On r-fold Partitions and a Certain Form of Infinite Products

by

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1. Let M be a set and $R_1,\dots,R_r \subset M \times M$ be equivalence relations among M. A structure $\mathbf{M}=(M;\ R_1,\dots,R_r)$ is called an *r-fold partition of set M* if $R_1 \subset \dots \subset R_r$. Let $\mathbf{M}=(M;\ R_1,\dots,R_r)$ and $\mathbf{M}'=(M';\ R_1',\dots,R_r')$ be two r-fold partitions of sets M and M' respectively. A bijection $\varphi: M \longrightarrow M'$ is called *isomorphism* if $\varphi(R_i)=R_i'$ for all $1 \leqslant i \leqslant r$. we then say that $\mathbf{M}$ and $\mathbf{M}'$ are *isomorphic* and denote $\mathbf{M} \cong \mathbf{M}'$. Thus " $\cong$ " is an equivalence relation. We write the set of all r-fold partitions of $\{1,\dots,n\}$ by $\tilde{P}(r;n)$ and the quotient set $\tilde{P}(r;n)/\cong$ by $P(r;n)$. Let us call an element of $P(r;n)$ an *r-fold partition of n.* we note Card $\tilde{P}(0;\ n)$ =Card $P(0;\ n)=1$, since in this case $\mathbf{M}$ is regarded as a no-structured set, and that Card $\tilde{P}(r;\ 0)$ =Card $P(r;0)$ =1, since in this case M is the empty set. An r-fold partition of $n \geqslant 1$ can be interpreted as a representation of n as the sum of any number of positive integral parts such that every part is closed $r-1$ tames by parenthe es.

Example. Let $M=\{1,\dots,6\}$ and $M/R_1=\{\{1\},\ \{2,\ 3\}\ \ \{4\},\ \{5,\ 6\}\}$, $M/R_2=\{\{1\},\ \{2,\ 3\},\ \{4,\ 5,\ 6\}\}$, $M/R_3=\{\{1\ \ 2,\ 3\},\ \{4,\ 5,\ 6\}\}$. Then $\mathbf{M}=(M;\ R_1,\ R_2,\ R_3)$ is a 3-fold partition of M. $\mathbf{M}$ has $6!/4=180$ different isomorphic 3-fold partitions in $\tilde{P}(3;\ 6)$ (see Fig.).

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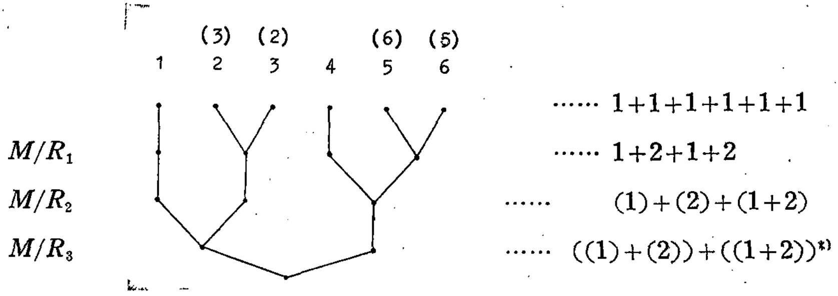

Fig. The structure tree of **M** and *) sum representation with two tames parentheses of **M**.

We can get same "structure tree" and "sum representation with $r-1$ times parentheses", for all elements of each isomorphic class of **M** $\in \tilde{P}(r;\,n)$ as above, without difference of order.

We define *the r-fold set partition function* and *the r-fold partition function* by

$$\tilde{p}(r;\,n) = \mathrm{Card}\,\tilde{P}(r;\,n)$$

and

$$p(r;\,n) = \mathrm{Card}\,P(r;\,n)$$

respectively. Under combinatorial consideration, we have

$$(1)\quad \tilde{p}(r;\,n) = \sum_{\substack{1\cdot s_1+2\cdot s_2+\cdots=n\\ s_1,s_2,\cdots\geqq 0}} \frac{n!\ \tilde{p}(r-1;\,1)^{s_1}\ \tilde{p}(r-1;\,2)^{s_2}\cdots\cdots}{(1!)^{s_1}\ s_1!\ (2!)^{s_2}\ s_2!\cdots\cdots}$$

and

$$(2)\quad p(r;\,n) = \sum_{\substack{1\cdot s_1+2\cdot s_2+\cdots=n\\ s_1,s_2,\cdots\geqq 0}} {}_{p(r-1;1)}H_{s_1}\ {}_{p(r-1;2)}H_{s_2}\cdots\cdots$$

TABLE OF $p(r;n)$

n	$p(r;n)$
1	1
2	$r+1$
3	$(1/2!)(r+1)(r+2)$
4	$(1/3!)(r+1)(r+2)(2r+3)$
5	$(1/4!)(r+1)(r+2)(5r^2+11r+12)$
6	$(1/5!)(r+1)(r+2)(16r^3+52r^2+92r+60)$
7	$(1/6!)(r+1)(r+2)(61r^4+252r^3+527r^2+600r+360)$
8	$(1/7!)(r+1)(r+2)(272r^5+1361r^4+3472r^3+5587r^2+5268r+2520)$

$p(r;n)$

n \ r	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10	11
3	1	3	6	10	15	21	28	36	45	55	66
4	1	5	14	30	55	91	140	204	285	385	506
5	1	7	27	75	170	336	602	1002	1575	2365	3421
6	1	11	58	206	571	1337	2772	5244	9237	15367	24368
7	1	15	111	518	1789	5026	12166	26328	52221	96613	168861
8	1	22	223	1344	5727	19193	54046	133476	297633	611644	1175845

for $r \geqslant 1$, where ${}_nH_m$ is the number of repeated combinations of choosing m objects from a collection of n distinct objects, namely

$$ {}_nH_m = \frac{(n+m-1)!}{m!(n-1)!} \,. $$

The numbers $\tilde{p}(r;\, n)$ have been treated by E. T. Bell [1]. He defines this numbers by

$$E(r;\, x) = \sum_{n=0}^{\infty} \tilde{p}(r;\, n) \frac{x^n}{n!}, \tag{3}$$

where
$$\begin{cases} E(0;\, x) = e^x \\ E(r;\, x) = \exp\,(E(r\text{-}1;\, x) - 1)\,. \end{cases}$$

Conversely we can easily see (3) from (1), using Faà di Bruno's formula

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{t=1}^{n} f^{(t)}(g(x)) \sum \frac{n!\,\{g'(x)\}^{s_1}\,\{g''(x)\}^{s_2}\cdots}{(1!)^{s_1}\, s_1!\, (2!)^{s_2}\, s_2!\cdots\cdots}$$

summed over $1\cdot s_1 + 2\cdot s_2 + \cdots = n$ and $s_1 + s_2 + \cdots = t$. $\tilde{p}(1;\, n)$ is well-known as the n-th Bell number. On the other hand, $p(n) = p(1;n)$ is the number of the usual partitions of n. The generating function of $p(n)$ was found by Euler, and is

$$F(1;\, x) = \sum_{n=0}^{\infty} p(n)\, x^n = \prod_{m=1}^{\infty} (1\text{-}x^m)^{-1}, \ |x| < 1.$$

Cayley [2] refered to the numbers $p(2;\, n)$ and found the generating function

$$F(2;\, x) = \sum_{n=0}^{\infty} p(2;\, n)\, x^n = \prod_{m=1}^{\infty} (1\text{-}x^m)^{-p(m)}\,.$$

More generally, we can derive the following

THEOREM 1. *Let* $\{a(n)\}_{n=1,2,\cdots}$ *be any complex number sequence and let*

$$(4) \qquad b(n) = \sum_{1.s_1+2.s_2+\cdots=n} \prod_{i=1}^{n} a(i)^{[s_i]},$$

where $x^{[s]} = x(x+1)(x+2)\cdots(x+s-1)/s!$, $s\geqslant 1$ *and* $x^{[0]}=1$. *Then the infinite product*

$$(\mathrm{A}) \qquad \prod_{m=1}^{\infty}(1-z^m)^{-a(m)} \equiv \prod_{m=1}^{\infty}\Big(1+\sum_{k=1}^{\infty} a(m)^{[k]} z^{mk}\Big)$$

is convergent in the formal power series ring $\mathbf{C}[[z]]$ *and is equal to*

$$(\mathrm{B}) \qquad 1+\sum_{n=1}^{\infty} b(n)z^n.$$

If $\sum_{n=1}^{\infty} a(n)\, z^n$ *has a positive or infinite radius* R *of convergence, then* (A) *is uniformly convergent in any compact subset of* $\{z;|z|<\min(1, R)\}$ *and is equal to* (B).

COROLLARY. $\{b(n)\}$ *holds the recurrence formula*

$$(5) \qquad n\, b(n) = \sigma(n) + \sigma(n-1)\, b(1) + \cdots + \sigma(1)\, b(n-1),$$

where $\sigma(n) = \sum_{d|n} d\cdot a(d)$.

Proof. we can get easily

$$\prod_{m=1}^{j}(1-z^m)^{-a(m)} = 1 + \sum_{n=1}^{\infty} b_j(n)\, z^n,$$

where

$$b_j(n) = \sum_{1\cdot s_1+\cdots+j\cdot s_j=n} \prod_{i=1}^{j} a(i)^{[s_i]} .$$

It is plain that $b_j(n) = b(n)$, for $n \leqslant j$. Thus we have

$$(6) \qquad (1 + \sum_{n=1}^{\infty} b(n)\ z^n) - \prod_{m=1}^{j} (1 - z^m)^{-a(m)}$$

$$= \sum_{n=j+1}^{\infty} (b(n) - b_j\ (n))\ z^n$$

and so for the valuation $\mathbf{o}(\sum_{n=0}^{\infty} \alpha_n z^n) = \min_{\alpha_n \neq 0} n$ ($\mathbf{o}(0) = +\infty$) ,

$$\mathbf{o}((1 + \sum_{n=1}^{\infty} b(n)\ z^n) - \prod_{m=1}^{j} (1 - z^m)^{-a(m)})$$

is greater than j and tends to infinity as $j \to \infty$ Hence (A) converges to (B) in $\mathbf{C}[[z]]$.

Assume that $\sum_{n=1}^{\infty} a(n)\ z^n$ has a positive or infinite radius R of convergence and D is a compact subset of $\{z;\ |z| < \min\ (1, R\ )\}$. we may show that

$$(C) \qquad \sum_{m=1}^{\infty} a(m) \log \frac{1}{1-z^m} = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} a(m) \frac{z^{mk}}{k}$$

converges unifomly in D instead of that (A) do. Let $\rho_0 = \max_{z \epsilon D} |z|$ and $\rho_0 < \rho_1 < \min\ (1,\ R)$, $\rho_0 = \theta \rho_1$ $(0 < \theta < 1)$. From the assumption, $|a(m)|\rho_1{}^m$ $(m=1,\ 2,\ \cdots)$ is bounded. Let $|a(m)|\rho_1{}^m < M$. Then we have a majorant

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{M\ \theta^{nk}}{k} = M \log \prod_{m=1}^{\infty} (1-\theta^m)^{-1}$$

$$= M \log F(1;\ \theta)$$

of (C). Hence (C) and so (A) converge unifomly in D. Since

$$|b(n)|\leqslant \sum_{1\cdot s_1+2\cdot s_2+\cdots=n} \prod_{i=1}^{n} |a(i)|^{[s_i]},$$

we have

$$1+\sum_{n=1}^{j}|b(n)||z|^n\leqslant \prod_{m=1}^{j}(1-|z|^m)^{-|a(m)|}$$

$$\leqslant \prod_{m=1}^{\infty}(1-|z|^m)^{-|a(m)|}.$$

Hence (B) is convergent in $|z|<\min(1, R)$. Now, since

$$\lim_{j\to\infty}\sum_{n=j+1}^{\infty} b(n)\ z^n = \lim_{j\to\infty}\sum_{n=j+1}^{\infty} b_j\ (n)\ z^n$$

$$= 0\ ,\ |z|<\min\ (1,R)$$

by (6) we have

$$1+\sum_{n=1}^{\infty} b(n)\ z^n = \prod_{m=1}^{\infty}(1-z^m)^{-a(m)},\ |z|<\min\ (1,\ R).$$

We shall show (5) without $R>0$. The map $d/dz:\mathbf{C}[[z]]\to\mathbf{C}[[z]]$, $(d/dz)(\sum_{n=0}^{\infty} \alpha_n z^n) = \sum_{n=1}^{\infty} n\,\alpha_n\, z^{n-1}$ is a derivation of $\mathbf{C}[[z]]$.

When we define $\log(1+F)$ and $(1+F)^{\alpha}$ by

$$\log(1+F) = F-\frac{F^2}{2}+\frac{F^3}{3}-+\cdots\quad F\in z\cdot\mathbf{C}[[z]]$$

and

$$(1+F)^{\alpha} = 1+\alpha F+\frac{\alpha(\alpha-1)}{2!}F^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}F^3+\ldots,$$

$$F\in z\cdot\mathbf{C}[[z]],\ \alpha\in\mathbf{C},$$

we get

(a) $$\log\{(1+F)\ (1+G)\} = \log(1+F)+\log(1+G)\ ,$$

(b) $$\lim_{k\to\infty}\log\ (1+F_k) = \log(1+F)\ ,\ \text{if}\ \lim_{k\to\infty}F_k = F\ ,$$

(c) $$(1-F)^{-\alpha} = 1+\sum_{n=1}^{\infty}\alpha^{[n]}F^n\ ,$$

(b) $$\log (1+F)^{\alpha} = \alpha \log (1+F)$$

(e) $$((1+F)^{\alpha})^{\beta} = (1+F)^{\alpha\beta}$$

(f) $$\log((1-F)^{-1}) = \sum_{n=1}^{\infty} \frac{F^n}{n} ,$$

(g) $$(1+F) \cdot (d/dz) \log(1+F) = (d/dz) F ,$$

for $F, G, F_k \in z \cdot \mathbf{C}[[z]]$ and α $\beta \in \mathbf{C}$. From (a), (b) and the fact that (B) coincides (A) in $\mathbf{C}[[z]]$, we have

$$\log(1+F) = \sum_{n=1}^{\infty} \log(1+ \sum_{k=1}^{\infty} a(m)^{[k]} z^{mk}) ,$$

where $F = \sum_{n=1}^{\infty} b(n) z^n$. From (c) $\sim$ (f), we have

$$\log(1+F) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} z^n .$$

By (g), we get

$$(1+ \sum_{n=1}^{\infty} b(n) z^n) \sum_{n=1}^{\infty} \sigma(n) z^n = \sum_{n=1}^{\infty} n\, b(n) z^n .$$

Hence we have (5). This completes the proof.

The theorem with (2) leads to

$$F(r; x) \equiv \sum_{n=0}^{\infty} p(r; n) x^n = \prod_{m=1}^{\infty} (1-x^m)^{-p(r-1;m)} , \ |x|<1.$$

And by the corollary we have

(7) $$p(r; n) = \frac{1}{n} \sum_{k=1}^{n} \sigma(r-1; k) p(r; n-k) , \ n \geqslant 1.$$

where $$\sigma(r-1; k) = \sum_{d|k} d \cdot p(r-1; d) ,$$

Bell [1] showed that $\tilde{P}(r; n)$ $n>1$, is a polynomial of degree $n-1$ in $\mathbf{Q}[r]$, and is divisible by $r+1$. We can show here the following

Proposition 1. *$p(r;\ n)$, $n \geqslant 1$, is a polynomial of degree $n-1$ in* $\mathbf{Q}\,[r]$, *and is divisible by $r+1$ (if $n \geqslant 2$) and by $r+2$ (if $n \geqslant 3$).*

Proof. By induction on n. Clearly $p(r;\ 1)$ is a polynomial of degree 0 in r. From (7) we have that

$$(8)\quad d(k;\ n) = p(k;\ n) - p(k-1;\ n)$$

$$= \frac{1}{n}\left\{\sum_{j=1}^{n-1} \sigma(k-1;j)p(k;\ n-j) + \sum_{\substack{d\mid n\\ d<n}} d\cdot p(k-1;\ d)\right\}$$

is a polynomial of degree $n-2$ in $\mathbf{Q}[k]$, if $p(k;\ j)$ is of degree $j-1$ for $j<n$. Hence $p(r;\ n) = 1 + \sum_{k=1}^{r} d(k;\ n)$ is a polynomial of degree $n-1$ in $\mathbf{Q}[r]$.

We can now regard (7) as a formula between polynomials in r. We have to show that $p(-1;\ n)=0$ $(n \geqslant 2)$ and $p(-2;\ n)=0$ $(n \geqslant 3)$. From (8)

$$p(-1;\ n) = 1 - \frac{1}{n}\left\{\sum_{j=1}^{n-1} \sum_{d\mid j} d\cdot p(-1;\ d) + \sum_{\substack{d\mid n\\ d<n}} d\cdot p(-1;\ d)\right\}$$

and so $p(-1;\ 2)=0$. By induction from 2,... $n-1$ to n we get

$p(-1;\ n) = 1 - (1/n)(\sum_{j=1}^{n-1} 1 + 1) = 0$. Similarly

$$p(-2;\ n) = -\frac{1}{n}\left\{\sum_{d\mid n-1} d\cdot p(-2;\ d) + \sum_{\substack{d\mid n\\ d<n}} d\cdot p(-2;\ d)\right\},n \leqslant 2$$

derives $p(-2;\ 2) = -1$ and $p(-2;\ n)=0$ $(n \geqslant 3)$.

Moreover we have

Proposition 2. *The polynomial $p(r;\ n) \in \mathbf{Q}[r]$ $n \geqslant 1$, has the leading coefficient $A_{n-1}/(n-1)!$, where A_k, $k \geqslant 0$, are positive integers and defined by*

$$\tan x + \sec x = \sum_{k=0}^{\infty} A_k \frac{x^k}{k!} ,$$

more precisely

$$2A_{k+1} = \sum_{i=0}^{k} \binom{k}{i} A_i\ A_{k-i}\ ,\ k \geqslant 1$$

with $A_0 = A_1 = 1$ (*see* E. Netto [5] § 63).

Proof. Let $p(r;\ n+1) = \frac{A_n}{n!} r^n + \ldots,\ A_n \in \mathbf{Q}$. Then

$$\sigma(k-1;\ j) = \sum_{d \mid j} d \cdot p(k-1;\ d) = j \cdot \frac{A_{j-1}}{(j-1)!} k^{j-1} + \ldots,$$

$$p(k;\ n-j) = \frac{A_{n-j-1}}{(n-j-1)!} k^{n-j-1} + \ldots,$$

$$d(k;\ n) = \left\{ \frac{1}{n} \sum_{j=1}^{n-1} \frac{jA_{j-1}A_{n-j-1}}{(j\text{-}1)!(n\text{-}j\text{-}1)!} \right\} k^{n-2} + \ldots\ ,\ n \geqslant 3.$$

Hence $p(r;\ n) = \left\{ \frac{1}{n(n-1)} \sum_{j=1}^{n-1} \frac{jA_{j-1}A_{n-j-1}}{(j\text{-}1)!(n\text{-}j\text{-}1)!} \right\} r^{n-1} + \ldots\ ,\ n \geqslant 3,$

since as well-known $\sum_{k=1}^{r} k^n = \frac{r^{n+1}}{n+1} + \cdots \in \mathbf{Q}[r]$.

Thus we have

$$A_n = \frac{1}{n+1} \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} A_j\ A_{n-1-j}$$

$$= \frac{1}{2} \sum_{j=0}^{n-1} \binom{n-1}{j} A_j\ A_{n-1-j}\ ,\ n \geqslant 2.$$

$p(r;\ 1) = 1$ and $p(r;\ 2) = r+1$ imply $A_0 = A_1 = 1$. This completes the proof.

Proposition 2 means that for fixed n

$$p(r;\ n) = \frac{A_{n-1}}{(n-1)!} r^{n-1} + O_n(r^{n-2}) \quad \text{as } r \to \infty.$$

On the other hand for fixed r, paticularly for $r=1$

$$p(n) = p(1; n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{\frac{2n}{3}})$$

is well-known (Hardy–Ramamujan [3]). For the case of $r=2$, the auther [4] proved recently the following

$$\log p(2; n) = \frac{\pi^2}{6} n(l(n)^{-1}+(\log n)^{-2}) + O(\frac{n \log \log n}{(\log n)^3})$$

$$\sim \frac{\pi^2 n}{6 \log n} ,$$

where $l(n)=\log n - (3/2)\log \log n + (1/2)\log(\pi^3/3)$.

2. Let $\{a(n)\}_{n=1,2,\cdots}$ be any complex number sequence and be the transformation such that

$$\mathbf{E}: \{a(n)\}\rightarrow\{b(n)\},$$

where $$b(n)= \sum_{1 \cdot s_1+2 \cdot s_2+\cdots=n} \prod_{i=1}^{n} a(i)^{[s_i]} .$$

we note that *if* $\{a(n)\}$ *is an integer sequence, then* $\{b(n)\}$ *is also an integer seguence.*

In this section we consider the inverse of transformation $\mathbf{E}$ and the converse of Theorem 1, that is following

THEOREM 2. *For any given complex number sequence* $\{b(n)\}_{n=1,2,\cdots}$ *let*

$$(9) \quad \sigma(n) = - \sum_{1 \cdot s_1+2 \cdot s_2+\cdots=n} (-1)^T \frac{n}{T} \binom{T}{s_1, \cdots, s_n} b(1)^{s_1} \ldots b(n)^{s_n},$$

where $T=s_1+\cdots+s_n$.

And let

$$(10) \qquad n\, a(n) = \sum_{d \mid n} \mu(n/d)\, \sigma(d),$$

where $\mu(n)$ *is the Möbius function. Then the transformaton* $\{b(n)\}\to\{a(n)\}$ *is the inverse of* $\mathbf{E}$. *If* $\{b(n)\}$ *is an integer sequence then* $\{a(n)\}$ *is also an integer sequence. If* $h(z)=1+\sum_{n=1}^{\infty} b(n)z^n$ *is regular and has no zero in* $|z|< R_0$ *then*

$$(11) \qquad h(z) = \prod_{m=1}^{\infty} (1 - z^m)^{-a(m)}, \quad |z|<\min(1, R_0).$$

And then right hand side of (11) *and* $\sum_{n=1}^{\infty} a(n)z^n$ *converge uniformly in any compact subset D of* $\{z\in\mathbf{C};\ |z| <\min(1\ R_0)\}$. *Moreover we have*

$$(12) \qquad m\, a(m) = \frac{1}{2\pi i}\int_{|z|=\rho} \Big(\sum_{\delta|m} \frac{\mu(m/\delta)}{z^\delta}\Big) \frac{h'(z)}{h(z)}\, dz\ ,$$

where ρ *is a positive number such that* $\rho<\min(1, R_0)$.

Proof. It is easy from (4) or (5) that the transformation $\mathbf{E}$ is invertible. It is also easy from (4) by mathematical induction that if $b(n)$ are integers for all $n=1, 2,\ldots$ then $\mathbf{E}^{-1} b(n)$ are also integers for all n. we shall show $\mathbf{E}^{-1}b(n)=a(n)$ for given $\{a(n)\}$ by (9) and (10). It is sufficient to show (5). From (9) we have

$$\begin{aligned}
& n\ b(n) - \sigma(n-1)b(1) - \ldots - \sigma(1)b(n-1) \\
&= n\ b(n) + \sum_{m=1}^{n-1} \sum_{1.s_1'+2.s_2'+\cdots=m} (-1)^{T'} \frac{m}{T'} \binom{T'}{s_{1'},\ldots,s_{m'}} \\
&\qquad\qquad \times\ b(1)^{s_1'}\cdots b(m)^{s_{m'}}\ b(n-m) \\
&= n\ b(n) + \sum_{1.s_1+2.s_2+\cdots+(n-1)s_{n-1}=n} b(1)^{s_1}\cdots b(n-1)^{s_{n-1}} \\
&\quad \times \sum_{\substack{1\leqslant m\leqslant n-1 \\ s_m\neq 0}} (-1)^{T-1} \frac{n-m}{T-1}\binom{T-1}{s_1,\ldots,s_m-1,\ldots,s_{n-1}},
\end{aligned}$$

where $T'=s_1' + s_2' +\cdots$ and $T=s_1+s_2+\cdots$. We have

$$\sum_{\substack{1\leqq m\leqq n-1 \\ s_m\neq 0}} (-1)^{T-1}\frac{n-m}{T-1}\binom{T-1}{s_1,\dots,s_m-1,\dots,s_{n-1}}$$

$$= -(-1)^T \frac{1}{T(T-1)}\binom{T}{s_1,\dots,s_{n-1}} \sum_{1\leqq m\leqq n-1} (n-m)s_m$$

$$= -(-1)^T \frac{n}{T}\binom{T}{s_1,\dots,s_{n-1}}$$

Thus we have

$$n\,b(n) - \sigma(n-1)b(1) - \dots - \sigma(1)b(n-1)$$

$$= - \sum_{1.s_1+\dots+n.s_n=n} (-1)^T \frac{n}{T}\binom{T}{s_1,\dots,s_n} b(1)^{s_1}\cdots b(n)^{s_n}$$

$$= \sigma(n).$$

The equation $\sigma(n)=\sum_{d|n} d\cdot a(d)$ is obtained from (10) by Möbius invertion formula. We must show (11) and (12). Since $h(z) = 1+\sum_{n-1}^{\infty} b(n)z^n$ is regular and has no zero in $|z|<R_0$

$$\sum_{n=1}^{\infty} \sigma(n)z^n = \frac{\sum_{n=1}^{\infty} n\,b(n)z^n}{1+\sum_{n=1}^{\infty} b(n)z^n} = \frac{z\,h'(z)}{h(z)} = g(z), \text{ say,} \tag{13}$$

is regular in $|z|<R_0$. Let $|z|<\rho<\min(1, R_0)$ and ζ be a complex number which has the absolute value ρ. Then we have

$$\frac{g(\zeta)}{\zeta-z} = \frac{g(\zeta)}{\zeta} + \sum_{n=1}^{\infty}\left(g(\zeta) \sum_{d|n} \frac{\mu(n/d)}{\zeta^{d+1}}\right) \frac{z^n}{1-z^n} .$$

This series is uniformly convergent on $|\zeta|=\rho$. Hence by Cauchy's theorem we have

$$g(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} \frac{g(\zeta)}{\zeta-z}\,d\zeta \tag{14}$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{1-z^n} \sum_{\delta|n} \mu(n/\delta) \left\{ \frac{1}{2\pi i}\int_{|\zeta|=\rho} \frac{g(\zeta)}{\zeta^{\delta+1}}\,d\zeta \right\}$$

$$= \sum_{n=1}^{\infty} n\,a(n)\,\frac{z^n}{1-z^n}\ , \quad |z|<\min(1,\ R_0).$$

The last Lambert series of (14) and so $\sum_{n=1}^{\infty} a(n)z^n$ converge uniformly in any compact subset D of $\{z\in \mathrm{C};\ |z|< \min(1,\ R_0)\}$. Since $\mathbf{E}\,a(n)=b(n)$, by Theorem 1 we have

$$h(z) = \prod_{m=1}^{\infty} (1 - z^m)^{-a(m)}$$

and right hand side converges uniformly in D. (14) leads (12). This completes the proof.

Remark. Let $\{b(n)\}$ and $\{\sigma(n)\}$ be two complex number sequences. Then we get that the follwing three equations are equivalent:

(i) $\quad n\ b(n) = \sigma(n)+\sigma(n-1)b(1)+\ldots+\sigma(1)b(n-1)$,

(ii) $\quad b(n) = \displaystyle\sum_{1\cdot s_1+2\cdot s_2+\cdots=n} \frac{\sigma(1)^{s_1}\cdots\cdots\sigma(n)^{s_n}}{1^{s_1}\cdot s_1!\cdots\cdots n^{s_n}\cdot s_n!}$,

(iii) $\quad \sigma(n) = -\displaystyle\sum_{1\cdot s_1+2\cdot s_2+\cdots=n} (-1)^T \frac{n}{T} \binom{T}{s_1,\ldots,s_n} b(1)^{s_1}\cdots\cdots b(n)^{s_n}$,

where $T = s_1+\cdots+s_n$.

The equivalency of (i) and (iii) is already shown. We shall show (ii). From (13) we have

$$h(z) = \exp\int_0^z \frac{g(z)}{z}\,dz\ , \tag{15}$$

where $h(z)=1+\sum_{n=1}^{\infty} b(n)z^n$ and $g(z)=\sum_{n=1}^{\infty} \sigma(n)z^n$, if $\sum_{n=1}^{\infty} \sigma(n)z^n$ has a positive or infinite radius R of convergence. Using Faà di Bruno's formula for (15) we get (ii), if $R>0$. Since $b(n)$ is determind only by $\sigma(1),\ldots,\ \sigma(n)$ from (i) in the case of $R=0$ we get (ii), considering the sequecne

$$\sigma(1) \ldots, \sigma(n), 0, 0, \ldots$$

instead of $\{\sigma(n)\}$.

Example 2・1. Let $\sigma(n)=n$. From (10), (15) and (ii) we have

$$\exp \frac{z}{1-z} = 1 + \sum_{n=1}^{\infty} z^n \sum_{n=1.s_1+2.s_2+\cdots} \frac{1}{s_1! \cdots s_n!}$$

$$= \prod_{m=1}^{\infty} (1-z^m)^{-\frac{1}{m}\sum_{d|m} \mu(m/d)\, d}$$

$$= \prod_{m=1}^{\infty} (1-z^m)^{-\varphi(m)/m} \qquad |z|<1,$$

where $\varphi(m)$ is the Euler's function.

Example 2・2. Let $\sigma(1)=1$, $\sigma(2)=\sigma(3)=\ldots=0$. In this case we have

$$e^z = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} = \prod_{m=1}^{\infty} (1-z^m)^{-\mu(m)/m}, \quad |z|<1.$$

Example 2・3. Let

$$h(z) = 1 + b_1 z + \ldots + b_n z^n \ (b_n \neq 0)$$
$$= (1-\alpha_1 z) \ldots\ldots (1-\alpha_n z).$$

In this case we have

$$h(z) = \prod_{m=1}^{\infty} (1-z^m)^{\frac{1}{m}\sum_{d|m} \mu(m/d)(-\sigma_d)},$$

$$|z|<\min(1, |\alpha|^{-1}, \ldots, |\alpha_n|^{-1}),$$

where

$$-\sigma_d = \alpha_1^d + \ldots + \alpha_n^d$$

$$= \sum_{1.s_1+\cdots+n.s_n=d} (-1)^T \frac{d}{T} \binom{T}{s_1,\ldots,s_n} b_1^{s_1}\cdots b_n^{s_n} ,$$

$T = s_1 + \ldots + s_n$.

Example 2・4.

$$\sin z = z \prod_{m=1}^{\infty} (1-z^m)^{\alpha(m)} , \quad |z|<1,$$

where

$$\alpha(m) = \frac{1}{m} \sum_{2d|m} \frac{22d}{(2d)!} B_d \cdot \mu(m/2d)$$

B_d is the d-th Bernoulli number, that is defined by

$$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} x^{2n} , \quad |x|<\pi.$$

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