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ON THE MULTIPLICATIVE PARTITION FUNCTION

By

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1. Introduction.

Let n be a positive integer. A *multiplicative partition* of the number n is a representation of n as the product of any number of integers that are greater than 1. Thus

$$24=2\cdot 12=3\cdot 8=4\cdot 6=2\cdot 2\cdot 6=2\cdot 3\cdot 4=2\cdot 2\cdot 2\cdot 3$$

has 7 multiplicative partitions (cf. the table annexed at the end of this paper). Let us denote the number of mutiplicative partitions of n by $X(n)$, namely

$$X(n)=\sum_{n=2^{l_2}3^{l_3}4^{l_4}\cdots,\ l_2,\ l_3,\ l_4,\cdots\geqq 0} 1 \qquad (n>1);$$

$X(1)$ is defined to be 1. This arithmetical function, we call it the multiplication partition function, was introduced by MacMahon [6] who noted that the function $X(n)$ has a generating function

$$G(s)\overset{\mathrm{def}}{=}\sum_{n=1}^{\infty}X(n)n^{-s}=\prod_{m=2}^{\infty}(1-m^{-s})^{-1}, \qquad \operatorname{Re}s>1. \tag{1}$$

Making use of this relation, Oppenheim [7], [8] found an asymptotic formula

$$\sum_{n\leqq x}X(n)=\frac{x\,e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}\left\{1+\sum_{k=1}^{N-1}\frac{\varepsilon_k}{(\log x)^{k/2}}+O_N\left(\frac{1}{(\log x)^{N/2}}\right)\right\},$$

where the ε_k are certain constants, for each N and all large x. He also obtained a better approximation

$$\sum_{n\leqq x}X(n)=x\sum_{k=0}^{\infty}d_k\frac{I_{k+1}(2\sqrt{\log x})}{\sqrt{\log x}^{\,k+1}}+O\left(x\frac{e^{\sqrt{\log x}}}{(\log x)^{3/8}}\right) \tag{2}$$

to the sum $\sum_{n\leqq x}X(n)$, where the $I_k(x)$ are modified Bessel functions, and the numbers d_k are the coefficients in the Taylor expansion

$$\frac{G(s)}{s}e^{-1/(s-1)}=\sum_{k=0}^{\infty}d_k(s-1)^k, \quad |s-1|<\frac{1}{2}. \tag{3}$$

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In this note, we shall prove (2) with sharper error term. This is the following

THEOREM. *We have*

(4) $$\sum_{n\leqq x} X(n)=x\sum_{k=0}^{\infty} d_k \frac{I_{k+1}(2\sqrt{\log x})}{\sqrt{\log x}^{k+1}}+O(x\, e^{-A\sqrt{\log x}}),$$

for any positive A, *and sufficiently large* $x\geqq x_0(A)$.

Concerning the function $G(s)$, we have immediately

(5) $$\log G(s)=\sum_{m=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}m^{-ks}=\sum_{k=1}^{\infty}\frac{1}{k}\{\zeta(ks)-1\},\qquad \mathrm{Re}\, s>1,$$

where $\zeta(s)$ is the Riemann zeta-function. This last series converges uniformly in any compact subset of the set $\{s\,;\mathrm{Re}\, s>0\}-\{1, 1/2, 1/3, \cdots\}$. The following lemma is due to Oppenheim [8].

LEMMA 1. *The function* $\log G(s)$ *is regular for* $s>0$ *except* $s=1/n$ $(n=1, 2, \cdots)$, *where there are simple poles of the function with respective residues* $1/n^2$ $(n=1, 2, \cdots)$. *In particular, near the point* $s=1$, *we have*

(6) $$\log G(s)=\frac{1}{s-1}+O(s-1).$$

By this lemma, we get the Taylor expansion (3) with

(7) $$d_0=1.$$

Moreover Estermann [3] showed that *the function* $G(s)$ *is singular at every point of the imaginary axis.*

In order to prove our theorem, in the next section we shall estimate the function

(8) $$\xi_1(x)=\sum_{n\leqq x}X(n)(x-n)=\int_1^x\xi_0(u)du,\qquad \text{where } \xi_0(x)=\sum_{n\leqq x}X(n),$$

using the theorem of Hardy and Littlewood (see Chandrasekharan [2]) that

(9) $$\zeta(s)=O\left(t^{4(1-\sigma)/\log(1/(1-\sigma))}\frac{\log t}{\log\log t}\right),$$

for $t\geqq 3$, uniformly for $63/64\leqq\sigma<1$, where $\sigma=\mathrm{Re}\, s$ and $t=\mathrm{Im}\, s$. This argument will lead us our estimate (4) of $\xi_0(x)$ in § 3. Finally in § 4, we shall give the numbers d_k in the Taylor expansion (3) an effective form.

2. Estimation of $\xi_1(x)$.

By Perron's formula, we have

(10) $$\xi_1(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{G(s)x^{s+1}}{s(s+1)}ds \qquad (c>1).$$

It is plain that

$$\sum_{k=2}^{\infty}\frac{1}{k}\{\zeta(ks)-1\}=O(1)$$

for $\sigma=\text{Re } s\geqq 2/3$ and all $t=\text{Im } s$. By (5) we have

(11) $$\log G(s)=\zeta(s)+O(1) \qquad (\sigma\geqq 2/3).$$

Let A_1 be any fixed positive number. From (9) we get that for $t\geqq 3$,

$$\zeta(s)=O_{A_1}\left(\frac{\log t}{\log\log t}\right)$$

uniformly for $1-\frac{A_1}{\log t}\leqq\sigma\leqq 1$. Thus in the region $t\geqq 3$, $1-\frac{A_1}{\log t}\leqq\sigma\leqq 1$, $2/3\leqq\sigma$ we have

(12) $$\log G(s)=O_{A_1}\left(\frac{\log t}{\log\log t}\right).$$

On the other hand for $\sigma\geqq 1$, $t\geqq 1$, we have

(13) $$\begin{aligned}|\log G(s)|&\leqq|\zeta(s)|+O(1)\\&\leqq\log t+O(1) \qquad \text{(see [2] p. 34)}.\end{aligned}$$

We now choose, for given $x>1$, the curve $C=C_1\cup C_2\cup C_3$ such that

$$\begin{cases} C_1=\left\{s\,;\,\sigma=1-\frac{A_1}{\log|t|},\ -\infty<t\leqq t_0\right\},\\ C_2=\{s\,;\,\sigma=\sigma_0,\ |t|\leqq t_0\},\\ C_3=\left\{s\,;\,\sigma=1-\frac{A_1}{\log t},\ t_0\leqq t<\infty\right\},\end{cases}$$

where

$$t_0=t_0(x)=e^{\sqrt{A_1\log x}},\quad \sigma_0=1-\frac{A_1}{\log t_0}.$$

The curve C is oriented by the parameter t. By (10), (12) and (13), we obtain

(14) $$\xi_1(x)=\operatorname*{Res}_{s=1}\frac{G(s)x^{s+1}}{s(s+1)}+\frac{1}{2\pi i}\int_C\frac{G(s)x^{s+1}}{s(s+1)}ds.$$

We divide the integral on the right-hand side into several parts. Let

$$\frac{1}{2\pi i}\int_C \frac{G(s)x^{s+1}}{s(s+1)}ds=E_1+E_2+\bar{E}_2+E_3+\bar{E}_3\,, \tag{15}$$

where

$$E_1=\frac{1}{2\pi}\int_{-3}^{3}\frac{G(s)x^{s+1}}{s(s+1)}dt \qquad (s=\sigma_0+ti)\,,$$

$$E_2=\frac{1}{2\pi}\int_{3}^{t_0}\frac{G(s)x^{s+1}}{s(s+1)}dt \qquad (s=\sigma_0+it)\,,$$

$$E_3=\frac{1}{2\pi i}\int_{C_3}\frac{G(s)x^{s+1}}{s(s+1)}ds\,,$$

and $\bar{E}_j$ ($j=2, 3$) are complex conjugates of E_j.

(i) Estimation of E_1. Let $s=\sigma_0+it$ ($|t|\leqq 3$). By lemma 1 we have

$$\log G(s)=\frac{1}{s-1}+O(1)\,,$$

$$\mathrm{Re}\log G(s)=\frac{\sigma_0-1}{(\sigma_0-1)^2+t^2}+O(1)\,.$$

Since $\sigma_0-1<0$, $G(s)=O(1)$. Thus we have for $x>1$,

$$E_1=O(x^{\sigma_0+1})=O(x^2e^{-\sqrt{A_1\log x}})\,. \tag{16}$$

(ii) Estimation of E_2. Let $s=\sigma_0+it$ ($3\leqq t\leqq t_0$). From (9), for sufficiently large $x\geqq x_1(A_1)$ we have

$$\begin{aligned}\zeta(s)&=O\left\{\exp\left(\frac{4A_1}{\log\log t_0-\log A_1}\right)\frac{\log t}{\log\log t}\right\}\\ &=O\left(\frac{\log t}{\log\log t}\right)=O\left(\frac{\log t_0}{\log\log t_0}\right)\\ &=O\left(\frac{\sqrt{A_1}}{\log A_1}\sqrt{\log x}\right)\end{aligned}$$

By (11), we get

$$E_2=O\left(x^2\exp\left\{-\left(1+O\left(\frac{1}{\log A_1}\right)\right)\sqrt{A_1\log x}\right\}\right).$$

Thus we have for sufficiently large A_1 and $x\geqq x_1(A_1)$,

$$E_2=O\left(x^2e^{-\frac{1}{2}\sqrt{A_1\log x}}\right). \tag{17}$$

(iii) Estimation of E_3. Let $\sigma=1-\dfrac{A_1}{\log t}$ ($t\geqq t_0$). The estimate (12) leads us to

(18) $$E_3=O\left(x^2\int_{t_0}^{\infty}t^{-3/2}dt\right)$$
$$=O\left(x^2e^{-\frac{1}{2}\sqrt{A_1\log x}}\right)$$

for sufficiently large $x\geqq x_2(A_1)$. By (14)-(18), we obtain the following

LEMMA 2.

(19) $$\xi_1(x)=\operatorname*{Res}_{s=1}\frac{G(s)x^{s+1}}{s(s+1)}+O(x^2e^{-A_2\sqrt{\log x}}),$$

for any positive A_2 *and sufficiently large* $x\geqq x_3(A_2)$.

3. Proof of the theorem.

Let

(20) $$U(x)\overset{\text{def}}{=}\operatorname*{Res}_{s=1}\frac{G(s)x^{s+1}}{s(s+1)}.$$

Then we have

(21) $$U'(x)=\operatorname*{Res}_{s=1}\frac{G(s)x^s}{s},\quad U''(x)=\operatorname*{Res}_{s=1}G(s)x^{s-1}.$$

Since $\xi_0(x)$ is an increasing function, we have

$$\frac{1}{h}\{\xi_1(x)-\xi_1(x-h)\}\leqq\xi_0(x)\leqq\frac{1}{h}\{\xi_1(x+h)-\xi_1(x)\},\ h>0,$$

by the definition (8). Suppose that

(22) $$h=h(x)>0,\quad h=o(x).$$

Then, by (19) we have

$$\xi_1(x\pm h)=U(x\pm h)+O(x^2e^{-A_1\sqrt{\log x}})$$

and

$$U(x\pm h)=U(x)\pm hU'(x)+\frac{h^2}{2}U''(x\pm\theta_\pm h),\quad 0<\theta_+,\ \theta_-<1.$$

Since

$$\pm\frac{1}{h}\{\xi_1(x\pm h)-\xi_1(x)\}=U'(x)\pm\frac{h}{2}U''(x\pm\theta_\pm h)+O\left(\frac{x^2}{h}e^{-A_2\sqrt{\log x}}\right),$$

we have

(23) $$\xi_0(x)=U'(x)+O\{hU''(x\pm\theta_\pm h)\}+O\left(\frac{x^2}{h}e^{-A_2\sqrt{\log x}}\right).$$

In connection with functions $U'(x)$, $U''(x)$, we can show

LEMMA 3.

(24) $$U'(x)=x\sum_{k=0}^{\infty}d_k\frac{I_{k+1}(2\sqrt{\log x})}{(\log x)^{(k+1)/2}},$$

(25) $$U''(x)\sim\frac{1}{2\sqrt{\pi}}\cdot\frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}}.$$

PROOF. By the definition of modified Bessel functions $I_n(x)$, we have

$$e^{\frac{1}{s-1}}x^{s-1}=\exp\left\{\sqrt{\log x}\left(\frac{1}{(s-1)\sqrt{\log x}}+(s-1)\sqrt{\log x}\right)\right\}$$

$$=\sum_{n=-\infty}^{\infty}I_n(2\sqrt{\log x})\{(s-1)\sqrt{\log x}\}^n$$

and

(26) $$\operatorname{Res} e^{1/(s-1)}x^{s-1}(s-1)^k=I_{-k-1}(2\sqrt{\log x})\sqrt{\log x}^{\,-k-1}$$
$$=\frac{I_{k+1}(2\sqrt{\log x})}{(\log x)^{(k+1)/2}}.$$

By (3) and (21), we get (24). Next we shall show (25). Let c_k be the constants such that

$$G(s)e^{-1/(s-1)}=\sum_{k=0}^{\infty}c_k(s-1)^k, \qquad |s-1|<1/2$$

(cf. Lemma 1). Then for some positive constant M,

$$|c_k|\leqq M^k \qquad (k=0, 1, 2, \cdots)$$

and we have

$$c_0=1.$$

By (21) and (26), we have

$$U''(x)=\frac{I_1(2\sqrt{\log x})}{\sqrt{\log x}}+E,$$

where

$$E=\frac{1}{2\pi i}\int_{|s|=\rho}e^{1/s}x^s\sum_{k=1}^{\infty}c_ks^kds \qquad (0<\rho<1/2).$$

If $M\rho<1$, we have

$$|E|\leqq\frac{M\rho^2}{1-M\rho}e^{1/\rho}x^\rho.$$

By taking $\rho=1/\sqrt{\log x}$, we obtain

$$U''(x)=\frac{I_1(2\sqrt{\log x})}{\sqrt{\log x}}+O\Big(\frac{e^{2\sqrt{\log x}}}{\log x}\Big).$$

Since we have, as is well known,

$$I_k(x)\sim\frac{e^x}{\sqrt{2\pi x}},$$

we get (25). This completes the proof.

By using this lemma with

$$h=xe^{-((A_2/2)+1)\sqrt{\log x}} \qquad \text{(see (22))}$$

(23) leads us to

$$\xi_0(x)=x\sum_{k=0}^{\infty}d_k\frac{I_{k+1}(2\sqrt{\log x})}{(\log x)^{(k+1)/2}}+O(xe^{-((A_2/2)-1)\sqrt{\log x}}).$$

Thus our theorem is proved.

REMARK. In our approximation (4), we may conjecture that the best order of the error term would be

$$O\Big(\sqrt{x}\,\frac{e^{\sqrt{\log x}}}{(\log x)^{3/4}}\Big),$$

for the reason that

$$\operatorname*{Res}_{s=1/2}\frac{G(s)x^s}{s}=\sqrt{x}\sum_{k=0}^{\infty}d'_k\frac{I_{k+1}(\sqrt{\log x})}{2^{k+1}(\log x)^{(k+1)/2}}$$

$$\sim\frac{d'_0}{2\sqrt{2\pi}}\sqrt{x}\,\frac{e^{\sqrt{\log x}}}{(\log x)^{3/4}},$$

where d'_k are defined by

$$\frac{G(s)}{s}e^{-\frac{1}{4(s-(1/2))}}=\sum_{k=0}^{\infty}d'_k\Big(s-\frac{1}{2}\Big)^k \qquad \Big(\Big|s-\frac{1}{2}\Big|<\frac{1}{6}\Big).$$

However, it seems very difficult to prove this.

4. The numbers d_k.

Let γ_n and α_n respectively denote the constants defined by

(27) $$\gamma_n=\lim_{N\to\infty}\Big(\sum_{\nu=1}^{N}\frac{\log^n\nu}{\nu}-\frac{\log^{n+1}N}{n+1}\Big) \qquad (n\geqq 0)$$

and

(28) $$\alpha_n=\sum_{m=1}^{n}(m-1)!\,S(n,m)\alpha_n^{(m)} \qquad (n>0),$$

where

$$\alpha_n^{(m)}=\begin{cases}\displaystyle\sum_{\nu=1}^{\infty}\frac{\log^n(\nu+1)}{\nu(\nu+1)}, & \text{if } m=1\,,\\ \displaystyle\sum_{\nu=1}^{\infty}\frac{\log^n(\nu+1)}{\nu^m}, & \text{if } m>1\,,\end{cases}$$

and integers $S(n, m)$ are Stirling numbers of the second kind, that is, defined by the identity

$$x^n=\sum_{m=0}^{n} S(n, m)x(x-1)\cdots(x-m+1)\,. \tag{29}$$

Then we have the following

PROPOSITION. *The numbers d_n can be represented in the form*

$$d_n=(-1)^n\sum_{m=0}^{n}\sum_{\substack{m=1\cdot m_1+2\cdot m_2+\cdots\\ m_1, m_2, \cdots\geqq 0}}\frac{\beta_1^{m_1}\beta_2^{m_2}\cdots}{(1\,!)^{m_1}m_1\,!\,(2\,!)^{m_2}m_2\,!\cdots}\,, \tag{30}$$

where $\beta_n=\gamma_n+\alpha_n$ $(n>0)$.

Thus we have

$$d_0=1\,,$$

$$d_1=-1-\beta_1=-2.18493\cdots\,,$$

$$d_2=1+\beta_1+\frac{1}{2}(\beta_2+\beta_1^2)=5.48422\cdots\,,$$

$$d_3=-1-\beta_1-\frac{1}{2}(\beta_2+\beta_1^2)-\frac{1}{6}(\beta_3+3\beta_2\beta_1+\beta_1^3)=-13.80378\cdots\,,$$

$$\cdots\cdots\cdots\cdots\cdots\,.$$

PROOF OF PROPOSITION. It is not difficult to see that β_n and γ_n can be defined alternatively by

$$\log G(s)=\frac{1}{s-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,!}\beta_n(s-1)^n \tag{31}$$

and

$$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n\,!}\gamma_n(s-1)^n \tag{32}$$

respectively. We obviously get (30) from (31). And also we have (27) from (32). The latter was found by Stieltjes, Jensen [5], and Briggs-Chowla [1]. Values $(-1)^n\gamma_n/n\,!$ have been calculated by Gram [4] with 16 decimals. We now have

$$\beta_n=\gamma_n+\alpha_n \qquad (n>0)\,,$$

where

$$\alpha_n=(-1)^n \sum_{k=2}^{\infty} k^{n-1}\zeta^{(n)}(k)=\sum_{\nu=1}^{\infty} \log^n \nu \sum_{k=2}^{\infty} k^{n-1}\nu^{-k}, \tag{33}$$

by (5), (31) and (32). It is enough to show (28) from the definition (33) of α_n. We have

$$\sum_{m=1}^{n} (m-1)!S(n, m)\alpha_n^{(m)}=\sum_{\nu=1}^{\infty} \log^n(\nu+1)\left\{\frac{1}{\nu(\nu+1)}+\sum_{m=2}^{n}(m-1)!S(n, m)\nu^{-m}\right\}$$

We may show, for all positive integers ν,

$$\sum_{k=2}^{\infty} k^{n-1}(\nu+1)^{-k}=\frac{1}{\nu(\nu+1)}+\sum_{m=2}^{n}(m-1)!S(n, m)\nu^{-m}. \tag{34}$$

This leads us to (28). Let $f_n(w)$ $(n=1, 2, \cdots)$ denote rational functions

$$\frac{w^2}{1+w}+\sum_{m=1}^{n}(m-1)!S(n, m)w^m,$$

and suppose $S(n, m)=0$ for integers m outside $0\leqq m\leqq n$. If $w=z/(1-z)$ then we have

$$\begin{aligned} f_n(w)&=\frac{z^2}{1-z}+\sum_{m=2}^{n}(m-1)!S(n, m)(z+z^2+\cdots)^m \\ &=\sum_{k=2}^{\infty} z^k \sum_{m=1}^{k}(m-1)!S(n, m)\frac{m(m+1)\cdots(k-1)}{(k-m)!} \\ &=\sum_{k=2}^{\infty} z^k \frac{1}{k}\sum_{m=1}^{n} S(n, m)k(k-1)\cdots(k-m+1) \\ &=\sum_{k=2}^{\infty} kz^{n-1}z^k \qquad (|z|<1), \end{aligned}$$

by (29). Thus we have (34), on putting $w=1/\nu$. This completes the proof.

REMARK. The author could not find Stieltjes's paper for γ_n, whereas Gram [4] referred to it.

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Table of $X(n)$ *)

a	$X(10a+b)$, $b=$ 0	1	2	3	4	5	6	7	8	9	$\sum_{n=1}^{10a+9} X(n)$
0		1	1	1	2	1	2	1	3	2	14
1	2	1	4	1	2	2	5	1	4	1	37
2	4	2	2	1	7	2	2	3	4	1	65
3	5	1	7	2	2	2	9	1	2	2	98
4	7	1	5	1	4	4	2	1	12	2	137
5	4	2	4	1	7	2	7	2	2	1	169
6	11	1	2	4	11	2	5	1	4	2	212
7	5	1	16	1	2	4	4	2	5	1	253
8	12	5	2	1	11	2	2	2	7	1	298
9	11	2	4	2	2	2	19	1	4	4	349
10	9	1	5	1	7	5	2	1	16	1	397
11	5	2	12	1	5	2	4	4	2	2	436
12	21	2	2	2	4	3	11	1	15	2	499
13	5	1	11	2	2	7	7	1	5	1	541
14	11	2	2	2	29	2	2	4	4	1	600
15	11	1	7	4	5	2	11	1	2	2	646
16	19	2	12	1	4	5	2	1	21	2	715
17	5	4	4	1	5	4	12	2	2	1	755
18	26	1	5	2	7	2	5	2	4	7	816
19	5	1	30	1	2	5	9	1	11	1	882
20	16	2	2	2	11	2	2	4	12	2	937
21	15	1	4	2	2	2	31	2	2	2	1000
22	11	2	5	1	19	9	2	1	11	1	1062
23	5	5	7	1	11	2	4	2	5	1	1105
24	38	1	4	7	4	4	5	2	7	2	1179
25	7	1	26	2	2	5	22	1	5	2	1252
26	11	4	2	1	21	2	5	2	4	1	1305
27	21	1	12	5	2	4	11	1	2	4	1368
28	21	1	5	1	4	5	5	2	47	2	1461
29	5	2	4	1	11	2	7	7	2	2	1504
30	26	2	2	2	12	2	11	1	11	2	1574
31	5	1	21	1	2	11	4	1	5	2	1628
32	30	2	5	2	29	4	2	2	7	2	1713
33	15	1	4	4	2	2	38	1	4	2	1786
34	11	2	11	3	7	5	2	1	11	1	1840
35	11	7	19	1	5	2	4	5	2	1	1897
36	52	2	2	4	11	2	5	1	12	4	1992
37	5	2	11	1	5	7	7	2	21	1	2054
38	11	2	2	1	45	5	2	4	4	1	2131
39	15	2	16	2	2	2	26	1	2	5	2204
40	29	1	5	2	4	12	5	2	21	1	2286
41	5	2	4	2	11	2	19	2	5	1	2339
42	36	1	2	4	7	4	5	2	4	5	2409
43	5	1	57	1	5	5	4	2	5	1	2495
44	21	9	5	1	11	2	2	2	30	1	2579
45	26	2	4	2	2	5	21	1	2	7	2651
46	11	1	15	1	12	5	2	1	26	2	2727
47	5	2	7	2	5	4	11	4	2	1	2770
48	64	2	2	5	9	2	19	1	7	2	2883
49	11	1	11	2	5	11	12	2	5	1	2944

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