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Didactic proofs for the surface area of a sphere and the chi-square density

Haruhiko Ogasawara

This note gives two didactic proofs associated with reverse problems in the surface area of a sphere and the probability density of the chi-square distribution. The corresponding preprints were submitted to ResearchGate (Ogasawara, 2022a, b). The author thanks Dr. Hans Stocker (Schaffhausen, Switzerland) for valuable discussions on the topics.

1. A stochastic derivation of the surface area of the (n-1)-sphere

Abstract: The surface area of the (n-1)-dimensional sphere in the n-dimensional Euclidean space is typically obtained by induction with a similar derivation for the volume of the n-dimensional ball in the same space. Huber (1982) obtained the volume of the ball using the gamma function without induction, followed by the surface area of the corresponding sphere. In this note, the surface area is first derived from the multivariate normal distribution using the polar coordinate system and the corresponding chi distribution for the radius. The volume of the ball is given by integration.

Keywords: Jacobians, multivariate normal distribution, chi distribution, probability density function (pdf), n-ball.

1.1 Introduction

Huber (1982, p. 301) stated that "Beginning students of analysis are often presented with a simple inductive derivation of n-sphere volume formulas" i.e., $V_n(r) = r^n \pi^{n/2} / \Gamma\{1+(n/2)\}$ for the n-dimensional ball of radius r in the n-dimensional Euclidian space, which is equal to the volume inside the (n-1)-dimensional sphere in the same space, whose surface area is given by $S_{n-1}(r) = 2\pi^{n/2} r^{n-1} / \Gamma(n/2)$, where $\Gamma(\cdot)$ is the gamma function. Huber first derived $V_n(r)$ using properties of $\Gamma(\cdot)$, followed by $S_{n-1}(r) = \mathrm{d}V_n(r)/\mathrm{d}\,r$. In this note, $S_{n-1}(r)$ is first obtained by the probability density functions (pdf's) of the chi and n-variate normal distributions.

1.2 Derivation of the surface area of a sphere and the volume of the corresponding ball

The pdf of the n-variate standard normal using the usual n Cartesian coordinates $\mathbf{x} = (x_1, ..., x_n)^{\mathrm{T}}$ $(-\infty < x_i < \infty,\ i = 1, ..., n)$ is $\phi_n(\mathbf{x}) = \exp(-\mathbf{x}^{\mathrm{T}}\mathbf{x}/2)/(2\pi)^{n/2}$. When the polar coordinate system with radial coordinate r $(0 \le r < \infty)$ and angular coordinates $\boldsymbol{\theta} = (\theta_1, .., \theta_{n-1})^{\mathrm{T}}$ $(0 \le \theta_i < \pi,\ i = 1, ..., n-2;\ 0 \le \theta_{n-1} < 2\pi)$ is employed, the pdf becomes

$$\psi_n(r, \boldsymbol{\theta}) = \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} J\{\mathbf{x} \to (r, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\},$$

where $r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$; $J\{\mathbf{x} \to (r, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\} = \det\{\mathrm{d}\,\mathbf{x}/\mathrm{d}(r, \boldsymbol{\theta}^{\mathrm{T}})\} = r^{n-1}g(\boldsymbol{\theta})$ is the Jacobian; and $g(\boldsymbol{\theta})$ is a known function of $\boldsymbol{\theta}$. When r is given, the marginal density at r is obtained by

$$\begin{aligned}\psi_n(r) &= \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} r^{n-1} g(\boldsymbol{\theta}) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-2} \mathrm{d}\theta_{n-1} \\ &= \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} S_{n-1}(r).\end{aligned}$$

It is known that the variable S^* corresponding to $s \equiv r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ is chi-square distributed with n degrees of freedom (df) whose pdf is given by $\dfrac{s^{(n/2)-1}\exp(-s/2)}{2^{n/2}\Gamma(n/2)}$. Then, the distribution of the square root of S^* is said to be chi distributed with n df. The pdf of the chi at r is given by

$$f_\chi(r\,|\,n) = \frac{s^{(n/2)-1}\exp(-s/2)}{2^{n/2}\Gamma(n/2)} \frac{\mathrm{d}s}{\mathrm{d}r} = \frac{r^{n-2}\exp(-r^2/2)}{2^{n/2}\Gamma(n/2)} 2r = \frac{r^{n-1}\exp(-r^2/2)}{2^{(n/2)-1}\Gamma(n/2)}.$$

Since $\psi_n(r) = f_\chi(r\,|\,n)$, we have

$$\frac{\exp(-r^2/2)}{(2\pi)^{n/2}} S_{n-1}(r) = \frac{r^{n-1}\exp(-r^2/2)}{2^{(n/2)-1}\Gamma(n/2)}$$

yielding the required surface area $S_{n-1}(r) = 2\pi^{n/2} r^{n-1}/\Gamma(n/2)$. The corresponding volume is derived by

$$\begin{aligned}V_n(r) &= \int_0^r S_{n-1}(t)\,\mathrm{d}t = \int_0^r \{2\pi^{n/2}t^{n-1}/\Gamma(n/2)\}\,\mathrm{d}t \\ &= 2\pi^{n/2}r^n/\{n\Gamma(n/2)\} = \pi^{n/2}r^n/\{(n/2)\Gamma(n/2)\} \\ &= \pi^{n/2}r^n/\Gamma\{(n/2)+1\},\end{aligned}$$

where the typos, easily found by the standard readership, in the expression

$$= 2\pi^{n/2} r^n / \{n\Gamma(n/2)\} = \pi^{n/2} r^n \{(n/2)\Gamma(n/2)\}$$
$$= \pi^{n/2} r^n \Gamma\{(n/2)+1\}$$

of the preprint have been corrected.

1.3 Remarks

The vector $\mathbf{x}$ is typically transformed to r and $\boldsymbol{\theta}$ as follows

$$x_i = r\left(\prod\nolimits_{j=1}^{i-1} \sin\theta_j\right)\cos\theta_i \ \ (i = 1, \ldots, n-1),$$
$$x_n = r\prod\nolimits_{j=1}^{n-1} \sin\theta_j .$$

Then, it can be shown that the Jacobian is written as

$$J\{\mathbf{x} \to (r, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\} = r^{n-1} g(\boldsymbol{\theta}) = r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2},$$

which can also be used to have the surface area

$$S_{n-1}(r) = \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} r^{n-1} g(\boldsymbol{\theta}) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-2} \mathrm{d}\theta_{n-1}$$
$$= r^{n-1} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-2} \mathrm{d}\theta_{n-1}$$

though this integral is somewhat tedious to obtain. Conversely, the result in the previous section gives an integral formula

$$\int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-2} \mathrm{d}\theta_{n-1} = 2\pi^{n/2} / \Gamma(n/2)$$

or equivalently

$$\int_0^{\pi} \cdots \int_0^{\pi} \sin^{m}\theta_1 \sin^{m-1}\theta_2 \cdots \sin\theta_m \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m = \pi^{m/2} / \Gamma\{(m/2)+1\} \ (m = 1, 2, \ldots),$$

where the former gives the surface area of the (n-1)-sphere with unit radius i.e., $S_{n-1}(1)$. For instance, when n = 2, we have the circumference $\int_0^{2\pi} \mathrm{d}\theta_{n-1} = 2\pi$ of a circle with unit radius. It is also found that the latter gives the volume of the m-ball with unit radius $V_m(1)$.

2. A simple geometric derivation of the chi-square density

Abstract: The probability density function (pdf) of the chi-square distribution with n degrees of freedom is derived using the pdf's of the multivariate distribution and its transformation under the polar coordinate system. The marginal density of the radial variable corresponding to the chi distribution is obtained by integrating the angular variables, which gives the factor of the known surface area of the (n-1)-dimensional sphere. The pdf of the chi-square is obtained by the square transformation of the chi distribution.

Keywords: sphere, Jacobian, multivariate normal distribution, chi

distribution, probability density function (pdf).

2.1 Introduction

The chi-square distributed variable denoted by S^* with n degrees of freedom (df) is defined as the sum of the squares of n independent standard normals $S^* = \mathbf{X}^{\mathrm{T}}\mathbf{X}$, where $\mathbf{X} = (X_1, \dots, X_n)^{\mathrm{T}}$ and $X_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)\ (i = 1, \dots, n)$. The probability density function (pdf) of the chi-square distribution with 1 df is given in two steps of variable transformation $X_1 \to Y_1(=|X_1|) \to S^*(=Y_1^2)$. The first step gives the half-normally distributed variable Y_1 whose pdf at y_1 is $\sqrt{2/\pi}\exp(-y_1^2/2)\ (0 \le y_1 < \infty)$. The second step yields the pdf of S^* at $s(=y_1^2)$ with the Jacobian $\mathrm{d}y_1/\mathrm{d}s = \mathrm{d}\sqrt{s}/\mathrm{d}s = 1/(2\sqrt{s})$ as

$$\sqrt{2/\pi}\exp(-s/2)/(2\sqrt{s}) = \exp(-s/2)/(\sqrt{2s\pi})$$
$$= s^{(1/2)-1}\exp(-s/2)/\{2^{1/2}\Gamma(1/2)\}\ (0 < s < \infty),$$

where $\Gamma(\cdot)$ is the gamma function. The above result is equal to the pdf of the gamma distribution with the same shape and rate parameters 1/2. The chi-square with n df is typically given by the closed property that the sum of independent gammas with equal rate parameters is gamma with the shape parameter being the sum of the n shape parameters. The closed property can directly be derived using beta integral or indirectly using the moment generating functions for n independent variables. Then, we have the pdf of the chi-square with n df as the gamma with the shape parameter n/2 and unchanged rate parameter 1/2 as

$$f_{\chi^2}(s\,|\,n) = \frac{s^{(n/2)-1}\exp(-s/2)}{2^{n/2}\Gamma(n/2)}\ (0 < s < \infty, n = 1;\ 0 \le s < \infty, n = 2,3,\dots).$$

2.2 A geometric derivation of the chi-square density

Recall that in the case of 1 df, we have the one-to-one correspondence in the variable transformation from X_1 to S^*. For the general case with n df, consider the variable transformation from **X** to those in the polar coordinate system, which typically takes the following relationship when $\mathbf{X} = \mathbf{x} = (x_1, \dots, x_n)^{\mathrm{T}}$:

$$x_i = r\left(\prod\nolimits_{j=1}^{i-1}\sin\theta_j\right)\cos\theta_i\ (i = 1, \dots, n-1), x_n = r\prod\nolimits_{j=1}^{n-1}\sin\theta_j$$
$$(0 \le r < \infty;\ 0 \le \theta_i < \pi,\ i = 1, \dots, n-2;\ 0 \le \theta_{n-1} < 2\pi),$$

where r and $\boldsymbol{\theta} = (\theta_1, .., \theta_{n-1})^{\mathrm{T}}$ are radial and angular coordinates, respectively with $r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$. Noting that the pdf of $\mathbf{X} = \mathbf{x}$ is $\phi_n(\mathbf{x}) = \exp(-\mathbf{x}^{\mathrm{T}}\mathbf{x}/2)/(2\pi)^{n/2}$, the pdf in the polar coordinate system is written by

$$\frac{\exp(-r^2/2)}{(2\pi)^{n/2}} J\{\mathbf{x} \to (r, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\},$$

where $J\{\mathbf{x} \to (r, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\} = \det\{\mathrm{d}\,\mathbf{x}/\mathrm{d}(r, \boldsymbol{\theta}^{\mathrm{T}})\} = r^{n-1} g(\boldsymbol{\theta})$ is the Jacobian and $g(\boldsymbol{\theta})$ is a function of $\boldsymbol{\theta}$. The marginal density of the radial variable at r is given from the above:

$$\begin{aligned} &\frac{\exp(-r^2/2)}{(2\pi)^{n/2}} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} r^{n-1} g(\boldsymbol{\theta}) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-2} \mathrm{d}\theta_{n-1} \\ &= \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} S_{n-1}(r) = \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} \frac{2\pi^{n/2} r^{n-1}}{\Gamma(n/2)} \\ &= \frac{r^{n-1} \exp(-r^2/2)}{2^{(n/2)-1}\Gamma(n/2)}, \end{aligned}$$

where $S_{n-1}(r) = 2\pi^{n/2} r^{n-1}/\Gamma(n/2)$ is the known formula for the surface area of the (n-1)-dimensional sphere in the n-dimensional Euclidian space (see e.g., Huber, 1982). Since $r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$, the above density is that of the square root of the chi-square or the chi distribution with n df denoted by $f_{\chi}(r \mid n) = r^{n-1} \exp(-r^2/2)/\{2^{(n/2)-1}\Gamma(n/2)\}$. Then, the pdf of the chi-square distributed variable S^* at $s = r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ is obtained as

$$f_{\chi^2}(s \mid n) = \frac{r^{n-1} \exp(-r^2/2)}{2^{(n/2)-1}\Gamma(n/2)} \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{s^{(n-1)/2} \exp(-s/2)}{2^{(n/2)-1}\Gamma(n/2)} \frac{1}{2\sqrt{s}} = \frac{s^{(n/2)-1} \exp(-s/2)}{2^{n/2}\Gamma(n/2)},$$

which is the required density.

2.3 Remarks

In the previous subsection the chi distribution was first obtained since the variable corresponds to the radial variable in the polar coordinate system. A direct transformation $\mathbf{x} \to \{s(=r^2), \boldsymbol{\theta}^{\mathrm{T}}\}^{\mathrm{T}}$ can also be used when desired. However, for this transformation the Jacobian becomes

$$J\{\mathbf{x} \to (s, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\} = \left(\det \frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}(r, \boldsymbol{\theta}^{\mathrm{T}})}\right) \frac{\mathrm{d}\,r}{\mathrm{d}\,s} = r^{n-1} g(\boldsymbol{\theta}) \frac{1}{2\sqrt{s}}.$$

When the above result is integrated over $\boldsymbol{\theta}$, we have $S_{n-1}(r)/(2\sqrt{s})$, which also gives the pdf of the chi-square. While the two methods seem to be comparable, the name of “chi-square” indicates the important role of the chi distribution corresponding to the radial variable.

References

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