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An expository supplement to the paper "Bias adjustment minimizing the asymptotic mean square error" with errata

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This expository article is to supplement Ogasawara (2013) with errata. Section 1 gives the asymptotic bias and added higher-order asymptotic variance of the normally distributed variable with a known coefficient of variation. This distribution is one of the simplest cases in the curved exponential family. Section 2 gives errata for the online version of Ogasawara (2013) and the corresponding print version unless corrected at the time of publication.

1. Supplement

Let $X_i \overset{\text{i.i.d.}}{\sim} N(\mu, c\mu^2)\ (i=1,\ldots,n),\ \mu>0$. For simplicity, c is set to be 1 as in Firth (1993). Define $L=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\,\mu}\exp\left\{-\frac{(x_i-\mu)^2}{2\mu^2}\right\}$ and $l=\log L$, then

$$\frac{\partial l}{\partial \mu}=\frac{\sum_{i=1}^{n}x_i^2}{\mu^3}-\frac{\sum_{i=1}^{n}x_i}{\mu^2}-\frac{n}{\mu},$$

$$-\frac{\partial^2 l}{\partial \mu^2}=\frac{3\sum_{i=1}^{n}x_i^2}{\mu^4}-\frac{2\sum_{i=1}^{n}x_i}{\mu^3}-\frac{n}{\mu^2}$$

and the Fisher information per observation becomes

$$i_{(\mu)}=\frac{3(\mu^2+\mu^2)}{\mu^4}-\frac{2\mu}{\mu^3}-\frac{1}{\mu^2}=\frac{3}{\mu^2}.$$

Define $u\equiv n^{-1}\sum_{i=1}^{n}x_i^2$, $m\equiv n^{-1}\sum_{i=1}^{n}x_i$, $\mathbf{y}\equiv(u,m)'$, $\mathrm{E}(\mathbf{y})=(2\mu^2,\mu)'\equiv\boldsymbol{\zeta}$. Let the first-order condition of the estimator $\hat{\mu}$ be

$$f \equiv n^{-1}\mu^3 \frac{\partial l}{\partial \mu} = u - m\mu - \mu^2 = 0 .$$

Then,

$$\frac{\partial f}{\partial \mu} = -m - 2\mu,\quad \frac{\partial^2 f}{\partial \mu^2} = -2,\quad \frac{\partial^3 f}{\partial \mu^3} = 0,\ \frac{\partial f}{\partial \mathbf{y}} = (1, -\mu)',$$

$$\frac{\partial^2 f}{\partial \mu \partial \mathbf{y}} = (0,-1)', \frac{\partial^2 f}{\partial \mu^2 \partial \mathbf{y}} = (0,0)',$$

$$\frac{\partial \mu}{\partial \boldsymbol{\zeta}} \equiv \frac{\partial \hat{\mu}}{\partial \mathbf{y}}\Big|_{\mathbf{y}=\boldsymbol{\zeta}} = -\left(\frac{\partial f}{\partial \mu}\right)^{-1} \frac{\partial f}{\partial \boldsymbol{\zeta}} = (\mu + 2\mu)^{-1}(1, -\mu)' = \frac{1}{3}(\mu^{-1}, -1)',$$

$$\frac{\partial^2 \mu}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}'} = -\left(\frac{\partial f}{\partial \mu}\right)^{-1} \left(\frac{\partial^2 f}{\partial \mu^2} \frac{\partial \mu}{\partial \boldsymbol{\zeta}} \frac{\partial \mu}{\partial \boldsymbol{\zeta}'} + \frac{\partial^2 f}{\partial \mu \partial \boldsymbol{\zeta}} \frac{\partial \mu}{\partial \boldsymbol{\zeta}'} + \frac{\partial \mu}{\partial \boldsymbol{\zeta}} \frac{\partial^2 f}{\partial \mu \partial \boldsymbol{\zeta}'} \right)$$

$$= (3\mu)^{-1} \left\{ (-2)\frac{1}{9}\begin{pmatrix} \mu^{-1} \\ -1 \end{pmatrix}(\mu^{-1},\ -1) + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\frac{1}{3}(\mu^{-1},\ -1) + \frac{1}{3}\begin{pmatrix} \mu^{-1} \\ -1 \end{pmatrix}(\ 0, -1) \right\}$$

$$= (3\mu)^{-1} \left\{ -\frac{2}{9}\begin{pmatrix} \mu^{-2} & -\mu^{-1} \\ -\mu^{-1} & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\mu^{-1}/3 & 1/3 \end{pmatrix} + \begin{pmatrix} 0 & -\mu^{-1}/3 \\ 0 & 1/3 \end{pmatrix} \right\}$$

$$= (3\mu)^{-1} \begin{pmatrix} -\frac{2}{9}\mu^{-2} & -\frac{1}{9}\mu^{-1} \\ -\frac{1}{9}\mu^{-1} & \frac{4}{9} \end{pmatrix} = \frac{1}{27}\begin{pmatrix} -2\mu^{-3} & -\mu^{-2} \\ -\mu^{-2} & 4\mu^{-1} \end{pmatrix},$$

$$\frac{\partial^3 \mu}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}' \partial \zeta_i} = -\left(\frac{\partial f}{\partial \mu}\right)^{-1} \left\{ \frac{\partial^3 f}{\partial \mu^3} \frac{\partial \mu}{\partial \boldsymbol{\zeta}} \frac{\partial \mu}{\partial \boldsymbol{\zeta}'} \frac{\partial \mu}{\partial \zeta_i} \right.$$

$$+ \frac{\partial^2 f}{\partial \mu^2} \left(\frac{\partial \mu}{\partial \boldsymbol{\zeta}} \frac{\partial^2 \mu}{\partial \boldsymbol{\zeta}' \partial \zeta_i} + \frac{\partial^2 \mu}{\partial \boldsymbol{\zeta} \partial \zeta_i} \frac{\partial \mu}{\partial \boldsymbol{\zeta}'} + \frac{\partial^2 \mu}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}'} \frac{\partial \mu}{\partial \zeta_i} \right)$$

$$+\frac{\partial^3 f}{\partial\mu^2\partial\zeta}\frac{\partial\mu}{\partial\zeta'}\frac{\partial\mu}{\partial\zeta_i}+\frac{\partial\mu}{\partial\zeta}\frac{\partial^3 f}{\partial\mu^2\partial\zeta'}\frac{\partial\mu}{\partial\zeta_i}+\frac{\partial^3 f}{\partial\mu^2\partial\zeta_i}\frac{\partial\mu}{\partial\zeta}\frac{\partial\mu}{\partial\zeta'}$$

$$\left.+\frac{\partial^2 f}{\partial\mu\partial\zeta}\frac{\partial^2\mu}{\partial\zeta'\partial\zeta_i}+\frac{\partial^2\mu}{\partial\zeta\partial\zeta_i}\frac{\partial^2 f}{\partial\mu\partial\zeta'}+\frac{\partial^2 f}{\partial\mu\partial\zeta_i}\frac{\partial^2\mu}{\partial\zeta\partial\zeta'}\right\}$$

$$=-\left(\frac{\partial f}{\partial\mu}\right)^{-1}\left\{\frac{\partial^2 f}{\partial\mu^2}\left(\frac{\partial\mu}{\partial\zeta}\frac{\partial^2\mu}{\partial\zeta'\partial\zeta_i}+\frac{\partial^2\mu}{\partial\zeta\partial\zeta_i}\frac{\partial\mu}{\partial\zeta'}+\frac{\partial^2\mu}{\partial\zeta\partial\zeta'}\frac{\partial\mu}{\partial\zeta_i}\right)\right.$$

$$\left.+\frac{\partial^2 f}{\partial\mu\partial\zeta}\frac{\partial^2\mu}{\partial\zeta'\partial\zeta_i}+\frac{\partial^2\mu}{\partial\zeta\partial\zeta_i}\frac{\partial^2 f}{\partial\mu\partial\zeta'}+\frac{\partial^2 f}{\partial\mu\partial\zeta_i}\frac{\partial^2\mu}{\partial\zeta\partial\zeta'}\right\}\ (i=1,2),$$

$$\frac{\partial^3\mu}{\partial\zeta\partial\zeta'\partial\zeta_1}=(3\mu)^{-1}\left[-2\frac{1}{81}\left\{\begin{pmatrix}\mu^{-1}\\-1\end{pmatrix}(-2\mu^{-3},-\mu^{-2})+\begin{pmatrix}-2\mu^{-3}\\-\mu^{-2}\end{pmatrix}(\mu^{-1},-1)\right.\right.$$

$$\left.\left.+\begin{pmatrix}-2\mu^{-3} & -\mu^{-2}\\-\mu^{-2} & 4\mu^{-1}\end{pmatrix}\mu^{-1}\right\}+\frac{1}{27}\left\{\begin{pmatrix}0\\-1\end{pmatrix}(-2\mu^{-3},-\mu^{-2})+\begin{pmatrix}-2\mu^{-3}\\-\mu^{-2}\end{pmatrix}(0,-1)\right\}\right]$$

$$=(3\mu)^{-1}\left\{-\frac{2}{81}\begin{pmatrix}-6\mu^{-4} & 0\\0 & 6\mu^{-2}\end{pmatrix}+\frac{1}{27}\begin{pmatrix}0 & 2\mu^{-3}\\2\mu^{-3} & 2\mu^{-2}\end{pmatrix}\right\}$$

$$=(3\mu)^{-1}\frac{1}{27}\begin{pmatrix}4\mu^{-4} & 2\mu^{-3}\\2\mu^{-3} & -2\mu^{-2}\end{pmatrix}=\frac{1}{81}\begin{pmatrix}4\mu^{-5} & 2\mu^{-4}\\2\mu^{-4} & -2\mu^{-3}\end{pmatrix},$$

$$\frac{\partial^3 \mu}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}' \partial \zeta_2} = (3\mu)^{-1}\left[-2\frac{1}{81}\left\{\begin{pmatrix}\mu^{-1}\\ -1\end{pmatrix}(-\mu^{-2}, 4\mu^{-1}) + \begin{pmatrix}-\mu^{-2}\\ 4\mu^{-1}\end{pmatrix}(\mu^{-1}, -1)\right.\right.$$

$$\left. + \begin{pmatrix}-2\mu^{-3} & -\mu^{-2}\\ -\mu^{-2} & 4\mu^{-1}\end{pmatrix}(-1)\right\}$$

$$\left. + \frac{1}{27}\left\{\begin{pmatrix}0\\ -1\end{pmatrix}(-\mu^{-2}, 4\mu^{-1}) + \begin{pmatrix}-\mu^{-2}\\ 4\mu^{-1}\end{pmatrix}(0, -1) + (-1)\begin{pmatrix}-2\mu^{-3} & -\mu^{-2}\\ -\mu^{-2} & 4\mu^{-1}\end{pmatrix}\right\}\right]$$

$$= (3\mu)^{-1}\left\{-\frac{2}{81}\begin{pmatrix}0 & 6\mu^{-2}\\ 6\mu^{-2} & -12\mu^{-1}\end{pmatrix} + \frac{1}{27}\begin{pmatrix}2\mu^{-3} & 2\mu^{-2}\\ 2\mu^{-2} & -12\mu^{-1}\end{pmatrix}\right\}$$

$$= (3\mu)^{-1}\frac{1}{27}\begin{pmatrix}2\mu^{-3} - 2\mu^{-2}\\ -2\mu^{-2} - 4\mu^{-1}\end{pmatrix} = \frac{1}{81}\begin{pmatrix}2\mu^{-4} - 2\mu^{-3}\\ -2\mu^{-3} - 4\mu^{-2}\end{pmatrix}.$$

Expand $\hat{\mu}$ about μ as

$$\hat{\mu} = \mu + \frac{\partial \mu}{\partial \boldsymbol{\zeta}'}(\mathbf{y} - \boldsymbol{\zeta}) + \frac{1}{2}\frac{\partial^2 \mu}{(\partial \boldsymbol{\zeta}')^{<2>}}(\mathbf{y} - \boldsymbol{\zeta})^{<2>} + \frac{1}{6}\frac{\partial^3 \mu}{(\partial \boldsymbol{\zeta}')^{<3>}}(\mathbf{y} - \boldsymbol{\zeta})^{<3>}$$

$$+ O_p(n^{-2}).$$

From

$$n\mathrm{E}\{(u - \mathrm{E}(u))^2\} = \mathrm{E}\{(X_i^2 - \mathrm{E}(X_i^2))^2\}$$

$$= \mathrm{E}[\{(X_i - \mu)^2 + 2\mu(X_i - \mu) - \mu^2\}^2]$$

$$= 3\mu^4 + 4\mu^4 + \mu^4 - 2\mu^4 = 6\mu^4,$$

$$n\mathrm{E}\{(u - \mathrm{E}(u))(m - \mathrm{E}(m))\} = \mathrm{E}\{2\mu(X_i - \mu))(X_i - \mu)\} = 2\mu^3,$$

$$n\mathrm{E}\{(m - \mathrm{E}(m))^2\} = \mu^2,$$

we have

$$n\mathrm{E}\{(\mathbf{y} - \boldsymbol{\zeta})(\mathbf{y} - \boldsymbol{\zeta})'\} = \begin{pmatrix}6\mu^4 & 2\mu^3\\ 2\mu^3 & \mu^2\end{pmatrix} \equiv \mathbf{\Omega} \quad \text{and}$$

$$\mathrm{E}(\hat{\mu}) = n^{-1}\frac{1}{2}\frac{\partial^2\mu}{(\partial\boldsymbol{\zeta})^{<2>}}\mathrm{E}\{(\mathbf{y}-\boldsymbol{\zeta})^{<2>}\} + O(n^{-2})$$

$$= n^{-1}\frac{1}{54}\mathrm{tr}\left\{\begin{pmatrix} -2\mu^{-3} - \mu^{-2} \\ -\mu^{-2} & 4\mu^{-1} \end{pmatrix}\begin{pmatrix} 6\mu^4 & 2\mu^3 \\ 2\mu^3 & \mu^2 \end{pmatrix}\right\} + O(n^{-2})$$

$$= n^{-1}\frac{\mu}{54}(-12-2-2+4) + O(n^{-2}) = -n^{-1}\frac{2}{9}\mu + O(n^{-2}),$$

which is found to be equal to the result of Firth (1993, p.35)

Further.

$$n^2\,\mathrm{E}\{(u-\mathrm{E}(u))^3\} = \mathrm{E}[\{(X_i-\mu)^2 - \mu^2 + 2\mu(X_i-\mu)\}^3]$$

$$= \mathrm{E}[\{(X_i-\mu)^2-\mu^2\}^3] + 6\mu\mathrm{E}[\{(X_i-\mu)^2-\mu^2\}^2(X_i-\mu)]$$

$$+12\mu^2\mathrm{E}[\{(X_i-\mu)^2-\mu^2\}(X_i-\mu)^2] + 8\mu^3\mathrm{E}\{(X_i-\mu)^3\}$$

$$= (15-3\times3+3-1)\mu^6 + 12\mu^2(3-1)\mu^4 = 32\mu^6,$$

where $n^2\mathrm{E}\{\cdot\}$ comes from the following properties: (1) the cumulant of the third power of a sample mean is n^{-3} times that of the sum of independent $X_i (i=1,\ldots,n)$ and (2) the cumulant of the sum is n times that of X_i.

$$n^2\,\mathrm{E}\{(u-\mathrm{E}(u))^2(m-\mathrm{E}(m))\}$$

$$= \mathrm{E}[\{(X_i-\mu)^2-\mu^2+2\mu(X_i-\mu)\}^2(X_i-\mu)]$$

$$= 4\mu\mathrm{E}[\{(X_i-\mu)^2-\mu^2\}(X_i-\mu)^2] = 4\mu(3-1)\mu^4$$

$$= 8\mu^5,$$

$$n^2\,\mathrm{E}\{(u-\mathrm{E}(u))(m-\mathrm{E}(m))^2\}$$

$$= \mathrm{E}[\{(X_i-\mu)^2-\mu^2+2\mu(X_i-\mu)\}(X_i-\mu)^2] = (3-1)\mu^4$$

$$= 2\mu^4,$$

$$n^2\,\mathrm{E}\{(m-\mathrm{E}(m))^3\} = 0.$$

The variance of $\hat{\mu}$ is

$$\mathrm{var}(\hat{\mu}) = n^{-1}\frac{\partial\mu}{\partial\boldsymbol{\zeta}'}\boldsymbol{\Omega}\frac{\partial\mu}{\partial\boldsymbol{\zeta}} + n^{-2}\left[\frac{1}{2}\mathrm{tr}\left(\frac{\partial^2\mu}{\partial\boldsymbol{\zeta}\partial\boldsymbol{\zeta}'}\boldsymbol{\Omega}\frac{\partial^2\mu}{\partial\boldsymbol{\zeta}\partial\boldsymbol{\zeta}'}\boldsymbol{\Omega}\right)\right.$$

$$+\left\{\frac{\partial\mu}{\partial\boldsymbol{\zeta}'}\otimes\frac{\partial^2\mu}{(\partial\boldsymbol{\zeta}')^{<2>}}\right\}n^2\mathrm{E}\{(\mathbf{y}-\boldsymbol{\zeta})^{<3>}\}$$

$$\left.+\frac{1}{3}\left\{\frac{\partial\mu}{\partial\boldsymbol{\zeta}'}\otimes\frac{\partial^3\mu}{(\partial\boldsymbol{\zeta}')^{<3>}}\right\}n^2\mathrm{E}\{(\mathbf{y}-\boldsymbol{\zeta})^{<4>}\}\right] + O_p(n^{-3}),$$

where the first term is

$$= n^{-1}\frac{1}{9}(\mu^{-1},-1)\begin{pmatrix}6\mu^4 & 2\mu^3\\ 2\mu^3 & \mu^2\end{pmatrix}\begin{pmatrix}\mu^{-1}\\ -1\end{pmatrix} = n^{-1}\frac{1}{9}(\mu^{-1},-1)(4\mu^3,\mu^2)'$$

$$= n^{-1}\frac{\mu^2}{3},$$

which is equal to the reciprocal of the Fisher information. The first term in $n^{-2}[\cdot]$ is

$$= \frac{1}{2}\times\frac{1}{27^2}\mathrm{tr}\left\{\begin{pmatrix}2\mu^{-1} & 1\\ 1 & -4\mu\end{pmatrix}\begin{pmatrix}6\mu^2 & 2\mu\\ 2\mu & 1\end{pmatrix}\begin{pmatrix}2\mu^{-1} & 1\\ 1 & -4\mu\end{pmatrix}\begin{pmatrix}6\mu^2 & 2\mu\\ 2\mu & 1\end{pmatrix}\right\}$$

$$= \frac{1}{2}\times\frac{1}{27^2}\mathrm{tr}\left\{\begin{pmatrix}14\mu & 5\\ -2\mu^2 & -2\mu\end{pmatrix}\begin{pmatrix}14\mu & 5\\ -2\mu^2 & -2\mu\end{pmatrix}\right\}$$

$$= \frac{1}{2}\times\frac{1}{27^2}\{14^2 - 5\times2 - 2\times5 + (-2)^2\}\mu^2 = \frac{90}{27^2}\mu^2 = \frac{10}{81}\mu^2.$$

The second term in $n^{-2}[\cdot]$ is

$$=\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^2\mu}{\partial\zeta_1^2}n^2\mathrm{E}\{(u-\mathrm{E}(u))^3\}$$

$$+\left(2\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^2\mu}{\partial\zeta_1\partial\zeta_2}+\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^2\mu}{\partial\zeta_1^2}\right)n^2\mathrm{E}\{(u-\mathrm{E}(u))^2(m-\mathrm{E}(m))\}$$

$$+\left(2\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^2\mu}{\partial\zeta_1\partial\zeta_2}+\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^2\mu}{\partial\zeta_2^2}\right)n^2\mathrm{E}\{(u-\mathrm{E}(u))(m-\mathrm{E}(m))^2\}$$

$$+\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^2\mu}{\partial\zeta_2^2}n^2\mathrm{E}\{(m-\mathrm{E}(m))^3\}$$

$$=\frac{\mu^{-1}}{3}\frac{-2\mu^{-3}}{27}32\mu^6+\left(2\times\frac{\mu^{-1}}{3}\frac{-\mu^{-2}}{27}+\frac{-1}{3}\times\frac{-2\mu^{-3}}{27}\right)8\mu^5$$

$$+\left(2\times\frac{-1}{3}\times\frac{-\mu^{-2}}{27}+\frac{\mu^{-1}}{3}\times\frac{4\mu^{-1}}{27}\right)2\mu^4$$

$$=\left(-\frac{64}{81}+\frac{6\times 2}{81}\right)\mu^2=-\frac{52}{81}\mu^2.$$

Recall $\mathbf{\Omega}=\begin{pmatrix}6\mu^4 & 2\mu^3\\ 2\mu^3 & \mu^2\end{pmatrix}$. Then, the third term in $n^{-2}[\cdot]$ is

$$=\left\{\frac{\partial\mu}{\partial\zeta'}\otimes\frac{\partial^3\mu}{(\partial\zeta')^{<3>}}\right\}\{\mathrm{vec}(\mathbf{\Omega})\}^{<2>}$$

$$=\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^3\mu}{\partial\zeta_1^3}\omega_{11}^2+\left(3\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^3\mu}{\partial\zeta_1^2\partial\zeta_2}+\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^3\mu}{\partial\zeta_1^3}\right)\omega_{11}\omega_{12}$$

$$+\left(3\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^3\mu}{\partial\zeta_2^2\partial\zeta_1}+\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^3\mu}{\partial\zeta_2^3}\right)\omega_{22}\omega_{12}$$

$$+\left(\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^3\mu}{\partial\zeta_1\partial\zeta_2^2}+\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^3\mu}{\partial\zeta_2\partial\zeta_1^2}\right)(\omega_{11}\omega_{22}+2\omega_{12}^2)$$

$$+\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^3\mu}{\partial\zeta_2^3}\omega_{22}^2$$

$$=\frac{\mu^{-1}}{3}\frac{4\mu^{-5}}{81}(6\mu^4)^2+\left(3\times\frac{\mu^{-1}}{3}\frac{2\mu^{-4}}{81}+\frac{-1}{3}\frac{4\mu^{-5}}{81}\right)6\mu^4 2\mu^3$$

$$+\left(3\times\frac{-1}{3}\frac{-2\mu^{-3}}{81}+\frac{\mu^{-1}}{3}\frac{-4\mu^{-2}}{81}\right)\mu^2 2\mu^3$$

$$+\left(\frac{\mu^{-1}}{3}\frac{-2\mu^{-3}}{81}+\frac{-1}{3}\frac{2\mu^{-4}}{81}\right)\{6\mu^4\mu^2+2(2\mu^3)^2\}+\frac{-1}{3}\frac{-4\mu^{-4}}{81}(\mu^2)^2$$

$$=\left\{\frac{48}{81}+\left(\frac{2}{81}-\frac{4}{3\times 81}\right)12+\left(\frac{2}{81}-\frac{4}{3\times 81}\right)2+\left(\frac{-4}{3\times 81}\right)14+\frac{4}{3\times 81}\right\}\mu^2$$

$$=\frac{1}{3\times 81}(48\times 3+2\times 12+2\times 2-56+4)\mu^2=\frac{1}{243}(144+32-56)\mu^2$$

$$=\frac{120}{243}\mu^2=\frac{40}{81}\mu^2.$$

The sum of the terms inside $n^{-2}[\cdot]$ becomes

$$=\left(\frac{10}{81}-\frac{52}{81}+\frac{40}{81}\right)\mu^2=-\frac{2}{81}\mu^2,$$

which gives

$$\mathrm{var}(\hat{\mu})=n^{-1}\mu^2-n^{-2}\frac{2}{81}\mu^2+O(n^{-3})$$

equal to the result of Firth (1993, p.35).

2. Errata

In Theorem1, the phrase "and $\beta_{\Delta 1}^{(\mathrm{U})} = 0$" at the end of the sentence should be deleted. That is, the theorem becomes

Theorem 1. *The* $\mathrm{MSE}(c_{\min}^{*(\mathrm{U})}\hat{\theta}^{(\mathrm{U})})$ *up to order* $O(n^{-2})$ *is greater than or equal to (smaller than) the corresponding* $\mathrm{MSE}(\hat{\theta}_{\mathrm{C}k\,\mathrm{min}})$ *when* $\theta_0^{-2}\beta_2^2 \le (>)\ \ \beta_1^{-2}\{\mathrm{acov}(\hat{\theta},\hat{\beta}_1)\}^2$, *where the equality occurs when* $\beta_1/\theta_0 = (\hat{\beta}_1/\hat{\theta}) + O(n^{-1}) = O(1)$.

In Subsection 3.1 Mean, the expression "$= \dfrac{n^{-1}\sigma^2}{1+n^{-1}(c_{\mathrm{V}}^{(\mathrm{L})})^2}$" in the last equation should be deleted. That is, the equation should be

$$\mathrm{MSE}(c_{(\mathrm{L})\mathrm{CV}}^{*}\hat{\theta}^{*}) < \mathrm{MSE}(\hat{\theta}^{*}) = \mathrm{MSE}(\bar{X}).$$

Equation (5.8) and the sentence including the equation should be as follows:

$$\begin{aligned}
&\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0 \\
&= -n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^{*} + \sum_{i=1}^{3}\boldsymbol{\Lambda}^{(i)}\mathbf{l}_0^{(i)} + n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^{*} - n^{-1}\boldsymbol{\Lambda}^{-1}\frac{\partial \mathbf{q}_0^{*}}{\partial \boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{(1)}\mathbf{l}_0^{(1)} \\
&\qquad - n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{J}_0^{(3)}\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^{*})\otimes\boldsymbol{\Lambda}^{-1}\}\mathbf{l}_0^{(1)} + O_p(n^{-2}) \qquad (5.8)\\
&= -n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^{*} + \sum_{i=1}^{3}\boldsymbol{\Lambda}^{(i)}\mathbf{l}_0^{(i)} - n^{-1}(\hat{\mathbf{L}}^{-1}\hat{\mathbf{q}}^{*} - \boldsymbol{\Lambda}^{-1}\mathbf{q}_0^{*})_{O_p(n^{-1/2})} + O_p(n^{-2}),
\end{aligned}$$

where $\mathbf{J}_0^{(3)} \equiv \mathrm{E}\left\{\dfrac{\partial^3 \bar{l}}{\partial\boldsymbol{\theta}_0(\partial\boldsymbol{\theta}_0{}')^{<2>}}\right\}$; $\hat{\mathbf{L}}$ is $\mathbf{L}$ using $\hat{\boldsymbol{\theta}}$ in place of $\boldsymbol{\theta}_0$; and $(\cdot)_{O_p(n^{-1/2})}$ indicates that $(\cdot)$ is of order $O_p(n^{-1/2})$ (see Ogasawara, 2010, Section 2.3).

The expressions $\mathbf{L}$ and $\mathbf{L}^{-1}$ should be $\hat{\mathbf{L}}$ and $\hat{\mathbf{L}}^{-1}$, respectively in the following passages: the sentence after that including (5.8), Equation (5.9) (two times), the paragraph after Theorem 4 (two times) and Equation (7.1).

References

Firth, D. (1993). Bias reduction of maximum likelihood estimates. *Biometrika, 80*, 27-38.

Ogasawara, H. (2013). Bias adjustment minimizing the asymptotic mean square error. To appear in *Communications in Statistics – Theory and Methods.* DOI:10.1080/03610926.2013.786788. Online published on April 25, 2013.