Stock Market Economy Under Uncertainty Part II: The Efficiency and Some Policy

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7. An Infinite Sequence of Temporary Equilibria

In the previous section, the existence of temporary equilibrium in each period is proven. Though each consumer lives for three periods, generations of consumers overlap each other and firms operate forever. The time horizon of the model studied here is actually infinite. The aim of this section is to consider the sequence of temporary equilibria over an infinite time horizon and to show the existence, which will be used for the discussion of efficiency in the next section. If a temporary equilibrium is illustrated as a snapshot of a real economy at the beginning of every period, an infinite sequence of such equilibria is a multi exposed picture lit by strobe light.

Let

$$R^{\infty} = \prod_{\tau=1}^{\infty} R^{L+J+2}(\tau)$$

denote the commodity space in an infinite time horizon model. The inner product of two vectors in R^{∞} is defined as

$$\boldsymbol{pz} = \sum_{\tau=1}^{\infty} p(\tau) z(\tau) = \sum_{\tau=1}^{\infty} \sum_{l=1}^{L+J+2} p_l(\tau) z_l(\tau)$$

where $\boldsymbol{p}, \boldsymbol{z} \in R^{\infty}$ [1]. The inner product is said to be well defined if the sequence $\{\sum_{\tau=1}^{T} p(\tau) z(\tau)\}_{T=1,2,\dots}$ has exactly one limit point. In the model

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1) Please notice that bald face letters $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \dots$ stand for vectors in R^{∞}.

here, the fact that the inner product may not be well defined causes difficulties both in establishing the existence of an equilibrium over an infinite horizon and in interpreting the meanings of such an equilbrium.

Let me explain some new notations. Consider a consumer i of generation t.

(7.1) $$\boldsymbol{x}^i=(\underline{0},\ldots,\underline{0}, x^i(t,t), x^i(t,t+1), x^i(t,t+2), \underline{0},\ldots) \in \boldsymbol{X}^i=\prod_{\tau=1}^{\infty} X^i(t,\tau)\subset R^{\infty},$$

denotes his consumption vector over an infinite time horizon. Note that the coordinates are zero except the three periods when he lives. It is easily checked that the consumption set $\boldsymbol{X}^i$ over an infinite horizon is convex and closed in the product topology and bounded from below if the assumption (a. 1) is satisfied. Let $\hat{\succsim}_i$ be the preference relation over $\boldsymbol{X}^i$ induced from $\succsim_i$ which is defined on $(X^i)^3$, by the the way in which

(7.2) $$\boldsymbol{x}^{i*}\hat{\succsim}_i\boldsymbol{x}^i$$
$$\text{iff } (x^{i*}(t,t), x^{i*}(t,t+1), x^{i*}(t,t+2))\succsim_i(x^i(t,t), x^i(t,t+1), x^i(t,t+2))$$

Clearly, the relation $\hat{\succsim}_i$ has all the properties that $\succsim_i$ posesses: convex, continuous, complete and monotone under the assumptions (b. 1) and (c). Hence, next lemma follows from Lemma 3. 2.

Lemma 7. 1

Under the assumptions (b. 1) and (c), there exists a continuous concave, strictly monotone and bounded function

$$\boldsymbol{u}^i : R^{\infty}\rightarrow R$$

given by

$$\boldsymbol{u}^i(\boldsymbol{x}^i)=u^i(x^i(t,t), x^i(t,t+1), x^i(t,t+2))$$

for $\boldsymbol{x}^i = (\underline{0}, \dots, \underline{0}, x^i(t,t), x^i(t,t+1), x^i(t,t+2), \underline{0}, \dots)$.

Under the assumption (a. 2),

$$\boldsymbol{x}_L^I = \sum_i \boldsymbol{x}_L^i < 0.$$

Next, consider firm j. As before, $(\bar{x}^j(t), \bar{y}^j(t+1))$ is a production plan of period t, an element of a closed and convex production possibility set $Y^j(t) \equiv Y^j(K^j(t)) \subset R^{2L}$ (see (d. 1)). The assumption (d. 2) saying that labor is necessary for production activity of each period implies

$$Y^j(t) \cap \{\xi \in R^{2L} | \xi \leq 0\} = \{0\} \quad \text{for all t.}$$

Let $n^j(t)$ and $b^j(t)$ be the number of shares newly offered and the units of borrowing made in the period t, $x^j(t) \in R_+^L$ the flow input purchased in period t (from which the output is produced in period t+1), and $\Delta K^j(t) \in R_+^L$ the new investment. Write the aggregate decision vector of the firm as

$$y^j(t) = (\{\bar{y}^j(t) - \bar{x}^j(t) - \Delta K^j(t)\}, \overbrace{0, \dots, 0}^{j-1}, n^j(t), \overbrace{0, \dots, 0}^{J-j+1}, b^j(t))$$
$$\in \hat{Y}^j(t) \subset R^{L+J+2}.$$

Following the argument of McKenzie (1959), the set $Y^j(t)$ of aggregate decisions of firm j in period t is a closed and convex cone with the vertex at the origin of R^{L+J+2} for the last $(J+2)$ components are fixed. Also the assumption (d. 2) means

$$\hat{Y}^j(t) \cap \{\xi \in R^{L+J+2} | \xi \leq 0\} = \{0\}.$$

Now, write the decision vector over an infinite time horizon as

$$(7.3) \quad \boldsymbol{y}^j = (y^j(1), y^j(2), \dots) \in \boldsymbol{Y}^j = \prod_{\tau=1}^{\infty} \hat{Y}^j(\tau).$$

Under the assumption (d. 1), the set $\boldsymbol{Y}^j$ of aggregate decision over an infinite time horizon is convex and closed in the product topology. And (d. 2) again guarantees

$$\boldsymbol{Y}^j \cap \{\xi \in \mathrm{R}^\infty \mid \xi \leq 0\} = \{0\}.$$

The following assumptions are made:

(g. 1)[2] there exist $\mu \geq 1$ and $\eta > 0$ such that for $\boldsymbol{x} \in \boldsymbol{X}^j$, $\tau = 1, 2, \ldots,$ and $l = 1, 2, \ldots, L+J+2$,

i) $|x_l(\tau)| \leq \mu^\tau \eta + \frac{\eta}{2}$

ii) the set $\{\boldsymbol{z} \in \mathrm{R}^\infty \mid \boldsymbol{z}_l(\tau) = 0$ if $\boldsymbol{x}_l(\tau) = 0, |\boldsymbol{z}_l(\tau)| \leq \mu^\tau \eta + \frac{\eta}{2}$ otherwise$\} \subset \boldsymbol{X}^i$

iii) let $\mathrm{C} = \{\boldsymbol{z} \in \mathrm{R}^\infty \mid -\mu^\tau \eta - \frac{\eta}{2} \leq \boldsymbol{z}(\tau) \leq \mu^\tau \eta\}$, then there exists $\boldsymbol{x}^* \in \boldsymbol{X}^i \setminus \mathrm{C}$ such that for any $\boldsymbol{x} \in \boldsymbol{X}^i \cap \mathrm{C}$

$$\boldsymbol{u}^i(\boldsymbol{x}^*) \geq \boldsymbol{u}^i(\boldsymbol{x})$$

iv) if $\boldsymbol{x}^i \in \{\boldsymbol{x}^i \in \boldsymbol{X}^i \mid \sum_i \boldsymbol{x}^i \leq \sum_j \boldsymbol{y}^j\}$ and $\boldsymbol{y}^j \in \{\boldsymbol{y}^j \in \boldsymbol{Y}^j \mid \sum_i \boldsymbol{x}^i \leq \sum_j \boldsymbol{y}^j\}$, then for any $\tau = 1, 2, \ldots$ $l = 1, \ldots, L+J+2$

$$\boldsymbol{x}_l(\tau) \leq \mu^\tau \eta$$

$$\boldsymbol{y}_l(\tau) \leq \mu^\tau \eta$$

(g. 2) there exist $\bar{\boldsymbol{x}}^i \in \boldsymbol{X}^i$ and $\bar{\boldsymbol{y}}^i \in \sum_j \boldsymbol{Y}^j$ such that $\bar{\boldsymbol{y}}^i - \bar{\boldsymbol{x}}^i > 0$.

(g. 3) there exists $\mathrm{T}_j \geq 0$ such that for any $\mathrm{t} \geq \mathrm{T}_j$

$${}^t\boldsymbol{Y}^j \equiv \{\boldsymbol{z} \equiv (\boldsymbol{y}^j(1), \ldots, \boldsymbol{y}^j(\mathrm{t}), \underline{0}, \ldots) \in \mathrm{R}^\infty\} \subset \boldsymbol{Y}^j$$

where $\boldsymbol{y}^j \equiv (\boldsymbol{y}^j(1), \boldsymbol{y}^j(2), \ldots) \in \boldsymbol{Y}^j$.

Theorem 7. 2

Under the assumptions (a. 1), (a. 2), (b. 1)–(b. 3), (d. 1), (d. 2) and (g. 1)–(g.3), there exists an infinite sequence of temporary equilibria

2) This assumption is first employed by Stigum (1973).

$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p})$ of consumption and production plans and price system such that

i) $\boldsymbol{x}^i$ is $\boldsymbol{u}^i$-maximizer in the i-th consumer's buget set,

ii) $\boldsymbol{y}^j$ is the return on share maximizing production plan subject to the production possibility set of firm j,

iii) $\sum_i \boldsymbol{x}^i = \sum_j \boldsymbol{y}^j$.

(Outline of proof) The idea of proof is borrowed from Stigum (1973); namely to find an equilibrium for an infinite time horizon model by taking a limit of equilibria for economies of finite time horizons. The proof is obtained in several steps:

1. to show the existence of equilibrium for an economy of a finite time horizon.
2. the return on share $\frac{\boldsymbol{p}^j \boldsymbol{y}^j}{\boldsymbol{n}^j}$ is well defined, which is equivalent to the fact that $\boldsymbol{p}\sum_j \boldsymbol{y}^j$ is well defined for given $\boldsymbol{n}=(\boldsymbol{n}^1,\dots,\boldsymbol{n}^J)$ [3].
3. the inner product $\boldsymbol{p}\,\boldsymbol{x}^i$ is well-defined.
4. $\boldsymbol{x}^i$ is an equilibrium consumption vector.
5. $\boldsymbol{y}^j$ is an equilibrium production vector.
6. to take a limit.

(Proof)

Step 1: $\boldsymbol{x}^{i*}$, $i=1,2,\dots$ be as in (g. 1) and take T^I so large that for any i and any $\boldsymbol{x}^i \in \boldsymbol{x}^i \cap C$

(7.4) $\boldsymbol{u}^i(\boldsymbol{x}^{*i}(1),\dots,\boldsymbol{x}^{*i}(T^I),\underline{0},\dots) > \boldsymbol{u}^i(\boldsymbol{x}^i)$.

3) For given $\{n^i(\tau)\}_{\tau=0,1,\dots}$ and $\{b^j(\tau)\}_{\tau=0,1,\dots}$ the maximization of return on share $[\tilde{p}(\tau)\tilde{y}^j(\tau)-b^j(\tau-1)]/\sum_{\nu=0}^{\tau-1} n^j(\nu)$ is equivalent to the maximization of $\tilde{p}(\tau)\tilde{y}^j(\tau)$ for each τ. Hence, Step 2 is more than necessary to show the condition ii) of equilibrium, but the step is necessary to prove Step 3.

Next, let

$$M=(M(1), M(2),\dots)$$

where

$$M_l(\tau)=\mu^{\tau}\eta+\frac{\eta}{2} \qquad \tau=1, 2,\dots \qquad l=1,\dots, L+J+2.$$

Let

$$C_M=\{z\in R^{\infty}\,|-2(I+J)M_l(\tau)\leq z_l(\tau)\leq M_l(\tau)\}$$

and suppose that each consumer i is required to choose his consumption vector x^i from $X^i\cap C_M$ and that each firm j is required to choose its production vector y^j from $Y^j\cap C_M$. Then it is easy to show that the decision correspondences of consumers and firms are convex and upper semicontinuous on the compact set

$$(7.5)\quad {}^{\nu}p=({}^{\nu}p\in R^{\infty}\,|\,{}^{\nu}p=(p(1),\dots,\ p(\nu), \underline{0},\dots)\geq\underline{0}\}.$$

Then, for large enough ν, there exists ${}^{\nu}p\geq\underline{0}$, ${}^{\nu}p\neq\underline{0}$ such that $x^i({}^{\nu}p)\in X^i\cap C_M$, $y^i({}^{\nu}p)\in Y^j\cap C_M$ and

$$(7.6)\quad \sum_i x^i({}^{\nu}p)\leq\sum_j y^j({}^{\nu}p)$$

$$(7.7)\quad \sum_i x_l^i({}^{\nu}p(\tau))=\sum_j y_l^j({}^{\nu}p(\tau)) \qquad \text{if} \quad {}^{\nu}p_l(\tau)>0$$

Then, there exists a subsequence of $\{[{}^{\nu}p,\ x^i({}^{\nu}p),\ y^j({}^{\nu}p)]\}_{\nu=1,\ 2,\dots}$ converging to $[p^0, x^i(p^0)]$ with $p^0\geq0$.

Step 2: Claim that

$$(7.8)\quad p^0\sum_j y^j(p^0)=\lim_{\nu\to\infty}{}^{\nu}p\sum_j y^j({}^{\nu}p)<\infty.$$

To establish (7. 8), suppose first that

$$(7.9)\quad \lim_{\nu\to\infty}\sup\ {}^{\nu}p\sum_j y^j({}^{\nu}p)<\infty$$

and take a subsequence of $\{{}^{\nu}\boldsymbol{p}\}$ (which is denoted by $\{{}^{\nu}\boldsymbol{p}\}$ as well) such that

$$(7.10)\quad \alpha=\lim_{\nu\to\infty} {}^{\nu}\boldsymbol{p}\sum_{j} \boldsymbol{y}^{j}({}^{\nu}\boldsymbol{p}) \quad (<\infty)$$

exists. Write β=the left hand side of (7. 8). Then, if (7. 8) does not hold, either

$$(7.11)\quad \alpha>\beta \quad \text{or}$$

$$(7.12)\quad \alpha<\beta.$$

Now, suppose (7. 11) is true and let $0<2\delta<(\alpha-\beta)$. Choose $\lambda_0\in(0,1)$ so small that for any i

$$(7.13)\quad \lambda_0\boldsymbol{p}^0(\boldsymbol{x}^{*i}(1),\dots, \boldsymbol{x}^{*i}(\mathrm{T}'),\underline{0},\dots)<\frac{[\alpha-(\beta+2\delta)](1-\lambda_0)}{I}$$

$$(7.14)\quad |\lambda_0\boldsymbol{x}^{*i}(\tau)|<\frac{\eta}{2} \quad \text{for} \quad \tau\leqq \mathrm{T}'$$

Also let

$$(7.15)\quad \rho^i_{\mathrm{T}}=\max_{x\in X^i\cap C}\{\boldsymbol{u}^i(\boldsymbol{x})-\boldsymbol{u}^i(\boldsymbol{x}(1),\dots, \boldsymbol{x}(\mathrm{T}), \underline{0},\dots)\}$$

and observe that ρ^i_T is non-negative and decreasing in T. Since $\boldsymbol{u}^i$ (.) is uniformly continuous on $\boldsymbol{x}^i\cap \mathrm{C}$,

$$\lim_{\mathrm{T}\to\infty}\sup\ \rho^i_{\mathrm{T}}=0.$$

Finally, let

$$\boldsymbol{x}^i(\lambda_0)=\lambda_0(\boldsymbol{x}^{*i}(1),\dots, \boldsymbol{x}^{*i}(\mathrm{T}'), \underline{0},\dots)+(1-\lambda_0)\boldsymbol{x}$$

and

$$(7.16)\quad \sigma^i=\min_{x\in X^i\cap C}\{\boldsymbol{u}^i(\boldsymbol{x}^i(\lambda_0))-\boldsymbol{u}^i(\boldsymbol{x})\}.$$

Take $\mathrm{T}_0>\mathrm{T}'$ so large and ε_0 so small that for any i

$$\rho^i_{\mathrm{T}_0}+\varepsilon_0<\sigma^i,$$

Choose $t_0 > T_0$ so that for any $\boldsymbol{x} \in \boldsymbol{X}^i \cap C_M$ and $t \geq t_0$

$$(7.17)\quad \boldsymbol{u}^i(\boldsymbol{x}(1),\dots,\boldsymbol{x}(T_0),\underline{0},\dots,\underline{0},\boldsymbol{x}(t+1),\dots) \geq \boldsymbol{u}^i(\boldsymbol{x}(1),\dots,\boldsymbol{x}(T_0),\underline{0},\dots)-\varepsilon_0.$$

Pick $t > t_0$ so large that

$$(7.18)\quad \sum_{\tau=T_0+1}^{t} {}^t\boldsymbol{p}(\tau)[\sum_j \boldsymbol{y}^j(\tau;\,{}^t\boldsymbol{p})] \geq \alpha-(\beta+\delta)$$

$$(7.19)\quad \lambda_0 {}^t\boldsymbol{p}(\boldsymbol{x}^{*i}(1),\dots,\boldsymbol{x}^{*i}(T^I),\underline{0},\dots) < \frac{[\alpha-(\beta+\delta)](1-\lambda_0)}{I}$$

Due to (7. 15)–(7. 17), for any i

$$\begin{aligned}(7.20)\quad & \boldsymbol{u}^i(\lambda_0(\boldsymbol{x}^{*i}(1),\dots,\boldsymbol{x}^{*i}(T^I),\underline{0},\dots)+(1-\lambda_0)(\boldsymbol{x}^{*i}({}^\nu\boldsymbol{p};1),\dots \\ & \qquad \boldsymbol{x}^i({}^\nu\boldsymbol{p};T_0),\underline{0},\dots,\underline{0},\boldsymbol{x}^i({}^\nu\boldsymbol{p};t+1),\dots) \\ & \geq \boldsymbol{u}^i(\boldsymbol{x}^i({}^\nu\boldsymbol{p};1),\dots,\boldsymbol{x}^i({}^\nu\boldsymbol{p};T_0),\underline{0},\dots,\underline{0},\boldsymbol{x}^i({}^\nu\boldsymbol{p};t+1),\dots)+\sigma^i \\ & \geq \boldsymbol{u}^i(\boldsymbol{x}^i({}^\nu\boldsymbol{p};1),\dots,\boldsymbol{x}^i({}^\nu\boldsymbol{p};T_0),\underline{0},\dots)-\varepsilon+\sigma^i \\ & \geq \boldsymbol{u}^i(\boldsymbol{x}^i({}^\nu\boldsymbol{p}))-\rho^i_{T_0}-\varepsilon+\sigma^i \\ & \geq \boldsymbol{u}^i(\boldsymbol{x}^i({}^\nu\boldsymbol{p}))\end{aligned}$$

On the other hand, choose i^* such that ${}^\nu\boldsymbol{p}\boldsymbol{x}^{i*} \geq {}^\nu\boldsymbol{p}\boldsymbol{x}^i$ over i. Then

$${}^\nu\boldsymbol{p}\boldsymbol{x}^{i*} \geq \frac{1}{I}\sum_i {}^\nu\boldsymbol{p}\boldsymbol{x}^i。$$

Hence, due to (7. 18) and (7. 19)

$$\begin{aligned}(7.21)\quad & {}^\nu\boldsymbol{p}\boldsymbol{x}^{i*}({}^\nu\boldsymbol{p})-{}^\nu\boldsymbol{p}[\lambda_0(\boldsymbol{x}^{*i*}(1),\dots,\boldsymbol{x}^{*i*}(T^I),\underline{0},\dots)+(1-\lambda_0) \\ & \qquad (\boldsymbol{x}^{i*}({}^\nu\boldsymbol{p};1),\dots,\boldsymbol{x}^{i*}({}^\nu\boldsymbol{p};T_0),\underline{0},\dots,\underline{0},\boldsymbol{x}^{i*}({}^\nu\boldsymbol{p};t+1),\underline{0},\dots)] \\ & \geq (1-\lambda_0)\frac{\alpha-(\beta+\delta)}{I}-\lambda_0{}^\nu\boldsymbol{p}(\boldsymbol{x}^{*i*}(1),\dots,\boldsymbol{x}^{*i*}(T^I),\underline{0},\dots)+\lambda_0{}^\nu\boldsymbol{p}\boldsymbol{x}^{i*}({}^\nu\boldsymbol{p}) \\ & \geq (1-\lambda_0)\frac{\alpha-(\beta+\delta)}{I}-\lambda_0{}^\nu\boldsymbol{p}(\boldsymbol{x}^{*i*}(1),\dots,\boldsymbol{x}^{*i*}(T^I),\underline{0},\dots) \\ & > 0\end{aligned}$$

Since $\lambda_0(\boldsymbol{x}^{*i*}(1),\ldots, \boldsymbol{x}^{*i*}(\mathrm{T}^I), \underline{0},\ldots)+(1-\lambda_0)(\boldsymbol{x}^{i*}({}^{\nu}\boldsymbol{p}; 1),\ldots, \boldsymbol{x}^{i*}({}^{\nu}\boldsymbol{p}; \mathrm{T}^I), \underline{0},\ldots, \underline{0}, \boldsymbol{x}^{i*}({}^{\nu}\boldsymbol{p}; \mathrm{t}+1),\ldots) \in \boldsymbol{X}^{i*}\cap \mathrm{C_M}$, (7. 20) and (7. 21) cannot hold simultaneously (assumptions (b. 1) and (c)), hence (7. 11) cannot be true. cannot be true.

Next, suppose that (7. 12) is true and let $\varepsilon=\beta-\alpha(>0)$. Choose large T^J so that

$$(7.22)\quad \sum_{\tau=1}^{T^J} {}^{\nu}\boldsymbol{p}(\tau)\sum_j \boldsymbol{y}^j(\tau) \geq \beta-\frac{\varepsilon}{8}$$

Pick $\nu>\mathrm{T}^J$ large enough such that

$$\sum_{\tau=1}^{T^J} {}^{\nu}\boldsymbol{p}(\tau)\sum_j \boldsymbol{y}^j(\tau) \geq \beta-\frac{\varepsilon}{4}$$

$${}^{\nu}\boldsymbol{p}\sum_j \boldsymbol{y}^j({}^{\nu}\boldsymbol{p}) < \alpha+\frac{\varepsilon}{4}$$

Then

$$\sum_{\tau=\mathrm{T}+1}^{t} {}^{\nu}\boldsymbol{p}\sum_j \boldsymbol{y}^j({}^{\nu}\boldsymbol{p}; \tau) < -\frac{\varepsilon}{2},$$

which contradicts to the condition ii) of competitive equilibrium. Hence, (7. 12) cannot hold.

Since the subsequence $\{{}^{\nu}\boldsymbol{p}\}$ of price vectors is arbitrary, (7. 8)' is true whenever (7. 9) holds. Suppose not. Redifine λ_0 so that

$$\lambda_0\boldsymbol{p}^0(\boldsymbol{x}^{*i*}(1),\ldots, \boldsymbol{x}^{*i*}(\mathrm{T}^I), \underline{0},\ldots) < \frac{(1-\delta)(1-\lambda_0)}{I}$$

and that (7. 14)–(7. 16) hold for any i. Choose $\mathrm{t}>\mathrm{T}_0$ large enough so

$$\lambda_0{}^{\nu}\boldsymbol{p}(\boldsymbol{x}^{*i}(1),\ldots, \boldsymbol{x}^{*i}(\mathrm{T}^I), \underline{0},\ldots) < \frac{1-\lambda_0}{I}$$

$$\sum_{\tau=\mathrm{T}_0+1}^{t} {}^{\nu}\boldsymbol{p}(\tau)\sum_j \boldsymbol{y}^j({}^{\nu}\boldsymbol{p}; \tau) \geq 1, \text{ for } \nu \geq \mathrm{t}.$$

Then for some i (call i^*), (7. 21) holds with $\dfrac{\alpha-(\beta+\delta)}{I}$ replaced with $\dfrac{1}{I}$

and (7. 20) holds for all i, This leads to the contradiction.

$p^0 \neq \underline{0}$ follows from (7. 7) and (7. 8).

Step 3: Claim that

(7.23) $p^0 x^i = \lim_{\nu \to \infty} {}^{\nu}p x^i({}^{\nu}p)$.

In order to show the claim, choose a subsequence of $\{{}^{\nu}p\}$ such that the the right hand side of (7. 23) converges for any i. Denote this subsequence $\{{}^{\nu}p\}$ as well and let ς be the limit for fixed i. Here

$$0 \leq {}^{\nu}p x^i({}^{\nu}p) \leq \sum_i {}^{\nu}p x^i({}^{\nu}p) = {}^{\nu}p \sum_j y^j({}^{\nu}p) < \infty.$$

Hence, $0 \leq \varsigma < \infty$ by (7. 8).

If (7. 23) does not hold, then either

(7.24) $\varsigma < p^0 x^i \equiv \kappa$, or

(7.25) $\varsigma > \kappa$

By the same argument in Step 2, (7. 25) is shown not to hold. On the other hand, by the definition of X^i (7. 24) cannot be true. Hence, (7. 23) is satisfied. Remember that (7. 23) implies the inner product $p^i x^i$ is well-defined.

Step 4: Claim that for any $x \in X^i \cap C_M$ such that $u^i(x) \geq u^i(x^i)$

(7.26) $p^0 x \geq p^0 x^i$

Take $x \in X^i \cap C_M$ and $u^i(x) > u^i(x^i)$. Obviously, such an x exists. Let $\varepsilon(>0)$ such that

$$u^i(x) = u^i(x^i) + \varepsilon$$

and choose T so large that

$$u^i(x(1), \ldots, x(T), x^i(T+1), \ldots) > u^i(x^i) + \frac{\varepsilon}{2}$$

and that for any arbitrary $\varepsilon_1 > 0$

$$\sum_{\tau=T+1}^{\infty} \boldsymbol{p}^0(\tau)\boldsymbol{x}^i(\tau) < \frac{\varepsilon_1}{4}.$$

Choose $T_0 > T$ so large that for any $t \geq T_0$

$$\sum_{\tau=T+1}^{t} {}^{\nu}\boldsymbol{p}(\tau)\boldsymbol{x}^i(\tau) < \varepsilon_1$$

and that

$$\boldsymbol{u}^i(\boldsymbol{x}^i({}^{\nu}\boldsymbol{p})) < \boldsymbol{u}^i(\boldsymbol{x}^i) + \frac{\varepsilon}{2}.$$

Then

$$\sum_{\tau=0}^{T} {}^{\nu}\boldsymbol{p}(\tau)\boldsymbol{x}(\tau) + \varepsilon > {}^{\nu}\boldsymbol{p}\boldsymbol{x}^i({}^{\nu}\boldsymbol{p})$$

hence

$$\sum_{\tau=0}^{T} \boldsymbol{p}^0(\tau)\boldsymbol{x}(\tau) + \varepsilon_1 \geqq \boldsymbol{p}^0\boldsymbol{x}^i$$

therefore

$$(7.27) \quad \boldsymbol{p}^0\boldsymbol{x} \geq \boldsymbol{p}^0\boldsymbol{x}^i$$

Next, suppose that $\boldsymbol{x} \in \boldsymbol{X}^i$, $\boldsymbol{x} \neq \boldsymbol{x}^i$ $\quad \boldsymbol{u}^i(\boldsymbol{x}) \geq \boldsymbol{u}^i(\boldsymbol{x}^i)$. Let
$\boldsymbol{x}(\lambda) = \lambda(\boldsymbol{x}^i(1), \ldots, \boldsymbol{x}^i(T), \underline{0}, \ldots) + (1-\lambda)\boldsymbol{x}$,

where T is chosen large enough so that

$$\boldsymbol{u}^i(\boldsymbol{x}^i(1), \ldots, \boldsymbol{x}^i(T), \underline{0}, \ldots) > \boldsymbol{u}^i(\boldsymbol{x}^i).$$

Then by (7. 27)

$$(7.28) \quad \boldsymbol{p}\boldsymbol{x}^i \leq \boldsymbol{p}^0\boldsymbol{x}(\lambda) = \lambda\boldsymbol{p}^0(\boldsymbol{x}^i(1), \ldots, \boldsymbol{x}^i(T), \underline{0}, \ldots) + (1-\lambda)\boldsymbol{p}^0\boldsymbol{x}$$

Since λ is chosen arbitrary, (7. 26) is true.

Step 5: Claim if $\boldsymbol{y} \in \boldsymbol{Y}^j \cap C_M$, then

$$(7.29)\quad \boldsymbol{p}^0 \boldsymbol{y} \leq \boldsymbol{p}^0 \boldsymbol{y}^j .$$

Suppose not and let $\varepsilon = \boldsymbol{p}^0 \boldsymbol{y} - \boldsymbol{p}^0 \boldsymbol{y}^j\ (>0)$. Then, there exists an integer T so large that

$$\sum_{\tau=0}^{T} \boldsymbol{p}^0(\tau)\boldsymbol{y}(\tau) > \boldsymbol{p}^0 \boldsymbol{y}^j + \frac{\varepsilon}{2}$$

Next, observe that for any i

$$(7.30)\quad \boldsymbol{p}^0 \boldsymbol{y}^i \leq \liminf_{\nu\to\infty} {}^{\nu}\boldsymbol{p}\boldsymbol{y}^j({}^{\nu}\boldsymbol{p})$$

$$\boldsymbol{p}^0 \sum_i \boldsymbol{x}^i_L \geq \liminf_{\nu\to\infty} {}^{\nu}\boldsymbol{p} \sum_i \boldsymbol{x}^i$$

From (7. 30), for an appropriately chosen subsequence of $\{{}^{\nu}\boldsymbol{p}\}$ (again call it $\{{}^{\nu}\boldsymbol{p}\}$),

$$(7.31)\quad \boldsymbol{p}^0 \sum_j \boldsymbol{y}^j = \lim_{\nu\to\infty} {}^{\nu}\boldsymbol{p} \sum_j \boldsymbol{y}^j({}^{\nu}\boldsymbol{p}).$$

Hence, the equality holds in (7. 30) for the selected subsequence. Then, for large ν,

$$(7.32)\quad \sum_{\nu=0}^{T} {}^{\nu}\boldsymbol{p}(\tau)\boldsymbol{y}^j(\tau) \geq \boldsymbol{p}^0 \boldsymbol{y}^j + \frac{\varepsilon}{4}$$

$$(7.33)\quad {}^{\nu}\boldsymbol{p}\boldsymbol{y}^j({}^{\nu}\boldsymbol{p}) \leq \boldsymbol{p}^0 \boldsymbol{y}^j + \frac{\varepsilon}{4}$$

which contradict the optimality of $\boldsymbol{y}^j({}^{\nu}\boldsymbol{p})$. Then, (7. 29) is valid.

It is true if $\boldsymbol{x} \in \boldsymbol{X}^i$ and $\boldsymbol{p}^0 \boldsymbol{x} = \boldsymbol{p}^0 \boldsymbol{x}^i$, then

$$(7.34)\quad \boldsymbol{u}^i(\boldsymbol{x}) \geq \boldsymbol{u}^i(\boldsymbol{x}^i)$$

because $\boldsymbol{x}^i$ is $\boldsymbol{u}^i$-maximizer.

Step 6: The preceding steps show that $(\boldsymbol{p}^0, \boldsymbol{x}^i, \boldsymbol{y}^j)$ is a competitive equilibrium in the economy with an infinite time horizon where each consumer has a consumption set $\boldsymbol{X}^i \cap C_M$ and each firm has a production possibility set $\boldsymbol{Y}^j \cap C$

By the standard procedure, it is easily shown that ($\boldsymbol{p}^0$, $\boldsymbol{x}^i$, $\boldsymbol{y}^j$) is a competitive equilibrium in an economy with consumption set $\boldsymbol{X}^i$ and production set $\boldsymbol{Y}^j$ as well.(g. e. d.)

8. The Efficiency

The aim of this section is to discuss the efficiency of a temporary equilibrium in a sequential market, of which the existence has been proven in the sections 6 and 7. Contrary to the traditional (static) general competitive equilibrium theory, an expectation of the future environment such as prices plays the central role in temporary equilibrium theory and causes some troubles in discussions of efficiency; since each consumer behaves so as to maximize his expected utility over his life time, his behavior of each period is interrelated to other periods through his expectation of future prices. So is the policy choice of a firm, since the objective of a firm is to maximize the expected return on a share held. One way to get around this difficulty in separating the behavior of agents in a particular period from other periods is to fix portfolios at the end of period and regard them as given.

In other words, it is equivalent to suppose that the forecast about future prices (=the disribution of expected prices) remains unchanged when the consumption vectors are reshuffled among consumers. Or at least, the influences of changes in the expectation, if any, cancel ont each other.

Because each consumer lives for three periods, four concepts of efficiency are introduced [see figure 8.1]; namely, one period- (or temporary-), two period-, three period-(or lifetime-) and an infinite horizon-(or longrun) efficiencies. One period efficiency of period t, say, is considered with given portfolios at the end of period t, which in turn implies that the purchased

Figure 8.1

GENERATION	Period t	Period t+1	Period t+2
t−2	...,...,[x^i, $\underline{0}$, 0: (t−2, t)]		
t−1	...,[x^i, s^i, b^i: (t−1, t)],	[x^i, $\underline{0}$, 0: (t−1, t+1)]	
t	[x^i, s^i, b^i: (t, t)],	[x^i, s^i, b^i: (t, t+1)],	[x^i, $\underline{0}$, 0: (t, t+2)]
t+1		[x^i, s^i, b^i: (t+1, t+1)],	[x^i, s^i, b^i: (t+1, t+2)],...
t+2			[x^i, s^i, b^i: (t+2, t+2)],...,...

input and investment are also given. Since the input of period (t−1) is a historical data, the output of period t, the result of input in period (t−1) is beyond consumers' control. Therefore, one period efficiency is defined in terms of current consumption vectors.

To consider two period efficiciency, say, over periods t and (t+1), the portfolios at the end of period (t+1) are fixed as before, hence the decision vectors of firms in period (t+1) are given. But, contrary to one period efficiency, the production vectors in period t ($n^j(t)$ and $b^j(t)$ hence $\tilde{x}^j(t)$ and $\Delta K^j(t)$) can be changed as well as consumption vectors. In two period efficiency, there are two chances of reshuffle: one in period t among consumption vectors $\tilde{x}^i(\tau, t)$, $\tau=t-2, t-1$ and t and production vectors $\tilde{x}^j(t)$ and $\Delta K^j(t)$ and the other in period (t+1) among consumption vectors only, $\tilde{x}^i(\tau, t)$, $\tau=t-1$, t and t+1. Three period efficiency is similarly defined as two period efficiency, though it has three chances of reshuffle instead of two. In this manner, one can consider four period-, five period-efficiency and so on by increasing the number of periods involved and finally, one gets an infinite time horizon efficiency as the limit.

Let me introduce the definitions of efficiencies formally.

1) One period efficiency

Given the portfolio $\{\underline{0}, 0, s^i(t-1, t), b^i(t-1, t), s^i(t, t), b^i(t, t)\}$ of period t (hence the behaviors in periods t+1 and t+2 expected by consumer i of generation t are fixed), a consumption plan $\{\tilde{x}^i(t-2, t), \tilde{x}^i(t-1, t), \tilde{x}^i(t, t)\}$ of period t is called to be *one period efficient* if there exists no other consumption plan derived by redistributing the original plan original plan such that at least one member of generation t is made better off without making anyone else worse off; that is if there does not exist a consumption plan $\{\tilde{x}^{i\prime}(t-2, t), \tilde{x}^{i\prime}(t-1, t), \tilde{x}^{i\prime}(t, t)\}$ of period t such that

a) $\sum_i \sum_{\tau=t-2}^{t} \tilde{x}^{i\prime}(\tau, t) \leq \sum_i \sum_{\tau=t-2}^{t} \tilde{x}^{i}(\tau, t)$

b) for any $i \in I(t-2)$,

$$u^i(\tilde{x}^i(t-2, t-2), \tilde{x}^i(t-2, t-1), \tilde{x}^{i\prime}(t-2, t))$$
$$\geq u^i(\tilde{x}^i(t-2, t-2), \tilde{x}^i(t-2, t-1), \tilde{x}^i(t-2, t))$$

where $\tilde{x}^i(t-2, t-1)$ and $\tilde{x}^i(t-2, t-1)$ are historically given.

c) for any $i \in I(t-1)$,

$$u^i(\tilde{x}^i(t-1, t-1), \tilde{x}^{i\prime}(t-1, t), {}_t\tilde{x}^i(t-1, t+1))$$
$$\geq u^i(\tilde{x}^i(t-1, t-1), \tilde{x}^i(t-1, t), {}_t\tilde{x}^i(t-1, t+1))$$

where $\tilde{x}^i(t-1, t-1)$ is historically given and the expected consumption ${}_t\tilde{x}^i(t-1, t+1)$ is fixed.

d) for any $i \in I(t)$

$$u^i(\tilde{x}^{i\prime}(t, t), {}_t\tilde{x}^i(t, t+1), {}_t\tilde{x}^i(t, t+2))$$
$$\geq u^i(\tilde{x}^i(t, t), {}_t\tilde{x}^i(t, t+1), {}_t\tilde{x}^i(t, t+2))$$

and the strict inequality holds for at least one i, where the expected consumption vectors ${}_t\tilde{x}^i(t, t+1)$ and ${}_t\tilde{x}^i(t, t+2)$ remain unchanged.

Remember that the reallocation takes place only among the consumption vectors $\tilde{x}^i(t-2, t)$, $\tilde{x}^i(t-1, t)$ and $\tilde{x}^i(t, t)$ of current period.

2) Two period efficiency

Given the portfolio $\{\underline{0}, 0, s^i(t, t+1), b^i(t, t+1), s^i(t+1, t+1), b^i(t+1, t+1)\}$ of period t+1, a consumption plan over two periods (periods t and t+1) $\{\tilde{x}^i(t-2, t), \tilde{x}^i(t-1, t), \tilde{x}^i(t, t), \tilde{x}^i(t-1, t+1), \tilde{x}^i(t, t+1), \tilde{x}^i(t+1, t+1)\}$ is said to be *two period efficient* if there exist no other consumption plans over these periods and modified input and investment plans in period t, keeping that of period t+1 unchanged and redistributing outputs within each period such that at least one member of generation t is made better off without making anyone else worse off; that is, there exists no alternative consumption plan $\{\tilde{x}^{i\prime}(t-2, t), \tilde{x}^{i\prime}(t-1, t), \tilde{x}^{i\prime}(t, t), \tilde{x}^{i\prime}(t-1, t+1), \tilde{x}^{i\prime}(t, t+1), \tilde{x}^{i\prime}(t+1, t+1)\}$ and associated input and investment plans $\{\tilde{x}^{j\prime}(t), \Delta K^{j\prime}(t)\}$ such that b) and

e) $\sum_i \sum_{\tau=t-2}^{t} \tilde{x}^{i\prime}(\tau, t) \leq \sum_j (\tilde{y}^j(t) - \tilde{x}^{j\prime}(t) - \Delta K^{j\prime}(t))$

where $(\tilde{x}^j(t-1), \tilde{y}^j(t)) \in Y^j(K^j(t))$ and $\tilde{x}^j(t-1)$ is historically given.

f) $\sum_i \sum_{\tau=t-1}^{t=1} \tilde{x}^{i\prime}(\tau, t+1) \leq \sum_j (\tilde{y}^{j\prime}(t+1) - {}_j\tilde{x}^j(t+1) - {}_j\Delta K^j(t+t))$

where $(\tilde{x}^{j\prime}(t), \tilde{y}^{j\prime}(t+1)) \in Y^j(K^j(t)+\Delta K^{j\prime}(t))$ and the expected vectors ${}_j\tilde{x}^j(t+1)$ and ${}_j\Delta K^j(t+)$ are fixed.

g) for any $i \in I(t-1)$,

$$u^i(\tilde{x}^i(t-1, t-1), \tilde{x}^{i\prime}(t-1, t), \tilde{x}^{i\prime}(t-1, t+1))$$
$$\geq u^i(\tilde{x}^i(t-1, t-1), \tilde{x}^i(t-1, t), \tilde{x}^i(t-1, t+1))$$

where $\tilde{x}^i(t-1, t-1)$ is historically given.

h) for any $i \in I(t)$,

$$u^i(\tilde{x}^{i\prime}(t, t), \tilde{x}^{i\prime}(t, t+t), {}_i\tilde{x}^i(t, t+2))$$
$$\geq u^i(\tilde{x}^i(t, t), \tilde{x}^i(t, t+1), {}_i\tilde{x}^i(t, t+2))$$

and at least one inequality holds strictly, where the expectad consumption vector ${}_i\tilde{x}^i(t, t+2)$ of period t+2 is fixed.

i) for any $i \in I(t+1)$,

$$u^i(\tilde{x}^{i\prime}(t+1, t+1), {}_i\tilde{x}^i(t+1, t+2), {}_i\tilde{x}^i(t+1, t+3))$$
$$\geq u^i(\tilde{x}^i(t+1, t+1), {}_i\tilde{x}^i(t+1, t+2), {}_i\tilde{x}^i(t+1, t+3))$$

where the expected consumption vectors ${}_i\tilde{x}^i$(t+1, t+2) and ${}_i\tilde{x}^i$(t+1, t+3) are fixed.

Note that the reallocations take place twice. In period t, the consunption vectors $\tilde{x}^i$(t−2, t), $\tilde{x}^i$(t−1, t), $\tilde{x}^i$(t, t) and the production vectors $\tilde{x}^j$(t), ΔK^j(t) are reshuffled. So are only the consumption vectors $\tilde{x}^i$(t−1, t+1), $\tilde{x}^i$ (t, t+1), $\tilde{x}^i$(t+1, t+1) in period t+1. Also note that the condition a) of one period efficiency is no longer meaningful hereafter.

3) Three period efficiency

Given the portfolio {$\underline{0}$, 0, s^i(t+1, t+2), b^i(t+1, t+2), s^i(t+2, t+2), b^i(t+2, t+2)} of period t+2, a consumption plan {$\tilde{x}^i$(t−2, t), $\tilde{x}^i$(t−1, t), $\tilde{x}^i$(t, t), $\tilde{x}^i$(t−1, t+1), $\tilde{x}^i$(t, t+1), $\tilde{x}^i$(t+1, t+1), $\tilde{x}^i$(t, t+2), $\tilde{x}^i$(t+1, t+2), $\tilde{x}^i$(t+2, t+2)} over three periods from t to t+2 is said to be *three period efficient* if there exist no other consumption plan covering these periods attainable through suitable reallocations of consumption vectors in three periods and production vectors in the first two periods such that at least one member of generation t is made better off without making anyone else worse off; that is, there exists no other plan {$\tilde{x}^{i\prime}$(t−2, t), $\tilde{x}^{i\prime}$(t−1, t), $\tilde{x}^{i\prime}$(t, t), $\tilde{x}^{i\prime}$(t−1, t+1), $\tilde{x}^{i\prime}$(t, t+1), $\tilde{x}^{i\prime}$(t+1, t+1), $\tilde{x}^{i\prime}$(t, t+2), $\tilde{x}^{i\prime}$(t+1, t+2), $\tilde{x}^{i\prime}$(t+2, t+2)} of consumption such that b), e), g) and

j) $\sum_i \sum_{\tau=t-1}^{t+1} \tilde{x}^{i\prime}(\tau, t+1) \leq \sum_j (\tilde{y}^{j\prime}(t+1) - \tilde{x}^{j\prime}(t+1) - \Delta K^{j\prime}(t+1))$,

where $(\tilde{x}^{j\prime}(t), \tilde{y}^{j\prime}(t+1)) \in Y^j(K^j(t) + \Delta K^{j\prime}(t))$.

k) $\sum_i \sum_{\tau=t}^{t+2} \tilde{x}^{i\prime}(\tau, t+2) \leq \sum_j (\tilde{y}^{j\prime}(t+2) - {}_j\tilde{x}^j(t+2) - {}_j\Delta K^j(t+2))$

where $(\tilde{x}^{j\prime}(t+1), \tilde{y}^{j\prime}(t+2)) \in Y^j(K^j(t) + \Delta K^{j\prime}(t) + \Delta K^{j\prime}(t+1))$

and the expected production plans ${}_j\tilde{x}^j$(t+2), ${}_j\Delta K^j$(t+2) are fixed.

l) for any $i \in I$(t),

$$u^i(\tilde{x}^{i\prime}(t, t), \tilde{x}^{i\prime}(t, t+t), \tilde{x}^{i\prime}(t, t+2))$$
$$\geq u^i(\tilde{x}^i(\tilde{x}^i(t, t), \tilde{x}^i(t, t+1), \tilde{x}^i(t, t+2))$$

and a strict inequality holds for at least one i.

m) for any $i \in I(t+1)$,

$$u^i(\tilde{x}^{i\prime}(t+1, t+1), \tilde{x}^{i\prime}(t+1, t+2), {}_i\tilde{x}^i(t+1, t+3))$$
$$\geq u^i(\tilde{x}^i(t+1, t+1), \tilde{x}^i(t+1, t+2), {}_i\tilde{x}^i(t+1, t+3)),$$

where the expected consumption ${}_i\tilde{x}^i(t+1, t+3)$ is fixed.

n) for any $i \in I(t+2)$,

$$u^i(\tilde{x}^{i\prime}(t+2, t+2), {}_i\tilde{x}^i(t+2, t+3), {}_i\tilde{x}^i(t+2, t+4))$$
$$\geq u(\tilde{x}^i(t+2, t+2), {}_i\tilde{x}^i(t+2, t+3), {}_i\tilde{x}^i(t+2, t+4))$$

where the expected consumptions ${}_i\tilde{x}^i(t+2, t+3)$ and ${}_i\tilde{x}^i(t+2, t+4)$ are given.

Note that there exists three chances of reshuffle: two are among both consumption and production vectors and one among just consumption vectors.

4) *An infinite time horizon efficiency*[4]

An infinite sequence of lifetime plans of consumption $\{(\tilde{x}^i(\tau, \tau), \tilde{x}^i(\tau, \tau+1), \tilde{x}^i(\tau, \tau+2))_{i \in I(\tau)}\}_{\tau=1,2,\ldots,\infty}$ is said to be *infinite time horizon efficient* if there exists no other sequence of consumption plans $\{(\tilde{x}^{i\prime}(\tau, \tau), \tilde{x}^{i\prime}(\tau, \tau+1), \tilde{x}^{i\prime}(\tau, \tau+2))_{i \in I(\tau)}\}_{\tau=1,2,\ldots,\infty}$ which is derivable by redistributing, within the same period, the original consumption plan and associated production plan such that at least one member of some generation is

4) After having written the text, I have found that in his overlapping generation model Starrett(1972) defines an infinite horizon efficiency (he just calls "efficiency", see p. 278) in the same way as I do. To my best knowledge, he is the first to give a precise definition.

made better off without making anyone else worse off; that is

o) for any t,

$$\sum_i \sum_{\tau=t-2}^{t} \tilde{x}^{i\prime}(\tau, t) \leq \sum_j (\tilde{y}^j(t) - \tilde{x}^{j\prime}(t) - \Delta K^{j\prime}(t)),$$

where $(\tilde{x}^j(t-1), \tilde{y}^j(t)) \in Y^j(K^j(t))$.

p) for any $i \in I(\tau)$,

$$u^i(\tilde{x}^{i\prime}(\tau, \tau), \tilde{x}^{i\prime}(\tau, \tau+1), \tilde{x}^{i\prime}(\tau, \tau+2))$$
$$\geq u^i(\tilde{x}^i(\tau, \tau), \tilde{x}^i(\tau, \tau+1), \tilde{x}^i(\tau, \tau+2)),$$

and at least one inequality holds strictly.

Note that the condition p) can be rewritten as follows since the infinite sequence of consumption plans can be represented by an infinite sequence $\{(u^i(\tau))_{i \in I(\tau)}\}_{\tau=1,2,\ldots,\infty}$ of expected utility $u^i(\tau) = u^i(x^i(\tau, \tau), \tilde{x}^i(\tau, \tau+1), \tilde{x}^i(\tau, \tau+2))$,

p′) for any $i \in I(\tau)$,

$$u^{i\prime}(\tau) \geq u^i(\tau)$$

and a strict inequality holds for at least one i.

An overlapping generation model of this kind is useful since the infiniteness can be discussed in a finite framework.

Theorem 8. 1

If each generation in an infinite sequence of consumption plans $\{(\tilde{x}^i(\tau, \tau), \tilde{x}^i(\tau, \tau+1), \tilde{x}^i(\tau, \tau+2))_{i \in I(\tau)}\}_{\tau=1,2,\ldots,\infty}$ enjoys three period efficiency, then such a sequence is infinite time horizon efficient.

(Proof) Suppose that the proposition does not hold. Then, there exists at least one member in some generation, say generation t, who can be made better off by a certain redistribution without making anyone else worse off. But it implies the existence of an infinite sequence $\{(\tilde{x}^{i\prime}(\tau, \tau), \tilde{x}^{i\prime}(\tau, \tau+1), \tilde{x}^{i\prime}(\tau, \tau+2))_{i\in I(\tau)}\}_{\tau=1,2,\ldots,\infty}$ where $\tilde{x}^{i\prime}(\tau, \kappa)=\tilde{x}^{i}(\tau, \kappa)$, $\kappa=\tau, \tau+1, \tau+2$, $\kappa\leq t-1$, $t+3\leq\kappa$ and e), j) and k) hold, which leads to the desired contradiction. (g. e. d.)

The following is an analogy to the well-known relationship between competitive equilibrium and efficient allocation in a (static) general competitive equilibrium theory.

Theorem 8. 2

A consumption allocation in a temporary equilibrium is one period efficient. With a given program of production, a feasible program of consumption is a temporary equilibrium if it is a one period efficient allocation.

(Proof) Suppose that $[x^i(t-2, t), x^i(t-1, t), x^i(t, t), y^j(t), p(t)]$ is a temporary equilibrium but not one period efficient, that is, there exists a consumption allocation of three generations of period t $\{\tilde{x}^{i\prime}(t-2, t), \tilde{x}^{i\prime}(t-1, t), \tilde{x}^{i\prime}(t, t)\}$ such that a)–d) hold, given $\{\underline{0}, 0, s^i(t-1, t), b^i(t-1, t), s^i(t, t), b^i(t, t)\}$. Then, there exists at least one consumer, call h, of generation t such that

$$(8.1)\quad u^h(\tilde{x}^{h\prime}(t, t), {}_h\tilde{x}^h(t, t+1), {}_h\tilde{x}^h(t, t+2))$$
$$>u^h(\tilde{x}^h(t, t), {}_h\tilde{x}^h(t, t+1), {}_h\tilde{x}^h(t, t+2)).$$

If $\tilde{p}(t)\ \tilde{x}^{h\prime}(t, t)>\tilde{p}(t)\ \tilde{x}^h(t, t)$, $\tilde{x}^{h\prime}(t, t)$ does not satisfy his budget constraint (3. 1) or (3. 1*). Hence, it cannot be a temporary equilibrium allocation. On the other hand, if $\tilde{p}(t)\ \tilde{x}^{h\prime}(t, t)\leq\tilde{p}(t)\ \tilde{x}^h(t, t)$, there is some consumer k such that $\tilde{p}(t)\ \tilde{x}^k(\tau, t)>\tilde{p}(t)\ \tilde{x}^{k\prime}(\tau, t)$, $\tau=t-2, t-1, t$ by a), for whom $\tilde{x}^{k\prime}(\tau, t)$ is not a temporary equilibrium allocation. Hence, there exists no allocation $\{\tilde{x}^{i\prime}(t-2, t), \tilde{x}^{i\prime}(t-1, t), \tilde{x}^{i\prime}(t, t)\}$ of consumers' vectors such that a)–d) are satisfied. Therefore, a temporary equilibrium is one period efficient.

For the proof of the latter part, let $\{x^i(t-2, t), x^i(t-1, t), x^i(t, t), y^j(t), p(t)\}$ be a feasible program and associated price system. Suppose that the consumption plan $\{\tilde{x}^i(t-2, t), \tilde{x}^i(t-1, t), \tilde{x}^i(t, t)\}$ is one period efficient with the given portfolio pattern. Then, for any price vector, there exists no other consumption plan $\{\tilde{x}^{i\prime}(t-2, t), \tilde{x}^{i\prime}(t-1, t), \tilde{x}^{i\prime}(t, t)\}$ such that a)–d) are satisfied, which implies the condition (i) of temporay equilibrium is satisfied. The feasibility means the existence of price vector $p'(t)$ such that the budget constraints (3. 1)–(3. 3) or (3. 1*)–(3. 3*) are satisfied for all the consumers. If this $\tilde{p}'(t)$ is not equal to the price vector $\tilde{p}(t)$ to maximize profits of firms, then the consumption plan $\{(\tilde{x}^i(t-2, t), \tilde{x}^i(t-1, t), \tilde{x}^i(t, t)\}$ associated with the price system $p(t)$ fails to be one period efficient at $p(t)$. since the dividend paid at $\tilde{p}'(t)$ is smaller than that at $\tilde{p}(t)$, Hence $\tilde{p}'(t)=\tilde{p}(t)$ and therefore the condition (ii) of temporary equilibrium is satisfied. Furthermore, if $q'(t)\neq q(t)$ and $r'(t)\neq r(t)$, the condition (vi) of feasible program fails to be satisfied because $q(t)$ and $r(t)$, together with $p(t)$, define the values of $n^j(t)$ and $b^j(t)$. Therefore, $q'(t)=q(t)$ and $r'(t)=r(t)$, hence $p'(t)=p(t)$. Then, the condition (iii) of a temporary equilibrium is satisfied. Finally, the conditions (iv) and (vii) of feasible program imply the market clearing. (g. e. d.)

Though the opposite is not always true, next proposition shows that a longrun efficiency implies shortrun ones.

Theorem 8. 3

If the given demand for assets coincides with that of the lifetime pattern, an efficiency in an infinite time horizon implies three-, hence two- and one-period efficiencies.

(Proof) Suppose that an infinite sequence $\{(\tilde{x}^i(\tau, \tau), \tilde{x}^i(\tau, \tau+1), \tilde{x}^i(\tau, \tau+2))_{i\in I(\tau)}\}_{\tau=1,2,\dots,\infty}$ of consumption plan is efficient in an infinite horizon but that the consumption plan $[\tilde{x}^i(t, t), \tilde{x}^i(t, t+1), \tilde{x}^i(t, t+2)]$ of consumer i of generation t in the sequence is not three period efficient given all other allocations. Then, there exists another plan for i, $[\tilde{x}^{i\prime}(t, t), \tilde{x}^{i\prime}(t, t+1), \tilde{x}^{i\prime}(t, t+2)]$ derived from the original plan by a certain redistribution such that

$$(8.2) \quad u^i(\tilde{x}^{i\prime}(t, t), \tilde{x}^{i\prime}(t, t+1), \tilde{x}^{i\prime}(t, t+2)) > u^i(\tilde{x}^i(t, t), \tilde{x}^i(t, t+1), \tilde{x}^i(t, t+2)).$$

But this contradicts the conditions o) and p) (or p′)) of an infinite efficiency.

Take periods t, t+1 and t+2 and generations t−2, t−1, t, t+1 and t+2 which participate in the markets in these periods. Suppose that three period efficiency holds for the generation t and consider the periods t and t+1. If the demands for portfolio at period t+1 $\{\underline{0}, 0, x^i(t, t+1), b^i(t, t+1), s^i(t+1, t+1), b^i(t+1, t+1)\}$ coincide with those of period t+1 in the three period efficient plan, then three period efficiency implies two period efficiency with the given asset demand patten. Otherwise, there exists another feasible plan $\{\tilde{x}^{i\prime}(t-2, t), \tilde{x}^{i\prime}(t-1, t), \tilde{x}^{i\prime}(t, t), \tilde{x}^{i\prime}(t-1, t+1), \tilde{x}^{i\prime}(t, t+1), \tilde{x}^{i\prime}(t+1, t+1)\}$ of consumption such that b), g), h) and i) are satisfied, which contradicts the definition of three period efficiency.

Similarly, two period efficiency implies one period efficiency, if the demands for assets $\{\underline{0}, 0, s^i(t-1, t), b^i(t-1, t), s^i(t, t), b^i(t, t)\}$ at period t coincide with those of the same period in two period efficient plan. (g.e.d.)

Since Samuelson (1958) pointed out a paradoxical result that there may exist a shortrun non-efficiency in an efficient infinite horizon program, a series of papers have appeared, most of which are based on pure exchange model. *Samuelson himself conjectured that the nonefficiency is* caused by negative rate of interest and cured by the introduction of money as a social contrivance. And Starrett (1972) generalized the conclusion, but this is not the case here because the interest rate is by definition positive in the sense of "biological theory of interest." According to the Gale (1973) criterion who calls a model *classic* (Fisher and Bohm-Bawerk) if aggregate assets are negative or *Samuelson* if positive, our model is classified in the Samuelson category. Also note that our model ignores the transition period and starts from "ecnomic time" of "biological time" or "days of Adam and Eve." Hence, the Impossibility Theorem of Samuelson (1958, p. 476) does not hold [see also Starrett (1972) and Gale (1973)], and the only obstacle to achieve efficiency is the wrong forecast. See also Takahashi (1980)

Theorem 8. 4

Suppose that all the price expectations by consumers are correct in the sense that all markets are cleared. Then, a two-(three- or an infinitely many) period feasible program each of which is one period efficient (hence, a temporary equilibrium) is two-(three- or an infinite horizon, respectively) efficient.

A temporary equilibrium is not necessarily two period-, three period- and hence an infinite horizon efficient because these concepts of efficiency depend on the forecast of future prices which is revised every period. In this sense, the discussion on the efficiency in a sequential market model is quite different from that in a traditional (static) general equilibrium theory.

Now, let me turn to the discussion of efficiency in an infinite sequence of temporary equilibria, which means that the model has an infinite dimensional commodity space. Next two theorems are analogous to the classical propositions of welfare economics.

Theorem 8. 5

A consumption program in an infinite horizon equilibrium is efficient in an infinite time horizon.

(Proof) The proof is completed in a few steps.

Step 1: Let $(\boldsymbol{x}^i, \boldsymbol{y}^j, \boldsymbol{p})$ be a competitive equilibrium over an infinite time horizon. Observe for any i

$$(8.3) \quad -\infty < \boldsymbol{p}\boldsymbol{x}^i_L \leq 0.$$

Now, claim that for $\boldsymbol{y}^j \in \boldsymbol{Y}^j$,

$$(8.4) \quad |\boldsymbol{p}\boldsymbol{y}^j| < \infty \text{ for any } j.$$

Suppose not, then there exists some j, say jo, such that

$$(8.5) \quad \boldsymbol{p}\boldsymbol{y}^{jo} = -\infty.$$

Since $\{\boldsymbol{y}^j\}$ is the return on share maximizer, then from (8. 3) there exists some i, denoted as io, such that

$$(8.6) \quad \boldsymbol{p}\boldsymbol{x}^{io} = -\infty.$$

By the assumption (b. i) and (c) (or Lemma 7. 1), there exists $\boldsymbol{x}^{io'} \in \boldsymbol{X}^{io}$ so that $\boldsymbol{u}^{io}(\boldsymbol{x}^{io'}) > \boldsymbol{u}^{io}(\boldsymbol{x}^{io})$. Hence, there is $t_0 > 0$ such that for any $t > t_0$

$$(8.7) \quad \boldsymbol{u}^{io}(\boldsymbol{x}^{io'}(1), \ldots, \boldsymbol{x}^{io'}(t), \underline{0}, \ldots) > \boldsymbol{u}^{io}(\boldsymbol{x}^{io}).$$

Write, for any $\lambda \in [0, 1]$,

$$\boldsymbol{x}(\lambda)=\lambda(\boldsymbol{x}^{io\prime}(1),\ldots, \boldsymbol{x}^{io\prime}(t), \underline{0},\ldots)+(1-\lambda)\boldsymbol{x}^{io}$$

then $\boldsymbol{x}(\lambda)\in \boldsymbol{X}^{io}$ by the assumption (a. 1). Moreover, by (8. 6) and (8. 7),

(8.8) $\boldsymbol{u}^{io}(\boldsymbol{x}(\lambda))>\boldsymbol{u}^{io}(\boldsymbol{x}^{io})$

(8.9) $\boldsymbol{px}(\lambda)=-\infty$,

which contradicts the fact that $\boldsymbol{x}^{io}$ is $\boldsymbol{u}^{io}$-maximizer in the budget set.

Step 2: Claim that $\boldsymbol{x}\in \boldsymbol{X}^{i}$ and $\boldsymbol{u}^{i}(\boldsymbol{x})\geq \boldsymbol{u}^{i}(\boldsymbol{x}^{i})$ imply

(8.10) $\boldsymbol{px}\geq \boldsymbol{px}^{i}$,

First, suppose that $\boldsymbol{x}\in \boldsymbol{X}^{i}$ and $\boldsymbol{u}^{i}(\boldsymbol{x})>\boldsymbol{u}^{i}(\boldsymbol{x}^{i})$. Then there exists $t_I>t_0$ such that for any $t>t_I$

(8.11) $\boldsymbol{u}^{i}(\boldsymbol{x}(1),\ldots, \boldsymbol{x}(t), \underline{0},\ldots)>\boldsymbol{u}^{i}(\boldsymbol{x}^{i})$.

It follows for $t>t_I$,

(8.12) $\boldsymbol{p}(\boldsymbol{x}(1),\ldots, \boldsymbol{x}(t), \underline{0},\ldots)>\boldsymbol{px}^{i}$.

Therefore,

(8.13) $\boldsymbol{px}\geq \boldsymbol{px}^{i}$ and $\boldsymbol{px}\neq \boldsymbol{px}^{i}$.

Next, suppose that $\boldsymbol{x}\in \boldsymbol{X}^{i}$ and $\boldsymbol{u}^{i}(\boldsymbol{x})\geq \boldsymbol{u}^{i}(\boldsymbol{x}^{i})$. Assume that the limit point d_x of $\{\sum_{\tau=1}^{t} \boldsymbol{p}(\tau)\boldsymbol{x}(\tau)\}_{t=1,2,\ldots,\infty}$ satisfies

$$d_x<\boldsymbol{px}^{i}.$$

Write

(8.14) $\boldsymbol{px}(\lambda)=\boldsymbol{p}\{\lambda(\boldsymbol{x}(1),\ldots, \boldsymbol{x}(t), \underline{0},\ldots)+(1-\lambda)d_x\}$

then $u^i(x(\lambda)) > u^i(x^i)$ and $[\lambda p(x(1),\dots, x(t), \underline{0},\dots) + (1-\lambda)d_x]$ is a limit point of a sequence $\{\sum_{\tau=1}^{t} p(\tau)x(\lambda;\tau)\}_{t=1,2,\dots,\infty}$. For sufficiently small $\lambda \in (0, 1)$,

(8.15) $\lambda p(x(1),\dots, x(t), \underline{0},\dots) + (1-\lambda)d_x < px^i$,

which contradicts (8. 13) and (8. 14). Therfore, (8. 10) must be true.

Step 3: Suppose that (x^{io}, y^{io}) satisfies $x^{io} \in X^i$ for any i, $y^{jo} \in Y^j$ for any j and

(8.16) $\sum_i x^{io} \leq \sum_j y^{jo}$

(8.17) $u^i(x^{io}) \gtrsim u^i(x^i)$

and at least one strict inequality holds. Denote him as consumer h. By (8.10) and (8.13), one can find limit points d_i of $\{\sum_{\tau=1}^{t} p(\tau)x^{io}(\tau)\}_{t=1,2,\dots,\infty}$ for any i such that

(8.18) $d_i \geq px^i$ for any $i \neq h$

(8.19) $d_h > px^h$.

Then

$$(8.20)\quad p(\sum_j y^{io}) \leq p(\sum_j y^j) = p(\sum_i x^i) = \sum_i (px^i)$$
$$< \sum_i d_i = \lim_{t\to\infty} \sum_{\tau=1}^{t} p(\tau) \sum_i x^{io}(\tau),$$

which leads to the desired contradiction. (g. e. d.)

Theorem 8. 6

Suppose that an infinite horizon consumption plan $\{x^{i*}\}$ is an efficient allocation in an infinite horizon associated with the production plan $\{y^{j*}\}$. Then, there exists a price system $p^* \neq 0$ such that

i) $p^* x^{i*} \leq p^* x^i$ for any $x^i \in X^i$ such that $u^i(x^i) \gtrsim u^i(x^{i*})$,

ii) $p^* y^{j*} \geq p^* y^j$ for any $y^j \in Y^j$ [5)]

iii) $\sum_i x^{i*} = \sum_j y^{j*}$.

(Proof) The condition (iii) of an infinite horizon competitive equilibrium follows from the feasibility of efficient allocation. Let

$$C^i = \{x^i \in X^i \mid u^i(x^i) \geq u^i(x^{i*})\}$$
$$\bar{C}^i = \{x^i \in X^i \mid u^i(x^i) > u^i(x^{i*})\}$$

and

$$\bar{C} = \bar{C}^1 + \sum_{i=2}^{I} C^i$$

By the concavity of u^i, $\bar{C}$ is convex. From the definition of infinite horizon efficiency, it follows that $z \in \bar{C}$ implies $z \notin \sum_j Y^j$. Therefore, $\bar{C}$ and $\sum_j Y^j$ are two disjoint convex sets. Hence, by the Hahn-Banach theorem, there exists $p^* \neq 0$ and a real number k such that

(8.21) $p^* z \leq k \qquad z \in \sum_j Y^j$

(8.22) $p^* z \geq k \qquad z \in \bar{C}$

Now, claim that $p^* z \geq k$. Since $\sum_i x^{i*} = \sum_j y^{j*}$ and $y^{j*} \in Y^j, x^{i*} \in X^i$. So $p^*(\sum_i x^{i*}) \leq k$. Suppose $x' \in \bar{C}$, then there exist $x^{1\prime} \in \bar{C}^1$ and $x^{i\prime} \in C^1$, $i=2, 3, \ldots, I$. Write $x' = \sum_i x^{i\prime}$ and let

$$x^i(\lambda) = \lambda x^{i*} + (1-\lambda) x^{i\prime} \quad \lambda \in (0, 1)$$
$$x(\lambda) = \sum_i x^i(\lambda).$$

Then $x^1(\lambda) \in \bar{C}^1$ and $x^i(\lambda) \in C^i$, hence $x(\lambda) \in \bar{C}$.

Suppose $p^* \sum_i x^{i*} < k$. From (8.22), $p^* z > p^* \sum_i x^{i*}$ for any $z \in \bar{C}$. In par-

5) This is equivalent to the return on share maximization which can be expressed as for given $\{n^j(\tau)\}_\tau$ and $\{b^j(\tau)\}_\tau$,

$$\tilde{p}^*(\tau) \tilde{y}^*(\tau) \geq \tilde{p}^*(\tau) \tilde{y}^j(\tau) \quad \text{for any } \tau.$$

ticular,

$$\boldsymbol{p}^*\boldsymbol{x}(\lambda) > \boldsymbol{p}^*\sum_i \boldsymbol{x}^{i*} \text{ for } \lambda \in (0, 1).$$

A desired conradiction follows from choosing λ small enough. Hence, $\boldsymbol{p}^*\sum_i \boldsymbol{x}^{i*} \not< k$, that is $\boldsymbol{p}^*\sum_i \boldsymbol{x}^{i*} \geq k$. On the other hand, $\boldsymbol{p}\sum_i \boldsymbol{x}^i \leq k$. Therefore, the claim is obtained. Now,

$$(8.23) \quad \boldsymbol{p}^*\boldsymbol{z} \leq \boldsymbol{p}^*\sum_i \boldsymbol{x}^{i*} \quad \text{any } \boldsymbol{z} \in \sum_j \boldsymbol{Y}^j$$

$$(8.24) \quad \boldsymbol{p}^*\boldsymbol{z} \geq \boldsymbol{p}^*\sum_i \boldsymbol{x}^{i*} \quad \text{any } \boldsymbol{z} \in \bar{\mathrm{C}}.$$

From (8.23), $\boldsymbol{p}^*\sum_j \boldsymbol{y}^j \leq \boldsymbol{p}^*\sum_i \boldsymbol{x}^{i*}$. $\sum_i \boldsymbol{x}^{i*} = \sum_j \boldsymbol{y}^{j*}$ leads to

$$(8.25) \quad \boldsymbol{p}^*\sum_j \boldsymbol{y}^j \leq \boldsymbol{p}^*\sum_j \boldsymbol{y}^{j*}.$$

Thus, the condition (ii) follows,

Similarly, from (8.24),

$$(8.26) \quad \boldsymbol{p}^*\boldsymbol{x}^1 \geq \boldsymbol{p}^*\boldsymbol{x}^{1*} \text{ for any } \boldsymbol{x}^1 \in \bar{\mathrm{C}}$$

$$(8.27) \quad \boldsymbol{p}^*\boldsymbol{x}^i \geq \boldsymbol{p}^*\boldsymbol{x}^{i*} \text{ for any } \boldsymbol{x}^i \in \mathrm{C}^i, i = 2, 3, \ldots, I.$$

Now, claim that for any $\boldsymbol{x}^1 \in \mathrm{C}^1$

$$\boldsymbol{p}^*\boldsymbol{x}^1 \geq \boldsymbol{p}^*\boldsymbol{x}^{1*}.$$

This is done by showing that $\boldsymbol{u}^1(\boldsymbol{x}^1) = \boldsymbol{u}^1(\boldsymbol{x}^{1*})$ implies the equality $\boldsymbol{p}^*\boldsymbol{x}^1 = \boldsymbol{p}^*\boldsymbol{x}^{1*}$. One can find a consumption vector $\boldsymbol{x}^{1''} \in \boldsymbol{X}^1$ such that $\boldsymbol{u}^i(\boldsymbol{x}^{1''}) \geq \boldsymbol{u}^1(\boldsymbol{x}^{1*}) \geq \boldsymbol{u}^1(\boldsymbol{x}^1)$. Let

$$(8.28) \quad \boldsymbol{x}^{1''}(\lambda) = \lambda\boldsymbol{x}^{1''} + (1-\lambda)\boldsymbol{x}^1, \quad \lambda \in (0, 1).$$

By the concavity of $\boldsymbol{u}^1$, $\boldsymbol{u}^1(\boldsymbol{x}^{1''}(\lambda)) \geq \boldsymbol{u}^1(\boldsymbol{x}^{1*}) \geq \boldsymbol{u}^1(\boldsymbol{x}^1)$ for $\lambda \in (0, 1)$. Then, by (8.26),

$$(8.29) \quad \boldsymbol{p}^*\boldsymbol{x}^{1''}(\lambda) \geq \boldsymbol{p}^*\boldsymbol{x}^{1*}.$$

Letting λ small, (8.29) gives,

(8.30) $\boldsymbol{p}^*\boldsymbol{x}^1 \geq \boldsymbol{p}^*\boldsymbol{x}^{1*}$ for any $\boldsymbol{x}^1 \in C^1$.

Thus, the condition (i) is obtained.

It remains to show that

(8.31) $\boldsymbol{u}^i(\boldsymbol{x}^{i*}) \geq \boldsymbol{u}^i(\boldsymbol{x}^i)$ for any $\boldsymbol{x}^i \in \boldsymbol{X}^i$ such that $\boldsymbol{p}^*\boldsymbol{x}^i \geq \boldsymbol{p}^*\boldsymbol{x}^{i*}$.

Suppose that $\boldsymbol{p}^*\boldsymbol{x}^i = \boldsymbol{p}^*\boldsymbol{x}^{i*}$ and $\boldsymbol{u}^i(\boldsymbol{x}^i) > \boldsymbol{u}^i(\boldsymbol{x}^{i*})$. Due to (g. 2), there exists $\boldsymbol{x}^{i\prime} \in \boldsymbol{X}^i$ such that $\boldsymbol{p}^*\boldsymbol{x}^i < \boldsymbol{p}^*\boldsymbol{x}^{i*}$. Now, (8.26) implies $\boldsymbol{u}^i(\boldsymbol{x}^{i\prime}) < \boldsymbol{u}^i(\boldsymbol{x}^{i*}) < \boldsymbol{u}^i(\boldsymbol{x}^i)$. Let

(8.32) $\boldsymbol{x}^{i\prime}(\lambda) = \lambda \boldsymbol{x}^{i\prime} + (1-\lambda)\boldsymbol{x}^i \qquad \lambda \in (0, 1)$.

Then, $\boldsymbol{p}^*\boldsymbol{x}^{i\prime}(\lambda) < \boldsymbol{p}^*\boldsymbol{x}^{i*}$. Choosing λ small enough leads to the desired

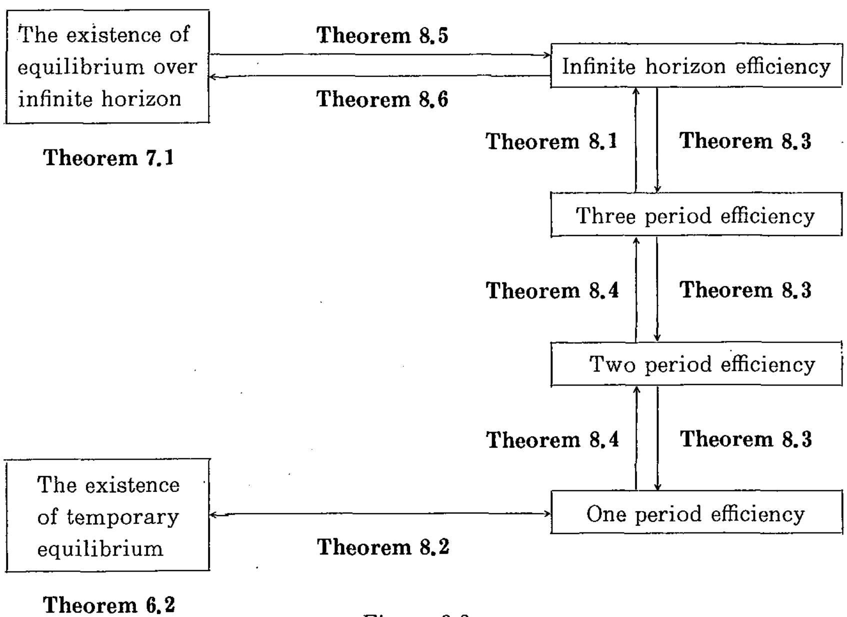

Figure 8.2

contradiction; $u^i(x^{i\prime}) \geq u^i(x^{i*})$ but $p^* x^{i\prime} < p^* x^{i*}$. Hence, the claim (8.31) is satisfied. (g. e. d.)

Figure 8.2 summarizes all the result which have been proven in sections 6,7 and 8.

9. Credit Rationing[6]

In this section, a model with credit rationing is studied. When the behavior is defined in section 5, the monetary authority is assumed to set the selling price $r^i(t)$ of bond and the lending one $r^j(t)$ of loan in order to clear the monetary market (the assumption (f)). But it might be more realistic to suppose that such a price mechanism has limitations (for example, the rate of interest on bond holding cannot be negative or below the level of liquidity trap).

Now, it is considered that the monetary market is guided by the price mechanism, but that the prices $r^i(t)$ and $r^j(t)$ can only move between their possible upper and lower bounds which are assumed to be given from the outside of the economy. Write them

$$\underline{r}^i(t) \leq r^i(t) \leq \bar{r}^i(t),$$
$$\underline{r}^j(t) \leq r^j(t) \leq \bar{r}^j(t).$$

Since the ranges of price change are restricted, some institutional arrangement should achieve the market clearing. Here, the monetary authority is assumed to announce the quantity signals, i. e., the upper limits $\bar{b}^i(t)$ on the bond purchase $b^i(\tau, t)$, $\tau = t-1$, t, and $\bar{b}^j(t)$ on the borrowing $b^j(t)$, in addition to the price signals $r^i(t)$ and $r^j(t)$. Note that the lower limits on $b^i(\tau, t)$ and $b^j(t)$ are zero by the hypothesis. Also note that the

6) This section is inspired by Dreze (1975).

monetary authority does not discriminate consumers by age; i. e.,

$$\bar{b}^i(t)=\bar{b}^i(t-1, t)=\bar{b}^i(t, t).$$

Then, each consumer decides his demand $b^i(\tau, t)$ for bond in order to maximize his expected utility (3.6) subject to the quantity constraint;

$$(9.1)\quad b^i(\tau, t)\leq\bar{b}^i(t), \quad \tau=t-1, t$$

in addition to the budget constraints (3.1)–(3.3) or (3.1*)–(3.3*). His budget correspondence (3.4) is replaced by;

$$(9.2)\quad B^{i*}: P\times \mathrm{R}^{J+3}\rightarrow X^i\subset \mathrm{R}^{L+J+2}$$

given by

$$B^{i*}(t-2, t)=B^i(t-2, t)$$
$$B^{i*}(t-1, t)=\{x^i(t-1, t)\in B^i(t-1, t)\,|\,(9.1)\text{ holds}\}$$
$$B^{i*}(t, t)=\{x^i(t, t)\in B^i(t, t)\,|\,(9.1)\text{ holds}\}$$

Lemma 9.1

The budget correspondence B^{i*} with quantity constraint defined by (9.2) is continuous at $(r^i(t), r^j(t), \bar{b}^i(t), \bar{b}^j(t))$ and non empty for $p\in$ int $\boldsymbol{P}$.

(Proof) Non emptyness follows from Lemma 3.1.

To show the continuity, define

$$\bar{B}^i(\tau, t)=\{x^i(\tau, t)\,|\,b^i(\tau, t)\leq b^i(t)\},\quad \text{where}\quad \tau=t-2, t-1, t.$$

Of course,

$$B^{i*}(\tau, t)=B^i(\tau, t)\cap\bar{B}^i(\tau, t).$$

Here, the continuity of $B^i(\tau, t)$ is proven in Lemma 3.1 and that of $\bar{B}^i(\tau,t)$ follows from the fact that it has non empty interior for all $\bar{b}^i(t)>0$. Then,

$B^{i*}(\tau, t)$ is upper semicontinuous as an intersection of continuous correspondences.

Hereafter, the time superscript is dropped without causing any confusion. To show the lower semicontinuity of B^{i*}, consider a converging sequence $\{(p^\nu, \bar{b}^{i\nu})\}_{\nu=1,2,\ldots}$ of signals in period t to $(p^0, \bar{b}^{i0})$, where $p^\nu \equiv (\tilde{p}^\nu, q^\nu, r^{i\nu}, r^{j\nu})$ and let

$$x^{i0} \equiv (\tilde{x}^{i0}, s^{i0}, b^{i0}, 0) \in B^{i*}(p^0, \bar{b}^{i0}).$$

Since B^{i*} is continuous, a sequence $\{x^{i*\nu}\}$ of consumer's demand can be chosen so as to satisfy that

$$x^{i*\nu} \equiv (\tilde{x}^{i*\nu}, s^{i*\nu}, b^{i*\nu}, 0) \in \bar{B}^i(\bar{b}^i),$$

that is, $b^{i*} \leq \bar{b}^i$, and that the sequence $\{x^{i*\nu}\}$ converges to x^{i0}.

If $x^{i0} \in \text{int } B^{i*}(p^0, \bar{b}^{i0})$, there exists a positive number ν' such that for all $\nu \geq \nu'$,

$$x^{i*\nu} \in B^i(p^\nu).$$

Indeed, if a pair of p^0 and x^{i0} satisfies (3.1)–(3.3), so does a pair of p^ν and $x^{i*\nu}$ for $\nu \geqq \nu'$.

Let a pair of p^0 and x^{i0} satisfy (3.1)–(3.3) with equality. Then, there exists $x^{i**} \in B^i(p^0)$ such that

$$x^{i**} \in \bar{B}^i(\bar{b}^{i0})$$

and that a pair of p and x^{i**} satisfies (3.1)–(3.3) for $\nu \geqq \nu'$.

Define a sequence $\{x^{i\nu}\}$ of consumer's behavior by:

a) for any ν such that $x^{i*\nu} \in B^i(p^\nu)$,

(1) a pair of p^ν and $x^{i\nu}$ satisfies (3.1)–(3.3) with equality,

(2) $x^{i\nu} = \lambda^\nu x^{i*\nu} + (1-\lambda^\nu) x^{i**\nu}$

b) otherwise

$$x^{i\nu}=x^{i*\nu}$$

Note that such $x^{i\nu}$ exists uniquely and $x^{i\nu}\in B^{i*}(p^{\nu}, \bar{b}^{i\nu})$.

Now, a pairwise convergence implies that a pair of p^0 and x^{i0} satisfies (3.1)−(3.3) with equality. But a pair of p^0 and $x^{i**\nu}$ satisfies them with strict inequality. Hence, $\{\lambda^{\nu}\}\rightarrow 1$ and then $\{x^i\}\rightarrow x^{i0}$. Therefore, B^{i*} is lower semicontinuous. (g. e. d.)

Similarly, each firm takes the quantity signal $\bar{b}^j(t)$ of credit rationing into account in addition to the price signal:

(9.3) $b^j(t)\leq\bar{b}^j(t)$

Then, firm's budget correspondence (4.5) is replaced by

(9.4) $B^{j*}(t)\colon P\times R\rightarrow R^{2L+2}$

defined as

$$B^{j*}(p(t), \bar{b}^j(t))=\{\rho^j(t)\equiv(\tilde{x}^j(t), \Delta K^j(t), n^j(t), b^j(t))\in B^j(p(t)) \mid (11.3) \text{ holds}\}.$$

Lemma 9.2

The budget correspondence (9.4) with credit rationing is continuous at $(r^i(t), r^j(t), \bar{b}^i(t), \bar{b}^j(t))$ under the assumption (d. 1).

(Proof) Similarly to Lemma 9.1. (g. e. d.)

As far as the behaviors of consumers and firms concerned there is no significant difference from previous sections, though there are additional constraints. On the other hand, there exists a noticable difference in the behavior of the monetary authority. It now performs a brand new function. It sends the quantity signals and rations the bond purchase or the borrowing if necessary. The followings are assumed on the the rationing scheme and replace (f) of section 5.

(h. 1) $b^i(\tau, t)=\bar{b}^i(t)$ for some i implies $b^j(t)<\bar{b}^j(t)$ for all j.

$b^j(t)=\bar{b}^j(t)$ for some j implies $b^i(\tau, t)<\bar{b}^i(t)$ for all i.

(h. 2) $\underline{r}^i(t)<r^i(t)$ implies $b^i(\tau, t)<\bar{b}^i(t)$ for all i.

$r^j(t)<\bar{r}^j(t)$ implies $b^j(t)<\bar{b}^j(t)$ for all j.

(h. 3) $\sum_{i\in I(t-1)} b^i(t-1, t)+\sum_{i\in I(t)} b^i(t, t)-\sum_{j\in J} b^i(t)=0.$

(h. 1), sometimes called the short side principle, means that the rationing may affect on either consumer(s) (i. e., the bond purchase) or firm(s) (i. e., the borrowing), but not simultaneously both sides. (h. 2) states that no quantity rationing is allowed unless price rigidities are binding. (h. 3) says that the rationing clears the market.

Using the bounds on the selling price of bond and borrowing price, define a compact set of auxiliary variables γ^i and γ^j,

$$\Gamma(t)=\{(\gamma^i, \gamma^j)\in R^2 | \underline{r}^i(t)-\varepsilon\leq\gamma^i\leq\bar{r}^i(t)+\varepsilon, \underline{r}^j(t)-\varepsilon\leq\gamma^j\leq\bar{r}^j(t)+\varepsilon\}.$$

Then, the rationing scheme is defined by

$$(9.5)\quad \begin{aligned} r^i(t)&=\min\{\bar{r}^i(t), \max\{\gamma^i, \underline{r}^i(t)\}\}\\ &=\max\{\underline{r}^i(t), \min\{\gamma^i, \bar{r}^i(t)\}\}\\ r^j(t)&=\min\{\bar{r}(t), \max\{\gamma^j, \underline{r}^j(t)\}\}\\ &=\max\{\underline{r}^j(t), \min\{\gamma^j, \bar{r}^j(t)\}\}\\ \bar{b}^i(t)&=\bar{r}^i(t)-\max\{\gamma^i, \bar{r}^i(t)\}\\ \bar{b}^j(t)&=\underline{r}^j(t)-\min\{\gamma^j, \underline{r}^j(t)\} \end{aligned}$$

Notice that there exists a one-to-one and continuous relation between (γ^i, γ^j) and the signals $(r^i(t), r^j(t), \bar{b}^i(t), \bar{b}^j(t))$ that the monetary authority announces.

Define a temporary equilibrium with credit rationing as an array of consumption vectors $x^i(\tau, t)$, $\tau = t-2, t-1, t$, production vectors $y^j(t)$, financial vectors $(n^j(t), b^j(t))$ and price vectors $p(t)$, given the quantity sigal $(\bar{b}^i(t), \bar{b}^j(t))$ such that

(i) $x^i(\tau, t)$ is a utility maximizer in $B^{i*}(\tau, t)$, $\tau = t-2, t-1, t$.

(ii) $y^j(t)$, $n^j(t)$ and $b^j(t)$ maximize expected return on share subject to $B^{j*}(t)$.

(iii) The markets of commodities, shares and money are cleared.

(iv) $\underline{r}^i(t) \leq r^i(t) \leq \bar{r}^i(t)$,

$\underline{r}^j(t) \leq r^j(t) \leq \bar{r}^j(t)$。

Theorem 9.3

Under assumptions (a. 1), (a. 2), (b. 1)–(b. 3), (c), (d. 1), (d. 2), (e. 1), (e. 2) and (h. 1)–(h. 3), there exists a temporary equilibrium with credit rationing.

(Proof) Similarly to Theorem 6.2 in the light of Lemmata 9.1 and 9.2. (g. e. d.)

10. The Monetary Policy in a Stationary State Path

A simple comparative statics is studied in this section. The question here is: what will happen on consumers' demands, firms' behaviours and prices if the monetary authority changes the pair $(r^i(t), r^j(t))$ of selling price of bond and lending price of loan still keeping the bond market to be cleared. So far, no assumption has been made on the size of generation. However, it is necessary to make some assumption about stationarity in order to study comparative statics. Though it is possible to assume, for example, that the size of generation $I(\tau)$ changes randamly from one period to another, it is less intersting to do so since the supply of labor force is elastically determined in a competitive labor market and hence the population size has only the secondary importance on the description of the model.

Hereafter, it is assumed that:

(i. 1) the size of generation grows exponentially at a constant rate, say γ, that is

$$\begin{aligned} I(t) &= \gamma I(t-1) \\ &= \gamma^2 I(t-2) \\ &\cdots\cdots\cdots\cdots \\ &= \gamma^t I(0), \quad \text{where} \quad I(0) \text{ is given.} \end{aligned}$$

(i. 2) each generation as a whole has an identical distribution of characteristics such as expectation, preference and endowment over the members of the generation.

(i. 3) if the rate of inflation observed in the past is constant, say the inflationary factor is θ, a consumer expects prices along the same rate of inflation.

Assuption (i. 1) implies that the composition ratio of generations at a given moment of time is constant and given by $1:\gamma:\gamma^2$ from the old, middel aged to the young. (i. 2) is made instead of supposing that all the members of each generation have the identical characteristics. Due to the stationary assumptions (i. 2) and (i. 3), the stationary state path can be discussed in terms of the "mean" variables. Namely, the assumption admits the existence of consumers with different characteristics and hence with different demand correspondences, but requires that consumer of a paticular characteristics can be found in every generation with the same probability. Hence, each generation repeats the same pattern of economic activities but the size. As concerning about an individual consumer, it does not matter to which generation he belongs but it does in which age of his life he is (i. e., whether he is young, middel aged or old) in a paticular period.

Let $\tilde{x}^i(\varsigma)$, $\varsigma=1, 2, 3$, be the mean consumption labor supply vector of ς-*th* age of life, where $\varsigma=1$ stands for the young, $\varsigma=2$ for the middle aged and $\varsigma=3$ for the old. Similarly, define $s^i(\varsigma)$, $b^i(\varsigma)$. Write

$$x^i(1)\equiv(\tilde{x}^i(1),\ s^i(1),\ b^i(1),\ 0)=x^i(t-2,\ t-2)=x^i(t-1,\ t-1)=x^i(t,\ t)$$
$$x^i(2)\equiv(\tilde{x}^i(2),\ s^i(2),\ b^i(2),\ 0)=x^i(t-2,\ t-1)=x^i(t-1,\ t)=x^i(t,\ t+1)$$
$$x^i(3)\equiv(\tilde{x}^i(3),\ s^i(3),\ b^i(3),\ 0)=x^i(t-2,\ t)=x^i(t-1,\ t+1)=x^i(t,\ t+2)$$
$$y^j(t)\equiv(\{\tilde{y}^j(t)-\tilde{x}^j(t)-\varDelta K^j(t)\},\ 0,\dots,\ 0,\ n^i(t),\ 0,\dots,\ 0,\ b^j(t))$$

The *feasible program* along the stationary state path is given by an array $(x^i(1), x^i(2), x^i(3), (\tilde{x}^j(t-1), \tilde{y}^j(t), \varDelta K^j(t), n^j(t), b^j(t))_j, p(t))$ such that

(10.1) $p(t)x^i(1)\leq 0$ for any i

where $x^i_L(1)\geq\tilde{\ell}$ and $s^i(1)\geq 0$.

(10.2) for $s^i(1)$ and $b^i(1)$,

$p(t)x^i(2)-d(t)s^i(1)-b^i(1)\leq 0$ for any i

where $x^i_L(2)\geq\tilde{\ell}$ and $s^i(1)+s^i(2)\geq 0$.

(10.3) for given $s^i(1)+s^i(2)$ and $b^i(t)$

$p(t)x^i(t)-\{d(t)+q(t)\}\{s^i(1)+s^i(2)\}-b^i(2)\leq 0$

where $x^i_L(3)\geq\tilde{\ell}$

(10.4) $\tilde{p}(t)\{\tilde{x}^j(t)+\varDelta K^j(t)\}-r^j(t)b^j(t)-q^j(t)n^j(t)\leq 0$.

A temporary equilibrium allocation in a stationary path $((x^i(t, t))_i$, $(x^i(t-1, t))_i$, $(x^i(t-2, t))_i$, $(y^j(t))_j$, $p(t))$ is called *a stationary state* *equilibrium* if

$$
\begin{aligned}
(10.5)\quad & x^i(t,t)=x^i(1)=(\tilde{x}^{i*}(1),\ s^{i*}(1),\ \theta^t b^{i*}(1),\ 0)\\
& x^i(t-1,t)=x^i(2)=(\tilde{x}^{i*}(2),\ s^{i*}(2),\ \theta^t b^{i*}(2),\ 0)\\
& x^i(t-2,t)=x^i(3)=(\tilde{x}^{i*}(3),\ s^{i*}(3),\ \theta^t b^{i*}(3),\ 0)\\
& y^j(t)=((\tilde{y}^{j*}-\tilde{x}^{j*}-\Delta K^{j*}),\ 0,\dots,\ 0,\ n^{j*},\ 0,\dots,\ 0,\ \theta^t b^{j*})\\
& p(t)=(\tilde{p}(t),\ q(t),\ r^i(t),\ r^j(t))\\
& \quad=(\theta^t\tilde{p}^*,\ \theta^t q^*,\ r^{i*},\ r^{j*})
\end{aligned}
$$

All real variables ($\tilde{x}^{i*}(\varsigma)$, $\tilde{s}^{i*}(\varsigma)$, $\varsigma=1, 2, 3$, and $\tilde{y}^{j*}$, $\tilde{x}^{j*}$, ΔK^{j*}, n^{j*}) and relative prices of commodities and shares are constant. As far as bond and loan, the selling price $r^i(t)$ and the lending price $r^j(t)$ are constant meanwhile the volume increases.

Theorem 10.1

There exists a sequence of stationary state equilibrium if the assumptions (i. 1)–(i. 3) are met.

(Proof) Define the set of (infinite) stationary path

$S=\{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p})|$ each component (x^i, y^j, p) satisfies the definition of stationary state (10.5)$\}$

This set is a subset of the non-empty set of all infinite time horizon equilibria. Clearly, there exists a trivial stationary state equilibrium such that the factor of inflation $\theta=1$ under the assumption (i. 3).

It remains to show that $S\neq\phi$ for $\theta\neq1$. This is done by recalling the fact that the consumption vector is homogeneious of degree zero with respect to price vectors of commodities and shares. (g. e. d.)

Though the homogenity of the demand (hence, excess demand) correspondence, the quantity theory of "money" does not hold. This is because the change in loan supply will cause the output change and secondly, the model has the time structure and contains uncertainty.

Consider a price adjustment process given by

(10.6) $p_k(t+1)=\max\ \{0,\ p_k(t)+\beta_k z_k(p(t))\}$

where $z_k(p(t))$ is the excess demand correspondence for $k=1,\ldots,$ $L+J$. $\beta_k>0$ is sufficiently small.

Theorem 10.2

A price adjustment process toward the stationary state path defined as (10.6) is globally stable, if the (strong) gross substitute prevails in the cosumers' demand correspondence.

(Proof) Since both the weak axiom of revealed prefernce and the uniqueness of equilibrium price vector are obtained under the gross substitute assumption, all the requirements of a stability theorem of Uzawa (1959–60, p. 185) are met. (g. e. d.)

Among many stationary state equilibria, a "golden rule path" may drow much attention. Now, consider a condition under which such a solution exists. At a golden rule equilibrium along the stationary state path,

$$r=\frac{1}{r^i}=\frac{1}{r^j}=\frac{q^j+d^j}{q^j}$$

Substitution gives a quadratic equation of $r=r^i=r^j$;

(10.7) $q(s(1)+s(2))r^2-(b(1)+qs(2))r-qs(1)=0.$

As the coefficient of r^2 is positive and that of r and the constant term are negative, there exists one positive root of r if the equation (10.7) has two real roots. A sufficient condition for (10.7) to have two real roots is

(10.8) $b^i(1)-2qs^i(1)\geq 0$

(10.9) $s^i(2)-2s^i(1)\geq 0$

since

(10.10) the discriminant $= b^i(1)'b^i(1)+q[s^i(1)+2s^i(1)][s^i(2)-2s^i(1)]q$
$+2qs^i(2)[b^i(1)-2qs^i(1)]$.

The meaning of conditions (10.8) and (10.9) is straightforward. (10.8) says that a young consumer buys more bonds than stocks. (10.9) says that in the middle age, people buys more shares than when young. One possible interpretation of these conditions is that during the young age when he earns only labor income, he can save a little hence he buys more bond relatively to shares since the bond is a safe asset in the sense that the redemption price is known at the time of purchase. On the other hand, in the middle aged period when he has four sources of income, he can afford to invest on shares which are considered risky assets.

Now, let me turn to a simple exercise of "monetary" policy. First, consider the tightening policy. This is done through the choice of a higher value for r^i(t) and lower one for r^j(t). For firms, the lower r^j(t) means the higher interest rate on loan, hence the demand for loan declines, which has two implications: one is the decrease in input to purchase $\tilde{x}^j$(t) and hence the decrease of output $\tilde{y}^j$(t+1) in the following period, and another is the impact on the (expected) profit of next period, which is negative as the production enjoys the non-decreasing return to scale. For consumers, the rise in r^i(t) means the the fall on the rate of return on bond, hence the decrease on the demand b^i(t) for bond as long as the price effect dominates the income effect. Then,

$$(10.11)\quad r^i(t)b^i(t)\begin{cases}\uparrow\\ \rightarrow\\ \downarrow\end{cases}\qquad \text{if}\quad \varepsilon_i\begin{Bmatrix}<\\ =\\ >\end{Bmatrix}1$$

where $\varepsilon_i=-\dfrac{r^i(t)}{b^i(t)}\dfrac{\partial b^i(t)}{\partial r^i(t)}$ is the elasticity of bond demand with respect to the selling price of bond.

Therefore, by the budget constraints (3.1)–(3.2) or (3.1*)–(3.2*),

$$(10.12)\quad x^i(\alpha, t) \text{ and } s^i(\alpha, t)\begin{cases}\uparrow\\ \rightarrow\\ \downarrow\end{cases}\qquad \text{if}\quad \varepsilon_i\begin{Bmatrix}<\\ =\\ >\end{Bmatrix}1$$

where α=t−1, t.

The almost same argument applies for the loosening policy, i. e., to make $r^i(t)$ lower and $r^j(t)$ higher. The only difference is the existence of possibility of public offering $n^j(t)$ of new shares, which makes the price of commodities further higher.

The results are summarized in Table 10.1.

Table 10.1

	The Tightening Policy			The Loosening Policy		
		ε_i			ε_i	
	<1	=1	>1	<1	=1	>1
Consumer						
$\tilde{x}^i$	↓	→	↑	↑	→	↓
s^i	↓	→	↑	↑	→	↓
Firm						
$\tilde{x}^j$		↓			↑	
$\varDelta K^j$		0			↑	
n^j		0			↑	
The Economy						
$\tilde{x}^i+\tilde{x}^j+\varDelta K^j$	↓	↓	?	↑	↑	?
p	↓	↓	?	↑	↑	?
q	↓	→	↑	?	↓	↓

References to Part II

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