 2$. Hence for any edge $e \in F$, there is an edge $e' \in F - \{e\}$ such that $\{e, e'\}$ is admissible in H, since otherwise Theorem 4.8 says that some $C' \in \mathcal{C}$ satisfies $d_H(s, C') = 1$, a contradiction. □

5 Algorithm

Based on the lower bounds in Section 3 and theorems about extensions and splittings in Section 4, we give an algorithm, called NAEC-AUG, for finding a feasible solution E' to k-NA-ECAP with $|E'| = opt_k(G, \mathcal{W})$, for a given area graph $(G, \mathcal{W})$ and an integer $k \geq 3$. It finds a feasible solution E' with $|E'| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil + 1$ if $(G, \mathcal{W})$ has property (P), $|E'| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ otherwise. The algorithm is based on edge-splitting operations and may resplit some edges which have been split off. For a graph $H = (V \cup \{s\}, E)$ and a designated vertex $s \notin V$, we say that H' is obtained from H by *hooking up* an edge $(u, v) \in E(H - s)$ at s, if we construct H' by replacing the edge (u, v) with two edges (s, u) and (s, v) in H.

The outline of algorithm NAEC-AUG is described as follows. In the first step, we first obtain a critical k-extension $H = (V \cup \{s\}, E \cup F_1)$ of a given $(G, \mathcal{W})$. Theorem 4.1 says that

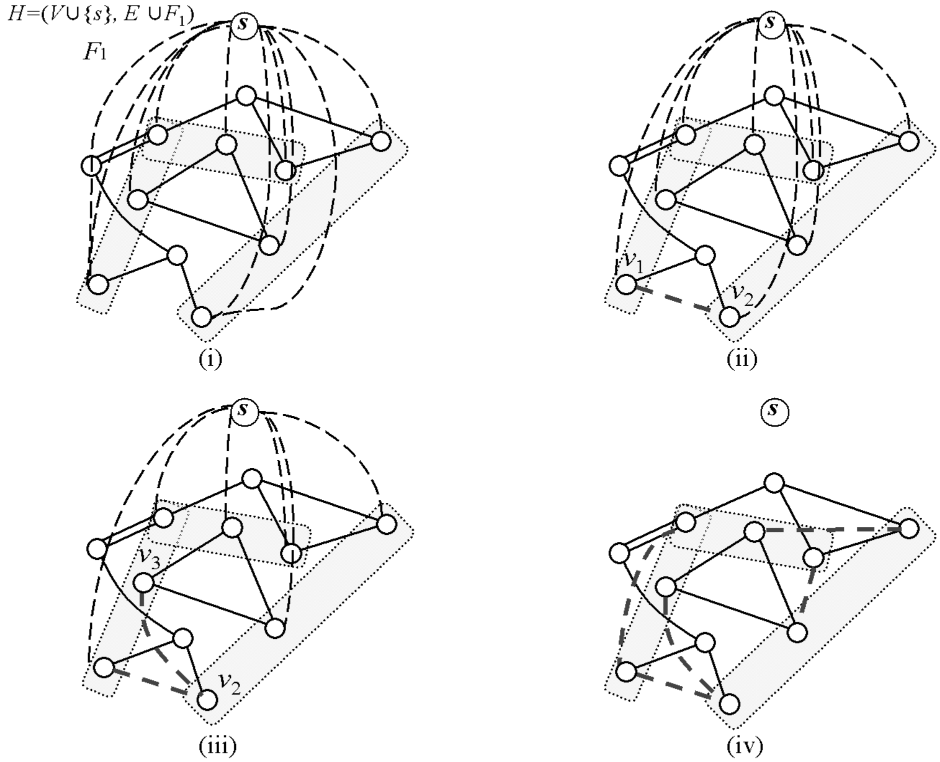

Figure 3: Computational process of algorithm NAEC-AUG applied to $k = 3$ and the area graph $(G, \mathcal{W})$ in Figure 1. The lower bound in Section 3 is $\lceil \alpha_3(G, \mathcal{W})/2 \rceil = 5$. (i) A critical k-extension $H = (V \cup \{s\}, E \cup F_1)$ of $(G, \mathcal{W})$ obtained by Step 1. Edges in F_1 are drawn as broken lines. Then $\lambda_H(v, W) \geq 3$ holds for every pair of $v \in V$ and $W \in \mathcal{W}$. (ii) $H_1 = (H - \{(s, v_1), (s, v_2)\}) \cup \{(v_1, v_2)\}$ obtained from H by an admissible splitting of (s, v_1) and (s, v_2). (iii) $H_2 = (H_1 - \{(s, v_2), (s, v_3)\}) \cup \{(v_2, v_3)\}$ obtained from H_1 by the admissible splitting of (s, v_2) and (s, v_3). (iv) H_3 obtained from H_2 by a complete admissible splitting at s. The graph $G_3 = H_3 - s$ is 3-NA-edge-connected.

$|F_1| = \alpha_k(G, \mathcal{W})$. If $|F_1|$ is odd, then we add an arbitrary one edge connecting s and a vertex in V to F_1. In the next step, we repeat admissible edge-splittings at s. If $(G, \mathcal{W})$ does not have property (P), then the algorithm finds a complete admissible splitting, and hence the set E^* of added edges satisfies $|E^*| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ and $\lambda(G + E^*, \mathcal{W}) \geq k$. If $(G, \mathcal{W})$ has property (P), then the algorithm finds such a complete splitting by adding one extra edge to G, and hence the obtained edge set E^* satisfies $|E^*| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil + 1$. In both cases, E^* is optimal by Lemmas 3.2 and 3.4.

More precisely, we describe the algorithm below, and introduce one theorem necessary to justify the algorithm, which will be proved in Section 6. An example of computational process of NAEC-AUG is shown in Figure 3.

Algorithm NAEC-AUG

Input: An area graph $(G = (V, E), \mathcal{W})$ and an integer $k \geq 3$.

Output: A set E^* of new edges with $\lambda(G + E^*, \mathcal{W}) \geq k$ and $|E^*| = opt_k(G, \mathcal{W})$.

Step 1: We find a critical k-extension $H = (V \cup \{s\}, E \cup F_1)$ of $(G, \mathcal{W})$. If $d_H(s)$ is odd, then we add to F_1 one extra edge between s and V.

Step 2: We continue to execute admissible edge-splittings at s until no pair of two edges incident to s is admissible. Let $H_2 = (V \cup \{s\}, E \cup E_2 \cup F_2)$ be the resulting graph, where $F_2 = E_{H_2}(s)$ and E_2 denotes the set of split edges.

If $F_2 = \emptyset$ holds, then halt after outputting $E^* := E_2$.

Otherwise $d_{H_2}(s) = 4$ holds and the graph $H_2 - s$ has two components C_1 and C_2 with $d_{H_2}(s, C_1) = 3$ and $d_{H_2}(s, C_2) = 1$, where $E_{H_2}(s, C_2) = \{(s, u^*)\}$ (they exist by Theorem 4.2). We have the following four cases (a) – (d).

(a) The vertex u^* is contained in no cut $X \subseteq C_2$ of type (A) with $d_{H_2}(X) = k$. Then after replacing (s, u^*) with a new edge (s, v) for some vertex $v \in C_1$ while preserving (4.1), execute a complete admissible splitting at s. Output the set E^* of all split edges, where $|E^*| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ holds.

(b) $E_2 \cap E(H_2[V - C_1]) \neq \emptyset$ holds. Then after hooking up one edge $e \in E_2 \cap E(H_2[V - C_1])$, execute a complete admissible splitting at s. Output the set E^* of all split edges, where $|E^*| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ holds.

(c) There is a set $E' \subseteq E_2$ of at most two split edges such that the graph H_3 resulting from hooking up the set E' of edges in H_2 has an admissible pair $\{(s, u^*), f\}$ for some $f \in E_{H_3}(s, V)$. After a complete admissible splitting at s in H_3, output the set E^* of all split edges, where $|E^*| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ holds.

(d) None of (a) – (c) holds. Then we can prove that $(G, \mathcal{W})$ has property (P). After adding one new edge e^* to $E_{H_2}(C_1, C_2)$, execute a complete admissible splitting at s in $H_2 + \{e^*\}$. Outputting the edge set $E^* := E_3 \cup \{e^*\}$, where E_3 denotes the set of all split edges and $|E^*| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil + 1$ holds. □

Figure 4 indicates that even in the case of $opt_k(G, \mathcal{W}) = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$, a greedy splitting in Step 2 may not construct an optimal solution unless hooking up operations are used. To justify the algorithm NAEC-AUG, it suffices to show the following theorem.

Theorem 5.1 *Let $(G, \mathcal{W})$ and H satisfy the assumption of Theorem 4.2, and $k \geq 3$ be an integer. Let H^* be a graph obtained by a sequence of admissible splittings at s from H such that $E_{H^*}(s, V) \neq \emptyset$ holds and no pair of two edge in $E_{H^*}(s, V)$ is admissible in H^*. Let C_1 and C_2 be two components in $H^* - s$ with $d_{H^*}(s, C_1) = 3$ and $d_{H^*}(s, C_2) = 1$ (they exist by Theorem 4.2).*

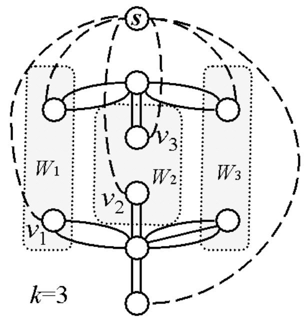

Figure 4: Illustration of a graph $H = (V \cup \{s\}, E \cup F_1)$ satisfying (4.1) for $k = 3$, where edges in F_1 are drawn by broken lines. If we first execute the admissible splitting of (s, v_2) and (s, v_3), then a complete splitting can be found. However, the resulting graph $H_1 = (H - \{(s, v_1), (s, v_2)\}) \cup \{(v_1, v_2)\}$ obtained from H by the admissible splitting of (s, v_1) and (s, v_2) has no admissible splitting pair at s.

Then if H^* satisfies one of the following conditions (a)–(c), then H has a complete admissible splitting at s after replacing at most one edge in $E_H(s, V)$. Otherwise $(G, \mathcal{W})$ has property (P).
(a) For $\{(s, u^*)\} = E_{H^*}(s, C_2)$, u^* is contained in no cut $X \subseteq C_2$ of type (A) with $d_{H^*}(X) = k$.
(b) $E_1 \cap E(H^*[V - C_1]) \neq \emptyset$ holds, where E_1 denotes the set of all split edges.
(c) There is a set $E' \subseteq E_1$ of at most two split edges such that the graph H' resulting from hooking up the set E' of edges in H^* has an admissible pair $\{(s, u^*), f\}$ for some $f \in E_{H'}(s, V)$.

PROOF: See Section 6. □

By Theorems 4.2 and 5.1, for the set E^* of edges obtained by algorithm NAEC-AUG, the graph $H^* = (V \cup \{s\}, E \cup E^*)$ satisfies (4.1), i.e., $d_{H^*}(X) \geq k$ for all cuts $X \subset V$ of type (A) or (B). By $d_{H^*}(s) = 0$, we have $d_{G+E^*}(X) = d_{H^*}(X) \geq k$ for all cuts $X \subset V$ of type (A) or (B). This implies that the graph $G + E^*$ is k-NA-edge-connected. By Theorems 4.1 and 5.1, we have $|E^*| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil + 1$ in the cases where an initial area graph $(G, \mathcal{W})$ has property (P), $|E^*| = \lceil \alpha_k(G, \mathcal{W})/2 \rceil$ otherwise. By Lemmas 3.2 and 3.4, we have $|E^*| = opt_k(G, \mathcal{W})$.

We analyze the time complexity of algorithm NAEC-AUG. It is not difficult to verify that it can be computed in polynomial time. In Step 1, we first add k edges between s and V so that H satisfies (4.1) and we attain F_1 minimal with respect to (4.1) as follows. For each vertex $v \in V$, after deleting all edges between s and v, we check whether the resulting graph H' satisfies (4.1) or not. By regarding k multiple edges as one edge with capacity k and using the maximum flow technique, we can compute in polynomial time $\lambda_{H'}(v, W)$ for a vertex $v \in V$ and an area $W \in \mathcal{W}$; we can check in polynomial time whether H' satisfies (4.1) or not. If (4.1) is violated, then we add $\max_{x \in V, W \in \mathcal{W}}\{k - \lambda_{H'}(x, W)\}$ edges between s and v in H'. In

Step 2, for each pair $\{u,v\} \subseteq V$, after splitting $\min\{d_H(s,u), d_H(s,v)\}$ pairs $\{(s,u),(s,v)\}$, we check whether the resulting graph H' satisfies (4.1) or not. If (4.1) is violated, then we hook up $\lceil \frac{1}{2} \max_{x\in V, W\in\mathcal{W}}\{k - \lambda_{H'}(x,W)\}\rceil$ pairs in H'. The procedures (a) – (d) can be also executed in polynomial time since the number of hooking up operations is $O(n^4)$.

By using further analysis, we can prove that it can be implemented to run in $O(n(n^2+k^3)(p+m+n\log n)\log k + pmn^2 \log(n^2/m))$, whose proof is omitted. As a result, this total complexity can be reduced to $O(m + n(k^3+n^2)(p+kn+n\log n)\log k + pkn^3\log(n/k))$ by applying the procedure to a sparse spanning subgraph of G with $O(kn)$ edges, where such sparsification takes $O(m+n\log n)$ time [16, 17].

Lemma 5.2 *Algorithm NAEC-AUG can be implemented to run in* $O(m+n(k^3+n^2)(p+kn+n\log n)\log k + pkn^3\log(n/k))$ *time.* □

Summarizing the argument given so far, Theorem 3.5 is now established.

6 Proof of Theorem 5.1

We first show the following property of minimal dangerous cuts.

Lemma 6.1 *Let* $H = (V\cup\{s\}, E\cup F)$ *be a* k*-extension of* $(G=(V,E),\mathcal{W})$ *and* $\{(s,u),(s,v)\}$ *be a pair of edges in* F *which is not admissible. Let* $Y \subset V$ *be a dangerous cut of type* (B) *with* $\{u,v\}\subseteq Y$ *such that no* $Y' \subset Y$ *with* $\{u,v\}\subseteq Y'$ *is a dangerous cut of type* (B)*. Then* Y *is unique.*

Proof: Assume by contradiction that there are two distinct dangerous cuts Y_1 and Y_2 of type (B) with $\{u,v\}\subseteq Y_1\cap Y_2$ such that no $Y'\subset Y_i$ with $\{u,v\}\subseteq Y'$ is a dangerous cut of type (B) for each $i=1,2$. From the minimality of Y_i, Y_1 and Y_2 cross each other in H. By Lemma 4.6, we have $d_H(s, Y_1\cap Y_2) = 1$, contradicting $\{u,v\}\subseteq Y_1\cap Y_2$. □

In this section, let G, H, and H^* satisfy the assumption of Theorem 5.1, E_1 be the set of all split edges in H^*, and $F^* = E_{H^*}(s,V)$ (note $H^* = (V\cup\{s\}, E\cup E_1\cup F^*)$). Since H^* has no admissible splitting pair at s, Theorem 4.2 implies that the graph $G^* = H^* - s$ has two components C_1 and C_2 with $d_{H^*}(s,C_1)=3$ and $d_{H^*}(s,C_2)=1$. Let $V[F^*]\cap C_1 = \{u_0,u_1,u_2\}$ and $V[F^*]\cap C_2 = \{u_3 = u^*\}$. An outline of the proof of Theorem 5.1 is given as follows. We first show that if at least one of the conditions (a) – (c) in Theorem 5.1 hold, then H^* can be modified to a graph H' by replacing or hooking up edges in $F^*\cup E_1$ so that H' has a complete admissible splitting. We next show that if none of (a) – (c) holds, then $(G,\mathcal{W})$ has property (P).

For a k-extension $H=(V\cup\{s\},E\cup F)$ of $(G=(V,E),\mathcal{W})$, a cut $X\subset V$ is called *critical* if X is of type (A) or (B) and we have $d_H(X)=k$. The next lemma shows the cases where (a) holds.

Lemma 6.2 *Let $k \geq 2$. If u_3 is contained in no critical cut of type (A) in H^*, then after replacing the edge (s, u_3) with a new edge (s, x) for some vertex $x \in C_1$, we can continue admissible edge-splittings until isolating s.*

PROOF: Assume that u_3 is contained in no critical cut of type (A) in H^*. Then u_3 is contained in a critical cut of type (B) or no critical cut. Let X_u denote a critical cut of type (B) with $u_3 \in X_u \subset V$ such that no cut $X' \subset X_u$ with $u_3 \in X'$ is critical of type (B) if exists, $X_u = V$ otherwise. Then $X_u \cap C_1 \neq \emptyset$ holds since otherwise $V - C_1$ is of type (B) and hence $d_{H^*}(V - C_1) \geq k \geq 2$ holds by (4.1), contradicting $d_{H^*}(V - C_1) = d_{H^*}(s, C_2) = 1$. Let $H_1 = (H^* - \{(s, u_3)\}) \cup \{(s, x)\}$ be the graph obtained from replacing the edge (s, u_3) with (s, x) with some $x \in X_u \cap C_1$ in H^*.

We claim that H_1 also satisfies (4.1). Assume by contradiction that H_1 violates (4.1). Then H^* has a critical cut $X' \subset V$ with $u_3 \in X' \cap X_u$ and $x \in X_u - X'$. Note that X' is of type (B) from the assumption of u_3. We have $X' - X_u \neq \emptyset$ from the minimality of X_u and hence X_u and X' cross each other in H^*. Now $X_u - X'$ (resp., $X' - X_u$) is of type (A) since it is disjoint with an area included in X' (resp., X_u) (note that both of X' and X_u are of type (B)). By (4.1), we have $d_{H^*}(X_u - X') \geq k$ and $d_{H^*}(X' - X_u) \geq k$. From $d_{H^*}(X_u) = d_{H^*}(X') = k$ and (2.1), we have $d_{H^*}(s, X_u \cap X') = 0$, contradicting $u_3 \in X_u \cap X'$.

All neighbors of s in H_1 are contained in a component in $H_1[V]$, and hence we can continue admissible edge-splittings until isolating s in H_1 by Theorem 4.2. $\square$

Next we show the cases where (b) or (c) hold in the following Lemmas 6.4 and 6.5, respectively. For $(s, u_0) \in F^*$, none of $\{(s, u_0), (s, u_i)\}$ for $i = 1, 2, 3$ is admissible in H^*. Hence Theorem 4.8 implies that H^* has two dangerous cuts Y_1 and Y_2 of type (A) with $C_1 = Y_1 \cup Y_2$ and $\{(s, u_0), (s, u_i)\} = E_{H^*}(s, Y_i)$ for $i = 1, 2$. By Lemma 4.6, we have $d_{H^*}(Y_1 - Y_2) = d_{H^*}(Y_2 - Y_1) = k$. Moreover, the graph H^* has the following property.

Lemma 6.3 *Let $k \geq 3$. $\lambda(G^*[C_1]) \geq 2$ holds.*

PROOF: $G^*[C_1]$ is connected and so assume by contradiction that $\lambda(G^*[C_1]) = 1$, i.e., there is a cut $X \subset C_1$ with $d_{G^*}(X, C_1 - X) = 1$. Let $d_{H^*}(s, X) \leq 1$ without loss of generality (note that $d_{H^*}(s, C_1) = 3$ holds). Since $Y_1 - Y_2$ is a cut of type (A) with $d_{H^*}(Y_1 - Y_2) = k$, we have $\lambda(G^*[Y_1 - Y_2]) \geq 2$ by Lemma 4.5. Similarly, $\lambda(G^*[Y_2 - Y_1]) \geq 2$ holds. Hence X cannot cross with any of $Y_1 - Y_2$ and $Y_2 - Y_1$. Moreover, by $d_{H^*}(s, X) \leq 1$, X can contain at most one of $Y_1 - Y_2$ and $Y_2 - Y_1$. This implies that $X \subseteq Y_1$ or $X \subseteq Y_2$ holds, contradicting that any cut X' of type (A) must satisfy $d_{H^*}(X') \geq k \geq 3$. $\square$

Lemma 6.4 *Let $k \geq 3$. If $E_1 \cap E(G^*[V - C_1]) \neq \emptyset$ holds, then after hooking up one edge in $E_1 \cap E(G^*[V - C_1])$, we can continue admissible edge-splittings until isolating s.*

Proof: Let $e_1 = (x, y) \in E_1 \cap E(G^*[V - C_1])$ and $H_1 = (H^* - \{(x, y)\}) \cup \{(s, x), (s, y)\}$ be the graph obtained from hooking up e_1 in H^*. We claim that

$$\{(s, u_0), f\} \text{ is admissible in } H_1 \text{ for some edge } f = (s, z) \text{ with } z \in \{u_3, x, y\}. \tag{6.1}$$

Note that also in H_1, neither $\{(s, u_0), (s, u_1)\}$ nor $\{(s, u_0), (s, u_2)\}$ is admissible. Assume by contradiction that (6.1) does not hold. Then there is no edge $f' \in E_{H_1}(s, V)$ such that $\{f', (s, u_0)\}$ is admissible in H_1. By $d_{H_1}(s, C_1) \geq 3$, the statement (ii) in Theorem 4.8 holds for the edge (s, u_0). This and Lemma 4.5 imply that all neighbors of s except exactly one neighbor are contained in a single component in $H_1[V]$, a contradiction. Therefore (6.1) holds.

Let H_2 be the graph obtained from splitting $\{(s, u_0), f\}$ in H_1. We here claim that there is a sequence of admissible edge-splittings in H_2 until the vertex s is isolated. Otherwise Theorem 4.2 says that $H_2[V]$ has two components $C'_1, C'_2 \subset V$ with $d_{H_2}(s, C'_1) = 3$ and $d_{H_2}(s, C'_2) = 1$. By $k \geq 3$ and Lemma 6.3, we have $\lambda(H_2[C'_1]) \geq 2$ (note that $d_{H_2}(s) = 4$ holds). Clearly, $\{u_0, u_1, u_2, z\} \subseteq C'_1$ hold. However, the edge (u_0, z) is a bridge in $H_2[C'_1]$, a contradiction. □

Lemma 6.5 *Let $k \geq 3$ and $E_1 \cap E(G^*[C_1]) \neq \emptyset$. Assume that there is a set E' of at most two split edges in $E_1 \cap E(G^*[C_1])$ in H^* such that the graph H_1 resulting from hooking up all edges in E' in H^* has an admissible pair $\{f, (s, u_3)\}$ for some edge $f \in E_{H_1}(s, V)$. Then in H_1, we can continue admissible edge-splittings until isolating s.*

Proof: Let $E'_1 \subseteq E_1 \cap E(G^*[C_1])$ denote a set of split edges with $|E'_1| \leq 2$ such that $\{f = (s, x), (s, u_3)\}$ is admissible for some $f \in E_{H_1}(s, V)$ in H_1, where H_1 results from hooking up all edges in E'_1 in H^*. Let $H_2 = (H_1 - \{(s, x), (s, u_3)\}) \cup \{(x, u_3)\}$. If $H_1[C_1]$ is connected, then all neighbors of s in H_2 are contained in a component in $H_2[V]$, and hence Theorem 4.2 implies that we can execute a complete splitting at s in H_2 while preserving (4.1).

We consider the case where $H_1[C_1]$ is not connected. By Lemma 6.3, $\lambda(G^*[C_1]) \geq 2$ holds. Hence we have $|E'_1| = 2$ and the number of components in $H_1[C_1]$ is two. Let C'_1, C'_2 be components in $H_1[C_1]$ with $d_{H_1}(s, C'_1) \geq d_{H_1}(s, C'_2)$ without loss of generality (note that $E'_1 = E_{G^*}(C'_1, C'_2)$ holds). Then by $d_{H^*}(s, C_1) = 3$, we have the following two possible cases (I) and (II). (I) $d_{H_1}(s, C'_1) = 5$ and $d_{H_1}(s, C'_2) = 2$ and (II) $d_{H_1}(s, C'_1) = 4$ and $d_{H_1}(s, C'_2) = 3$.

In both cases of (I) and (II), all neighbors of s in H_2 are contained in at most two components in $H_2[V]$ from the construction of H_2. By $d_{H_2}(s) = 6$, Theorem 4.2 says that H_2 always has an admissible pair $\{(s, v'), (s, v'')\} \subseteq E_{H_2}(s, V)$. Let H_3 denote the graph obtained from splitting (s, v') and (s, v'') in H_2. If H_3 also has an admissible pair at s, then the lemma is proved. Otherwise Theorem 4.2 says that $H_3[V]$ has two components C^*_1 and C^*_2 with $d_{H_3}(s, C^*_1) = 3$, and $d_{H_3}(s, C^*_2) = 1$. Then in both cases of (I) and (II), $E(H_3[C^*_2])$ contains the split edges f^* with $f^* = (x, u_3)$ or $f^* = (v', v'')$. Lemma 6.4 implies that after hooking up the edge f^*, we can execute a complete splitting at s while preserving (4.1), which proves the lemma. □

We finally show that if none of (a) – (c) holds, then $(G, \mathcal{W})$ has property (P) by the following two lemmas.

Lemma 6.6 *Let H^* satisfy none of (a) – (c) in Theorem 5.1. Then for every vertex $v \in V[E_1] \cup (V[F^*] - \{s\})$, there is a cut $X_v \subset V$ with $v \in X_v$ satisfying the following (i) and (ii).*
(i) $d_{H^*}(X_v) = k$ and X_v is of type (A).
(ii) If $v \in C_1$ holds, then there is a cut $Y_v \supseteq X_v$ of type (B) with $u_3 \in Y_v$, $d_{H^*}(Y_v) \leq k+1$, and $E(G^*[Y_v]) \cap E_1 = \emptyset$.

Proof: Since (a) does not hold, the lemma for $v = u_3$ holds. Since (b) does not hold, we have $V[E_1] \cup (V[F^*] - \{s, u_3\}) \subseteq C_1$. We first show the lemma for vertices in $V[F^*] - \{s, u_3\}$ and next show it for vertices in $V[E_1]$.

Let Y_i' be a dangerous cut containing two vertices u_i and u_3 in H^* such that no cut $Y'' \subset Y_i'$ with $\{u_i, u_3\} \subseteq Y''$ is dangerous in H^* for $i = 0, 1, 2$. Lemma 6.1 says that Y_i' is unique (note that any dangerous cut Y containing u_3 and a vertex in C_1 is of type (B) since $H^*[Y]$ is disconnected). By Lemma 4.3, we have $d_{H^*}(s, V - Y_i') \geq d_{H^*}(s, Y_i') - 1$ and hence $Y_i' \cap \{u_0, u_1, u_2\} = \{u_i\}$. Hence for any pair $\{i, j\} \subset \{0, 1, 2\}$ with $i \neq j$, Y_i' and Y_j' cross each other in H^*. By Lemma 4.6, both of $Y_i' - Y_j'$ and $Y_j' - Y_i'$ are of type (A) and we have $d_{H^*}(Y_i' \cap Y_j', V \cup \{s\} - Y_i' - Y_j') = d_{H^*}(s, u_3) = 1$,

$$d_{H^*}(Y_i' - Y_j') = d_{H^*}(Y_j' - Y_i') = k, \ u_i \in Y_i' - Y_j', \text{ and } u_j \in Y_j' - Y_i'.$$

Then we claim that

$$E(G^*[Y_i']) \cap E_1 = \emptyset \tag{6.2}$$

holds for $i = 0, 1, 2$. Assume by contradiction that $E(G^*[Y_i']) \cap E_1 \neq \emptyset$ holds. Let $(y_1, y_2) \in E(G^*[Y_i']) \cap E_1$ and $H_1 = (H^* - \{(y_1, y_2)\}) \cup \{(s, y_1), (s, y_2)\}$ be the graph obtained by hooking up the edge (y_1, y_2) in H^*. Then $\{(s, u_i), (s, u_3)\}$ is admissible in H_1, because any cut Y with $\{u_i, u_3\} \subseteq Y$ which is dangerous in H^* satisfies $Y \supseteq Y_i'$ by the minimality and uniqueness of Y_i' and hence $d_{H_1}(Y) = d_{H^*}(Y) + 2$, and hence it is non-dangerous in H_1. This contradicts that H^* does not satisfy (c). Hence (6.2) holds for each $i = 0, 1, 2$. Therefore the cuts $X_{u_i} = Y_i' - Y_j'$ and $Y_{u_i} = Y_i'$ for $i = 0, 1, 2$ prove the lemma for any vertex $v \in V[F^*] - \{s, u_3\}$.

Finally, we prove the lemma for any vertex $v \in V[E_1]$. Let $(x_1, x_2) \in E_1$ and $H_2 = (H^* - \{(x_1, x_2)\}) \cup \{(s, x_1), (s, x_2)\}$ be the graph obtained by hooking up the edge (x_1, x_2) in H^*. Since (c) does not hold, neither $\{(s, u_3), (s, x_1)\}$ nor $\{(s, u_3), (s, x_2)\}$ is admissible in H_2. Let $Y_i' \subset V$ denote a dangerous cut with $\{u_3, x_i\} \subseteq Y_i'$ in H_2 such that no $Y'' \subset Y_i'$ with $\{u_3, x_i\} \subseteq Y''$ is dangerous in H_2 for $i = 1, 2$. Lemma 6.1 says that Y_i' is unique. Since $\{(s, x_1), (s, x_2)\}$ is clearly admissible in H_2, we have $x_1 \in Y_1' - Y_2'$ and $x_2 \in Y_2' - Y_1'$, and Y_1' and Y_2' cross each other in H_2. By Lemma 4.6, both of $Y_1' - Y_2'$ and $Y_2' - Y_1'$ are of type (A) and we have

$d_{H_2}(Y_1' \cap Y_2', V \cup \{s\} - Y_1' - Y_2') = d_{H_2}(s, u_3) = 1$ and

$$d_{H_2}(Y_1') = d_{H^*}(Y_1'),\ d_{H_2}(Y_1' - Y_2') = d_{H^*}(Y_1' - Y_2') = k,\ x_1 \in Y_1' - Y_2',$$
$$d_{H_2}(Y_2') = d_{H^*}(Y_2'),\ d_{H_2}(Y_2' - Y_1') = d_{H^*}(Y_2' - Y_1') = k,\ x_2 \in Y_2' - Y_1'.$$

Moreover, similarly to the arguments about vertices in $V[F^*] - \{s, u_3\}$, by using the assumption that (c) does not hold, we see that $E(G^*[Y_i']) \cap E_1 = \emptyset$ holds for $i = 1, 2$. Therefore the cuts $X_{x_i} = Y_i' - Y_j'$ and $Y_{x_i} = Y_i'$ for $\{i, j\} = \{1, 2\}$ prove the lemma for any vertex $v \in V[E_1]$. □

Lemma 6.7 *Let H^* satisfy none of (a) – (c) in Theorem 5.1. Then G has property* (P).

Proof: By Lemma 6.6, for any $v \in V[E_1] \cup (V[F^*] - s)$, there are two cuts X_v and Y_v of Lemma 6.6 (i) and (ii). Then let X_v be a cut such that no cut $X' \subset X_v$ with $v \in X'$ satisfies this property. Moreover, we can assume that $Y_v \supseteq V - C_1$ holds. This follows since for any cut Y_v satisfying Lemma 6.6(ii), we have $X_v \subseteq Y_v \cup (V - C_1)$, $d_{H^*}(Y_v \cup (V - C_1)) \leq d_{H^*}(Y_v) \leq k+1$ (by $d_{H^*}(V - C_1) = d_{H^*}(s, u_3) = 1$ and $u_3 \in Y_v$), and $E(G^*[Y_v \cup (V - C_1)]) \cap E_1 = \emptyset$ (since (b) does not hold), which implies that the cut $Y_v \cup (V - C_1)$ also satisfies Lemma 6.6(ii). Let $\mathcal{X}$ be the family of all cuts X_v, $v \in V[E_1] \cup (V[F^*] - s)$ such that $\bigcup_{X \in \mathcal{X}} X$ includes $V[E_1] \cup (V[F^*] - s)$ and no cut $X_v \in \mathcal{X}$ satisfies $X_v \subset X$ for some $X \in \mathcal{X}$, and $\mathcal{Y}$ be the family of the corresponding cuts Y_v. We will show that $\alpha_k(G, \mathcal{W})$ is even and the family $\mathcal{X}$ is a subpartition of V satisfying $\sum_{X \in \mathcal{X}}(k - d_G(X)) = \alpha_k(G, \mathcal{W})$ and (P1)–(P3).

We claim that

$$\mathcal{X} \text{ is a subpartition of } V. \tag{6.3}$$

Otherwise there are two cuts $X_u, X_v \in \mathcal{X}$ which cross each other in H^*. Since both of X_u and X_v are of type (A), by (4.1) and (4.2), we have $d_{H^*}(X_u - X_v) \geq k$ and $d_{H^*}(X_v - X_u) \geq k$. From $d_{H^*}(X_u) = d_{H^*}(X_v) = k$ and (2.1), we have $d_{H^*}(X_u - X_v) = d_{H^*}(X_v - X_u) = k$ and $d_{H^*}(X_u \cap X_v, (V \cup \{s\}) - X_u - X_v) = 0$. Then $u \in X_u - X_v$ holds, since if $u \in X_u \cap X_v$ holds, then there is a split edge $(u, u') \in E_1$ with $u' \in X_u \cup X_v$ by $d_{H^*}(X_u \cap X_v, (V \cup \{s\}) - X_u - X_v) = 0$, which contradicts $E(G^*[X_u]) \cap E_1 = \emptyset = E(G^*[X_v]) \cap E_1$. From $d_{H^*}(X_u - X_v) = k$ and $u \in X_u - X_v$, the cut $X_u - X_v$ contradicts the minimality of X_u.

By (6.3) and $E(G^*[X]) \cap E_1 = \emptyset$ for every $X \in \mathcal{X}$, we have $\sum_{X \in \mathcal{X}}(k - d_G(X)) = \alpha_k(G, \mathcal{W})$, and $\alpha_k(G, \mathcal{W})$ is even by $\alpha_k(G, \mathcal{W}) = 2|E_1| + |F^*|$. Moreover, $\mathcal{X}$ is a subpartition of V satisfying (P1) and (P2) by taking $X_1 = X_{u_3}$. Now for every $Y \in \mathcal{Y}$ which does not cross with any $X_i \in \mathcal{X}$ in H^*, we have $\sum_{X' \in \mathcal{X}, X' \subseteq Y}(k - d_G(X')) \leq (k+1) - d_G(Y)$ by $E(G^*[Y]) \cap E_1 = \emptyset$. Therefore, in order to show that $\mathcal{X}$ satisfies (P3), we finally prove that for any $X_u \in \mathcal{X}$ with $u \neq u_3$, there is a cut $Y_w \in \mathcal{Y}$ with $X_u \subseteq Y_w$ such that for any cut $X \in \mathcal{X}$, Y_w and X do not cross each other in H^*. For this, it suffices to show that

$$\begin{aligned}&\text{if there is a cut } Y_u \in \mathcal{Y} \text{ which crosses with some } X_v \in \mathcal{X} \text{ in } H^*,\\&\text{then } v \neq u_3 \text{ and } Y_u \subseteq Y_v \text{ hold.}\end{aligned} \tag{6.4}$$

Since each $Y \in \mathcal{Y}$ satisfies $X_{u_3} \subseteq V - C_1 \subseteq Y$, $v \neq u_3$ holds. Assume by contradiction that $Y_u - Y_v \neq \emptyset$ holds. By $X_v - Y_u \neq \emptyset \neq X_v \cap Y_u$, Y_u and Y_v cross each other in H^*. By Lemma 4.6, $Y_v - Y_u$ is of type (A) and we have $d_{H^*}(Y_u - Y_v) = d_{H^*}(Y_v - Y_u) = k$ and $d_{H^*}(s, u_3) = d_{H^*}(Y_u \cap Y_v, V \cup \{s\} - Y_u - Y_v) = 1$. Note that $X_v - (Y_v - Y_u) \neq \emptyset$ holds since X_v and Y_u cross each other in H^*. We have $v \in X_v - Y_u$ since if $v \in Y_u \cap Y_v$ holds, then there is a split edge $(v, v') \in E_1 \cap E(G^*[Y_u \cup Y_v])$ by $E_{H^*}(Y_u \cap Y_v, V \cup \{s\} - Y_u - Y_v) = \{(s, u_3)\}$, which contradicts $E(G^*[Y_u]) \cap E_1 = \emptyset = E(G^*[Y_v]) \cap E_1$. By $v \in X_v - Y_u$, we have $X_v \cap (Y_v - Y_u) \neq \emptyset$. Moreover, $(Y_v - Y_u) - X_v \neq \emptyset$ holds since if $Y_v - Y_u \subseteq X_v$ holds, then the cut $Y_v - Y_u$ of type (A) contradicts the minimality of X_v by $d_{H^*}(Y_v - Y_u) = k$, $v \in Y_v - Y_u$, and $X_v - (Y_v - Y_u) \neq \emptyset$. This means that X_v and $Y_v - Y_u$ cross each other in H^*. Now $d_{H^*}(X_v \cap (Y_v - Y_u), V \cup \{s\} - X_v - (Y_v - Y_u)) > 0$ holds since $E(G^*[Y_v]) \cap E_1 = \emptyset$ implies that the edge $(v, v') \in E_1 \cup F^*$ satisfies $v' \notin Y_v$. By applying Lemma 4.6, we have $d_{H^*}(X_v) = k + 1$, contradicting $d_{H^*}(X_v) = k$ (note that both of X_v and $Y_v - Y_u$ are of type (A)). Hence (6.4) holds. □

7 Conclusion

In this paper, we considered the problem of asking to augment a given area graph $(G = (V, E), \mathcal{W})$ by adding the minimum number of new edges such that the resulting graph becomes k-NA-edge-connected. We showed that the problem in the case of $k \geq 3$ can be solved in polynomial time. The time complexity of our algorithm is $O(m + n(k^3 + n^2)(p + kn + n \log n) \log k + pkn^3 \log(n/k))$, where $n = |V|$, $m = |\{\{u, v\} | (u, v) \in E\}|$, and $p = |\mathcal{W}|$. This paper treated the cases where the connectivity requirement between a vertex $v \in V$ and an area $W \in \mathcal{W}$ is unique. It is a future work to consider the problems with general connectivity requirements.

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