A Time Sequential Variable-Threat Nash's Bargaining Problem*

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Abstract

Throughout a finite number of periods two players observe an independent nonnegative random variable one by one. Each time they observe the random variable, if only one player announces the observed value to be acceptable, he receives it actually. If both players announce so, each of them receives the observed value with probability 1/2. How do they behave to maximize their own total expected payoff if they can announce 'accept' limited number of times and can talk about their decision completely at each observation? We formulate this problem into a time sequential variable-threat Nash's bargaining one. An important property of an optimal strategy is investigated.

1. Introduction

A finite sequence of independent nonnegative random variables appear one by one before two players. Each time they observe a random variable, they can announce the observed value to be acceptable. If only one of them announces, he receives it actually. If both of them announce, each of them

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receives it with probability 1/2. How do they behave to maximize their own total expected payoff if they can announce 'accept' limited number of times and can talk about their decision completely at each observation?

Many authors considered game versions of one-sided sequential decision problems such as a secretary problem, a stochastic sequential assignment problem, and a sequential allocation problem. For example, zero-sum game versions were considered in [6], [11], [14], and [16]. [6] is concerned with a generalization of a stochastic sequential assignment problem. [11] and [16] are ones of a sequential allocation problem. Non-zero-sum game versions were considered in [2], [4], [5], [9], [10], [12], and [15]. [2], [9], and [15] derived Nash equilibrium strategies in the context of secretary problem. [5] dealt with the situation where two players had signed a treaty. [10] and [12] introduced the concept of equilibrium neutral functions. [3] and [13] discussed multilateral sequential games. Thus, so far, non-cooperative time sequential games were extensively investigated. Although non-cooperative features are important aspects of social affairs, cooperative features (especially the possibility of correlated strategies) are also important. In [1] Nash's bargaining solution was used in the selection of a trajectory for a spacecraft. Cooperative features play a more important role than non-cooperative ones in, for example, Japan-United States trade issues and Japan-United States Security Cooperation. Of course the actual problem like this is beyond our analysis because of its complexity. In this paper we treat a rather simple situation, formulate it into a time sequential variable-threat Nash's bargaining problem, and investigate a property of an optimal strategy.

An outline of the paper is as follows: In Section 2 we formulate the problem. In Section 3 a property of an optimal strategy is considered. In Section 4 some numerical examples are discussed.

2. Model and Formulation

A finite sequence $\{X_n;\ n=N,\ N-1,\ldots,1\}$ of independent nonnegative random variables which express a monetary value appear one by one before two players. Each time they observe a random variable, they can announce the observed value to be acceptable. If only one player announces, he receives it actually. If both players announce, each of them receives it with probability 1/2. Player $i(i=1,2)$ can announce 'accept' at most M_i times. How do they behave to maximize their own total expected payoff if they can talk about their decision completely at each observation? We formulate this problem into a time sequential two-person variable-threat Nash's bargaining one. Nash's bargaining problem is a sort of two-person non-zero-sum game [7, 8].

We define what we mean by optimal procedures in the following backward way. Let $(n,\ m_1,\ m_2;\ x)$ denote the state where there are n periods remaining, $X_n=x$ is observed, and player $i(i=1,\ 2)$ can announce 'accept' at most m_i times. If two players face the state $(1,\ m_1,\ m_2;\ x)$, then they play a two-person variable-threat Nash's bargaining game whose payoff matrices are given as follows:

$$\begin{matrix} & \text{reject} & \text{accept} \\ \text{reject} & (0,\ 0) & (0,\ x) \\ \text{accept} & (x,\ 0) & (x/2,\ x/2) \end{matrix} \quad \text{if } m_1>0 \text{ and } m_2>0,$$

$$\begin{matrix} \text{reject} & \text{accept} \\ ((0,\ 0) & (0,\ x)) \end{matrix} \quad \text{if } m_1=0 \text{ and } m_2>0,$$

$$\begin{matrix} \text{reject} & (0,\ 0) \\ \text{accept} & (x,\ 0) \end{matrix} \quad \text{if } m_1>0 \text{ and } m_2=0.$$

Let $(V_n^1(m_1,\ m_2;\ x),\ V_n^2(m_1,\ m_2;\ x))$ be the optimal Nash arbitrated payoff of the state $(n,\ m_1,\ m_2;\ x)$ and $\bar{V}_n^i(m_1,\ m_2)=E[V_n^i(m_1,\ m_2;\ X_n)](i$

=1, 2). If two players face the state $(n+1, m_1, m_2; x)$, then they play a two-person variable-threat Nash's bargaining game whose payoff matrices are given as follows:

$$
\begin{array}{ccc}
 & \text{reject} & \text{accept} \\
\text{r.} & (\bar{V}_n^1(m_1, m_2), \bar{V}_n^2(m_1, m_2)) & (\bar{V}_n^1(m_1, m_2-1), x+\bar{V}_n^2(m_1, m_2-1)) \\
\text{a.} & (x+\bar{V}_n^1(m_1-1, m_2), \bar{V}_n^2(m_1-1, m_2)) & (x/2+\bar{V}_n^1(m_1-1, m_2-1), x/2+\bar{V}_n^2(m_1-1, m_2-1))
\end{array}
\quad \text{if } m_1>0 \text{ and } m_2>0,
$$

$$
\begin{array}{cc}
\text{reject} & \text{accept} \\
((0, \bar{V}_n^2(0, m_2)) & (0, x+\bar{V}_n^2(0, m_2-1)))
\end{array}
\quad \text{if } m_1=0 \text{ and } m_2>0,
$$

$$
\begin{array}{cc}
\text{reject} & (\bar{V}_n^1(m_1, 0), 0) \\
\text{accept} & (x+\bar{V}_n^1(m_1-1, 0), 0)
\end{array}
\quad \text{if } m_1>0 \text{ and } m_2=0.
$$

If we solve these bargaining problems recursively, we can derive the optimal strategies.

Suppose that during a specified period free passes for movies are sent to a well-matched couple one by one. The couple are assumed to have common interest in movies. So the payoff of each movie may be regarded as the same to the couple, and it may be expressed as a nonnegative random variable. Each time a free pass is sent, the couple must decide to accept it or not at once. Each of the couple is restricted to announce 'accept' in number in terms of, say, the total number of his own holidays in that period. If only one of them decides to accept the offered movie, he does go to it and acquires the payoff of it. If both of them decide to accept it, each of them goes to it actually with probability 1/2. The value 1/2 reflects their fairness to each other for forthcoming opportunities. Each of the couple is assumed to be interested in the sum of the payoffs that are assigned to the movies he actually sees, not any function of them.

The couple are so fair but each of them is interested in only his own total payoff. So in order to raise his own gain each of the couple uses his own advantage (the remaining number of announcing 'accept') at each decision point to the extent that his opponent does not feel that he is unfair. Therefore it is resonable in this case to assume that an optimal procedure is determined in the backward way described in the above paragraphs and that they come to an agreement of their commitment to Nash's solution at each decision point. Thus the model discussed in the above paragraphs may apply to this situation.

3. Property of an Optimal Strategy

Note that optimal strategies are derived if we solve the games recursively mentioned in the last section. But in generel we cannot solve them explicitly. In this section an important property of an optimal strategy is examined.

First, it is obvious that $\bar{V}_n^1(m_1, m_2)=\bar{V}_n^2(m_2, m_1)$ and only the cases of $m_1 \geqq m_2$ need to be considered from the symmetry of the problem. Define v_n as follows:

$$v_1(m)=\begin{cases} EX_1 & \text{for } m=1, 2, \ldots, \\ 0 & \text{for } m=0, \end{cases}$$

$$v_n(m)=\begin{cases} E[\max\{X_n+v_{n-1}(m-1),\ v_{n-1}(m)\}] \\ \qquad = T_{F_n}(v_{n-1}(m)-v_{n-1}(m-1))+v_{n-1}(m) & \text{for } m=1, 2, \ldots, \\ 0 & \text{for } m=0, \end{cases}$$

for $n=2, 3, \ldots$, where $T_F(z)=\int_z^\infty (x-z)dF(x)$ and F_n is c. d. f. of X_n. $v_n(m)$ is the maximum payoff of a player when there are n periods remaining, he can announce 'accept' at most m times, and his opponent cannot announce 'accept' at all. One may conjecture that the sum of $\bar{V}_n^1(m_1, m_2)$ and

$\bar{V}_n^2(m_1, m_2)$ is equal to $v_n(m_1+m_2)$ because of the symmetry of the game. The following Property is concerned with the conjecture, and its proof is given in Appendix 1. In the following Property

$$P_1=(\bar{V}_{n-1}^1(m_1, m_2), \bar{V}_{n-1}^2(m_1, m_2)), P_2(x)=(x+\bar{V}_{n-1}^1(m_1-1, m_2), \bar{V}_{n-1}^2(m_1-1, m_2)),$$

$$P_3(x)=(\bar{V}_{n-1}^1(m_1, m_2-1), x+\bar{V}_{n-1}^2(m_1, m_2-1)), \text{ and}$$

$$P_4(x)=(x/2+\bar{V}_{n-1}^1(m_1-1, m_2-1), x/2+\bar{V}_{n-1}^2(m_1-1, m_2-1)).$$

Property. If all the five relations as

$$\begin{cases}(1)\ \bar{V}_{n-1}^1(m_1, m_2)-\bar{V}_{n-1}^1(m_1-1, m_2)\leqq\bar{V}_{n-1}^2(m_1, m_2)-\bar{V}_{n-1}^2(m_1, m_2-1)\\(2)\ \bar{V}_{n-1}^1(m_1, m_2-1)-\bar{V}_{n-1}^1(m_1-1, m_2-1)\leqq\bar{V}_{n-1}^2(m_1-1, m_2)-\bar{V}_{n-1}^2(m_1-1, m_2-1) \text{ for } m_1\geqq m_2,\end{cases}$$

$$(3)\ \bar{V}_{n-1}^1(m_1, m_2)+\bar{V}_{n-1}^2(m_1, m_2)=v_{n-1}(m_1+m_2) \quad \text{for all } m_1, m_2,$$

$$\begin{cases}(4)\ \bar{V}_{n-1}^1(m_1, m_2)\leqq\bar{V}_{n-1}^1(m_1, m_2-1)\\(5)\ \bar{V}_{n-1}^2(m_1, m_2)\leqq\bar{V}_{n-1}^2(m_1-1, m_2)\end{cases} \quad \text{for all } m_1, m_2,$$

are satisfied, then the following results hold in the state

$(n, m_1, m_2; x)(m_1\geqq m_2>0)$:

(i) If $0\leqq x\leqq v_{n-1}(m_1+m_2)-v_{n-1}(m_1+m_2-1)$, then P_1 is an optimal threat point and the optimal Nash arbitrated one, that is,

$$\begin{cases}V_n^1(m_1, m_2; x)=\bar{V}_{n-1}^1(m_1, m_2),\\V_n^2(m_1, m_2; x)=\bar{V}_{n-1}^2(m_1, m_2),\end{cases}$$

and (reject, reject) is optimal.

(ii) If $v_{n-1}(m_1+m_2)-v_{n-1}(m_1+m_2-1)\leqq x\leqq\bar{V}_{n-1}^1(m_1, m_2)-\bar{V}_{n-1}^1(m_1-1, m_2)+\bar{V}_{n-1}^2(m_1-1, m_2)-\bar{V}_{n-1}^2(m_1, m_2)$, then P_1 is an optimal threat point, the optimal Nash arbitrated point is on the segment $P_2(x)P_3(x)$ and is given by

$$\begin{cases} V_n^1(m_1,\ m_2;\ x)=x/2+\bar{V}_{n-1}^1(m_1,\ m_2) \\ \qquad -(v_{n-1}(m_1+m_2)-v_{n-1}(m_1+m_2-1))/2, \\ V_n^2(m_1,\ m_2;\ x)=x/2+\bar{V}_{n-1}^2(m_1,\ m_2) \\ \qquad -(v_{n-1}(m_1+m_2)-v_{n-1}(m_1+m_2-1))/2. \end{cases}$$

Furthermore the strategy as (accept, reject) with probability α an (reject, accept) with probability $1-\alpha$ is optimal, where

$$\alpha=(x/2+\bar{V}_{n-1}^1(m_1,\ m_2)-\bar{V}_{n-1}^1(m_1,\ m_2-1)-(v_{n-1}(m_1+m_2)-v_{n-1}(m_1+m_2-1))/2)/(x+\bar{V}_{n-1}^1(m_1-1,\ m_2)-\bar{V}_{n-1}^1(m_1,\ m_2-1)).$$

(iii) If $\bar{V}_{n-1}^1(m_1,\ m_2)-\bar{V}_{n-1}^1(m_1-1,\ m_2)+\bar{V}_{n-1}^2(m_1-1,\ m_2)-\bar{V}_{n-1}^2(m_1,\ m_2)\leqq x\leqq\bar{V}_{n-1}^1(m_1-1,\ m_2-1)-\bar{V}_{n-1}^1(m_1-1,\ m_2-1)+\bar{V}_{n-1}^2(m_1-1,\ m_2)-\bar{V}_{n-1}^2(m_1-1,\ m_2-1)$,

then $P_2(x)$ is an optimal threat point and the optimal Nash arbitrated one, that is,

$$\begin{cases} V_n^1(m_1,\ m_2;\ x)=x+\bar{V}_{n-1}^1(m_1-1,\ m_2), \\ V_n^2(m_1,\ m_2;\ x)=\bar{V}_{n-1}^2(m_1-1,\ m_2), \end{cases}$$

and (accept, reject) is optimal.

(iv) If $x\geqq\bar{V}_{n-1}^1(m_1-1,\ m_2-1)-\bar{V}_{n-1}^1(m_1-1,\ m_2)+\bar{V}_{n-1}^2(m_1-1,\ m_2)-\bar{V}_{n-1}^2(m_1-1,\ m_2-1)$,

then $P_4(x)$ is an optimal threat point, the optimal Nash arbitrated point which is on the segment $P_2(x)P_3(x)$ is given by

$$\begin{cases} V_n^1(m_1,\ m_2;\ x)=x/2+\bar{V}_{n-1}^1(m_1-1,\ m_2-1) \\ \qquad +(v_{n-1}(m_1+m_2-1)-v_{n-1}(m_1+m_2-2))/2, \end{cases}$$

$$\begin{cases} V_n^2(m_1, m_2; x) = x/2 + \bar{V}_{n-1}^2(m_1-1, m_2-1) \\ \qquad + (v_{n-1}(m_1+m_2-1) - v_{n-1}(m_1+m_2-2))/2, \end{cases}$$

and (accept, reject) with probability β and (reject, accept) with probability $1-\beta$ is optimal, where

$$\beta = (x/2 + \bar{V}_{n-1}^1(m_1-1, m_2-1) - \bar{V}_{n-1}^1(m_1, m_2-1) + (v_{n-1}(m_1+m_2-1) - v_{n-1}(m_1+m_2-2))/2)/(x + \bar{V}_{n-1}^1(m_1-1, m_2) - \bar{V}_{n-1}^1(m_1, m_2-1)).$$

Furthermore $\bar{V}_n^1(m_1, m_2) + \bar{V}_n^2(m_1, m_2) = v_n(m_1+m_2)$.

Remark 1. By Property, note that an optimal pair of threat strategies is given within the pure strategies (accept, accept), (reject, reject), (accept, reject), and (reject, accept), not in the mixed strategies of them because of a special structure of the model.

This property says that if the five relations mentioned above are satisfied, then the sum of two players' expected payoff is given by a usual one-person sequential allocation problem. Each player's share depends on their limitations of announcing 'accept'. The five relations state the following: The marginal expected payoff of one player is equal to or smaller than that of the other if the former can announce 'accept' more times than the latter ((1) and (2)). The expected payoff of one player is a nonincreasing function of the other's limitation of announcing 'accept' ((4) and (5)). As for the five relations we have shown in Property that if the five relations (1), (2), (3), (4), and (5) are satisfied for $n=k-1$, so is the relation (3) for $n=k$. We will show in the following that the relations (4) and (5) are also satisfied for $n=k$. (See Appendix 2 for its proof.)

Remark 2. If the five relations (1), (2), (3), (4), and (5) in Property are satisfied for $n=k-1$, (4) and (5) are satisfied for $n=k$.

We have proved that if the five relations in Property are satisfied for $n=k-1$, (3), (4), and (5) are satisfied for $n=k$. Unfortunately we have

Table 1.

state	optimal strategy
$(1, 1, 0;\ x)$	(accept, ——)
$(1, 1, 1;\ x)$	(accept, reject) with probability 1/2 (reject, accept) with probability 1/2, (or (accept, accept))
$(2, 1, 1;\ x)$	(accept, reject) with probability 1/2 (reject, accept) with probability 1/2
$(2, 2, 1;\ x)$	for $0 \leqq x \leqq EX_1$ (accept, reject), for $x \geqq EX_1$ (accept, reject) with probability $x/(2x-EX_1)$ (reject, accept) with probability $(x-EX_1)/(2x-EX_1)$, (or (accept, accept))
$(2, 2, 2;\ x)$	(accept, reject) with probability 1/2 (reject, accept) with probability 1/2, (or (accept, accept))
$(3, 1, 1;\ x)$	for $0 \leqq x \leqq EX_2 - T_{F_2}(EX_1)$ (reject, reject), for $x \geqq EX_2 - T_{F_2}(EX_1)$ (accept, reject) with probability 1/2 (reject, accept) with probability 1/2
$(3, 2, 1;\ x)$	for $0 \leqq x \leqq EX_2 - T_{F_2}(EX_1)$ (accept, reject) with probability $\alpha = (x-(EX_1+T_{F_2}(EX_1)))/(2x-(EX_2+EX_1))$ (reject, accept) with probability $1-\alpha$, for $EX_2 - T_{F_2}(EX_1) \leqq x \leqq EX_1 + T_{F_2}(EX_1)$ (accept, reject), for $x \geqq EX_1 + T_{F_2}(EX_1)$ (accept, reject) with probability $\beta = (x-(EX_2-T_{F_2}(EX_1)))/(2x-(EX_2+EX_1))$ (reject, accept) with probability $1-\beta$
$(3, 2, 2;\ x)$	(accept, reject) with probability 1/2 (reject, accept) with probability 1/2, (or (accept, accept))

not been able to prove (1) and (2) for $n=k$ till now.

Let us solve the problem backward for $n=1$, 2, and 3. The result is listed in Table 1.

The expected payoff functions are given as follows:

$\bar{V}_1^1(1, 0)=EX_1, \quad \bar{V}_1^2(1, 0)=0.$

$\bar{V}_1^1(1, 1)=\bar{V}_1^2(1, 1)=EX_1/2.$

$\bar{V}_2^1(1, 1)=\bar{V}_2^2(1, 1)=(EX_2+EX_1)/2.$

$\bar{V}_2^1(2, 1)=EX_2+EX_1-(T_{F_2}(EX_1)+EX_1)/2, \ \bar{V}_2^2(2, 1)=(T_{F_2}(EX_1)+EX_1)/2.$

$\bar{V}_2^1(2, 2)=\bar{V}_2^2(2, 2)=(EX_2+EX_1)/2.$

$\bar{V}_3^1(1, 1)=\bar{V}_3^2(1, 1)=[(EX_2+EX_1)+T_{F_3}(EX_2-T_{F_2}(EX_1))]/2.$

$$\bar{V}_3^1(2, 1)=[(EX_3+EX_2+EX_1)+\int_0^a(EX_2-T_{F_2}(EX_1))dF_3(x)+\int_a^b xdF_3(x)+\int_b^\infty(EX_1+T_{F_2}(EX_1))dF_3(x)]/2,$$

(where $a=EX_2-T_{F_2}(EX_1)$ and $b=EX_1+T_{F_2}(EX_1)$)

$\bar{V}_3^2(2, 1)=(EX_3+EX_2+EX_1)-\bar{V}_3^1(2, 1).$

$\bar{V}_3^1(2, 2)=\bar{V}_3^2(2, 2)=(EX_3+EX_2+EX_1)/2.$

Tedious calculation shows that the five relations in Property are satisfied for $n=2$, 3, and 4.

4. Numerical Examples

We have shown in Property and Remark 2 that if (1), (2), (3), (4), and (5) in Property are satisfied for $n=k-1$, (3), (4), and (5) are satisfied for $n=k$. We discuss whether (1) and (2) hold or not through four simple numerical examples. The answer is affirmative, that is, the five relations (1), (2), (3), (4), and (5) hold up to $n=20$ for the following four examples: (1) X_n has an identical uniform distribution on the interval (0, 1). (2, 3, and 4) X_n is distributed as in Tables 2, 3, and 4.

Table 2. $\mathrm{pr}\{X_n=x\}=P(x)$

x	0.0	0.25	0.5	0.75	1.0
$P(x)$	0.4	0.1	0.1	0.1	0.3

Table 3. $\mathrm{pr}\{X_n=x\}=P(x)$

x	0.0	0.25	0.5	0.75	1.0
$P(x)$	0.1	0.3	0.3	0.2	0.1

Table 4. $\mathrm{Pr}\{X_n=x\}=P_n(x)$

x	0.0	0.25	0.5	0.75	1.0
$P_1(x)$	0.8	0.1	0.1	0.0	0.0
$P_2(x)$	0.7	0.2	0.1	0.0	0.0
$P_3(x)$	0.7	0.1	0.2	0.0	0.0
$P_4(x)$	0.3	0.2	0.1	0.0	0.4
$P_5(x)$	0.1	0.3	0.4	0.2	0.0
$P_6(x)$	0.1	0.1	0.4	0.1	0.3
$P_7(x)$	0.1	0.0	0.5	0.3	0.1
$P_8(x)$	0.1	0.0	0.2	0.1	0.6
$P_9(x)$	0.0	0.1	0.3	0.0	0.6
$P_{10}(x)$	0.0	0.3	0.1	0.1	0.5
$P_{11}(x)$	0.2	0.1	0.2	0.1	0.4
$P_{12}(x)$	0.2	0.2	0.1	0.1	0.4
$P_{13}(x)$	0.1	0.2	0.2	0.2	0.3
$P_{14}(x)$	0.1	0.0	0.3	0.1	0.5
$P_{15}(x)$	0.2	0.7	0.0	0.1	0.0
$P_{16}(x)$	0.2	0.7	0.0	0.1	0.0
$P_{17}(x)$	0.1	0.6	0.3	0.0	0.0
$P_{18}(x)$	0.3	0.5	0.0	0.1	0.1
$P_{19}(x)$	0.2	0.4	0.3	0.1	0.0
$P_{20}(x)$	0.7	0.1	0.0	0.2	0.0

5. Conclusion

We considered a time sequential two-person variable-threat Nash's bargaining problem. We derived a sufficient condition under which the optimal strategy had an intuitively appealing property, that is, the sum of two players' expected payoff is given by solving a usual one-person sequential allocation problem and each player's share depends on their limitations of announcing 'accept'. The validity of this condition is open in general, but it was shown that the condition is satisfied in four numerical examples.

Appendix 1. Proof of Property:

First we examine the order of endpoints of intervals given in (i), (ii), (iii), and (iv). The monotonicity of v_n with respect to m is obvious by definition. Thus, we have by (3) and (5)

$$v_{n-1}(m_1+m_2)-v_{n-1}(m_1+m_2-1)\leqq\bar{V}^1_{n-1}(m_1,\ m_2)$$
$$-\bar{V}^1_{n-1}(m_1-1,\ m_2)+\bar{V}^2_{n-1}(m_1-1,\ m_2)-\bar{V}^2_{n-1}(m_1,\ m_2).$$

We have also, by (1), (2), and (3),

$$\bar{V}^1_{n-1}(m_1,\ m_2)-\bar{V}^1_{n-1}(m_1-1,\ m_2)+\bar{V}^2_{n-1}(m_1-1,\ m_2)-\bar{V}^2_{n-1}(m_1,\ m_2)$$
$$\leqq\bar{V}^1_{n-1}(m_1-1,\ m_2-1)-\bar{V}^1_{n-1}(m_1-1,\ m_2)+\bar{V}^2_{n-1}(m_1-1,\ m_2)-\bar{V}^2_{n-1}$$
$$(m_1-1,\ m_2-1).$$

For convenience, let $\xi_1(P)$ and $\xi_2(P)$ denote the first and the second component of P, respectively. Note that P_1 is on the line $r_1+r_2=v_{n-1}(m_1+m_2)$, $P_2(x)$ and $P_3(x)$ are on the line $r_1+r_2=x+v_{n-1}(m_1+m_2-1)$, and $P_4(x)$ is on the line $r_1+r_2=x+v_{n-1}(m_1+m_2-2)\,(v_{n-1}(m_1+m_2)\geqq v_{n-1}(m_1+m_2-1)\geqq v_{n-1}(m_1+m_2-2))$. Also that $P_2(x)$ moves on a horizontal line as x becomes large, $P_3(x)$ on a vertical line, and $P_4(x)$ on a line with slope 1.

From now on we prove the statements (i), (ii), (iii), and (iv). In each proof we use the fact that (see Fig. 1) a nonextreme Pareto-optimal point A is the Nash arbitrated one for every threat point which lies on the segment having the negative slope of the Pareto-optimal segment containing A and that an extreme Pareto-optimum B is the Nash arbitrated point for

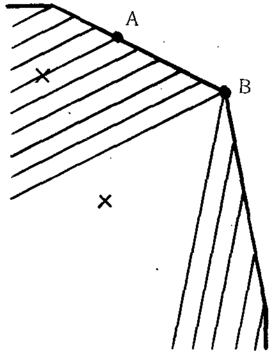

Fig. 1

every threat point lying inside the angle made by two segments each of which has each of the negative slopes of two Pareto-optimal segments containing B respectively.

(i) In this case, P_1 is above the segment $P_2(x)P_3(x)$. Since $\xi_2(P_1)\leqq \xi_2(P_2(x))$ and $\xi_1(P_1)\leqq\xi_1(P_3(x))$ by (4) and (5), a typical situation is expressed in Fig. 2. Obviously P_1 is Pareto-optimal. We show that P_1 is an optimal threat point and therefore the optimal Nash arbitrated one. If player 2 uses 'reject' as his threat strategy, then the arbitrated point moves on segment $P_1P_2(x)$ at player 1's disposal. Player 1 wishes to maximize its first component. So he prefers to announce 'reject'. If player 1 uses 'reject' as his threat strategy, then the arbitrated point moves on

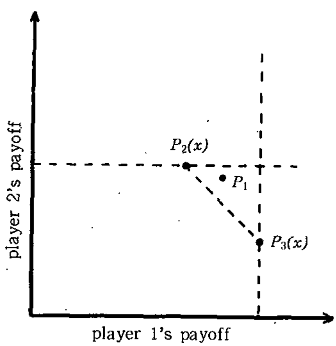

Fig. 2

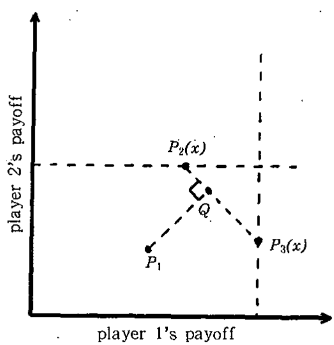

Fig. 3

segment $P_1P_3(x)$ at player 2's disposal. Since player 2 wishes to maximize its second component, he prefers 'reject'. Therefore (reject, reject) is an optimal pair of threat strategies.

(ii) In this case $\xi_2(P_2(x))-\xi_2(P_1)\leqq\xi_1(P_2(x))-\xi_1(P_1)$ and P_1 is below the segment $P_2(x)P_3(x)$. Since $\xi_1(P_3(x))-\xi_1(P_1)\geqq\xi_2(P_2(x))-\xi_2(P_1)$ by (1), Fig. 3 denotes a typical situation. Note that any point on the segment $P_2(x)P_3(x)$ is Pareto-optimal. If player 2 uses 'reject' as his threat strategy, then the arbitrated point moves on the segment $P_2(x)Q$ at player 1's disposal. So player 1 uses 'reject'. If player 1 uses 'reject' as his threat strategy, then the arbitrated point moves on the segment $P_3(x)Q$ at player 2's disposal. So player 2 use 'reject'. Thus (reject, reject) is an optimal pair of threat strategies. And the optimal Nash arbitrated point is given by Q. Further if we identify a point with a vector, Q is given by the following form; $Q=\alpha P_2(x)+(1-\alpha)P_3(x)$.

(iii) Since $\xi_2(P_2(x))-\xi_2(P_1)\leqq\xi_1(P_2(x))-\xi_1(P_1)$ and $\xi_2(P_2(x))-\xi_2(P_4(x))\geqq\xi_1(P_2(x))-\xi_1(P_4(x))$, a typical situation is given in Fig. 4. It is obvious that $P_2(x)$ is Pareto-optimal. If player 2 uses 'reject' as his threat strategy, then the threat point moves on the segment $P_1P_2(x)$ at player 1's disposal. Among corresponding arbitrated points player 1 pre-

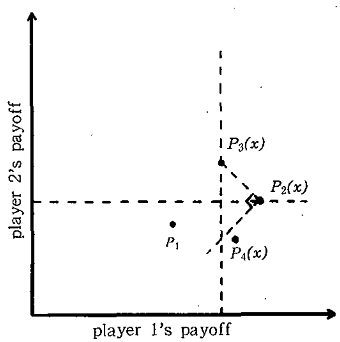

Fig. 4

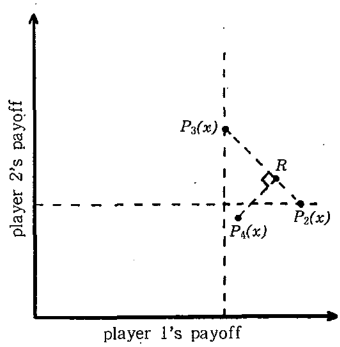

Fig. 5

fers $P_2(x)$. So player 1 uses 'accept'. If player 1 uses 'accept' as his threat strategy, then the threat point moves on the segment $P_2(x)P_4(x)$ at player 2's disposal. So player 2 uses 'reject'. Thus $P_2(x)$ is an optimal threat point and therefore the optimal Nash arbitrated one.

(iv) Since $\xi_2(P_2(x))-\xi_2(P_4(x))\leqq\xi_1(P_2(x))-\xi_1(P_4(x))$ and $\xi_1(P_2(x))\geqq\xi_1(P_3(x))$(by $\xi_1(P_3(x))-\xi_1(P_4(x))=\xi_1(P_3(0))-\xi_1(P_4(0))-x/2 \leqq\xi_2(P_2(0))-\xi_2(P_4(0))-x/2=\xi_2(P_2(x))-\xi_2(P_4(x))\leqq\xi_1(P_2(x))-\xi_1(P_4(x))$), a typical situation is given in Fig. 5. It is obvious that any point on the segment $P_2(x)P_3(x)$ is Pareto-optimal. If player 2 uses 'accept' as his threat strategy, then the arbitrated point moves on the segment $RP_3(x)$ at player 1's disposal. So player 1 prefers to announce 'accept'. If player 1 uses 'accept' as his threat strategy, then the arbitrated point moves on the segment $RP_2(x)$ at player 2's disposal. So player 2 prefers 'accept'. Thus $P_4(x)$ is an optimal threat point and R is the optimal Nash arbitrated point. And $R=\beta P_2(x)+(1-\beta)P_3(x)$.

The last statement that $\bar{V}_n^1(m_1, m_2)+\bar{V}_n^2(m_1, m_2)=v_n(m_1+m_2)$ is obvious by the above argument. Q. E. D.

Appendix 2. Proof of Remark 2:

It is sufficient to show

(6) $V_k^1(m_1, m_2; x)\leqq V_k^1(m_1, m_2-1; x)$,

(7) $V_k^2(m_1, m_2; x)\leqq V_k^2(m_1-1, m_2; x)$,

First we show (6). It should be noted that $V_k^1(m_1, m_2; x)$ and $V_k^1(m_1, m_2-1; x)$ are piecewise linear continuous nondecreasing functions of x of the following form:

$$\begin{cases} \bar{V}_{k-1}^1(m_1, m_2), \\ x/2+\bar{V}_{k-1}^1(m_1, m_2)- \\ \qquad (v_{k-1}(m_1+m_2)-v_{k-1}(m_1+m_2-1))/2, \end{cases}$$

$$(8)\quad V_k^1(m_1, m_2;\ x)=\begin{cases} x+\bar{V}_{k-1}^1(m_1-1,\ m_2), \\ x/2+\bar{V}_{k-1}^1(m_1-1,\ m_2-1) \\ \qquad +(v_{k-1}(m_1+m_2-1)-v_{k-1}(m_1+m_2-2))/2. \end{cases}$$

$$(9)\quad V_k^1(m_1,\ m_2-1;\ x)=\begin{cases} \bar{V}_{k-1}^1(m_1,\ m_2-1), \\ x/2+\bar{V}_{k-1}^1(m_1,\ m_2-1)-(v_{k-1}(m_1 \\ \qquad +m_2-1)-v_{k-1}(m_1+m_2-2))/2, \\ x+\bar{V}_{k-1}^1(m_1-1,\ m_2-1), \\ x/2+\bar{V}_{k-1}^1(m_1-1,\ m_2-2)+(v_{k-1}(m_1 \\ \qquad +m_2-2)-v_{k-1}(m_1+m_2-3))/2, \end{cases}$$

In the above expression we omit a corresponding interval of x where one of the right quantities is equal to the left quantity. If we show every term of the right hand side of (8) is equal to or smaller than the corresponding term of (9), then (6) holds. The first, third, and fourth term of (8) are equal to or smaller than the corresponding terms of (9) respectively by the assumption and the concavity of $v_{k-1}(\cdot)$. We will show that the second term of (8) is equal to or smaller than that of (9).

$$\begin{aligned} &[x/2+\bar{V}_{k-1}^1(m_1,\ m_2-1)-(v_{k-1}(m_1+m_2-1)-v_{k-1}(m_1+m_2-2))/2] \\ &-[x/2+\bar{V}_{k-1}^1(m_1,\ m_2)-(v_{k-1}(m_1+m_2)-v_{k-1}(m_1+m_2-1))/2] \\ &=[(\bar{V}_{k-1}^1(m_1-1,\ m_2-1)-\bar{V}_{k-1}^2(m_1-1,\ m_2-1)-\bar{V}_{k-1}^1(m_1,\ m_2)+\bar{V}_{k-1}^2(m_1, \\ &m_2))+2(\bar{V}_{k-1}^2(m_1-1,\ m_2-1)-\bar{V}_{k-1}^2(m_1,\ m_2-1))]/2\geqq 0. \end{aligned}$$

Therefore (6) holds.

To show (7), it should be also noted that $V_k^2(m_1,\ m_2;\ x)$ and $V_k^2(m_1-1,\ m_2;\ x)$ are piecewise linear continuous nondecreasing functions of x of the following form:

$$\begin{cases} \bar{V}_{k-1}^2(m_1,\ m_2), \\ x/2+\bar{V}_{k-1}^2(m_1,\ m_2)-(v_{k-1}(m_1+m_2)-v_{k-1} \end{cases}$$

$$(10)\quad V_k^2(m_1, m_2;\ x)=\begin{cases} \qquad (m_1+m_2-1))/2, \\ \bar{V}_{k-1}^2(m_1-1,\ m_2), \\ x/2+\bar{V}_{k-1}^2(m_1-1,\ m_2-1)+(v_{k-1}(m_1 \\ \qquad +m_2-1)-v_{k-1}(m_1+m_2-2))/2. \end{cases}$$

$$(11)\quad V_k^2(m_1-1, m_2;\ x)=\begin{cases} \bar{V}_{k-1}^2(m_1-1,\ m_2), \\ x/2+\bar{V}_{k-1}^2(m_1-1,\ m_2)-(v_{k-1}(m_1 \\ \qquad +m_2-1)-v_{k-1}(m_1+m_2-2))/2, \\ \bar{V}_{k-1}^2(m_1-2,\ m_2), \\ x/2+\bar{V}_{k-1}^2(m_1-2,\ m_2-1)+(v_{k-1}(m_1 \\ \qquad +m_2-2)-v_{k-1}(m_1+m_2-3))/2. \end{cases}$$

Since the third term of the right hand side of (10) is equal to the first term of the right hand side of (11), it is sufficient to show that the fourth term of the right hand side of (10) is equal to or smaller than that of (11), which is obvious by the assumption and the concavity of $v_{k-1}(\cdot)$.

Q. E. D.

References

[1] Bonnardeaux, J., Dolait, J–P., and Dyer, J. S.: The Use of the Nash Bargaining Model in Trajectory Selection. *Management Science*, Vol. 22(1976), 766–777.

[2] Fushimi, M.: The Secretary Problem in a Competitive Situation. *Journal of the Operations Research Society of Japan*, Vol. 24(1981), 350–359.

[3] Kadane, J. B.: Reversibility of a Multilateral Sequential Game: Proof of a Conjecture of Sakaguchi. *Ibid*, Vol. 21(1978), 509–516.

[4] Kurano, M., Yasuda, M. and Nakagami, J.: Multi-Variate Stopping Problem with a Majority Rule. *Ibid*, Vol. 23(1980), 205–223.

[5] Maschler, M.: The Inspector's Non-Constant-Sum Game: Its Dependence on a System of Detectors. *Naval Research Logistics Quarterly*, Vol. 14(1967), 275–290.

[6] Nakai, T.: A Time Sequential Game Related to the Sequential Stochastic Assignment Problem. *Journal of the Operations Research Society of Japan*, Vol. 25(1982), 129–139.

[7] Nash, J.: Two-Person Cooperative Games, *Econometrica*, Vol. 21 (1953), 128–140.

[8] Owen, G.: *Game Theory*. W. B. Saunders Company, (Philadelphia), 1968.

[9] Presman, E. L. and Sonin, I. M.: Equilibrium Points in a Game Related to the Best Choice Problem. *Theory of Probability and its Applications*, Vol. 20(1975), 770–781.

[10] Sakaguchi, M.: Optimal Stopping in Sampling from a Bivariate Distribution. *Journal of the Operations Research Society of Japan*, Vol. 16 (1973), 186–200.

[11] ―――――: A Sequential Allocation Game for Targets with Varying Values. *Ibid*, Vol. 20(1977), 182–193.

[12] ―――――: When to Stop: Randomly Appearing Bivariate Target Values. *Ibid*, Vol. 21(1978), 45–57.

[13] ―――――: A Bilateral Sequential Game for Sums of Bivariate Random Variables. *Ibid*, Vol. 21(1978), 486–508.

[14] ―――――: Note: A Time-Sequential Game for Sums of Random Variables. *Ibid*, Vol. 23(1980), 81–90.

[15] ―――――: Non-Zero-Sum Games Related to the Secretary Problem. *Ibid*, Vol. 23(1980), 287–293.

[16] Thomas, M. U. and Nisgav, Y.: An Infiltration Game with Time Dependent Payoff. *Naval Research Logistics Quarterly*, Vol. 23(1976), 297–302.