ON THE OPTIMAL ECONOMIC GROWTH UNDER UNCERTAINTY

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Abstract

This paper studies the optimal economic growth under labor force supply uncertainty. The model is the finite horizon neoclassical aggregative model of optimal accumulation with random rate of increase in the labor force supply. The main object of the analysis is to investigate the effects of increased uncertainty on the optimal program of capital accumulation. Whether the planning boad facing increased uncertainty should increase or decrease the optimal rate of accumulation depends on the attitude of the representative individual toward risk aversion. It is shown that according to possible alternative terminal conditions different results concerning the above question are obtained.

1. Introduction

In the middle of the nineteen-sixties Samuelson, Koopmans, Cass among others studied the problem of optimal capital accumulation in the neoclassical one-sector economy. This theory is not only the resurgence of the classical Ramsey's problem but also the cornerstone of subsequent extensions of the model in various directions. Many authors have extended the model to the analysis of optimal growth in an economy with money, social overhead capital, externalities, exhaustible resources, heterogeneous capital goods, international trade, capital movements and so on.

In the present paper I shall be concerned with another important extension, that is, the theory of optimal growth under uncertainty. The

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theory of optimal accumulation under uncertainty was first originated by Phelps [14] as the theory of optimal saving of an individual consumer facing capital risk. Since Phelps' pathbreaking work the influence on the optimal individual saving and portfolio decision of uncertainty about capital return, wage income, span of life and commodity prices has been extensively studied by many theorists. On the other hand, the problem of optimal growth in a macroeconomy with uncertainty has also been studied in parallel with the theory of optimal stochastic growth of the consumer.

I shall study the theory of optimal economic growth under uncertainty, in particular, under labor force supply uncertainty. The model employed for analysis is the neoclassical one-sector model of optimal growth with random rate of labor force growth.

According to the introduction of uncertainty the objective criterion is replaced by the maximization of the expected total utility over a planning horizon, since in a stochastic environment one must think all variables random. Then, two important questions stem from this kind of model. One is a problem pertaining to the properties of probability distributions of the random variables through time. In this field contributions by Brock and Mirman [3], Bourguignon [2], Merton [12], and Danthine and Donaldson [6, 7] are remarkable. Another is a problem concerning the effects on the optimal policies of increased uncertainty of the distributions of the original random variable in terms of Rothschild and Stiglitz's mean preserving spread ([16, 17]). In what follows, I shall be concerned only with the latter problem.

The effects of increased labor force supply uncertainty in the optimal stochastic growth model have not yet been examined in the previous work,[1] whereas the effects of increased uncertainty about other kinds of random factors have been examined by Hamada, Mirman, Leland, Bismut, McCabe and Sibley, and Calvo. In general, the effects of variable uncertainty (i. e., change in the degree of uncertainty) on the optimal accumulation strategies depend on the psychological attitude of the people

1) In the static economic model the effects of increased labor force supply uncertainty were analyzed by Rothenberg and Smith [15].

or the representative man toward risk or risk aversion. So I would seek to obtain exact results for the problem posed above thereby to reinforce the insight of previous authors and further to derive a new result. Through the analysis the importance of the terminal conditions of the model is pointed out.

In order to explain the background of our question I would briefly refer to Merton's study of stochastic economic growth that incorporates explicitly the labor force supply uncertainty in the model. His model is formulated as a continuous-time stochastic neoclassical growth model. His model starts from a diffusion process type population dynamics. Its process is expressed by a stochastic differential equation

(1–1) $dL(t)=nL(t)dt+\sigma L(t)dz(t), \ (n>\sigma^2)$

where L represents the population size or labor force supply, dz stands for a Wiener process, and nL and σ^2L^2 imply the instantaneous mean and variance per unit time respectively.

$L(t)/L_0$ has a log-normal distribution with

(1–2) $E_0[log\frac{L(t)}{L_0}]=(n-\frac{1}{2}\sigma^2)t$

and

(1–3) $Var[log\frac{L(t)}{L_0}]=\sigma^2 t,$

where $L_0=L(0)$.

Combining (1–1) with an accumulation equation

(1–4) $dK(t)=F(K(t), L(t))dt-C(t)dt$

yields the accumulation equation in terms of per capita amounts

(1–5) $dk=[f(k)-nk-c+\sigma^2k]dt-\sigma kdz,$

where K, C stand for the aggregate fixed capital stock and aggregate consumption respectively, and $F(\ )$ represents the linear homogeneous aggregative production function, and $k=K/L$, $c=C/L$, $f(\ )=F(\ )/L$ respectively. In the derivation of the equation (1–5) famous Ito's lemma is used.

The optimal growth model in the above described economy has a perverse characteristic. Namely, maximizing the expected total utility

$$E_0[\int_0^{\infty} u(c(t))e^{-\beta t}dt]$$

by choosing the optimal consumption policies <$c(k)$> subject to the stochastic accumulation equation, (1-5), the initial condition, $k(0)=k_0=$given constant, and the transversality condition, $lim_{t\to\infty}E_t[u'(c)e^{-\beta t}k]=0$, gives rise to a serious mathematical difficulty (In above expressions $u(\ \ )$ is a strictly concave utility function and β is a discount rate). To avoid this trouble Merton assumes $\beta=0$, and the model therefore reduces to the undiscounted Ramsey type model with stochastic labor force growth. The optimal consumption function $c(k)$ in this stochastic Ramsey type optimal growth model with the simplified objective function $E_0[\int_0^{\infty} u(c)dt]$ and the transversality condition $lim_{t\to\infty}E_t[u'(c)k]=0$ satisfies a first order ordinary differential equation

$$(1\text{-}6)\quad 0=u(c(k))+[f(k)-nk-c(k)+\sigma^2k]u'(c(k))+\frac{1}{2}\sigma^2k^2u''(c(k))c'(k).$$

(To avoid the divergence of the utility integral assume here that $u(c)$ and/or $f(k)$ are bounded from above.) To obtain a closed form solution from this equation is impossible, and unfortunately to examine the effects of an increase in variance σ^2 on the optimal consumption $c(k)$ is also in fact difficult.

On the other hand, if we transform Merton's model into the discrete-time model, then, as will be shown later, the discounted finite-horizon case can easily be handled, and especially the effects of increased uncertainty can also be investigated. There are serious differences between the continuous-time optimal stochastic growth model and the discrete-time one. I shall therefore utilize the discrete-time model which has a greater advantage for my purpose in the present paper.

Before proceeding to the formulation and analysis of the model I briefly refer to the previous studies. Hamada [8] analyzes the balanced growth in an economy with random return on the foreign investment. The social welfare is expressed by the expected value of a quadratic utility function of per capita consumption. It is shown that an increase in uncertainty decreases the optimal level of foreign investment. Mirman [13] deals with the uncertainty of production technology in a simple two-period model. It is shown that the third derivative of the utility function plays an important role in the analysis of the effects of increased uncertainty. Leland [10] considers the optimal growth in an economy with random fixed technology coefficient. The utility functions are specified to the iso-elastic form. His main result is that an increased uncertainty decreases (increases) the optimal consumption as the elasticity of marginal utility (i. e., the measure of the relative risk aversion) is greater (smaller) than unity. Bismut [1] introduces the random disturbance into the accumulation process (i. e., the process of transformation of saving into investment). He obtains the somewhat simple result which asserts that an increased uncertainty increases the optimal consumption. McCabe and Sibley [11] investigate the optimal growth in an open economy facing uncertain export revenue. Their conclusion is that an increased uncertainty decreases (increases) the optimal consumption as the third derivative of the utility function is positive (negative). Calvo [4] examines the effects of increased uncertainty on the optimal consumption strategies within the framework of the Rawlsian maxi-min accumulation model with random technology coefficient. It is shown that an increased uncertainty decreases the optimal consumption consistent in the intergenerational equity. The studies reviewed above are our main knowledge on the theory of optimal economic growth under uncertainty with the discussion of the effects of increased uncertainty on the optimal accumulation policies.

The paper is organized as follows. In Section 2 the first model (refered to as the *optimal accumulation program with initially prescribed terminal capital*) is formulated. On the basis of the expected utility maximization hypothesis since Daniel Bernoulli maximizing the expected value of the total discounted per capita utility arising from the consumption over a finite planning horizon in a neoclassical growing economy with uncertain future labor force supply is attempted. The terminal capital-labor ratio as the destination of the system is fixed at the outset. Reflecting the risk aversion behavior of the people the one-period utility function is assumed to have the concavity. The optimal program of capital accumulation is derived by Bellman's dynamic programming techniques and its properties are described.

In Section 3 the effects of increased uncertainty on the optimal accumulation policies are examined. The result is that a mean preserving increase in uncertainty decreases (increases) the optimal consumption per head when the third derivative of the utility function is positive (negative).

In Section 4 the second model (refered to as the *optimal accumulation program with endogenously determined terminal capital*) is introduced. This model differs from the previous one in the following two respects. One is that the valuation function of the terminal capital-labor ratio is added to the social welfare function and therefore the optimal terminal capital-labor ratio is not prescribed at the outset but is determined endogenously in the process of optimization. Another is that the utility functions of both current consumption and the terminal capital are specified to the iso-elastic form. As is well known in this class of utility functions the Arrow-Pratt measure of the relative risk aversion is constant for any feasible value of argument. The optimal policies and their properties are derived.

In Section 5 the effects of increased uncertainty for this model are examined. The result is that an increased uncertainty decreases (increases) the optimal per capita consumption when the relative risk aversion index is smaller than one or greater than two (the index is greater than one but smaller than two).

2. Optimal Accumulation Program with Initially Prescribed Terminal Capital I : The Model and Derivation of the Optimal Policies

In this section the first model is formulated. As described before the discrete-time model is adopted.

Consider a growing economy where a single good Q_t (Net National Product) is produced by the fixed capital stock K_t and the labor force L_t. It is assumed that the economy has the neoclassical structure and therefore that the labor force is always fully employed and the planned saving is always equal to the planned investment. The technological relationship among Q_t, K_t and L_t is expressed by the smoothly substitutable production function

(2–1) $Q_t = F(K_t, L_t)$.

It is assumed that in the production process the law of constant returns to scale prevails, i. e., $F(\ )$ is homogeneous of degree one. It is further assumed that the technological knowledge for the economy is unchanged, i. e., $F(\ )$ is a time-invariant function. The capital accumulation process is expressed by

(2–2) $K_{t+1} = K_t + Q_t - C_t$,

where C_t represents consumption. For simplicity capital depreciation is ignored. It is assumed that Q_t, K_t, L_t and C_t are the beginning-of-periods amounts. The labor force growth process is expressed by

(2–3) $L_{t+1} = (1+n_t)L_t, \qquad -1 < n_t < +\infty$

where n_t represents a rate of growth of the labor force. Combining these equations yields the accumulation equation in terms of per capita amounts

(2–4) $k_{t+1} = \frac{1}{1+n_t}(k_t + f(k_t) - c_t)$,

where $k_t = K_t/L_t$, $f(k_t) = F(K_t, L_t)/L_t = F(k_t, 1) = Q_t/L_t$, $c_t = C_t/L_t$. It is assumed that $f(k)$ is well-behaved, i. e.,

(2-5) $f' > 0,\ f'' < 0,\ f'(0) = +\infty,\ \lim_{k \to \infty} f'(k) = 0$.

The per capita fixed capital stock k_t starts from a given initial value, i. e.,

(2–6) $k_0 = given, \qquad 0 < k_0 < +\infty$.

Since k_t must be non-negative,

(2–7) $0 \leqq c_t \leqq k_t + f(k_t)$

must be satisfied during the planning horizon. The current consumption c_t generates an instantaneous utility $u(c_t)$. In this and the next sections the utility function $u(\ )$ is not specified to any particular form. It is assumed that $u(\ )$ has properties

(2–8) $u'(0) = +\infty,\ u' > 0,\ u'' < 0$.

The concavity of $u(\ )$ represents the risk aversion attitude of the representative individual.

The central planning boad must decide the optimal consumption/

investment allocation of the net national product at each time period. Our criterion for intertemporal optimization is the maximization of the expected total discounted utility of the per capita consumption over a given planning horizon

$$(2\text{-}9)\quad E_0[\sum_{t=0}^{T} R^t u(c_t)] \qquad t=0,\ 1,\ \ldots,\ T$$

through the choice of a feasible consumption sequence $<c_0, c_1, \ldots, c_T>$. E_t denotes the conditional expectation operator, conditional on k_t and n_t. R $(0<R<1)$ represents the discount factor, i. e., $(1+\text{rate of discount})^{-1}$. T is a planning horizon. $<n_0, n_1, \cdots, n_T>$ is a sequence of independently but not necessarily identically distributed random variables assumed to be observable at the beginning of each time period. Finally the model is closed by introducing the terminal condition

$$(2\text{–}10)\quad k_{T+1}=A=\textit{non-random constant.} \qquad 0<A<+\infty$$

In sum the model is as follows.

$$\begin{cases} \underset{c_0, c_1, \cdots, c_T}{\text{Maximize}}\ E_0[\sum_{t=0}^{T} R^t u(c_t)] \\ \text{subject to } k_{t+1}=\frac{1}{1+n_t}(k_t+f(k_t)-c_t), \\ \qquad 0\leqq c_t \leqq k_t+f(k_t), \\ \qquad k_0=\textit{given}, \qquad 0<k_0<+\infty \\ \qquad k_{T+1}=A=\text{non-random constant}, \qquad 0<A<+\infty \\ \qquad <n_0, n_1, \cdots, n_T>: \text{a sequence of independently but not} \\ \qquad \text{necessarily identically distributed random variables obser-} \\ \qquad \text{vable at the beginning of each time period,} \quad -1<n_t<+\infty \\ \qquad t=0,\ 1, \cdots,\ T. \end{cases}$$

I would call this model the *optimal accumulation program with initially prescribed terminal capital.* Evidently, this model is the (discrete-time) stochastic generalization of Cass' deterministic model of optimal growth (Cass [5]).

As is shown below the optimal per capita consumption at each time period is a function of the per capita fixed capital stock at the same time period. Therefore, I shall write the optimal consumption per head at time t as $c_t(k_t)$ throughout the paper. A sequence of optimal consumption policies $<c_0(k_0), c_1(k_1),\cdots, c_T(k_T),>$ is derived by the backward recursive procedure of Bellman's dynamic programming techniques. From the terminal condition (2–10) the optimal per capita consumption at the final stage, $c_T(k_T)$, is given by

(2–11) $c_T(k_T)=(k_T+f(k_T))-(1+n_T)A.$

The remaining optimal policies $<c_0(k_0), c_1(k_1),\cdots, c_{T-1}(k_{T-1})>$ satisfy the first order difference equation

$$\text{(2–12)}\quad u'(c_t(k_t))=\frac{1+f'(k_{t+1})}{1+n_t}RE_t[u'(c_{t+1}(k_{t+1}))].\quad t=0,\ 1,\cdots,\ T-1$$

That is to say, the accumulation path $<k_0=$given, $k_1, k_2,\cdots, k_{T-1}, k_T, A=$ prescribed$>$ satisfying the Euler equation analogue (2–12) is the optimal trajectory of the system among feasible ones.

Properties of the optimal policies are next described. The optimal consumption is an increasing function of the accumulated capital, i. e.,

PROPOSITION 1.

(2–13) (*i*) $c_T'(k_T)=1+f'(k_T)$,

(2–14) (*ii*) $0<c_t'(k_t)<1+f'(k_t)$. $\quad t=0,\ 1,\cdots,\ T-1$

Proof. (*i*) Differentiating (2–11) with respect to k_T yields(2–13).

(*ii*)$c_{t-1}(k_{t-1})$ must satisfy

$$\text{(2–15)}\quad u'(c_{t-1}(k_{t-1}))=\frac{1+f'(k_t)}{1+n_{t-1}}RE_{t-1}[u'(c_t(k_t))].$$

$$t=1,\ 2,\cdots,T$$

Differentiating this equation with respect to k_{t-1} yields

$$\text{(2–16)}\quad u''(c_{t-1})c_{t-1}'(k_{t-1})=\frac{1}{(1+n_{t-1})^2}(1+f'(k_{t-1})-c_{t-1}'(k_{t-1}))$$
$$[f''(k_t)\ RE_{t-1}[u'(c_t(k_t))]$$
$$+(1+f'(k_t))RE_{t-1}[u''(c_t)c_t'(k_t)]].$$

This implies that if $0<c_t'(k_t)<1+f'(k_t)$ then $0<c_{t-1}'(k_{t-1})<1+f'(k_{t-1})$. For $t=T$, substituting (2–13), (2–16) implies sign$[c_{T-1}'(k_{T-1})]=$sign$[1+f'(k_{T-1})-c_{T-1}'(k_{T-1})]$, i. e., $0<c_{T-1}'(k_{T-1})<1+f'(k_{T-1})$. The backward induction on t establishes (2–14). Q. E. D.

The proposition states that the marginal propensity to consume out of capital is positive and is not greater than one plus the marginal productivity of capital.

The social welfare (the expected total utility per head over the planning horizon) maximized by using the optimal accumulation policies is also a function of the capital-labor ratio. I denote the expected total utility per head from time t on when the optimal policies are used by $J_t(k_t)$. That is to say, $J_t(k_t)$ is defined by

$$(2\text{–}17)\quad J_t(k_t)=\max_{c_t,\cdots,c_T} E_t[\sum_{s=t}^{T} R^{s-t}u(c_s)]$$
$$=E_t[\sum_{s=t}^{T} R^{s-t}u(c_s(k_s))].$$
$$t=0,\ 1,\cdots,\ T;\qquad s=t,\ t+1,\cdots,\ T$$

The function $J_t(k_t)$ satisfies the basic recursion relation or the functional recursion equation

$$(2\text{–}18)\quad J_t(k_t)=\max_{c_t}\ [u(c_t)+RE_t[J_{t+1}(k_{t+1})]]$$
$$=u(c_t(k_t))+RE_t[J_{t+1}((1+n_t)^{-1}(k_t+f(k_t)-c_t(k_t)))]$$
$$t=0,\ 1,\cdots,\ T-1$$

The maximized social welfare at the last stage, $J_T(k_T)$, is given by

$$(2\text{–}19)\quad J_T(k_T)=u(c_T(k_T))$$
$$=u((k_T+f(k_T))-(1+n_T)A).$$

In the process of recursive calculation one immediately knows the relations between the optimal consumption $c_t(k_t)$ and the associated social welfare $J_t(k_t)$

$$(2\text{–}20)\quad u'(c_t(k_t))=\frac{1}{1+n_t}RE_t[J'_{t+1}(k_{t+1})]$$
$$t=0,\ 1,\cdots,\ T-1$$

and

$$(2\text{–}21)\quad J'_t(k_t)=u'(c_t(k_t))(1+f'(k_t)).$$
$$t=0,\ 1,\cdots,\ T$$

Differentiating (2–21) with respect to k_t yields

$$(2\text{–}22)\quad J''_t(k_t)=u''(c_t)c'_t(k_t)(1+f'(k_t))+u'(c_t)f''(k_t).$$

Checking the signs of (2–21) and (2–22) establishes

PROPOSITION 2.

(2–23) $J_t'(k_t)>0, \qquad t=0, 1,\cdots, T$

(2–24) $J_t''(k_t)<0. \qquad t=0, 1,\cdots, T$

Namely, the social welfare associated with the optimal policies is an increasing and strictly concave function of the accumulated capital.

3. Continued II: The Effects of Increased Uncertainty on the Optimal Accumulation Policies

I now proceed to the analysis of the influence of an increase in the degree of uncertainty upon the welfare maximizing multi-period plan of capital accumulation. Let $<n_0^*, n_1^*,\cdots, n_T^*>$ be a sequence of n_t's under some original distributions, and let $<n_0^{**}, n_1^{**},\cdots, n_T^{**}>$ be the one under more uncertain distributions which differ from the original sequence by mean preserving spreads. Moreover, let $<c_0^*(k_0), c_1^*(k_1),\cdots, c_T^*(k_T)>$ be the sequence of the optimal amounts of per capita consumption under the original distributions, and let $<c_0^{**}(k_0), c_1^{**}(k_1),\cdots, c_T^{**}(k_T)>$ be the one under the more uncertain distributions. Then, how should the authorities facing more uncertain situation change the optimal consumption/investment allocation of the net national product at each time period? The answer to the question is as follows.

PROPOSITION 3.

(3–1) (*i*) $c_T^{**}(k_T)=c_T^*(k_T)$.

(3–2) (*ii*) $c_t^{**}(k_t)\lesseqgtr c_t^*(k_t)$ as $u'''\gtreqless 0. \qquad t=0, 1,\cdots, T-1$

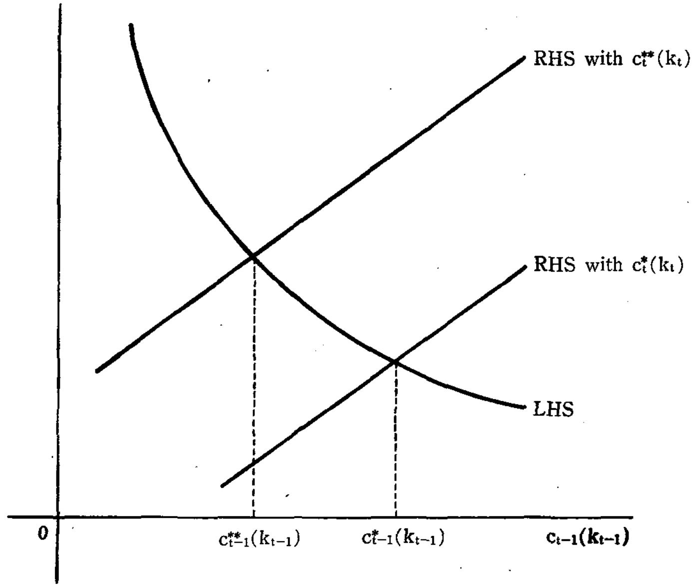
Proof.[2] (*i*) (2–11) immediately leads

(3–1) $c_T^{**}(k_T)=c_T^*(k_T)=(k_T+f(k_T))-(1+n_T)A.$

(*ii*) The proof is inductive. For a while, suppose that $u'''>0$. Consider (2–15). The LHS of this equation, $u'(c_{t-1})$, is a decreasing and strictly convex function of c_{t-1}, since $u''<0$ and $u'''>0$ by the assumption. Differ-

2) The method of proof is the same as the one used in Hartman [9] dealing with the optimal investment decision of a competitive firm under price and wage uncertainty.

Fig. 1. The Relation between Optimal Per Capita Consumptions at Adjacent Time Periods: The Case where $c_t^{**}(k_t) < c_t^{*}(k_t)$



entiating the RHS of the equation with respect to c_{t-1} yields

$$-(1+n_{t-1})^{-2}[f''(k_t)RE_{t-1}[u'(c_t)]+(1+f'(k_t))RE_{t-1}[u''(c_t)c_t'(k_t)]].$$

Since this expression is positive, the RHS is an increasing function of c_{t-1}. (Whether this is convex or concave is indeterminate, since an assumption about the sign of f''' is not introduced.) Figure 1 illustrates the both sides of (2–15) in the case where $c_t^{**}(k_t) < c_t^{*}(k_t)$. The value of abscissa of the point of intersection of the both curves is the optimal per capita consumption at time t–1. The figure shows that $c_{t-1}^{**}(k_{t-1}) < c_{t-1}^{*}(k_{t-1})$. It is obvious that if $c_t^{**}(k_t) > c_t^{*}(k_t)$ then $c_{t-1}^{**}(k_{t-1}) > c_{t-1}^{*}(k_{t-1})$. Therefore, we have the important backward recursive relation between optimal amounts of per capita consumption at adjacent time periods

$$(*)\quad c_t^{**}(k_t) \lesseqgtr c_t^{*}(k_t) \Rightarrow c_{t-1}^{**}(k_{t-1}) \lesseqgtr c_{t-1}^{*}(k_{t-1}).$$

$$t=1,\ 2,\cdots,\ T$$

(Note that for the case $u'''<0$ this relation holds as well.)

$c_{T-1}(k_{T-1})$ must satisfy

$$(3\text{-}3)\quad u'(c_{T-1}(k_{T-1}))=\frac{1+f'(k_T)}{1+n_{T-1}}RE_{T-1}[u'(c_T(k_T))]$$
$$=\frac{1+f'(k_T)}{1+n_{T-1}}RE_{T-1}[u'(k_T+f(k_T)-(1+n_T)A)]$$

Differentiating $u'(c_T(k_T))$ twice with respect to the random variable n_T yields $A^2u'''(c_T(k_T))>0$. Hence, $u'(c_T(k_T))$ is a strictly convex function of n_T. It follows, therefore, that

$$(3\text{-}4)\quad E_{T-1}[u'(k_T+f(k_T)-(1+n_T^{**})A)]$$
$$>E_{T-1}[u'(k_T+f(k_T)-(1+n_T^{*})A],$$

since the expected value of a function of a random variable increases (decreases) with a mean preserving increase in uncertainty when the function is strictly convex (concave). As in Figure 1 illustrate the equation (3–3) corresponding to n_T^{**} and n_T^{*} to get

$$(3\text{-}5)\quad c_{T-1}^{**}(k_{T-1})<c_{T-1}^{*}(k_{T-1}).$$

(The figure is omitted.)

It is obvious that the proposition holds when $u'''=0$ or <0. Q. E. D.

4. Optimal Accumulation Program with Endogenously Determined Terminal Capital I : The Model and Derivation of the Optimal Policies

This section deals with the second model with an alternative terminal condition. In the model the objective function to be maximized is modified as

$$(4\text{-}1)\quad E_o[\sum_{t=0}^{T} R^t u(c_t)+R^{T+1}\varphi(k_{T+1})],\qquad t=0,\ 1,\cdots,\ T$$

where

$$(4\text{-}2)\quad u(c_t)=\begin{cases}\frac{1}{1-a}c_t^{1-a}, & a\neq 1,\ a>0\\ \log c_t, & a=1\end{cases}$$

and

$$(4\text{-}3)\quad \varphi(k_{T+1})=\begin{cases}\frac{b}{1-a}k_{T+1}^{1-a}, & a\neq 1,\ a>0\\ b\log k_{T+1}, & a=1.\end{cases}$$

The accumulation equation and the initial condition are the same as the foregoing model. It is assumed in this model that $<n_0, n_1, \cdots, n_T>$ is a sequence of independently but not necessarily identically distributed random variables assumed to be observable at the end of each time period. Therefore, E_t in this model means the expectation operator, conditional on k_t and n_{t-1}. $\varphi(k_{T+1})$ is the valuation function of the terminal capital per head. Both the utility functions $u(\ )$ and $\varphi(\ )$ are assumed to belong to a class of the constant relative risk aversion (iso-elastic) utility functions. The social welfare is the expected value of total discounted utility arising from both the current per capita consumption stream and the terminal per capita fixed capital stock. I would call this model the *optimal accumulation program with endogenously determined terminal capital,* since in the model the optimal terminal capital-labor ratio is endogenously determined along with the decision of the optimal terminal consumption per head.

The model is summarized as follows.

$$\begin{cases} \underset{c_0, c_1, \cdots, c_T}{\text{Maximize}}\ E_0\left[\sum_{t=0}^{T} R^t u(c_t) + R^{T+1}\varphi(k_{T+1})\right] \\ u(c_t) = \begin{cases} \frac{1}{1-a} c_t^{1-a}, & a \neq 1,\ a>0 \\ \log c_t, & a=1 \end{cases} \qquad \varphi(k_{T+1}) = \begin{cases} \frac{b}{1-a} k_{T+1}^{1-a}, & a\neq 1,\ a>0 \\ b \log k_{T+1}, & a=1 \end{cases} \\ \text{subject to } k_{t+1} = \frac{1}{1+n_t}(k_t + f(k_t) - c_t), \\ \qquad 0 \leqq c_t \leqq k_t + f(k_t), \\ \qquad k_0 = given, \quad 0 < k_0 < +\infty \\ \qquad <n_0, n_1, \cdots, n_T> : \text{a sequence of independently but not necessarily identically distributed random variables observable at the end of each time period,} \quad -1 < n_t < +\infty \\ \qquad t = 0, 1, \cdots, T \end{cases}$$

As before relying upon the backward recursive manner of the dynamic programming techniques yields the optimal accumulation policies. The optimal per capita consumption at the last stage is given by

(4–4) $c_T(k_T)=\frac{z_T}{1+z_T}(k_T+f(k_T)),\qquad$ *for all* $a>0$

where

(4–5) $z_T=\begin{cases}[bRE_T[(1+n_T)^{a-1}]]^{-1/a}, & a\neq 1\\ [bR]^{-1}. & a=1\end{cases}$

The social welfare as of time t maximized by using the optimal policies is defined by

$$(4\text{–}6)\quad J_t(k_t)=\max_{c_t,\cdots,\,c_T} E_t[\sum_{s=t}^{T} R^{s-t}u(c_s)+R^{T+1-t}\varphi(k_{T+1})]$$

$$=E_t[\sum_{s=t}^{T} R^{s-t}u(c_s(k_s))+R^{T+1-t}\varphi((1+n_T)^{-1}(k_T+f(k_T)-c_T(k_T)))].$$

$$t=0,\ 1,\cdots,\ T;\qquad s=t,\ t+1,\cdots,\ T$$

Furthermore, the function $J_t(k_t)$ satisfies the fundamental recursion relation

$$(4\text{–}7)\quad J_t(k_t)=\max_{c_t}\ [u(c_t)+RE_t[J_{t+1}(k_{t+1})]]$$

$$=u(c_t(k_t))+RE_t[J_{t+1}((1+n_t)^{-1}\ (k_t+f(k_t)-c_t(k_t)))].$$

$$t=0,\ 1,\cdots,\quad T-1$$

$J_T(k_T)$ is given by

$$(4\text{–}8)\quad J_T(k_T)=\begin{cases}\left(\frac{z_T}{1+z_T}\right)^{-a}\frac{1}{1-a}(k_T+f(k_T))^{1-a}, & a\neq 1\\ \left(\frac{z_T}{1+z_T}\right)^{-1}\log\ (k_T+f(k_T))+\log\frac{z_T}{1+z_T}+z_T^{-1}\log\frac{1}{1+z_T} & \\ -z_T^{-1}E_T[\log\ (1+n_T)]. & a=1\end{cases}$$

A sequence of remaining optimal consumption policies $<c_0(k_0),\ c_1(k_1),\cdots,c_{T-1}(k_{T-1})>$ is given as solutions to the first order difference equation

$$(4\text{–}9)\quad c_t(k_t)^{-a}=RE_t[\frac{1}{1+n_t}J'_{t+1}(k_{t+1})]$$

$$=RE_t[\frac{1+f'(k_{t+1})}{1+n_t}c_{t+1}(k_{t+1})^{-a}].\qquad \textit{for all } a>0$$

$$t=0,\ 1,\cdots,\ T-1$$

Another expression of the relation between $J_t(k_t)$ and $c_t(k_t)$ is given by

(4–10) $J'_t(k_t)=(1+f'(k_t))c_t(k_t)^{-a}.\qquad$ *for all* $a>0\qquad t=0,\ 1,\cdots,\ T$

The function $c_t(k_t)$ representing the optimal consumption policies has the following properties.

PROPOSITION 1′.

(4–11) $0<c_t'(k_t)<1+f'(k_t)$. $t=0, 1, \cdots, T$

(Proof is omitted.) Proposition 2 holds without any modification for this model.

5. Continued II : The Effects of Increased Uncertainty on the Optimal Accumulation Policies

The following proposition states the effects of increased uncertainty on the optimal program of capital accumulation. Namely, when we go from any sequence $<n_0^*, n_1^*, \cdots, n_T^*>$ to a more uncertain sequence $<n_0^{**}, n_1^{**}, \cdots, n_T^{**}>$ leaving the means unchanged the original optimal consumption policies $<c_0^*(k_0), c_1^*(k_1), \cdots, c_T^*(k_T)>$ must be revised to new policies $<c_0^{**}(k_0), c_1^{**}(k_1), \cdots, c_T^{**}(k_T)>$. The result is sensitive to the value of the measure of relative risk aversion.

PROPOSITION 4.

$$(5\text{–}1)\quad c_t^{**}(k_t)\begin{cases} <c_t^*(k_t) & as\ 0<a<1\ or\ a>2, \\ >c_t^*(k_t) & as\ 1<a<2, \\ =c_t^*(k_t) & as\ a=1\ or\ a=2. \end{cases}$$

$$t=0, 1, \cdots, T$$

Proof. In (4–5) for the case $a \neq 1$, $(1+n_T)^{a-1}$ is a strictly convex (concave) function of n_T as $0<a<1$ or $a>2$ $(1<a<2)$. Since the expected value of a function of a random variable increases (decreases) with a mean preserving increase in uncertainty when the function is strictly convex (concave), we have

$$(5\text{–}2)\quad \begin{cases} z_T^{**}<z_T^*,\ i.\ e.,\ c_T^{**}(k_T)<c_T^*(k_T) & as\ 0<a<1\ or\ a>2, \\ z_T^{**}>z_T^*,\ i.\ e.,\ c_T^{**}(k_T)>c_T^*(k_T) & as\ 1<a<2, \\ z_T^{**}=z_T^*,\ i.\ e.,\ c_T^{**}(k_T)=c_T^*(k_T) & as\ a=2, \end{cases}$$

where

$$(5\text{–}3)\quad z_T^*(*)=[bRE_T[(1+n_T^*(*))^{a-1}]]^{-1/a}.$$

Next, (4–5) for the Bernoulli logarithmic case $a=1$ implies $z_T^{**}=z_T^*$, *i.e.*, $c_T^{**}(k_T)=c_T^*(k_T)$.

Since the backward recursion relation of the adjacent optimal consumption policies, (*), holds in this model as well, the proposition is established by the inductive reasoning. Q. E. D.

6. Conclusions

In this paper I have introduced the labor force supply uncertainty into the neoclassical aggregative (one-sector) model of optimal economic growth. The accumulation program which maximizes the expected total discounted per capita utility over a given planning horizon is considered to analyze the implications of uncertain future labor force supply. By applying the dynamic programming techniques the optimal consumption policies are derived. Both the optimal consumption and the associated social welfare are increasing functions of the accumulated fixed capital. The main object of the analysis is to investigate the effects of mean preserving increase in uncertainty on the optimal current consumption/investment strategies. I have considered this question under the alternative terminal conditions and have obtained the different exact results.

I have confirmed the importance of the third derivative of the utility function, u''', for the *optimal accumulation program with initially prescribed terminal capital* that can be seen as the discrete-time stochastic Cass model of optimal growth. Namely, I have verified that when $u'''>0$ (<0) an increased uncertainty increases (decreases) the optimal investment in the fixed capital stock.

On the other hand, to derive the clear-cut results I have assumed the constant relative risk aversion (iso-elastic) utility functions for the *optimal accumulation program with endogenously determined terminal capital.* I have found the significance of both $a=1$ and $a=2$ as critical values (a=constant measure of the relative risk aversion or elasticity of the marginal utility). Namely, when $0<a<1$ or $a>2$ an increased uncertainty increases the optimal investment, whereas when $1<a<2$ an increased uncertainty decreases the optimal investment. These conditions have not yet been known in the previous literature, while the conditions $a \gtrless 1$ have been well known since Phelps.

In general, the rate of economic expansion of a country is governed by the rate of capital accumulation (or propensity to save), the rate of population (or labor force) increase, the rate of technical progress and the rate of growth of the international trade, capital movements and aid, given the institutional and organizational arrangements. In an actual economy these factors always undergo random disturbances. In this sense the theory of economic planning should be reinforced by the stochastic control theory. In every country the forecasting of future population and labor force is really very hard. In the forecasting demographic specialists' views seldom coincide. In such an uncertain situation the central planning boad is obliged to revise the optimal plan of accumulation in comparison with the certainty case with ideal perfect foresight. My theoretical analysis in this paper might contribute to thinking of this difficult but important problem.

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