

## Multiplication operators, integration operators and companion operators on weighted Bloch space

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**Abstract.** Let  $g$  be an analytic function on the open unit disk  $D$  in the complex plane  $\mathbf{C}$ . We will study the following operator

$$I_g(h)(z) := \int_0^z h'(\zeta)g(\zeta)d\zeta, \quad J_g(h)(z) := \int_0^z h(\zeta)g'(\zeta)d\zeta$$

on the Bloch space. In this paper, we will study the boundedness and compactness of  $I_g$  on the  $\alpha$ -Bloch space, and the boundedness and compactness of products of  $I_g$  and  $J_g$  defined on the  $\alpha$ -Bloch space. And we will get the relationship of multiplication operator  $M_g$  and the operators  $I_g, J_g$  defined on the  $\alpha$ -Bloch space.

*Key words:* multiplication operator, integration operator, Bloch space, boundedness, compactness.

### 1. Introduction

Let  $D = \{z \in \mathbf{C} : |z| < 1\}$  denote the open unit disk in the complex plane  $\mathbf{C}$  and let  $\partial D = \{z \in \mathbf{C} : |z| = 1\}$  denote the unit circle. Let  $H(D)$  denote the space of analytic functions on  $D$ . For  $1 \leq p < +\infty$ , the Lebesgue space  $L^p(D, dA)$  is defined to be the Banach space of Lebesgue measurable functions on the open unit disk  $D$  with

$$\|f\|_{L^p(dA)} := \left( \int_D |f(z)|^p dA(z) \right)^{1/p} < +\infty,$$

where  $dA(z)$  is the normalized area measure on  $D$ . The Bergman space  $L_a^p(D)$  is defined to be the subspace of  $L^p(D, dA)$  consisting of analytic functions. For  $0 < p < +\infty$ , the Hardy space  $H^p$  is defined to be the Banach space of analytic functions  $f$  on  $D$  with

$$\|f\|_p := \left( \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < +\infty.$$

The space of analytic functions on  $D$  of bounded mean oscillation,

denoted by  $BMOA$ , consists of functions  $f$  in  $H^2$  for which

$$\|f\|_{BMOA} := |f(0)| + \sup_{z \in D} \|f \circ \varphi_z - f(z)\|_2 < +\infty.$$

Let  $\alpha \geq 1$ . Then the  $\alpha$ -Bloch space  $B^\alpha$  of  $D$  is defined to be the space of analytic functions  $f$  on  $D$  such that

$$\|f\|_{B^\alpha} := |f(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < +\infty.$$

And the little  $\alpha$ -Bloch space of  $D$ , denoted  $B_0^\alpha$ , is the closed subspace of  $B^\alpha$  consisting of functions  $f$  with  $(1 - |z|^2)^\alpha f'(z) \rightarrow 0$  ( $|z| \rightarrow 1^-$ ).

Note that  $B^1$ ,  $B_0^1$  are the Bloch space  $B$ , the little Bloch space  $B_0$ , respectively.

Let  $\omega$  be analytic on  $\{\zeta: |1 - \zeta| < 1\}$ . Suppose that  $|\omega(1 - |z|^2)| \rightarrow 0$  as  $z \in D$  and  $|z| \rightarrow 1^-$ . Then the weighted Bloch space  $B_\omega$  of  $D$  is defined to be the space of analytic functions  $f$  on  $D$  such that

$$\|f\|_{B_\omega} := |f(0)| + \sup_{z \in D} |\omega(1 - |z|^2)| |f'(z)| < +\infty.$$

For  $g$  analytic on  $D$ , the operator  $J_g$  is defined by the following:

$$J_g(h)(z) := \int_0^z h(\zeta) g'(\zeta) d\zeta.$$

If  $g(z) = z$ , then  $J_g$  is the integration operator. If  $g(z) = \log 1/(1 - z)$ , then  $J_g$  is the Cesàro operator. And we also define the companion operator  $I_g$ , the multiplication operator  $M_g$  by the following:

$$I_g(h)(z) := \int_0^z g(\zeta) h'(\zeta) d\zeta, \quad M_g(h)(z) := g(z)h(z).$$

Let  $X$  be a Banach space. For an analytic function  $g$  on  $D$ ,  $g$  is a multiplier for  $X$  if  $gX \subset X$ , i.e.  $fg \in X$  for all  $f \in X$ . By the closed-graph theorem,  $gX \subset X$  if and only if the multiplication operator  $M_g$  is bounded on  $X$ . Let  $S: X \rightarrow X$  be a linear operator. Then the operator  $S$  is said to be compact operator if for every bounded sequence  $\{x_n\}$  in  $X$ ,  $\{S(x_n)\}$  has a convergent subsequence. On the other hand, the operator  $S$  is said to be weakly compact operator if for every bounded sequence  $\{x_n\}$  in  $X$ ,  $\{S(x_n)\}$  has a weakly convergent subsequence. Then the operator  $S$  is weakly compact operator if and only if  $S^{**}(X^{**}) \subset X$  where  $S^{**}$  be the second adjoint of  $S$  and  $X$  is identified with its image under the natural

embedding into its second dual  $X^{**}$ .

In [5], Ch. Pommerenke showed that  $J_g$  is a bounded operator on Hardy space  $H^2$  if and only if  $g$  is in  $BMOA$ , and this result was extended to the other Hardy space  $H^p$   $1 \leq p < +\infty$  in [1]. In [2], A. Aleman and A.G. Siskakis studied the operator  $J_g$  defined on the weighted (radial weight) Bergman space. Recently, in [7], A.G. Siskakis and R. Zhao studied the operator  $J_g$  defined on  $BMOA$ .

In [9], we showed the following result about the operator  $J_g$  defined on the Bloch space  $B$ .

**Theorem A** *For  $g$  analytic on  $D$ , the operator  $J_g$  is bounded on  $B$  if and only if*

$$\sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| < +\infty,$$

*and the operator  $J_g$  is compact on  $B$  if and only if*

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| = 0.$$

*Let  $\alpha > 1$ . Then the operator  $J_g$  is bounded on  $B^\alpha$  if and only if  $g \in B$ , and the operator  $J_g$  is compact on  $B^\alpha$  if and only if  $g \in B_0$ .*

In this paper, we will study the boundedness and compactness of  $I_f$  defined on the  $\alpha$ -Bloch space. And we will give the relationship between multiplication operator  $M_g$  and the operators  $I_g, J_g$ . In some cases, it is advantageous to think of  $I_g$  and  $J_g$  as distant cousins of Hankel and Toeplitz operators, respectively. In [8], K. Stroethoff and D. Zheng studied products of Hankel and Toeplitz operators. So we will also study the boundedness and compactness of products of  $I_f$  and  $J_g$ .

Throughout this paper,  $C$  will denote positive constant whose value is not necessary the same at each occurrence.

## 2. The operators $I_f$ defined on the $\alpha$ -Bloch space

In this section, we study the boundedness and compactness of  $I_f$  on the  $\alpha$ -Bloch space.

**Lemma B** *Let  $\alpha \geq 1$ . Then there exist  $h_1, h_2 \in B^\alpha$  such that*

$$|h'_1(z)| + |h'_2(z)| \geq \frac{1}{(1 - |z|)^\alpha} \quad (z \in D).$$

*Proof.* See Proposition 5.4 in [6]. □

**Theorem 2.1** *Let  $\alpha \geq 1$  and  $f$  be an analytic function on  $D$ . Then the operator  $I_f$  is bounded on  $B^\alpha$  if and only if  $\sup_{z \in D} |f(z)| < +\infty$ .*

*Proof.* Let  $\alpha \geq 1$ . Let  $f$  be an analytic function on  $D$ . If  $f \in H^\infty$ , it is trivial that  $I_f$  is bounded on  $B^\alpha$ . To prove the converse, suppose that  $I_f$  is bounded on  $B^\alpha$ . By Lemma B, there exist  $h_1, h_2 \in B^\alpha$  such that

$$|h'_1(z)| + |h'_2(z)| \geq \frac{1}{(1 - |z|)^\alpha},$$

for all  $z \in D$ . So for any  $z \in D$ , we have

$$\begin{aligned} |f(z)| &\leq (1 - |z|^2)^\alpha (|h'_1(z)| + |h'_2(z)|) |f(z)| \\ &\leq \sup_{z \in D} (1 - |z|^2)^\alpha |h'_1(z)| |f(z)| + \sup_{z \in D} (1 - |z|^2)^\alpha |h'_2(z)| |f(z)| \\ &= \|I_f h_1\|_{B^\alpha} + \|I_f h_2\|_{B^\alpha} \\ &\leq \|I_f\| \|h_1\|_{B^\alpha} + \|I_f\| \|h_2\|_{B^\alpha} < +\infty. \end{aligned}$$

Hence we have  $\sup_{z \in D} |f(z)| < +\infty$ . □

**Corollary 2.2** *For  $g$  analytic on  $D$ , the following are equivalent:*

- (i)  $gB \subset B$ ;
- (i)'  $gB_0 \subset B_0$ ;
- (ii) Both  $I_g$  and  $J_g$  are bounded operators on  $B$ .
- (ii)' Both  $I_g$  and  $J_g$  are bounded operators on  $B_0$ .
- (iii)  $g \in H^\infty$ ,  $\sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| < +\infty$ .

*Let  $\alpha > 1$ . For  $g$  analytic on  $D$ , the following are equivalent:*

- (i)  $gB^\alpha \subset B^\alpha$ ;
- (i)'  $gB_0^\alpha \subset B_0^\alpha$ ;
- (ii)  $I_g$  is bounded operator on  $B^\alpha$ ;
- (ii)'  $I_g$  is bounded operator on  $B_0^\alpha$ ;
- (iii)  $g \in H^\infty$ .

*Proof.* The above equivalences of (i), (i)', (iii) were proved by [3] and [12].

The other equivalences are immediate consequences of Theorem 2.1 and Theorem 1 in [3].  $\square$

**Theorem 2.3** *Let  $\alpha \geq 1$ . Let  $f$  be an analytic function on  $D$ . Then the operator  $I_f$  is compact on  $B^\alpha$  if and only if  $f \equiv 0 \dots (*)$ .*

*Proof.* Let  $\alpha = 1$ . Let  $f$  be an analytic function on  $D$ . Since  $|h(z)| \leq C \|h\|_B \log(1/(1 - |z|^2))$  for  $h \in B$ , the unit ball of  $B$  is a normal family of analytic functions. By normal family arguments,  $I_f$  is compact operator on  $B$  if and only if every sequence  $\{h_n\}$  in  $B$  with  $\|h_n\|_B \leq 1$  and  $h_n \rightarrow 0$  ( $n \rightarrow +\infty$ ) uniformly on compact subsets of  $D$  has a subsequence  $\{h_{n_k}\}$  in  $B$  such that  $\|I_f h_{n_k}\|_B \rightarrow 0$  ( $k \rightarrow +\infty$ ).

We show that every sequence which goes to zero has a subsequence such that the condition  $(*)$  holds when the limit is taken over that sequence. This implies that the condition  $(*)$  holds.

Suppose that the operator  $I_f$  is compact on  $B$ . Let  $a_n \rightarrow a \in \partial D$  and consider the test functions  $h_n(z) := \log(1/(1 - \overline{a_n}z))$ ,  $h(z) := \log(1/(1 - \overline{a}z))$ . Then  $h_n \rightarrow h$  uniformly on compact subsets of  $D$ . Using the fact  $|c + d|^2 \leq 2|c|^2 + 2|d|^2$  and the subharmonicity of  $|f(z)|$ ,

$$\sup_{a \in D} \int_{D(a,r)} \frac{dA(z)}{(1 - |z|^2)^2} < \infty,$$

we have

$$\begin{aligned} & |a_n|^2 |f(a_n)|^2 \\ & \leq C \frac{|a_n|^2}{(1 - |a_n|^2)^2} \int_{D(a_n,r)} |f(z)|^2 dA(z) \\ & \leq CK \int_{D(a_n,r)} \left| \left( \log \frac{1}{1 - \overline{a_n}z} \right)' \right|^2 |f(z)|^2 dA(z) \\ & = CK \int_{D(a_n,r)} (1 - |z|^2)^2 \left| \left( \log \frac{1}{1 - \overline{a_n}z} \right)' \right|^2 |f(z)|^2 \frac{dA(z)}{(1 - |z|^2)^2} \\ & \leq CK \sup_{z \in D(a_n,r)} (1 - |z|^2)^2 \left| \left( \log \frac{1}{1 - \overline{a_n}z} \right)' \right|^2 |f(z)|^2 \\ & \quad \times \int_{D(a_n,r)} \frac{dA(z)}{(1 - |z|^2)^2} \end{aligned}$$

$$\begin{aligned}
&\leq 2CK \int_{D(a_n, r)} \frac{dA(z)}{(1-|z|^2)^2} \\
&\quad \times \sup_{z \in D(a_n, r)} \left| \left( \log \frac{1}{1-\bar{a}_n z} \right)' - \left( \log \frac{1}{1-\bar{a}z} \right)' \right|^2 |f(z)|^2 (1-|z|^2)^2 \\
&\quad + 2CK \int_{D(a_n, r)} \frac{dA(z)}{(1-|z|^2)^2} \\
&\quad \times \sup_{z \in D(a_n, r)} \left| \left( \log \frac{1}{1-\bar{a}z} \right)' \right|^2 |f(z)|^2 (1-|z|^2)^2 \\
&\leq 2CK \int_{D(a_n, r)} \frac{dA(z)}{(1-|z|^2)^2} \|I_f(h_n - h)\|_B^2 \\
&\quad + 2CK \int_{D(a_n, r)} \frac{dA(z)}{(1-|z|^2)^2} \\
&\quad \times \sup_{z \in D(a_n, r)} \left| \log \left( \frac{1}{1-\bar{a}z} \right)' \right|^2 |f(z)|^2 (1-|z|^2)^2 \\
&=: M_1 + M_2.
\end{aligned}$$

By the compactness of  $I_f$ , we have  $M_1 \rightarrow 0$  ( $n \rightarrow \infty$ ). Since  $B_0$  is a subspace of  $B$  and they share the same norm, the compactness of  $I_f$  on  $B$  implies its compactness on  $B_0$ . Hence we see that  $I_f$  is weakly compact on  $B_0$ . Since  $(B_0)^{**} = B$  and  $I_f^{**} = I_f$ , by using the fact of the introduction, we have  $I_f(B) \subset B_0$ . Thus we have  $I_f(h) \in B_0$ . Thus we have

$$\begin{aligned}
M_2 &= \sup_{z \in D(a_n, r)} \left| \left( \log \frac{1}{1-\bar{a}z} \right)' \right|^2 |f(z)|^2 (1-|z|^2)^2 \\
&= \sup_{z \in D(a_n, r)} \left( (1-|z|^2) |(I_f(h))'(z)| \right)^2 \\
&= \sup_{z \in D} \left( \chi_{D(a_n, r)}(z) (1-|z|^2) |(I_f(h))'(z)| \right)^2.
\end{aligned}$$

Hence we have  $M_2 \rightarrow 0$  ( $n \rightarrow +\infty$ ). So we have  $\lim_{|a_n| \rightarrow 1^-} |f(a_n)| = 0$ . Since  $f \in H^\infty$ , thus we see  $f \equiv 0$ . The proof of the converse is trivial.

Let  $\alpha > 1$ . Let  $f$  be an analytic function on  $D$ . Since  $|h(z)| \leq C \|h\|_{B^\alpha} (1-|z|^2)^{1-\alpha}$  for  $h \in B^\alpha$ , the unit ball of  $B^\alpha$  is a normal family of analytic functions. By normal family arguments,  $I_f$  is a compact operator on  $B^\alpha$  if and only if every sequence  $\{h_n\}$  in  $B^\alpha$  with  $\|h_n\|_{B^\alpha} \leq 1$  and  $h_n \rightarrow 0$  ( $n \rightarrow +\infty$ ) uniformly on compact subsets of  $D$  has a subse-

quence  $\{h_{n_k}\}$  in  $B^\alpha$  such that  $\|I_f h_{n_k}\|_{B^\alpha} \rightarrow 0$  ( $k \rightarrow +\infty$ ).

We show that every sequence which goes to zero has a subsequence such that the condition (\*) holds when the limit is taken over that sequence. This implies that the condition (\*) holds.

Suppose that the operator  $I_f$  is compact on  $B^\alpha$ . Let  $a_n \rightarrow a \in \partial D$  and consider the test functions  $h_{a_n}(z) := (1 - \overline{a_n}z)^{1-\alpha}$ ,  $h_a(z) := (1 - \overline{a}z)^{1-\alpha}$ . Then  $h_{a_n}(z) \rightarrow h_a(z)$  uniformly on compact subsets of  $D$ . Then we have by using  $\sup_{a \in D} \int_{D(a,r)} dA(z)/(1 - |z|^2)^2 < \infty$

$$\begin{aligned}
 & |a_n|^2 |f(a_n)|^2 \\
 & \leq C \int_{D(a_n,r)} \frac{dA(z)}{(1 - |z|^2)^2} \sup_{z \in D(a_n,r)} (1 - |z|^2)^{2\alpha} |h'_{a_n}(z)|^2 |f(z)|^2 \\
 & \leq 2C \int_{D(a_n,r)} \frac{dA(z)}{(1 - |z|^2)^2} \\
 & \quad \times \sup_{z \in D(a_n,r)} (1 - |z|^2)^{2\alpha} |h'_{a_n}(z) - h'_a(z)|^2 |f(z)|^2 \\
 & \quad + 2C \int_{D(a_n,r)} \frac{dA(z)}{(1 - |z|^2)^2} \sup_{z \in D(a_n,r)} (1 - |z|^2)^{2\alpha} |h'_a(z)|^2 |f(z)|^2 \\
 & \leq 2C \int_{D(a_n,r)} \frac{dA(z)}{(1 - |z|^2)^2} \|I_f(h_{a_n} - h_a)\|_{B^\alpha}^2 \\
 & \quad + 2C \int_{D(a_n,r)} \frac{dA(z)}{(1 - |z|^2)^2} \sup_{z \in D(a_n,r)} (1 - |z|^2)^{2\alpha} |h'_a(z)|^2 |f(z)|^2 \\
 & =: N_1 + N_2.
 \end{aligned}$$

By the compactness of  $I_f$ , we have  $N_1 \rightarrow 0$  ( $n \rightarrow \infty$ ). Since  $B_0^\alpha$  is a subspace of  $B^\alpha$  and they share the same norm, the compactness of  $I_f$  on  $B^\alpha$  implies its compactness on  $B_0^\alpha$ . Hence we see that  $I_f$  is weakly compact on  $B_0^\alpha$ . Since  $(B_0^\alpha)^{**} = B^\alpha$  (see [12]) and  $I_f^{**} = I_f$ , by using the fact of the introduction, we have  $I_f(B^\alpha) \subset B_0^\alpha$ . Thus we have  $I_f(h_a) \in B_0^\alpha$ . Thus we have

$$\begin{aligned}
 N_2 & = \sup_{z \in D(a_n,r)} (1 - |z|^2)^{2\alpha} |h'_a(z)|^2 |f(z)|^2 \\
 & = \sup_{z \in D(a_n,r)} \left( (1 - |z|^2)^\alpha |(I_f(h_a))'(z)| \right)^2 \\
 & = \sup_{z \in D} \left( \chi_{D(a_n,r)}(z) (1 - |z|^2)^\alpha |(I_f(h_a))'(z)| \right)^2.
 \end{aligned}$$

Hence we have  $N_2 \rightarrow 0$  ( $n \rightarrow +\infty$ ). So we have  $\lim_{|a_n| \rightarrow 1^-} |f(a_n)| = 0$ . Since  $f \in H^\infty$ , thus we see  $f \equiv 0$ . The proof of the converse is trivial.  $\square$

### 3. The operators $I_f$ defined on the weighted Bloch space

In this section, we study the boundedness of  $I_f$  on the weighted Bloch space  $B_\omega$ . And we will give the relationship between multiplication operator  $M_g$  and the operators  $I_g, J_g$  defined on the weighted Bloch space  $B_\omega$ . The examples of the weighted Bloch space  $B_\omega$  are the  $\alpha$ -Bloch space and  $\{f \in H(D) : \sup_{z \in D} (1 - |z|^2)(\log(1/(1 - |z|^2)))|f'(z)| < \infty\}$ , and so on.

**Theorem 3.1** *Let  $0 < r < +\infty$ . Let  $\omega$  be analytic on  $D$  and non-vanishing on  $\{\zeta : |1 - \zeta| < 1\}$ . Suppose that  $\sup_{z, a \in D} |\omega(1 - |z|^2)| / |\omega(1 - \bar{a}z)| < \infty$ , and that for any  $a \in D$  there is a constant  $C > 0$  (independent of  $a$ ) such that  $|\omega(1 - \bar{a}z)| / |\omega(1 - |z|^2)| \leq C$  for all  $z \in D(a, r)$ . Let  $f$  be an analytic function on  $D$ . Then the operator  $I_f$  is bounded on  $B_\omega$  if and only if*

$$\sup_{z \in D} |f(z)| < +\infty.$$

*Proof.* Let  $f$  be an analytic function on  $D$ . Suppose that  $\|f\|_\infty = \sup_{z \in D} |f(z)| < +\infty$ . Then

$$\|I_f(g)\|_{B_\omega} = \sup_{z \in D} |\omega(1 - |z|^2)| |g'(z)f(z)| \leq \|f\|_\infty \|g\|_{B_\omega}.$$

Hence we see that  $I_f$  is bounded on  $B_\omega$ .

Next, we prove the converse. Suppose that  $I_f$  is bounded on  $B_\omega$ . Put  $h_a(z) := \int_0^z 1/\omega(1 - \bar{a}\eta) d\eta$ . Then it is clear that  $h_a \in B_\omega$  for all  $a \in D$  because of the assumption  $\sup_{z, a \in D} |\omega(1 - |z|^2)| / |\omega(1 - \bar{a}z)| < +\infty$ . Since

$$\sup_{a \in D} \int_{D(a, r)} \frac{dA(z)}{(1 - |z|^2)^2} < +\infty \quad \text{and} \quad \frac{1}{|\omega(1 - |z|^2)|} \leq \frac{C}{|\omega(1 - \bar{a}z)|}$$

for all  $z \in D(a, r)$ , for any  $a \in D$

$$\begin{aligned} & |f(a)|^2 \\ & \leq K \int_{D(a, r)} \frac{1}{|\omega(1 - |z|^2)|^2} |\omega(1 - |z|^2)|^2 |f(z)|^2 \frac{dA(z)}{(1 - |z|^2)^2} \\ & \leq KC^2 \int_{D(a, r)} \frac{1}{|\omega(1 - \bar{a}z)|^2} |\omega(1 - |z|^2)|^2 |f(z)|^2 \frac{dA(z)}{(1 - |z|^2)^2} \end{aligned}$$

$$\begin{aligned}
 &= KC^2 \int_{D(a,r)} |(h_a(z))'|^2 |\omega(1 - |z|^2)|^2 |f(z)|^2 \frac{dA(z)}{(1 - |z|^2)^2} \\
 &\leq KC^2 \int_{D(a,r)} \frac{dA(z)}{(1 - |z|^2)^2} \sup_{z \in D(a,r)} |(h_a(z))'|^2 |\omega(1 - |z|^2)|^2 |f(z)|^2 \\
 &\leq KC^2 \int_{D(a,r)} \frac{dA(z)}{(1 - |z|^2)^2} \|I_f h_a\|_{B_\omega}^2 \\
 &\leq KC^2 \int_{D(a,r)} \frac{dA(z)}{(1 - |z|^2)^2} \|I_f\|^2 \|h_a\|_{B_\omega}^2 \\
 &\leq KC^2 \int_{D(a,r)} \frac{dA(z)}{(1 - |z|^2)^2} \|I_f\|^2 \sup_{a \in D} \|h_a\|_{B_\omega}^2 < +\infty.
 \end{aligned}$$

Hence we see  $\sup_{z \in D} |f(z)| < +\infty$ . □

We proved the following proposition in [9].

**Proposition 3.2** *Let  $0 < r < +\infty$ . Let  $\omega$  be analytic and non-vanishing on  $\{\zeta : |1 - \zeta| < 1\}$ . Suppose that  $\sup_{z \in D} |\omega(1 - |z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1 - s^2)|} < +\infty$  and  $\sup_{z, a \in D} \frac{|\omega(1 - |z|^2)|}{|\omega(1 - \bar{a}z)|} < +\infty$ , and  $\int_0^{|z|} \frac{ds}{|\omega(1 - s^2)|} < +\infty$  for any  $z \in D$  and  $\int_0^{|z|} \frac{ds}{|\omega(1 - s^2)|} \rightarrow \infty$  ( $|z| \rightarrow 1^-$ ), and that for any  $a \in D$  there is a constant  $C > 0$  (independent of  $a$ ) such that  $|\frac{\omega(1 - \bar{a}z)}{\omega(1 - |z|^2)}| \leq C$  for all  $z \in D(a, r)$ , and that there is a constant  $K > 0$  (independent of  $z$ ) such that  $\int_0^{|z|} \frac{ds}{|\omega(1 - s^2)|} \leq K |\int_0^z \frac{1}{\omega(1 - \bar{z}\eta)} d\eta|$  for all  $z \in D$ . Then the operator  $J_g$  is bounded on  $B_\omega$  if and only if*

$$\|g\|_W := \sup_{z \in D} |\omega(1 - |z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1 - s^2)|} |g'(z)| < +\infty.$$

*Proof.* See [9]. □

**Corollary 3.3** *Let  $0 < r < +\infty$ . Let  $\omega$  be as Proposition 3.2. Then for  $g$  analytic on  $D$ , the following are equivalent:*

- (i)  $gB_\omega \subset B_\omega$ ;
- (ii) Both  $I_g$  and  $J_g$  are bounded operators on  $B_\omega$ ;
- (iii)  $g \in H^\infty$ ,  $\sup_{z \in D} |\omega(1 - |z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1 - s^2)|} |g'(z)| < +\infty$ .

*Proof.* The equivalence of (ii) and (iii) is an immediate consequence of Theorem 3.1 and Proposition 3.2. If (ii) holds, it is trivial that (i) holds.

So it suffices to prove that (i) implies (ii). In fact, suppose that  $gB_\omega \subset B_\omega$ . For arbitrary  $a \in D$ , let  $h_a(z) := \int_0^z \frac{1}{\omega(1-\bar{a}\eta)} d\eta$ . And let  $k_a(z) := h_a(z) - h_a(a)$ . Then we see that  $k'_a(z) = \frac{1}{\omega(1-\bar{a}z)}$ ,  $k_a(a) = 0$  and  $k_a \in B_\omega$  because of  $\sup_{z, a \in D} \frac{|\omega(1-|z|^2)|}{|\omega(1-\bar{a}z)|} < +\infty$ . So for  $g$  analytic on  $D$ , we have  $(gk_a)'(a) = g(a) \frac{1}{\omega(1-|a|^2)}$ . By using the boundedness of  $M_g$  on  $B_\omega$  and the subharmonic property of  $|\omega(1-\bar{a}z)| |(gk_a)'(z)|$  and the assumption that for any  $a \in D$  there is a constant  $C > 0$  (independent of  $a$ ) such that  $\frac{|\omega(1-\bar{a}z)|}{|\omega(1-|z|^2)|} \leq C$  for all  $z \in D(a, r)$ ,

$$\begin{aligned} |g(a)| &= |\omega(1-|a|^2)| |(gk_a)'(a)| \\ &\leq K \int_{D(a,r)} |\omega(1-\bar{a}z)| |(gk_a)'(z)| \frac{dA(z)}{(1-|z|^2)^2} \\ &\leq KC \int_{D(a,r)} |\omega(1-|z|^2)| |(gk_a)'(z)| \frac{dA(z)}{(1-|z|^2)^2} \\ &\leq KC \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \|M_g k_a\|_{B_\omega} \\ &\leq KC \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \|M_g\| \|k_a\|_{B_\omega} \\ &\leq KC \sup_{a \in D} \int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2} \|M_g\| \sup_{a \in D} \|k_a\|_{B_\omega} < +\infty. \end{aligned}$$

Hence we have  $g \in H^\infty$ . By Theorem 3.1,  $g \in H^\infty$  if and only if  $I_g$  is bounded on  $B_\omega$ . So by the boundedness of  $I_g$  and  $M_g$  on  $B_\omega$ , we see that  $J_g$  is bounded on  $B_\omega$ .  $\square$

**Remark** Carefully examining the proof of the above corollary, we see that the equivalence of (i) and (ii) can be proved by using the assumption of  $\omega$  in Theorem 3.1 only.  $\square$

Based on Corollary 3.3, we furthermore make the following conjecture.

**Conjecture** Let  $0 < r < +\infty$ . Let  $\omega$  be as Proposition 3.2. And suppose that  $\frac{|\omega(1-|z|^2)|}{1-|z|^2} \rightarrow 0$  ( $|z| \rightarrow 1^-$ ). Then for  $g$  analytic on  $D$ , the following are equivalent:

- (i)  $gB_\omega \subset B_\omega$ ;
- (ii)  $I_g$  is a bounded operator on  $B_\omega$ ;
- (iii)  $g \in H^\infty$ .

Supposing the assumption that there is a constant  $C > 0$  (independent of  $z$ ) such that  $|\omega(1 - |z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \leq C(1 - |z|^2)$  for all  $z \in D$  which is stronger than the assumption  $\frac{|\omega(1-|z|^2)|}{1-|z|^2} \rightarrow 0$  ( $|z| \rightarrow 1^-$ ), we see that the above conjecture holds.

**4. The product of the operators  $I_f$  and  $J_g$  defined on the  $\alpha$ -Bloch space**

In this section, we study the boundedness and compactness of products of  $I_f$  and  $J_g$  defined on the  $\alpha$ -Bloch space.

**Theorem 4.1** *Let  $f$  be an analytic function on  $D$  and  $g$  be an analytic function on  $D$ . Suppose that  $\sup_{z \in D} |f(z)| < +\infty$ . Then the operator  $I_f J_g$  is bounded on  $B$  if and only if*

$$\sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| |f(z)| < +\infty.$$

Let  $\alpha > 1$ . Then the operator  $I_f J_g$  is bounded on  $B^\alpha$  if and only if

$$\sup_{z \in D} (1 - |z|^2) |g'(z)| |f(z)| < +\infty.$$

*Proof.* Let  $\alpha = 1$ . Let  $f$  be a function on  $D$ . First, suppose that

$$\sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| |f(z)| < +\infty.$$

Then we see

$$I_f J_g h(z) = \int_0^z h(\zeta) g'(\zeta) f(\zeta) d\zeta.$$

Since  $|h(z)| \leq C \|h\|_B \log \frac{1}{1-|z|^2}$  for  $h \in B$ , we have

$$\begin{aligned} & (1 - |z|^2) |(I_f J_g h)'(z)| \\ &= (1 - |z|^2) |h(z) g'(z) f(z)| \\ &\leq \sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| |f(z)| \|h\|_B. \end{aligned}$$

Hence

$$\|I_f J_g h\|_B \leq \sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| |f(z)| \|h\|_B.$$

To prove the converse, we suppose that  $I_f J_g$  is bounded on  $B$ . For  $a \in D$ , it is clear that the test function  $h(z) := \log(1 - \bar{a}z) \in B$ . Since  $I_f J_g$  is bounded on  $B$ , for any  $z \in D$

$$\begin{aligned} & (1 - |z|^2) |\log(1 - \bar{a}z)| |g'(z)| |f(z)| \\ &= (1 - |z|^2) |h(z)| |g'(z)| |f(z)| \\ &\leq \|I_f J_g h\|_B \leq \|I_f J_g\| \|h\|_B \leq K < +\infty. \end{aligned}$$

Applying  $z = a$ , we have

$$(1 - |a|^2) \left( \log \frac{1}{1 - |a|^2} \right) |g'(a)| |f(a)| \leq K,$$

for any  $a \in D$ . Hence we have

$$\sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| |f(z)| < +\infty.$$

In the case of  $\alpha > 1$ , we can prove it as well. So we omit it.  $\square$

**Theorem 4.2** *Let  $f$  be function on  $D$  and  $g$  be an analytic function on  $D$ . Suppose that  $\sup_{z \in D} |f(z)| < +\infty$ . Then the operator  $I_f J_g$  is compact on  $B$  if and only if*

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| |f(z)| = 0.$$

*Let  $\alpha > 1$ . Then the operator  $I_f J_g$  is compact on  $B^\alpha$  if and only if*

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) |g'(z)| |f(z)| = 0.$$

*Proof.* Let  $\alpha = 1$ . Then we have

$$\begin{aligned} \|I_f J_g h\|_B &\leq \sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| |f(z)| \|h\|_B \\ &\leq \sup_{z \in D} |f(z)| \sup_{z \in D} (1 - |z|^2) \left( \log \frac{1}{1 - |z|^2} \right) |g'(z)| \|h\|_B. \end{aligned}$$

By using the fact that  $A_1 A_2$  is compact operator on  $B$  for any bounded operator  $A_1$  on  $B$  and compact operator  $A_2$  on  $B$  and the fact  $(I_f J_g)^{**} = I_f J_g$ , we can prove this theorem as well as the proof of Theorem 2.3. In the case of  $\alpha > 1$ , we can prove it as well.  $\square$

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