On Zame's Example of a Production Set in an Infinite Dimensional Commodity Space

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1. Introduction

In their recent paper, Khan-Peck [1989] proved a theorem which could be quite useful in checking the interiority of a production set in an infinite dimensional commodity space. The theorem says that in an ordered Banach space, order boundedness below of supporting functionals for a closed convex set containing the negative orthant implies that the set has a non-empty interior. To illustrate the usefulness of the result, Khan-Peck [1989] also studied a production set discussed in the fourth example of Zame [1987]. According to their claim, the production set has a non-empty interior by the application of their theorem. If this is correct, there is an open problem in Zame [1987]. For the example of Zame [1987] is used to illustrate the usefulness of his equilibrium existence theorem for production economies where production sets may have empty interiors. The claim by Khan-Peck [1989] is important since it requires reexamination of the usefulness of Zame's existence theorem.

We must point out, however, that the production set discussed

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by Khan-Peck [1989] is somewhat different from Zame's. Furthermore, it is impossible to apply the Khan-Peck theorem to the original production set. Thus we must study Zame's example itself. Our claim is contrary to that of Khan-Peck : Zame's production set has an empty interior ! In the next section, we will provide the proof.

2. Zame's Example

Khan-Peck [1989] studied the following production set.

$$Y = \{ y \in \ell^1 \mid y(T+1) < \Sigma_{t=1}^{T+1} y(t)\gamma^{T-t} \text{for all } T \},$$
$$\text{where} \quad 0 < \gamma < 1.$$

Of course, ℓ^1 denotes the set of summable sequences. This set seems somewhat different from Zame's original production set. Here, we have corrected some typos in Khan-Peck [1989]. For $y \in \ell^1$, $\| y \|$ denotes its ℓ^1-norm. The topological dual $(\ell^1)^*$ of the space 1^1 can be identified with the set ℓ^∞ of bounded sequences.

. For $f \in (\ell^1)^*$, $\| f \|$ denotes its dual norm. For $f \in (\ell^1)^*$ and for $C \subset \ell^1$, let $a(f, C) = \sup\{ f(x) x \in C \}$. If ℓ^1 is interpreted as the commodity space and ℓ^1 as the price space, the measure a(f, C) could be useful in expressing bounds for marginal rates of substitution. Khan-Peck then claim the following.

Claim. Let $f \in \ell^\infty$, $f > 0$, $\| f \| = 1$ and $a(f, Y) < \infty$. Then $f_1 = 1$.

Certainly this implies the condition (ii) in Theorem 1 in Khan-Peck [1989]. But the condition (i), $-\ell^1_+ \subset Y$, is not satisfied. Indeed, we have that $(-1, 0, 0, \ldots) \subset Y$. Therefore, Theorem 1 is not applicable.

Now, let us go back to the original example of Zame.

$$Y = \{y \in l^1 : \Sigma_{t=1}^{T} y(t)\gamma^{T-t} \leqq 0 \text{ for all } T \geqq 1\},$$
$$\text{where} \quad 0 < \gamma < 1.$$

The number γ is interpreted as the depreciation factor for the single good. Thus the production set expresses a simple storage activity of the single good which depreciates at a constant rate.

Our main goal is to show that the norm interior of the production set Y is empty, which is contrary to the assertion of Khan-Peck [1989].

Suppose that there is a norm interior point e of Y. Then, there is $\varepsilon > 0$ such that $\| y-e \|_1 < \varepsilon$ implies $y \in Y$. Let $\delta > 0$ be such that $0 < \delta < \varepsilon(1-\gamma)/4$. Since $e \in l^1$, there is T_o such that $t \geqq T_o$ implies $e(t) > -\delta$. Let $A_T(y) = \Sigma_{t=1}^{T} y(t)\gamma^{T-t}$ for any $y \in l^1$ and T. Then, $y \in Y$ can be written as $A_T(y) \leqq 0$ for all T.

Take T_1 such that $\gamma^{T_1} A_{T_1}(e) > -\delta(1-r)$, which is possible by $0 < \gamma < 1$

Now, I have the following inequality.

$$A_{T_0+T_1}(e) = \Sigma_{t=1}^{T_0+T_1} e(t)\gamma^{T_0+T_1-t}$$

$$= \sum_{t=T_0+1}^{T_0+T_1} e(t)\gamma^{T_0+T_1-t} + \gamma^{T_1} A_{T_0}(e)$$

$$> -\delta(1-\gamma^{T_1})/(1-\gamma) - \delta(1-\gamma)$$
$$> -2\delta/(1-\gamma).$$

Define $y \in \ell^1$ by $y(t) = e(T_0+T_1) + \varepsilon/2$ for $t = T_0+T_1$ and $y(t) = e(t)$ for other values of t.

Then, $\|y-e\|_1 < \varepsilon$ so $y \in Y$ must hold.

But, it follows from the inequality above that

$$A_{T_0+T_1}(y) = A_{T_0+T_1}(e) + \varepsilon/2 > -2\delta(1-\gamma) + \varepsilon/2 > 0.$$

This is a contradiction. Therefore, the norm interior of Y is empty. This completes the proof.

Clearly, the depreciation rate $\gamma < 1$ plays a crucial role in the above argument. To put it another way, let us consider the case that the single good is completely durable, i.e. $\gamma = 1$. Then, it is easy to see that the production set Y has a non - empty norm interior. For example, $(-1, 0, 0, \dots) \in \text{int } Y$. If the good is only incompletely durable, then given stocks of the good will eventually vanish. Hence, arbitrarily small addition of output in the distant future makes the production plan technically infeasible.

References

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