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# Supplement to the paper "Asymptotic properties of the Bayes modal estimators of item parameters in item response theory"

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This note is to supplement Ogasawara (2013), and gives asymptotic expansions for the reciprocals of the estimated asymptotic standard errors used for studentized estimators and a simplification of Theorem 6.

**(a)** $\boldsymbol{t}$

Define $\hat{i}_{\mathrm{W}}^{\alpha\alpha} = (\hat{\mathbf{I}}_{\mathrm{W}}^{-1})_{(\alpha\alpha)}$. Then, for $t = N^{1/2}(\hat{\alpha}_{\mathrm{W}} - \alpha_0)/(\hat{i}_{\mathrm{W}}^{\alpha\alpha})^{1/2}$,

$$
\begin{aligned}
(\hat{i}_{\mathrm{W}}^{\alpha\alpha})^{-1/2} &= (i_0^{\alpha\alpha})^{-1/2} + \frac{1}{2}(i_0^{\alpha\alpha})^{-3/2}\sum_{a\ge b}\frac{2-\delta_{ab}}{2}\{\mathbf{I}_0^{-1}(\mathbf{E}_{ab}+\mathbf{E}_{ba})\mathbf{I}_0^{-1}\}_{(\alpha\alpha)}\frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0{}'}(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0) \\
&\quad + \frac{3}{8}(i_0^{\alpha\alpha})^{-5/2}\left[\sum_{a\ge b}\frac{2-\delta_{ab}}{2}\{\mathbf{I}_0^{-1}(\mathbf{E}_{ab}+\mathbf{E}_{ba})\mathbf{I}_0^{-1}\}_{(\alpha\alpha)}\frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0{}'}(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)\right]^2 \\
&\quad - \frac{1}{4}(i_0^{\alpha\alpha})^{-3/2}\sum_{a\ge b}\sum_{c\ge d}\frac{1}{4}(2-\delta_{ab})(2-\delta_{cd}) \\
&\qquad \times 2\{\mathbf{I}_0^{-1}(\mathbf{E}_{ab}+\mathbf{E}_{ba})\mathbf{I}_0^{-1}(\mathbf{E}_{cd}+\mathbf{E}_{dc})\mathbf{I}_0^{-1}\}_{(\alpha\alpha)}\frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0{}'}(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)\frac{\partial i_{0cd}}{\partial \boldsymbol{\alpha}_0{}'}(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0) \\
&\quad + \frac{1}{4}(i_0^{\alpha\alpha})^{-3/2}\sum_{a\ge b}\frac{2-\delta_{ab}}{2}\{\mathbf{I}_0^{-1}(\mathbf{E}_{ab}+\mathbf{E}_{ba})\mathbf{I}_0^{-1}\}_{(\alpha\alpha)}\frac{\partial^2 i_{0ab}}{(\partial \boldsymbol{\alpha}_0{}')^{<2>}}(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)^{<2>} \\
&\quad + O_p(N^{-3/2}) \\
&\equiv (i_0^{\alpha\alpha})^{-1/2} + \mathbf{i}_0^{(1)}{}'(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0) + \mathbf{i}_0^{(2)}{}'(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)^{<2>} + O_p(N^{-3/2}) \\
&= (i_0^{\alpha\alpha})^{-1/2} + \mathbf{i}_0^{(1)}{}'(N^{-1}\boldsymbol{\eta}_0 + N^{-1/2}\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)} + N^{-1}\boldsymbol{\Lambda}^{(2)}\mathbf{m}^{(2)}) \\
&\qquad + \mathbf{i}_0^{(2)}{}'(N^{-1/2}\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)})^{<2>} + O_p(N^{-3/2})
\end{aligned}
\qquad \text{(S.1)}
$$

$$
\begin{aligned}
&= (i_0^{\cdot\alpha\alpha})^{-1/2} + N^{-1/2}\mathbf{i}_0^{(1)}{}'\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)} + N^{-1}\mathbf{i}_0^{(1)}{}'\boldsymbol{\eta}_0 \\
&\qquad + N^{-1}\{\mathbf{i}_0^{(1)}{}'\boldsymbol{\Lambda}^{(2)}\mathbf{m}^{(2)} + \mathbf{i}_0^{(2)}{}'(\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)})^{<2>}\} + O_p(N^{-3/2}) \\
&\equiv (i_0^{\cdot\alpha\alpha})^{-1/2} + N^{-1}\eta_{t0} + \sum_{j=1}^{2} N^{-j/2}\boldsymbol{\lambda}_t^{(j)}{}'\mathbf{m}^{(j)} + O_p(N^{-3/2}),
\end{aligned}
$$

where $\sum_{a \ge b}(\cdot) = \sum_{a=1}^{q}\sum_{b=1}^{a}(\cdot)$, $\delta_{ab}$ is the Kronecker delta, $\mathbf{E}_{ab}$ is a square matrix of an appropriate size whose $(a, b)$th element is 1 with other ones being zero, and $\boldsymbol{\Lambda}^{(j)}(j=1,2)$ are multivariate versions of $\boldsymbol{\lambda}^{(j)}{}'(j=1,2)$, respectively (see (8.2)).

In (S.1),

$$
\begin{aligned}
\mathbf{i}_0^{(1)} &= \frac{1}{2}(i_0^{\cdot\alpha\alpha})^{-3/2}\sum_{a \ge b}(2-\delta_{ab})\, i_0^{\cdot\alpha a} i_0^{\cdot\alpha b}\frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0}, \\
\mathbf{i}_0^{(2)} &= \frac{3}{8}(i_0^{\cdot\alpha\alpha})^{-5/2}\left[\sum_{a \ge b}(2-\delta_{ab})\, i_0^{\cdot\alpha a} i_0^{\cdot\alpha b}\frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0}\right]^{<2>} \\
&\quad -\frac{1}{8}(i_0^{\cdot\alpha\alpha})^{-3/2}\sum_{a \ge b}\sum_{c \ge d}(2-\delta_{ab})(2-\delta_{cd})\sum_{(abcd)}^{4} i_0^{\cdot\alpha a} i_0^{\cdot bc} i_0^{\cdot d\alpha}\frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0}\otimes\frac{\partial i_{0cd}}{\partial \boldsymbol{\alpha}_0} \qquad \text{(S.2)} \\
&\quad +\frac{1}{4}(i_0^{\cdot\alpha\alpha})^{-3/2}\sum_{a \ge b}(2-\delta_{ab})\, i_0^{\cdot\alpha a} i_0^{\cdot\alpha b}\frac{\partial^2 i_{0ab}}{(\partial \boldsymbol{\alpha}_0)^{<2>}},
\end{aligned}
$$

$\eta_{t0} = \mathbf{i}_0^{(1)}{}'\boldsymbol{\eta}_0$, $\boldsymbol{\lambda}_t^{(1)}{}' = \mathbf{i}_0^{(1)}{}'\boldsymbol{\Lambda}^{(1)}$, $\boldsymbol{\lambda}_t^{(2)}{}' = \mathbf{i}_0^{(1)}{}'\boldsymbol{\Lambda}^{(2)} + \mathbf{i}_0^{(2)}{}'\boldsymbol{\Lambda}^{(1)<2>}$, where $\sum_{(abcd)}^{4}$ is the sum of four terms considering the symmetric cases for $a$, $b$, c and $d$, and noting $\mathbf{I}_0 \equiv \sum_{k=1}^{K}\frac{1}{\pi_{0k}}\frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0}\frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0{}'}$ with $\pi_{0k} \equiv (\boldsymbol{\pi}_0)_k \equiv \{\boldsymbol{\pi}(\boldsymbol{\alpha}_0)\}_k$, we have

$$\frac{\partial i_{0ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K}\left( -\frac{1}{\pi_{0k}^2}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}} + \sum_{(ab)}^{2}\frac{1}{\pi_{0k}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}} \right),$$

$$\begin{aligned}\frac{\partial^2 i_{0ab}}{\partial \alpha_{0c}\partial \alpha_{0d}} &= \sum_{k=1}^{K}\left\{ \frac{2}{\pi_{0k}^3}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d}} \right. \\ &\quad -\frac{1}{\pi_{0k}^2}\left( \sum_{(ab)}^{2}\sum_{(cd)}^{2}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d}} + \frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0c}\partial \alpha_{0d}} \right) \\ &\quad + \frac{1}{\pi_{0k}}\left( \sum_{(ab)}^{2}\frac{\partial^3 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}\partial \alpha_{0d}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}} + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0d}} \right. \\ &\quad \left.\left. + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0d}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0c}} \right)\right\}\end{aligned} \qquad \text{(S.3)}$$

$(a,b,c,d = 1,\ldots,q)$.

**(b)** $t_H$

Using $\hat{\mathbf{H}}_{\mathrm{W}}$ (see (4.2b)), define $\hat{h}_{\mathrm{W}}^{\alpha\alpha}$, $\mathbf{H}_0$, $\hat{\mathbf{D}}_{\mathrm{W}}$, $\hat{d}_{\mathrm{W}}^{\alpha\alpha}$ and $d_0^{\alpha\alpha}$ in the following equations:

$$t_H \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}} - \alpha_0)}{\{(-\hat{\mathbf{H}}_{\mathrm{W}}^{-1})_{(\alpha\alpha)}\}^{1/2}} \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}} - \alpha_0)}{(-\hat{h}_{\mathrm{W}}^{\alpha\alpha})^{1/2}},$$

$$-\hat{h}_{\mathrm{W}}^{\alpha\alpha} = \left[\left\{ \sum_{k=1}^{K} p_k \left( -\frac{1}{\hat{\pi}_k}\frac{\partial^2 \hat{\pi}_k}{\partial \hat{\boldsymbol{\alpha}}_{\mathrm{W}}\partial \hat{\boldsymbol{\alpha}}_{\mathrm{W}}'} + \frac{1}{\hat{\pi}_k^2}\frac{\partial \hat{\pi}_k}{\partial \hat{\boldsymbol{\alpha}}_{\mathrm{W}}}\frac{\partial \hat{\pi}_k}{\partial \hat{\boldsymbol{\alpha}}_{\mathrm{W}}'} \right)\right\}^{-1}\right]_{(\alpha\alpha)}, \qquad \text{(S.4)}$$

$$\hat{\mathbf{D}}_{\mathrm{W}} = -\hat{\mathbf{H}}_{\mathrm{W}},\ \mathbf{D}_0 = -\mathbf{H}_0 = \sum_{k=1}^{K}\pi_{\mathrm{T}k}\left( -\frac{1}{\pi_{0k}}\frac{\partial^2 \pi_{0k}}{\partial \boldsymbol{\alpha}_0 \partial \boldsymbol{\alpha}_0'} + \frac{1}{\pi_{0k}^2}\frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0}\frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0'} \right).$$

Then,

$$\begin{aligned}(-\hat{h}_{\mathrm{W}}^{\alpha\alpha})^{-1/2} &= (\hat{d}_{\mathrm{W}}^{\alpha\alpha})^{-1/2} = (d_0^{\alpha\alpha})^{-1/2} \\ &\quad + \frac{1}{2}(d_0^{\alpha\alpha})^{-3/2}\sum_{a \ge b}(2-\delta_{ab})d_0^{\alpha a}d_0^{\alpha b}\frac{\partial d_{0ab}}{\partial(\boldsymbol{\pi}_{\mathrm{T}}', \boldsymbol{\alpha}_0')}\{(\mathbf{p} - \boldsymbol{\pi}_{\mathrm{T}})', (\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)'\}'\end{aligned} \qquad \text{(S.5)}$$

$$+\left[\frac{3}{8}(d_0^{\alpha\alpha})^{-5/2}\left\{\sum_{a\ge b}(2-\delta_{ab})d_0^{\alpha a}d_0^{\alpha b}\frac{\partial d_{0ab}}{\partial(\boldsymbol{\pi}_T',\boldsymbol{\alpha}_0')}\right\}^{<2>}\right.$$

$$-\frac{1}{8}(d_0^{\alpha\alpha})^{-3/2}\sum_{a\ge b}\sum_{c\ge d^*}(2-\delta_{ab})(2-\delta_{cd^*})$$

$$\times\sum_{(abcd^*)}^{4}d_0^{\alpha a}d_0^{bc}d_0^{d^*\alpha}\frac{\partial d_{0ab}}{\partial(\boldsymbol{\pi}_T',\boldsymbol{\alpha}_0')}\otimes\frac{\partial d_{0cd^*}}{\partial(\boldsymbol{\pi}_T',\boldsymbol{\alpha}_0')}$$

$$\left.+\frac{1}{4}(d_0^{\alpha\alpha})^{-3/2}\sum_{a\ge b}(2-\delta_{ab})d_0^{\alpha a}d_0^{\alpha b}\frac{\partial^2 d_{0ab}}{\{\partial(\boldsymbol{\pi}_T',\boldsymbol{\alpha}_0')\}^{<2>}}\right]\{(\mathbf{p}-\boldsymbol{\pi}_\mathrm{T})',(\hat{\boldsymbol{\alpha}}_\mathrm{W}-\boldsymbol{\alpha}_0)'\}'^{<2>}$$

$$\equiv(d_0^{\alpha\alpha})^{-1/2}+\mathbf{d}_0^{(1)}{}'\{(\mathbf{p}-\boldsymbol{\pi}_\mathrm{T})',(\hat{\boldsymbol{\alpha}}_\mathrm{W}-\boldsymbol{\alpha}_0)'\}'$$

$$+\mathbf{d}_0^{(2)}{}'\{(\mathbf{p}-\boldsymbol{\pi}_\mathrm{T})',(\hat{\boldsymbol{\alpha}}_\mathrm{W}-\boldsymbol{\alpha}_0)'\}'^{<2>}+O_p(N^{-3/2})$$

$$=(d_0^{\alpha\alpha})^{-1/2}+\mathbf{d}_0^{(1)}{}'\{N^{-1/2}\mathbf{m}^{(1)}{}',(N^{-1}\boldsymbol{\eta}_0+N^{-1/2}\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)}+N^{-1}\boldsymbol{\Lambda}^{(2)}\mathbf{m}^{(2)})'\}'$$

$$+\mathbf{d}_0^{(2)}{}'\{N^{-1/2}\mathbf{m}^{(1)}{}',N^{-1/2}(\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)})'\}'^{<2>}+O_p(N^{-3/2}).$$

Decomposing $\mathbf{d}_0^{(j)}(j=1,2)$ using subvectors as

$$\mathbf{d}_0^{(1)}=(\mathbf{d}_0^{(1\mathrm{A})}{}',\mathbf{d}_0^{(1\mathrm{B})}{}')' \text{ and } \mathbf{d}_0^{(2)}=(\mathbf{d}_0^{(2\mathrm{AA})}{}',\mathbf{d}_0^{(2\mathrm{AB})}{}',\mathbf{d}_0^{(2\mathrm{BA})}{}',\mathbf{d}_0^{(2\mathrm{BB})}{}')', \quad \text{(S.6)}$$

(S.5) becomes

$$=(d_0^{\alpha\alpha})^{-1/2}+N^{-1/2}(\mathbf{d}_0^{(1\mathrm{A})}{}'+\mathbf{d}_0^{(1\mathrm{B})}{}'\boldsymbol{\Lambda}^{(1)})\mathbf{m}^{(1)}+N^{-1}\mathbf{d}_0^{(1\mathrm{B})}{}'\boldsymbol{\eta}_0$$

$$+N^{-1}\{\mathbf{d}_0^{(1\mathrm{B})}{}'\boldsymbol{\Lambda}^{(2)}+\mathbf{d}_0^{(2\mathrm{AA})}{}'+2\mathbf{d}_0^{(2\mathrm{AB})}{}'(\mathbf{I}_{(K)}\otimes\boldsymbol{\Lambda}^{(1)})+\mathbf{d}_0^{(2\mathrm{BB})}{}'\boldsymbol{\Lambda}^{(1)<2>}\}\mathbf{m}^{(1)<2>}$$

$$+O_p(N^{-3/2})$$

$$\equiv(d_0^{\alpha\alpha})^{-1/2}+N^{-1}\eta_{H0}+\sum_{j=1}^{2}N^{-j/2}\boldsymbol{\lambda}_H^{(j)}{}'\mathbf{m}^{(j)}+O_p(N^{-3/2}), \quad \text{(S.7)}$$

where

$$\eta_{H0}=\mathbf{d}_0^{(1\mathrm{B})}{}'\boldsymbol{\eta}_0,\ \boldsymbol{\lambda}_H^{(1)}{}'=\mathbf{d}_0^{(1\mathrm{A})}{}'+\mathbf{d}_0^{(1\mathrm{B})}{}'\boldsymbol{\Lambda}^{(1)},$$

$$\boldsymbol{\lambda}_H^{(2)}{}'=\mathbf{d}_0^{(1\mathrm{B})}{}'\boldsymbol{\Lambda}^{(2)}+\mathbf{d}_0^{(2\mathrm{AA})}{}'+2\mathbf{d}_0^{(2\mathrm{AB})}{}'(\mathbf{I}_{(K)}\otimes\boldsymbol{\Lambda}^{(1)})+\mathbf{d}_0^{(2\mathrm{BB})}{}'\boldsymbol{\Lambda}^{(1)<2>} \quad \text{(S.8)}$$

and $\mathbf{I}_{(K)}$ is the $K\times K$ identity matrix.

In (S.5),

$$\frac{\partial d_{0ab}}{\partial \pi_{\mathrm{T}k^*}} = -\frac{1}{\pi_{0k^*}} \frac{\partial^2 \pi_{0k^*}}{\partial \alpha_{0a} \partial \alpha_{0b}} + \frac{1}{\pi_{0k^*}^2} \frac{\partial \pi_{0k^*}}{\partial \alpha_{0a}} \frac{\partial \pi_{0k^*}}{\partial \alpha_{0b}},$$

$$\frac{\partial d_{0ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K} \pi_{\mathrm{T}k} \left( -\frac{2}{\pi_{0k}^3} \frac{\partial \pi_{0k}}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}}{\partial \alpha_{0c}} + \frac{1}{\pi_{0k}^2} \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0b}} \frac{\partial \pi_{0k}}{\partial \alpha_{0c}} \right.$$
$$\left. + \sum_{(ab)}^{2} \frac{1}{\pi_{0k}^2} \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0c}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} - \frac{1}{\pi_{0k}} \frac{\partial^3 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0b} \partial \alpha_{0c}} \right),$$

$$\frac{\partial^2 d_{0ab}}{\partial \pi_{\mathrm{T}k^*} \partial \pi_{\mathrm{T}l}} = 0,$$

$$\frac{\partial^2 d_{0ab}}{\partial \pi_{\mathrm{T}k^*} \partial \alpha_{0c}} = -\frac{2}{\pi_{0k^*}^3} \frac{\partial \pi_{0k^*}}{\partial \alpha_{0a}} \frac{\partial \pi_{0k^*}}{\partial \alpha_{0b}} \frac{\partial \pi_{0k^*}}{\partial \alpha_{0c}} + \frac{1}{\pi_{0k^*}^2} \frac{\partial^2 \pi_{0k^*}}{\partial \alpha_{0a} \partial \alpha_{0b}} \frac{\partial \pi_{0k^*}}{\partial \alpha_{0c}}$$
$$+ \frac{1}{\pi_{0k^*}^2} \sum_{(ab)}^{2} \frac{\partial^2 \pi_{0k^*}}{\partial \alpha_{0a} \partial \alpha_{0c}} \frac{\partial \pi_{0k^*}}{\partial \alpha_{0b}} - \frac{1}{\pi_{0k^*}} \frac{\partial^3 \pi_{0k^*}}{\partial \alpha_{0a} \partial \alpha_{0b} \partial \alpha_{0c}},$$

$$\frac{\partial^2 d_{0ab}}{\partial \alpha_{0c} \partial \alpha_{0d^*}} = \sum_{k=1}^{K} \pi_{\mathrm{T}k} \left\{ \frac{6}{\pi_{0k}^4} \frac{\partial \pi_{0k}}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}}{\partial \alpha_{0d^*}} \right.$$
$$- \frac{2}{\pi_{0k}^3} \left( \sum_{(abcd^*)}^{4} \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0c}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}}{\partial \alpha_{0d^*}} + \frac{\partial \pi_{0k}}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0c} \partial \alpha_{0d^*}} \right. \quad \text{(S.9)}$$
$$\left. + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0b}} \frac{\partial \pi_{0k}}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}}{\partial \alpha_{0d^*}} \right)$$
$$+ \frac{1}{\pi_{0k}^2} \left( \sum_{(abcd^*)}^{4} \frac{\partial^3 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0b} \partial \alpha_{0c}} \frac{\partial \pi_{0k}}{\partial \alpha_{0d^*}} + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0b}} \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0c} \partial \alpha_{0d^*}} \right.$$
$$\left. + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0c}} \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b} \partial \alpha_{0d^*}} + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0d^*}} \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b} \partial \alpha_{0c}} \right)$$
$$\left. - \frac{1}{\pi_{0k}} \frac{\partial^4 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0b} \partial \alpha_{0c} \partial \alpha_{0d^*}} \right\}$$

$(a, b, c, d^* = 1, \ldots, q; k^*, l = 1, \ldots, K)$.

**(c)** $t_G$

Using $\hat{\mathbf{G}}_{\mathrm{W}}$ (see (4.2c)), define $\hat{g}_{\mathrm{W}}^{\alpha\alpha}$, $\mathbf{G}_0$ and $g_0^{\alpha\alpha}$ as

$$t_G \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}}-\alpha_0)}{\{(\hat{\mathbf{G}}_{\mathrm{W}}^{-1})_{(\alpha\alpha)}\}^{1/2}} \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}}-\alpha_0)}{(\hat{g}_{\mathrm{W}}^{\alpha\alpha})^{1/2}},$$

$$\mathbf{G}_0 = \sum_{k=1}^{K} \frac{\pi_{\mathrm{T}k}}{\pi_{0k}^2} \frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0} \frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0{}'} \qquad \text{(S.10)}$$

$$\text{and } g_0^{\alpha\alpha} = (\mathbf{G}_0^{-1})_{(\alpha\alpha)}.$$

Then,

$$\begin{aligned}
&(\hat{g}_{\mathrm{W}}^{\alpha\alpha})^{-1/2} = (g_0^{\alpha\alpha})^{-1/2} \\
&+\frac{1}{2}(g_0^{\alpha\alpha})^{-3/2}\sum_{a\ge b}(2-\delta_{ab})g_0^{\alpha a}g_0^{\alpha b}\frac{\partial g_{0ab}}{\partial(\boldsymbol{\pi}_{\mathrm{T}}{}',\boldsymbol{\alpha}_0{}')}\{(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})',(\hat{\boldsymbol{\alpha}}_{\mathrm{W}}-\boldsymbol{\alpha}_0)'\}' \qquad \text{(S.11)}\\
&+\Bigg[\frac{3}{8}(g_0^{\alpha\alpha})^{-5/2}\left\{\sum_{a\ge b}(2-\delta_{ab})g_0^{\alpha a}g_0^{\alpha b}\frac{\partial g_{0ab}}{\partial(\boldsymbol{\pi}_T{}',\boldsymbol{\alpha}_0{}')}\right\}^{<2>} \\
&-\frac{1}{8}(g_0^{\alpha\alpha})^{-3/2}\sum_{a\ge b}\sum_{c\ge d^*}(2-\delta_{ab})(2-\delta_{cd^*}) \\
&\quad\times\sum_{(abcd^*)}^{4} g_0^{\alpha a}g_0^{bc}g_0^{d^*\alpha}\frac{\partial g_{0ab}}{\partial(\boldsymbol{\pi}_T{}',\boldsymbol{\alpha}_0{}')}\otimes\frac{\partial g_{0cd^*}}{\partial(\boldsymbol{\pi}_T{}',\boldsymbol{\alpha}_0{}')} \\
&+\frac{1}{4}(g_0^{\alpha\alpha})^{-3/2}\sum_{a\ge b}(2-\delta_{ab})g_0^{\alpha a}g_0^{\alpha b}\frac{\partial^2 g_{0ab}}{\{\partial(\boldsymbol{\pi}_T{}',\boldsymbol{\alpha}_0{}')\}^{<2>}}\Bigg]\{(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})',(\hat{\boldsymbol{\alpha}}_{\mathrm{W}}-\boldsymbol{\alpha}_0)'\}'^{<2>} \\
&\equiv (g_0^{\alpha\alpha})^{-1/2} + \mathbf{g}_0^{(1)}{}'\{(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})',(\hat{\boldsymbol{\alpha}}_{\mathrm{W}}-\boldsymbol{\alpha}_0)'\}' \\
&\qquad + \mathbf{g}_0^{(2)}{}'\{(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})',(\hat{\boldsymbol{\alpha}}_{\mathrm{W}}-\boldsymbol{\alpha}_0)'\}'^{<2>} + O_p(N^{-3/2}) \\
&= (g_0^{\alpha\alpha})^{-1/2} + \mathbf{g}_0^{(1)}{}'\{N^{-1/2}\mathbf{m}^{(1)}{}',(N^{-1}\boldsymbol{\eta}_0 + N^{-1/2}\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)} + N^{-1}\boldsymbol{\Lambda}^{(2)}\mathbf{m}^{(2)})'\}' \\
&\qquad + \mathbf{g}_0^{(2)}{}'\{N^{-1/2}\mathbf{m}^{(1)}{}',N^{-1/2}(\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)})'\}'^{<2>} + O_p(N^{-3/2}).
\end{aligned}$$

Decomposing $\mathbf{g}_0^{(j)}\,(j=1,2)$ using subvectors as $\mathbf{g}_0^{(1)} = (\mathbf{g}_0^{(1A)}{}',\mathbf{g}_0^{(1B)}{}')'$ and $\mathbf{g}_0^{(2)} = (\mathbf{g}_0^{(2AA)}{}',\mathbf{g}_0^{(2AB)}{}',\mathbf{g}_0^{(2BA)}{}',\mathbf{g}_0^{(2BB)}{}')'$, (S.11) becomes

$$
\begin{aligned}
&= (g_0^{aa})^{-1/2} + N^{-1/2}(\mathbf{g}_0^{(1\mathrm{A})}{}' + \mathbf{g}_0^{(1\mathrm{B})}{}'\boldsymbol{\Lambda}^{(1)})\mathbf{m}^{(1)} + N^{-1}\mathbf{g}_0^{(1\mathrm{B})}{}'\boldsymbol{\eta}_0 \\
&+ N^{-1}\{\mathbf{g}_0^{(1\mathrm{B})}{}'\boldsymbol{\Lambda}^{(2)} + \mathbf{g}_0^{(2\mathrm{AA})}{}' + 2\mathbf{g}_0^{(2\mathrm{AB})}{}'(\mathbf{I}_{(K)} \otimes \boldsymbol{\Lambda}^{(1)}) + \mathbf{g}_0^{(2\mathrm{BB})}{}'\boldsymbol{\Lambda}^{(1)<2>}\}\mathbf{m}^{(1)<2>} \\
&+ O_p(N^{-3/2}) \\
&\equiv (g_0^{aa})^{-1/2} + N^{-1}\eta_{G0} + \sum_{j=1}^{2} N^{-j/2}\boldsymbol{\lambda}_G^{(j)}{}'\mathbf{m}^{(j)} + O_p(N^{-3/2}),
\end{aligned}
\tag{S.12}
$$

where

$$
\begin{aligned}
&\eta_{G0} = \mathbf{g}_0^{(1\mathrm{B})}{}'\boldsymbol{\eta}_0,\ \ \boldsymbol{\lambda}_G^{(1)}{}' = \mathbf{g}_0^{(1\mathrm{A})}{}' + \mathbf{g}_0^{(1\mathrm{B})}{}'\boldsymbol{\Lambda}^{(1)}, \\
&\boldsymbol{\lambda}_G^{(2)}{}' = \mathbf{g}_0^{(1\mathrm{B})}{}'\boldsymbol{\Lambda}^{(2)} + \mathbf{g}_0^{(2\mathrm{AA})}{}' + 2\mathbf{g}_0^{(2\mathrm{AB})}{}'(\mathbf{I}_{(K)} \otimes \boldsymbol{\Lambda}^{(1)}) + \mathbf{g}_0^{(2\mathrm{BB})}{}'\boldsymbol{\Lambda}^{(1)<2>}
\end{aligned}
\tag{S.13}
$$

In (S.11),

$$
\begin{aligned}
&\frac{\partial g_{0ab}}{\partial \pi_{\mathrm{T}k^*}} = \frac{1}{\pi_{0k^*}^2}\frac{\partial \pi_{0k^*}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k^*}}{\partial \alpha_{0b}}, \\
&\frac{\partial g_{0ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K}\pi_{\mathrm{T}k}\left(-\frac{2}{\pi_{0k}^3}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}} + \frac{1}{\pi_{0k}^2}\sum_{(ab)}^{2}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\right), \\
&\frac{\partial^2 g_{0ab}}{\partial \pi_{\mathrm{T}k^*}\partial \pi_{\mathrm{T}l}} = 0, \\
&\frac{\partial^2 g_{0ab}}{\partial \pi_{\mathrm{T}k^*}\partial \alpha_{0c}} = -\frac{2}{\pi_{0k^*}^3}\frac{\partial \pi_{0k^*}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k^*}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k^*}}{\partial \alpha_{0c}} + \frac{1}{\pi_{0k^*}^2}\sum_{(ab)}^{2}\frac{\partial^2 \pi_{0k^*}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k^*}}{\partial \alpha_{0b}}, \\
&\frac{\partial^2 g_{0ab}}{\partial \alpha_{0c}\partial \alpha_{0d^*}} = \sum_{k=1}^{K}\pi_{\mathrm{T}k}\left\{\frac{6}{\pi_{0k}^4}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d^*}}\right. \\
&\qquad -\frac{2}{\pi_{0k}^3}\left(\sum_{(abcd^*)}^{4}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d^*}} + \frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0c}\partial \alpha_{0d^*}}\right) \\
&+\frac{1}{\pi_{0k}^2}\left(\sum_{(ab)}^{2}\frac{\partial^3 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}\partial \alpha_{0d^*}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}} + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0d^*}}\right. \\
&\qquad \left.\left. + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0d^*}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0c}}\right)\right\}
\end{aligned}
\tag{S.14}
$$

$(a, b, c, d^* = 1, \ldots, q; k^*, l = 1, \ldots, K)$.

**(d)** $t_R$

Define $\hat{r}_{\mathrm{W}\alpha\alpha}$ as

$$t_R \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}}-\alpha_0)}{\left(\frac{\partial \hat{\alpha}_{\mathrm{W}}}{\partial \mathbf{p}'}\hat{\mathbf{\Omega}}_{\mathrm{T}}\frac{\partial \hat{\alpha}_{\mathrm{W}}}{\partial \mathbf{p}}\right)^{1/2}} \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}}-\alpha_0)}{\hat{r}_{\mathrm{W}\alpha\alpha}^{1/2}}, \qquad \text{(S.15)}$$

where $\frac{\partial \hat{\alpha}_{\mathrm{W}}}{\partial \mathbf{p}} = \frac{\partial \alpha_{\mathrm{W}}(\mathbf{p})}{\partial \mathbf{p}}\big|_{\mathbf{p}=\mathbf{p}}$.

Recalling $\frac{\partial \alpha_{\mathrm{W}}}{\partial \boldsymbol{\pi}_{\mathrm{T}}} = \frac{\partial \alpha_{\mathrm{W}}(\mathbf{p})}{\partial \mathbf{p}}\big|_{\substack{\alpha=\alpha_{\mathrm{W}}\\ \mathbf{p}=\boldsymbol{\pi}_{\mathrm{T}}}} = \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}} + N^{-1}\frac{\partial \alpha_{\Delta\mathrm{W}}}{\partial \boldsymbol{\pi}_{\mathrm{T}}} + O(N^{-2})$ (see (3.4)),
and using the definitions

$$r_{\mathrm{W}\alpha\alpha} \equiv \frac{\partial \alpha_{\mathrm{W}}}{\partial \boldsymbol{\pi}_{\mathrm{T}}'}\mathbf{\Omega}_{\mathrm{T}}\frac{\partial \alpha_{\mathrm{W}}}{\partial \boldsymbol{\pi}_{\mathrm{T}}}, \quad r_{0\alpha\alpha} \equiv \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}'}\mathbf{\Omega}_{\mathrm{T}}\frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \qquad \text{(S.16)}$$

and $r_{\Delta\mathrm{W}0} \equiv \frac{\partial \alpha_{\Delta\mathrm{W}}}{\partial \boldsymbol{\pi}_{\mathrm{T}}'}\mathbf{\Omega}_{\mathrm{T}}\frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}}$,

we have

$$\begin{aligned}\hat{r}_{\mathrm{W}\alpha\alpha}^{-1/2} &= r_{\mathrm{W}\alpha\alpha}^{-1/2} - \frac{1}{2}r_{\mathrm{W}\alpha\alpha}^{-3/2}\frac{\partial r_{\mathrm{W}ab}}{\partial \boldsymbol{\pi}_{\mathrm{T}}'}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}}) + \frac{3}{8}r_{\mathrm{W}\alpha\alpha}^{-5/2}\left(\frac{\partial r_{\mathrm{W}ab}}{\partial \boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})^{<2>} \\ &\quad - \frac{1}{4}r_{\mathrm{W}\alpha\alpha}^{-3/2}\frac{\partial^2 r_{\mathrm{W}ab}}{(\partial \boldsymbol{\pi}_{\mathrm{T}}')^{<2>}}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})^{<2>} + O_p(N^{-3/2}),\end{aligned} \qquad \text{(S.17)}$$

where

$$\begin{aligned} r_{\mathrm{W}\alpha\alpha} &= r_{0\alpha\alpha} + 2N^{-1}r_{\Delta\mathrm{W}0} + O(N^{-2}), \\ r_{\mathrm{W}\alpha\alpha}^{-1/2} &= r_{0\alpha\alpha}^{-1/2} - N^{-1}r_{0\alpha\alpha}^{-3/2}r_{\Delta\mathrm{W}0} + O(N^{-2}), \\ r_{\mathrm{W}\alpha\alpha}^{-j/2} &= r_{0\alpha\alpha}^{-j/2} + O(N^{-1}) \ \ (j=3,5), \\ \frac{\partial r_{\mathrm{W}ab}}{\partial \boldsymbol{\pi}_{\mathrm{T}}} &= \frac{\partial r_{0ab}}{\partial \boldsymbol{\pi}_{\mathrm{T}}} + O(N^{-1}), \quad \frac{\partial^2 r_{\mathrm{W}ab}}{(\partial \boldsymbol{\pi}_{\mathrm{T}})^{<2>}} = \frac{\partial^2 r_{0ab}}{(\partial \boldsymbol{\pi}_{\mathrm{T}})^{<2>}} + O(N^{-1}). \end{aligned} \qquad \text{(S.18)}$$

From the above results, (S.17) becomes

$$= r_{0\alpha\alpha}^{-1/2} - N^{-1} r_{0\alpha\alpha}^{-3/2} r_{\Delta \mathrm{W}0} - \frac{N^{-1/2}}{2} r_{0\alpha\alpha}^{-3/2} \frac{\partial r_{0ab}}{\partial \boldsymbol{\pi}_{\mathrm{T}}{}'} \mathbf{m}^{(1)}$$

$$+ N^{-1} \left\{ \frac{3}{8} r_{0\alpha\alpha}^{-5/2} \left( \frac{\partial r_{0\alpha\alpha}}{\partial \boldsymbol{\pi}_{\mathrm{T}}{}'} \right)^{<2>} - \frac{1}{4} r_{0\alpha\alpha}^{-3/2} \frac{\partial^2 r_{0\alpha\alpha}}{(\partial \boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}} \right\} \mathbf{m}^{(1)<2>} + O_p(N^{-3/2}) \qquad \text{(S.19)}$$

$$\equiv r_{0\alpha\alpha}^{-1/2} + N^{-1} \eta_{R0} + \sum_{j=1}^{2} N^{-j/2} \boldsymbol{\lambda}_R^{(j)}{}' \mathbf{m}^{(j)} + O_p(N^{-3/2}),$$

where $\eta_{R0} = -r_{0\alpha\alpha}^{-3/2} r_{\Delta \mathrm{W}0}$, $\boldsymbol{\lambda}_R^{(1)} = -\frac{1}{2} r_{0\alpha\alpha}^{-3/2} \frac{\partial r_{0\alpha\alpha}}{\partial \boldsymbol{\pi}_{\mathrm{T}}}$

and $\boldsymbol{\lambda}_R^{(2)} = \frac{3}{8} r_{0\alpha\alpha}^{-5/2} \left( \frac{\partial r_{0\alpha\alpha}}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \right)^{<2>} - \frac{1}{4} r_{0\alpha\alpha}^{-3/2} \frac{\partial^2 r_{0\alpha\alpha}}{(\partial \boldsymbol{\pi}_{\mathrm{T}})^{<2>}}$.

In (S.19), using $\frac{\partial \boldsymbol{\Omega}_{\mathrm{T}}}{\partial \pi_{\mathrm{T}j}} = \mathbf{E}_{jj} - \boldsymbol{\pi}_{\mathrm{T}} \mathbf{e}_j{}' - \mathbf{e}_j \boldsymbol{\pi}_{\mathrm{T}}{}'$ where $\mathbf{e}_j$ is the vector whose $j$-th element is 1 with other ones being zero, we have

$$\frac{\partial r_{0\alpha\alpha}}{\partial \pi_{\mathrm{T}j}} = 2 \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \boldsymbol{\pi}_{\mathrm{T}}{}'} \boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}} + \left( \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}j}} \right)^2 - 2 \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}j}} \boldsymbol{\pi}_{\mathrm{T}}{}' \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}},$$

$$\frac{\partial^2 r_{0\alpha\alpha}}{\partial \pi_{\mathrm{T}j} \partial \pi_{\mathrm{T}k}} = 2 \frac{\partial^3 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \pi_{\mathrm{T}k} \partial \boldsymbol{\pi}_{\mathrm{T}}{}'} \boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}} + 2 \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \boldsymbol{\pi}_{\mathrm{T}}{}'} \boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}k} \partial \boldsymbol{\pi}_{\mathrm{T}}} \qquad \text{(S.20)}$$

$$+ \sum_{(jk)}^{2} 2 \left( \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \pi_{\mathrm{T}k}} \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}k}} - \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \boldsymbol{\pi}_{\mathrm{T}}{}'} \boldsymbol{\pi}_{\mathrm{T}} \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}k}} \right)$$

$$- 2 \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}j}} \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}k}} - 4 \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \pi_{\mathrm{T}k}} \boldsymbol{\pi}_{\mathrm{T}}{}' \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \quad (j, k = 1, \ldots, q).$$

**(e)** $\eta_{\text{B-C}V0}$

The quantity $\eta_{V0}$ is the generic expression for $\eta_{t0}(=\mathbf{i}_0^{(1)}{}'\boldsymbol{\eta}_0)$, $\eta_{H0}(=\mathbf{d}_0^{(1\text{B})}{}'\boldsymbol{\eta}_0)$, $\eta_{G0}(=\mathbf{g}_0^{(1\text{B})}{}'\boldsymbol{\eta}_0)$ and $\eta_{R0}\left(=-r_{0\alpha\alpha}^{-3/2}r_{\Delta\text{W}0}=-r_{0\alpha\alpha}^{-3/2}\frac{\partial\alpha_{\Delta\text{W}}}{\partial\boldsymbol{\pi}_\text{T}'}\boldsymbol{\Omega}_\text{T}\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_\text{T}}\right)$. The corresponding $\eta_{\text{B-C}V0}$ is generically defined for $\eta_{\text{B-C}t0}(=\mathbf{i}_0^{(1)}{}'\boldsymbol{\eta}_{\text{B-C}0})$, $\eta_{\text{B-C}H0}(=\mathbf{d}_0^{(1\text{B})}{}'\boldsymbol{\eta}_{\text{B-C}0})$, $\eta_{\text{B-C}G0}(=\mathbf{g}_0^{(1\text{B})}{}'\boldsymbol{\eta}_{\text{B-C}0})$ and $\eta_{\text{B-C}R0}\left(=r_{0\alpha\alpha}^{-3/2}\frac{\partial\beta_{\text{ML}1}}{\partial\boldsymbol{\pi}_\text{T}'}\boldsymbol{\Omega}_\text{T}\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_\text{T}}\right)$.

**(f) A simplification of Theorem 6.**

$\kappa_2(w_{\text{B-P}}^*)$ can be simplified as follows. Using (8.20), (8.19) becomes

$$\frac{\partial\boldsymbol{\alpha}_{\text{B-P}}}{\partial\boldsymbol{\pi}_\text{T}'}=-\boldsymbol{\Lambda}^{-1}\frac{\partial^2 l_{\text{ML}}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\pi}_\text{T}'}-N^{-1}\frac{\partial\boldsymbol{\beta}_{\text{ML}1}}{\partial\boldsymbol{\pi}_\text{T}'}+O(N^{-2})$$
$$\equiv\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_\text{T}'}+N^{-1}\frac{\partial\boldsymbol{\alpha}_{\Delta\text{B-P}}}{\partial\boldsymbol{\pi}_\text{T}'}+O(N^{-2}),$$

which gives

$$\kappa_2(w_{\text{B-P}}^*)=\beta_{\text{ML}2}+N^{-1}\left(\beta_{\text{ML}\Delta2}+2\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_\text{T}'}\boldsymbol{\Omega}_\text{T}\frac{\partial\alpha_{\Delta\text{B-P}}}{\partial\boldsymbol{\pi}_\text{T}}\right)+O(N^{-2})$$

$$\left(\beta_{\text{B-P}2}=\beta_{\text{ML}2},\ \beta_{\text{B-P}\Delta2}=\beta_{\text{B-CML}\Delta2}=\beta_{\text{ML}\Delta2}-2\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_\text{T}'}\boldsymbol{\Omega}_\text{T}\frac{\partial\beta_{\text{ML}1}}{\partial\boldsymbol{\pi}_\text{T}}\right).$$

That is, we find that the asymptotic cumulants of $\hat{\alpha}_{\text{B-P}}$ up to the fourth order and the higher-order asymptotic variance are identical to those of $\hat{\alpha}_{\text{B-CML}}$ or the bias-corrected ML estimator, respectively.

## Reference

Ogasawara, H. (2013). Asymptotic properties of the Bayes modal estimators of item parameters in item response theory. *Computational Statistics, 28* (6), 2559-2583.