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Expository supplement I to the paper "Asymptotic expansions for the estimators of Lagrange multipliers and associated parameters by the maximum likelihood and weighted score methods"

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This article gives the first half of an expository supplement to Ogasawara (2015).

1. Asymptotic expansions of the estimators with restrictions for model identification

In this section, the estimator $\hat{\boldsymbol{\theta}}_{\mathrm{W}}$ with restrictions for model identification is dealt with. Recall that in this case $\hat{\boldsymbol{\eta}}_{\mathrm{W}} = \mathbf{0}$. Setting $\hat{\boldsymbol{\eta}}_{\mathrm{W}} = \mathbf{0}$ in (A1.4), we have

$$\mathbf{0} = \begin{pmatrix} \dfrac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \\ \mathbf{0} \end{pmatrix}_{O_p(n^{-1/2})} + \begin{pmatrix} n^{-1}\mathbf{q}_0^* \\ \mathbf{0} \end{pmatrix}_{O(n^{-1})} + \left\{ \boldsymbol{\Lambda}_0^* \begin{pmatrix} \hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0 \\ \mathbf{0} \end{pmatrix} \right\}_{O_p(n^{-1/2})}$$

$$+ \left\{ \begin{pmatrix} \mathbf{L}_0 - \boldsymbol{\Lambda}_0 \\ \mathbf{O} \end{pmatrix} (\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0) \right\}_{O_p(n^{-1})} + \begin{Bmatrix} n^{-1} \dfrac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0'} (\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0) \\ \mathbf{0} \end{Bmatrix}_{O_p(n^{-3/2})} \quad \text{(S1.1)}$$

$$+ \left\{ \frac{1}{2} \begin{pmatrix} \dfrac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<2>}} \end{pmatrix} (\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)^{<2>} \right\}_{O_p(n^{-1})}$$

$$+\left\{\frac{1}{6}\begin{pmatrix}\dfrac{\partial^4 \bar{l}}{\partial\boldsymbol{\theta}_0(\partial\boldsymbol{\theta}_0')^{<3>}}\\ \dfrac{\partial^3\mathbf{h}_0}{(\partial\boldsymbol{\theta}_0')^{<3>}}\end{pmatrix}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<3>}\right\}_{O_p(n^{-3/2})} + O_p(n^{-2}).$$

From the above result,

$$\begin{aligned}
\begin{pmatrix}\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0\\ \mathbf{0}\end{pmatrix} = &-\left\{\boldsymbol{\Lambda}_0^{*-1}\begin{pmatrix}\dfrac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\\ \mathbf{0}\end{pmatrix}\right\}_{O_p(n^{-1/2})} -\left\{n^{-1}\boldsymbol{\Lambda}_0^{*-1}\begin{pmatrix}\mathbf{q}_0^*\\ \mathbf{0}\end{pmatrix}\right\}_{O(n^{-1})}\\
&-\left\{\boldsymbol{\Lambda}_0^{*-1}\begin{pmatrix}\mathbf{L}_0-\boldsymbol{\Lambda}_0\\ \mathbf{O}\end{pmatrix}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)\right\}_{O_p(n^{-1})}\\
&-\left\{n^{-1}\boldsymbol{\Lambda}_0^{*-1}\begin{pmatrix}\dfrac{\partial\mathbf{q}_0^*}{\partial\boldsymbol{\theta}_0'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)\\ \mathbf{0}\end{pmatrix}\right\}_{O_p(n^{-3/2})}\\
&-\left\{\frac{1}{2}\boldsymbol{\Lambda}_0^{*-1}\begin{pmatrix}\mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)})\\ \dfrac{\partial^2\mathbf{h}_0}{(\partial\boldsymbol{\theta}_0')^{<2>}}\end{pmatrix}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<2>}\right\}_{O_p(n^{-1})}\\
&-\left\{\frac{1}{2}\boldsymbol{\Lambda}_0^{*-1}\begin{pmatrix}\mathbf{J}_0^{(3)}-\mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)})\\ \mathbf{O}\end{pmatrix}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<2>}\right\}_{O_p(n^{-3/2})}\\
&-\left\{\frac{1}{6}\boldsymbol{\Lambda}_0^{*-1}\begin{pmatrix}\mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(4)})\\ \dfrac{\partial^3\mathbf{h}_0}{(\partial\boldsymbol{\theta}_0')^{<3>}}\end{pmatrix}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<3>}\right\}_{O_p(n^{-3/2})} + O_p(n^{-2}).
\end{aligned} \tag{S1.2}$$

Recalling $\mathbf{M}\equiv\mathbf{L}_0-\boldsymbol{\Lambda}_0$, Equation (S1.2) gives the following result:

$$
\begin{aligned}
\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0 = &-\left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)_{O_p(n^{-1/2})} - (n^{-1}\boldsymbol{\Lambda}_0^{(11)}\mathbf{q}_0^*)_{O(n^{-1})} \\
&+\left\{\boldsymbol{\Lambda}_0^{(11)}\mathbf{M}\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right\}_{O_p(n^{-1})} \\
&-\left\{\frac{1}{2}(\boldsymbol{\Lambda}_0^{(11)}\ \ \boldsymbol{\Lambda}_0^{(12)})\begin{pmatrix} \mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<2>}} \end{pmatrix}\left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)^{<2>}\right\}_{O_p(n^{-1})} \\
&+(n^{-1}\boldsymbol{\Lambda}_0^{(11)}\mathbf{M}\boldsymbol{\Lambda}_0^{(11)}\mathbf{q}_0^*)_{O_p(n^{-3/2})} - \left(\boldsymbol{\Lambda}_0^{(11)}\mathbf{M}\boldsymbol{\Lambda}_0^{(11)}\mathbf{M}\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)_{O_p(n^{-3/2})} \\
&+\left\{\frac{1}{2}\boldsymbol{\Lambda}_0^{(11)}\mathbf{M}(\boldsymbol{\Lambda}_0^{(11)}\ \ \boldsymbol{\Lambda}_0^{(12)})\begin{pmatrix} \mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<2>}} \end{pmatrix}\left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)^{<2>}\right\}_{O_p(n^{-3/2})} \\
&+\left(n^{-1}\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0 '}\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)_{O_p(n^{-3/2})} \\
&-\Bigg[_{(\mathrm{A})}\ (\boldsymbol{\Lambda}_0^{(11)}\boldsymbol{\Lambda}_0^{(12)})\begin{pmatrix} \mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<2>}} \end{pmatrix}\Bigg[_{(\mathrm{B})}\left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right) \\
&\quad \otimes\left\{n^{-1}\boldsymbol{\Lambda}_0^{(11)}\mathbf{q}_0^* - \boldsymbol{\Lambda}_0^{(11)}\mathbf{M}\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right. \\
&\quad \left. +\frac{1}{2}(\boldsymbol{\Lambda}_0^{(11)}\boldsymbol{\Lambda}_0^{(12)})\begin{pmatrix} \mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<2>}} \end{pmatrix}\left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)^{<2>}\right\}\Bigg]_{(\mathrm{B})}\Bigg]_{(\mathrm{A})O_p(n^{-3/2})}
\end{aligned}
\qquad \text{(S1.3)}
$$

$$-\left\{\frac{1}{2}\boldsymbol{\Lambda}_0^{(11)}(\mathbf{J}_0^{(3)}-\mathrm{E}_\mathrm{T}(\mathbf{J}_0^{(3)}))\left(\boldsymbol{\Lambda}_0^{(11)}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right\}_{O_p(n^{-3/2})}$$

$$+\left\{\frac{1}{6}(\boldsymbol{\Lambda}_0^{(11)}\ \boldsymbol{\Lambda}_0^{(12)})\begin{pmatrix}\mathrm{E}_\mathrm{T}(\mathbf{J}_0^{(4)})\\ \dfrac{\partial^3\mathbf{h}_0}{(\partial\boldsymbol{\theta}_0{}')^{<3>}}\end{pmatrix}\left(\boldsymbol{\Lambda}_0^{(11)}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\right)^{<3>}\right\}_{O_p(n^{-3/2})}+O_p(n^{-2})$$

$$\equiv\sum_{i=1}^{3}\boldsymbol{\Lambda}_\mathrm{W}^{(i)}\mathbf{l}_0^{(i)}-n^{-1}\boldsymbol{\Lambda}_0^{(11)}\mathbf{q}_0^{*}+O_p(n^{-2})$$

$$(\boldsymbol{\Lambda}_\mathrm{W}^{(i)}=O(1),\ \mathbf{l}_0^{(i)}=O_p(n^{-i/2}),\ \ i=1,2,3).$$

In (S1.3),

$$\boldsymbol{\Lambda}_\mathrm{W}^{(1)}\mathbf{l}_0^{(1)}=\boldsymbol{\Lambda}_\mathrm{ML}^{(1)}\mathbf{l}_0^{(1)}=-\boldsymbol{\Lambda}_0^{(11)}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0} \tag{S1.4}$$

where $\boldsymbol{\Lambda}_\mathrm{W}^{(1)}=\boldsymbol{\Lambda}_\mathrm{ML}^{(1)}=-\boldsymbol{\Lambda}_0^{(11)}$, $\mathbf{l}_0^{(1)}=\partial\bar{l}/\partial\boldsymbol{\theta}_0$,

$$\boldsymbol{\Lambda}_\mathrm{W}^{(2)}\mathbf{l}_0^{(2)}=\boldsymbol{\Lambda}_\mathrm{ML}^{(2)}\mathbf{l}_0^{(2)}$$

$$=\boldsymbol{\Lambda}_0^{(11)}\mathbf{M}\boldsymbol{\Lambda}_0^{(11)}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}-\frac{1}{2}(\boldsymbol{\Lambda}_0^{(11)}\boldsymbol{\Lambda}_0^{(12)})\begin{pmatrix}\mathrm{E}_\mathrm{T}(\mathbf{J}_0^{(3)})\\ \dfrac{\partial^2\mathbf{h}_0}{(\partial\boldsymbol{\theta}_0{}')^{<2>}}\end{pmatrix}\left(\boldsymbol{\Lambda}_0^{(11)}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>} \tag{S1.5}$$

$$=(\boldsymbol{\Lambda}_\mathrm{ML}^{(2-1)}\ \boldsymbol{\Lambda}_\mathrm{ML}^{(2-2)})\left\{\mathrm{v}(\mathbf{M})'\otimes\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'},\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)^{<2>}\right\}',$$

where $\boldsymbol{\Lambda}_\mathrm{ML}^{(2)}=(\boldsymbol{\Lambda}_\mathrm{ML}^{(2-1)}\ \boldsymbol{\Lambda}_\mathrm{ML}^{(2-2)})$ with $\boldsymbol{\Lambda}_\mathrm{ML}^{(2-1)}$ and $\boldsymbol{\Lambda}_\mathrm{ML}^{(2-2)}$ being $q\times\{q^2(q+1)/2\}$ and $q\times q^2$ submatrices, respectively,

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(2-1)})_{\bullet(ijk^*)} = \sum\nolimits_{(ij)}^{2} (\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} \frac{2-\delta_{ij}}{2} (\boldsymbol{\Lambda}_0^{(11)})_{jk^*}$$

$$(1 \le i \le j \le q; k^* = 1,\ldots,q),$$

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(2-2)})_{\bullet(ij)} = -\frac{1}{2}(\boldsymbol{\Lambda}_0^{(11)}\ \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<2>}} \end{pmatrix} \{(\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} \otimes (\boldsymbol{\Lambda}_0^{(11)})_{\bullet j}\}$$

$$(i, j = 1,\ldots,q),$$

$$\boldsymbol{\Lambda}_{\mathrm{W}}^{(3)} \mathbf{l}_0^{(3)} = \boldsymbol{\Lambda}_{\mathrm{ML}}^{(3)} \mathbf{l}_0^{(3)} + \boldsymbol{\Lambda}_{\mathrm{W}}^{(\Delta)} n^{-1} \mathbf{l}_0^{(\mathrm{W})}, \tag{S1.6}$$

$$\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3)} \mathbf{l}_0^{(3)} = (\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-1)}\ \boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-2)}\ \boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-3)}\ \boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-4)})$$

$$\times \left\{ \mathrm{v}(\mathbf{M})'^{<2>} \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}, \mathrm{v}(\mathbf{M})' \otimes \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}\right)^{<2>}, \right.$$

$$\left. \mathrm{vec}\{(\mathbf{J}_0^{(3)} - \mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)})\}' \otimes \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}\right)^{<2>}, \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}\right)^{<3>} \right\}',$$

$$\boldsymbol{\Lambda}_{\mathrm{W}}^{(\Delta)} n^{-1} \mathbf{l}_0^{(\mathrm{W})} = (\boldsymbol{\Lambda}_{\mathrm{W}}^{(\Delta-1)}\ \boldsymbol{\Lambda}_{\mathrm{W}}^{(\Delta-2)}) n^{-1} \left\{ \mathrm{v}(\mathbf{M})', \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right\}',$$

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-1)})_{\bullet(ijk^*l^*m)} = -\sum_{(ij)}^{2} \sum_{(k^*l^*)}^{2} (\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} \frac{2-\delta_{ij}}{2} (\boldsymbol{\Lambda}_0^{(11)})_{jk^*} \frac{2-\delta_{k^*l^*}}{2} (\boldsymbol{\Lambda}_0^{(11)})_{l^*m}$$

$$(1 \le i \le j \le q; 1 \le k^* \le l^* \le q; m = 1,\ldots,q),$$

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-2)})_{\bullet(ijk^*l^*)} = \sum_{(ij)}^{2} \frac{1}{2} (\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} \frac{2-\delta_{ij}}{2}$$

$$\times \left\{ (\boldsymbol{\Lambda}_0^{(11)} \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathrm{E_T}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<2>}} \end{pmatrix} \{ (\boldsymbol{\Lambda}_0^{(11)})_{\bullet k^*} \otimes (\boldsymbol{\Lambda}_0^{(11)})_{\bullet l^*} \} \right\}_j$$

$$+ (\boldsymbol{\Lambda}_0^{(11)} \ \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathrm{E_T}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<2>}} \end{pmatrix} \left[(\boldsymbol{\Lambda}_0^{(11)})_{\bullet k^*} \otimes \sum_{(ij)}^{2} \left\{ (\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} \frac{2-\delta_{ij}}{2} (\boldsymbol{\Lambda}_0^{(11)})_{jl^*} \right\} \right]$$

$(1 \le i \le j \le q; k^*, l^* = 1, \ldots, q)$,

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-3)})_{\bullet(ijk^*l^*m)} = -\frac{1}{2} (\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} (\boldsymbol{\Lambda}_0^{(11)})_{jl^*} (\boldsymbol{\Lambda}_0^{(11)})_{k^*m}$$

$(i, j, k^*, l^*, m = 1, \ldots, q)$,

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-4)})_{\bullet(ijk^*)} = -\frac{1}{2} (\boldsymbol{\Lambda}_0^{(11)} \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathrm{E_T}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<2>}} \end{pmatrix} \Big[_{(\mathrm{A})} \ (\boldsymbol{\Lambda}_0^{(11)})_{\bullet i}$$

$$\otimes \Big\{_{(\mathrm{B})} \ (\boldsymbol{\Lambda}_0^{(11)} \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathrm{E_T}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<2>}} \end{pmatrix} \{ (\boldsymbol{\Lambda}_0^{(11)})_{\bullet j} \otimes (\boldsymbol{\Lambda}_0^{(11)})_{\bullet k} \} \ \Big\}_{(\mathrm{B})} \Big]_{(\mathrm{A})}$$

$$+ \frac{1}{6} (\boldsymbol{\Lambda}_0^{(11)} \ \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathrm{E_T}(\mathbf{J}_0^{(4)}) \\ \dfrac{\partial^3 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0 ')^{<3>}} \end{pmatrix} \{ (\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} \otimes (\boldsymbol{\Lambda}_0^{(11)})_{\bullet j} \otimes (\boldsymbol{\Lambda}_0^{(11)})_{\bullet k^*} \}$$

$(i, j, k^* = 1, \ldots, q)$,

$$(\mathbf{\Lambda}_{\mathrm{W}}^{(\Delta-1)})_{\bullet(ij)} = \sum_{(ij)}^{2} (\mathbf{\Lambda}_0^{(11)})_{\bullet i} \frac{2-\delta_{ij}}{2} (\mathbf{\Lambda}_0^{(11)} \mathbf{q}_0^*)_j$$

$(1 \le i \le j \le q)$,

$$(\mathbf{\Lambda}_{\mathrm{W}}^{(\Delta-2)})_{\bullet i} = \mathbf{\Lambda}_0^{(11)} \frac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0{}'} (\mathbf{\Lambda}_0^{(11)})_{\bullet i}$$

$$-(\mathbf{\Lambda}_0^{(11)}\ \mathbf{\Lambda}_0^{(12)}) \begin{pmatrix} \mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)}) \\ \dfrac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0{}')^{<2>}} \end{pmatrix} \{(\mathbf{\Lambda}_0^{(11)})_{\bullet i} \otimes (\mathbf{\Lambda}_0^{(11)} \mathbf{q}_0^*)\}$$

$(i = 1, \ldots, q)$.

Equation (S1.3) is alternatively written as

$$\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0 = \sum_{i=1}^{3} \mathbf{\Lambda}_{\mathrm{ML}}^{(i)} \mathbf{l}_0^{(i)} + \mathbf{\Lambda}_{\mathrm{W}}^{(\Delta)} n^{-1} \mathbf{l}_0^{(\mathrm{w})} - n^{-1} \mathbf{\Lambda}_0^{(11)} \mathbf{q}_0^* + O_p(n^{-2})$$
$$= \hat{\boldsymbol{\theta}}_{\mathrm{ML}} - \boldsymbol{\theta}_0 + \mathbf{\Lambda}_{\mathrm{W}}^{(\Delta)} n^{-1} \mathbf{l}_0^{(\mathrm{W})} - n^{-1} \mathbf{\Lambda}_0^{(11)} \mathbf{q}_0^* + O_p(n^{-2}). \tag{S1.7}$$

2. Properties of augmented matrices

Define column-wise q blocks of a matrix $\mathbf{A}$ as $\mathbf{A}_{(\bullet j)}$ $(j = 1, \ldots, q)$ i.e., $\mathbf{A} = (\mathbf{A}_{(\bullet 1)}\ \mathbf{A}_{(\bullet 2)} \cdots \mathbf{A}_{(\bullet q)})$. Similarly, define row-wise p blocks of $\mathbf{A}$ as $\mathbf{A}_{(i\bullet)}$ $(i = 1, \ldots, p)$ i.e., $\mathbf{A} = (\mathbf{A}_{(1\bullet)}{}'\ \mathbf{A}_{(2\bullet)}{}' \cdots \mathbf{A}_{(p\bullet)}{}')'$. Let $\mathbf{B} = (\mathbf{B}_{(\bullet 1)}\ \mathbf{B}_{(\bullet 2)})$, where $\mathbf{B}_{(\bullet 1)}$ and $\mathbf{B}_{(\bullet 2)}$ are $b \times q$ and $b \times r$ submatrices, respectively; $q > r$, $q = r$ or $q < r$. Post-multiply $\mathbf{B}$ by $\mathbf{G} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}$. Then, **BG** becomes $(\mathbf{B}_{(\bullet 1)} + \mathbf{B}_{(\bullet 2)}\mathbf{C}' : \mathbf{B}_{(\bullet 2)})$ with the colon being used to show separation for clarity. Matrix **B** is restored from **BG** by subtracting $\mathbf{B}_{(\bullet 2)}\mathbf{C}'$ from the first column-wise block of **BG**, which is given by $\mathbf{G}^* = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ -\mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}$. Since $\mathbf{B}\mathbf{G}\mathbf{G}^* = \mathbf{B}$, $\mathbf{G}^* = \mathbf{G}^{-1}$. That is, we have elementary results:

Lemma S1. $\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ -\mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}$ *and*

$\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{C} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & -\mathbf{C} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix}$.

Let $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}$ be a non-singular $(q+r)\times(q+r)$ matrix, where the $q\times q$ diagonal block **B** may be asymmetric and singular.

Lemma S2. *The first column-wise and row-wise blocks of* $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ *are equal to those of* $\begin{pmatrix} \mathbf{B}+\mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$*, respectively. Equivalently, the two matrices are different only in their second diagonal blocks.*

Proof. Since $\left\{\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}\right\}^{-1} = \begin{pmatrix} \mathbf{B}+\mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$,

$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}\begin{pmatrix} \mathbf{B}+\mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$. Using Lemma S1,

$\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ -\mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B}+\mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$. Noting that pre-multiplication of a matrix by $\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ -\mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}$ does not change the first row-wise brock of the multiplied matrix, we find that the first row-wise blocks of the inverted matrices are equal. Similarly, from

$\left\{\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{D} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix}\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}\right\}^{-1} = \begin{pmatrix} \mathbf{B}+\mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ we have

$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B}+\mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{D} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix}$. Using Lemma S1,

$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I}_{(q)} & -\mathbf{D} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix} = \begin{pmatrix} \mathbf{B}+\mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$, which shows that the first column-wise blocks of the inverted matrices are equal. Q.E.D.

When $\mathbf{B}$ is non-singular and when $q \geq r$, $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ is straightforwardly obtained by the formula of the inverse of a partitioned square matrix

$\begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{FJGE}^{-1} & -\mathbf{E}^{-1}\mathbf{FJ} \\ -\mathbf{JGE}^{-1} & \mathbf{J} \end{pmatrix}$, where $\mathbf{J} = (\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1}$ with the assumption of the existence of the inverses, which gives

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{B}^{-1} & \mathbf{B}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{B}^{-1} & -(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1} \end{pmatrix}.$$

When $\mathbf{B}$ is possibly singular, and $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}$ with $q \geq r$ and $\mathbf{B}+\mathbf{DC}'$ are non-singular, we have

Lemma S3. *Let* $\mathbf{A} = \mathbf{B}+\mathbf{DC}'$. *Then,*

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \end{pmatrix}.$$

Proof. Using the result in the proof of Lemma S2, and the formula of the inverse of a partitioned matrix,

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix} \begin{pmatrix} \mathbf{B}+\mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & -(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \end{pmatrix}. \quad \text{Q.E.D.}$$

In the case of the augmented information matrix per observation $\mathbf{I}_0^*$ with $q > r$, we have

Theorem S1. *Let* $\mathbf{I}_0^* = \begin{pmatrix} \mathbf{I}_0 & -\mathbf{H}_0 \\ -\mathbf{H}_0' & \mathbf{O} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{I}_0 & \mathbf{C} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{I}_0^{(11)} & \mathbf{I}_0^{(12)} \\ \mathbf{I}_0^{(21)} & \mathbf{I}_0^{(22)} \end{pmatrix}^{-1}$ *, where* $\mathbf{I}_0$ *may be singular;* $\mathbf{A} = \mathbf{I}_0 + \mathbf{CC}'$ *and* $\mathbf{C}'\mathbf{A}^{-1}\mathbf{C}$ *are non-singular. Then,*

$$\mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1} \end{pmatrix}.$$

Proof. Use Lemma S3 with $\mathbf{B} = \mathbf{I}_0$ and $\mathbf{D} = \mathbf{C}$. Q.E.D.

It is known that the asymptotic covariance matrix of $(n^{1/2}\hat{\boldsymbol{\theta}}_W{}', n^{-1/2}\hat{\boldsymbol{\eta}}_W{}')'$ under correct model specification is given by $\mathbf{I}_0^{*-1}\begin{pmatrix} \mathbf{I}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\mathbf{I}_0^{*-1}$, whose alternative expression (see Silvey, 1959, Lemma 6; Jennrich, 1974; Lee, 1979) is given by the following:

Corollary S1. *When* $\mathbf{I}_0$ *in the augmented matrix may be singular,*

$$\mathbf{I}_0^{*-1}\begin{pmatrix} \mathbf{I}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{I}_0^{(11)} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_0^{(22)} \end{pmatrix}.$$

Proof.

$$\mathbf{I}_0^{*-1}\begin{pmatrix} \mathbf{I}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} \end{pmatrix}(\mathbf{A} - \mathbf{CC}')$$

$$\times\left(\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} \quad \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\right)$$

$$= \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{O} \\ \mathbf{O} & (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1} - \mathbf{I}_{(r)} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{I}_0^{(11)} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_0^{(22)} \end{pmatrix} \quad \text{Q.E.D.}$$

When $\mathbf{I}_0$ is non-singular, since $\mathbf{I}_0^{*-1}$ is obtained by the formula of the inverse of a partitioned matrix as

$$\mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{I}_0 & \mathbf{C} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_0^{-1} - \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} & \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1} \\ (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} & -(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1} \end{pmatrix},$$ it

follows that

$$\mathbf{I}_0^{*-1}\begin{pmatrix}\mathbf{I}_0 & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\mathbf{I}_0^{*-1}=\begin{pmatrix}\mathbf{I}_0^{-1}-\mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} & \mathbf{O}\\ \mathbf{O} & (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}\mathbf{I}_0^{(11)} & \mathbf{O}\\ \mathbf{O} & -\mathbf{I}_0^{(22)}\end{pmatrix},$$

which is also known (Aitchison & Silvey, 1958, Theorem 2).

The result $\mathbf{I}_0^{(11)}\mathbf{I}_0\mathbf{I}_0^{(11)}=\mathbf{I}_0^{(11)}$ in Corollary S1 shows that $\mathbf{I}_0$ is a generalized inverse of $\mathbf{I}_0^{(11)}$. However, the reverse is not true (Jennrich, 1974), which is shown as follows.

$$\mathbf{I}_0\mathbf{I}_0^{(11)}\mathbf{I}_0=(\mathbf{A}-\mathbf{CC}')\{\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1}\}(\mathbf{A}-\mathbf{CC}')$$

$$=\mathbf{A}-2\mathbf{CC}'+\mathbf{CC}'\mathbf{A}^{-1}\mathbf{CC}'$$

$$-(\mathbf{A}-\mathbf{CC}')\mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1}(\mathbf{A}-\mathbf{CC}')$$

$$=\mathbf{A}-2\mathbf{CC}'+\mathbf{CC}'\mathbf{A}^{-1}\mathbf{CC}'$$

$$-\{\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'-2\mathbf{CC}'+\mathbf{CC}'\mathbf{A}^{-1}\mathbf{CC}'\}$$

$$=\mathbf{A}-\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'$$

$$=\mathbf{I}_0+\mathbf{C}[\mathbf{I}_{(r)}-\{\mathbf{C}'(\mathbf{I}_0+\mathbf{CC}')^{-1}\mathbf{C}\}^{-1}]\mathbf{C}'$$

$$<\mathbf{I}_0,$$

which indicates that $\mathbf{I}_0^{(11)}$ is not a generalized inverse of $\mathbf{I}_0$.

Noting that Theorem S1 holds when $\mathbf{I}_0$ is non-singular as well as when $\mathbf{I}_0$ is singular, we have

Corollary S2. *When* $\mathbf{I}_0$ *is non-singular and* $\mathbf{A}=\mathbf{I}_0+\mathbf{CC}'$,

$$\begin{pmatrix}\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)}-(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}\mathbf{I}_0^{-1}-\mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} & \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\\ (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} & -(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\end{pmatrix}.$$

Proof. An alternative direct proof of Corollary S2 is given as follows. First, we derive the equality of the second diagonal blocks of the above equation. Since

$$\mathbf{C'A^{-1}C = C'(I_0 + CC')^{-1}C}$$
$$\mathbf{= C'\{I_0^{-1} - I_0^{-1}C(C'I_0^{-1}C + I_{(r)})^{-1}C'I_0^{-1}\}C}$$
$$\mathbf{= C'I_0^{-1}C - C'I_0^{-1}C(C'I_0^{-1}C + I_{(r)})^{-1}(C'I_0^{-1}C + I_{(r)} - I_{(r)})}$$
$$\mathbf{= C'I_0^{-1}C(C'I_0^{-1}C + I_{(r)})^{-1},}$$
$$\mathbf{I_{(r)} - (C'A^{-1}C)^{-1} = I_{(r)} - (C'I_0^{-1}C + I_{(r)})(C'I_0^{-1}C)^{-1}}$$
$$\mathbf{= -(C'I_0^{-1}C)^{-1}.}$$

The equality of the lower-left (or upper-right) blocks is given as follows.

$$\mathbf{(C'A^{-1}C)^{-1}C'A^{-1}}$$
$$\mathbf{= \{I_{(r)} + (C'I_0^{-1}C)^{-1}\}C'\{I_0^{-1} - I_0^{-1}C(C'I_0^{-1}C + I_{(r)})^{-1}C'I_0^{-1}\}}$$
$$\mathbf{= \{I_{(r)} + (C'I_0^{-1}C)^{-1}\}[C'I_0^{-1} - \{I_{(r)} - (C'I_0^{-1}C + I_{(r)})^{-1}\}C'I_0^{-1}]}$$
$$\mathbf{= \{I_{(r)} + (C'I_0^{-1}C)^{-1}\}(C'I_0^{-1}C + I_{(r)})^{-1}C'I_0^{-1}}$$
$$\mathbf{= \{I_{(r)} + (C'I_0^{-1}C)^{-1}\}[\,(C'I_0^{-1}C)^{-1}}$$
$$\mathbf{\quad - (C'I_0^{-1}C)^{-1}\{(C'I_0^{-1}C)^{-1} + I_{(r)}\}^{-1}(C'I_0^{-1}C)^{-1}\,]C'I_0^{-1}}$$
$$\mathbf{= \{I_{(r)} + (C'I_0^{-1}C)^{-1}\}\{(C'I_0^{-1}C)^{-1} + I_{(r)}\}^{-1}(C'I_0^{-1}C)^{-1}C'I_0^{-1}}$$
$$\mathbf{= (C'I_0^{-1}C)^{-1}C'I_0^{-1}.}$$

The equality of the first diagonal blocks is given as follows.

$$\mathbf{A^{-1} - A^{-1}C(C'A^{-1}C)^{-1}C'A^{-1}}$$
$$\mathbf{= A^{-1} - A^{-1}C(C'I_0^{-1}C)^{-1}C'I_0^{-1}}$$
$$\mathbf{= I_0^{-1} - I_0^{-1}C(C'I_0^{-1}C + I_{(r)})^{-1}C'I_0^{-1}}$$
$$\mathbf{\quad - \{I_0^{-1} - I_0^{-1}C(C'I_0^{-1}C + I_{(r)})^{-1}C'I_0^{-1}\}C(C'I_0^{-1}C)^{-1}C'I_0^{-1}}$$
$$\mathbf{= I_0^{-1} - I_0^{-1}C(C'I_0^{-1}C)^{-1}C'I_0^{-1}.} \quad \text{Q.E.D.}$$

A different expression of the result of Lemma S3 when **B** is non-singular is given from the result before Lemma S3 as

Lemma S4. *When* **B** *is non-singular and* $\mathbf{A}=\mathbf{B}+\mathbf{DC}'$ *as in Lemma S3,*

$$\begin{pmatrix}\mathbf{B} & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}=\begin{pmatrix}\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)}-(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}\mathbf{B}^{-1}-\mathbf{B}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{B}^{-1} & \mathbf{B}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\\ (\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{B}^{-1} & -(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\end{pmatrix}.$$

Let **C** and **D** in Lemmas S2, S3 and S4 be partitioned as $\mathbf{C}=(\mathbf{C}_{(\cdot1)}\ \mathbf{C}_{(\cdot2)})$ and $\mathbf{D}=(\mathbf{D}_{(\cdot1)}\ \mathbf{D}_{(\cdot2)})$, where $\mathbf{C}_{(\cdot1)}$ and $\mathbf{D}_{(\cdot1)}$ are $q\times s$ submatrices, and $\mathbf{C}_{(\cdot2)}$ and $\mathbf{D}_{(\cdot2)}$ are $q\times t$ submatrices with $s+t=r$ after possible simultaneous permutations of the columns of **C** and **D**. Assume that $\mathbf{B}+\mathbf{D}_{(\cdot2)}\mathbf{C}_{(\cdot2)}{}'$ is non-singular, where **B** may or may not be singular. The quantity $t\,(0<t<r)$ is not necessarily the minimum value to yield the non-singular $\mathbf{B}+\mathbf{D}_{(\cdot2)}\mathbf{C}_{(\cdot2)}{}'$ when **B** is singular.

Lemma S5. *The first and second row-wise blocks of* $\begin{pmatrix}\mathbf{B} & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}$ *and* $\begin{pmatrix}\mathbf{B}+\mathbf{D}_{(\cdot2)}\mathbf{C}_{(\cdot2)}{}' & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}$ *corresponding to the first q rows and the following s rows are equal. Similarly, the first and second column-wise blocks of the matrices corresponding to the first q columns and the following s columns are equal.*

Proof. From the identity,

$$\left\{\begin{pmatrix}\mathbf{B} & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O}\\ \mathbf{C}_{(\cdot2)}{}' & \mathbf{O} & \mathbf{I}_{(t)}\end{pmatrix}\right\}^{-1}=\begin{pmatrix}\mathbf{B}+\mathbf{D}_{(\cdot2)}\mathbf{C}_{(\cdot2)}{}' & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1},$$

we have $$\begin{pmatrix}\mathbf{B} & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}=\begin{pmatrix}\mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O}\\ \mathbf{C}_{(\cdot2)}{}' & \mathbf{O} & \mathbf{I}_{(t)}\end{pmatrix}\begin{pmatrix}\mathbf{B}+\mathbf{D}_{(\cdot2)}\mathbf{C}_{(\cdot2)}{}' & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}$$

Similarly, from the identity

$$\left\{\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{D}_{(\cdot 2)} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix}\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}\right\}^{-1} = \begin{pmatrix} \mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{C}_{(\cdot 2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1},$$

using Lemma S1 we have

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & -\mathbf{D}_{(\cdot 2)} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} = \begin{pmatrix} \mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{C}_{(\cdot 2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$$

The above results show the required equalities. Q.E.D.

Define $\mathbf{A}_{(2)} = \mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{C}_{(\cdot 2)}'$. Then, using Lemma S5, we have

Theorem S2. *The three submatrices other than the second diagonal block on the right-hand side of the identity*

$$\begin{pmatrix} \mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{C}_{(\cdot 2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \mathbf{A}_{(2)}^{-1} - \mathbf{A}_{(2)}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1}\mathbf{C}' & \mathbf{A}_{(2)}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}_{(2)}^{-1} & -(\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1} \end{pmatrix}$$

are unchanged irrespective of the choice of $\mathbf{C}_{(\cdot 2)}$ *and* $\mathbf{D}_{(\cdot 2)}$.

Theorem S3. *Theorem S2 holds even if* $\mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{C}_{(\cdot 2)}'$ *is replaced by* $\mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{E}\mathbf{C}_{(\cdot 2)}'$, *where* **E** *is a non-singular* $t \times t$ *matrix.*

Proof. From the identity

$$\left\{\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{E}\mathbf{C}_{(\cdot 2)}' & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix}\right\}^{-1} = \begin{pmatrix} \mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{E}\mathbf{C}_{(\cdot 2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1},$$ we obtain

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{E}\mathbf{C}_{(\cdot 2)}' & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix}\begin{pmatrix} \mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{E}\mathbf{C}_{(\cdot 2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}.$$ Similarly,

$$\left\{\begin{pmatrix}\mathbf{I}_{(q)} & \mathbf{O} & \mathbf{D}_{(\cdot 2)}\mathbf{E}\\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{(t)}\end{pmatrix}\begin{pmatrix}\mathbf{B} & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}\right\}^{-1} = \begin{pmatrix}\mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{E}\mathbf{C}_{(\cdot 2)}' & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}$$ gives

$$\begin{pmatrix}\mathbf{B} & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1} = \begin{pmatrix}\mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{E}\mathbf{C}_{(\cdot 2)}' & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{I}_{(q)} & \mathbf{O} & \mathbf{D}_{(\cdot 2)}\mathbf{E}\\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{(t)}\end{pmatrix}$$. These two results show that $\begin{pmatrix}\mathbf{B} & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}$ and $\begin{pmatrix}\mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{E}\mathbf{C}_{(\cdot 2)}' & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}$ are the same except their second diagonal blocks. Q.E.D.

Crowder's (1984, Lemma 6) result for the augmented matrix derived in a different way is a special case of Theorem S3 when $\mathbf{B}=\mathbf{I}_0$ and $\mathbf{D}=\mathbf{C}=-\mathbf{H}_0$. Note that the arbitrariness of $\mathbf{E}$ corresponds to the arbitrary expression of the restriction $\mathbf{Eh}=\mathbf{0}$ with $\mathbf{E}$ being non-singular.

Using Lemmas S1, S5 and Theorem S2, we have

Corollary S3. *When* **B** *is possibly singular,*

$$\begin{pmatrix}\mathbf{B} & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1} = \begin{pmatrix}\mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O}\\ \mathbf{C}_{(\cdot 2)}'\mathbf{O} & & \mathbf{I}_{(t)}\end{pmatrix}\begin{pmatrix}\mathbf{B}+\mathbf{D}_{(\cdot 2)}\mathbf{C}_{(\cdot 2)}' & \mathbf{D}\\ \mathbf{C}' & \mathbf{O}\end{pmatrix}^{-1}$$

$$= \begin{pmatrix}\mathbf{A}_{(2)}^{-1}-\mathbf{A}_{(2)}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1}\mathbf{C}' & \mathbf{A}_{(2)}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1}\\ (\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}_{(2)}^{-1} & \begin{pmatrix}\mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}_{(t)}\end{pmatrix}-(\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1}\end{pmatrix}.$$

Equating the results of Lemma S3 and Corollary S3, we find that

$$\begin{pmatrix}\mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}_{(t)}\end{pmatrix}-(\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1} = \mathbf{I}_{(r)}-(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}$$ or equivalently

$$(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}-(\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1} = \begin{pmatrix}\mathbf{I}_{(s)} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}$$. When **B** is non-singular.

$$\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} - (\mathbf{C}'\mathbf{A}_{(2)}^{-1}\mathbf{D})^{-1} = -(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}.$$

3. Applications of the general formulas to Example 2.1 (Example 1.2)
3.1 Non-studentized estimators

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \equiv n^{-1}\left(\frac{\sum_{i=1}^{n_1} x_{1i} - n_1\theta_{01}}{\theta_{01}(1-\theta_{01})}, \frac{\sum_{i=1}^{n_2} x_{2i} - n_2\theta_{02}}{\theta_{02}(1-\theta_{02})} \right)'$$

$$= n^{-1}\left(\frac{m_{11} - n_1\theta_{01}}{\theta_{01}(1-\theta_{01})}, \frac{m_{21} - n_2\theta_{02}}{\theta_{02}(1-\theta_{02})} \right)',$$

$$\frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0'} = n^{-1}\begin{bmatrix} \frac{-n_1}{\theta_{01}(1-\theta_{01})} - \frac{m_{11} - n_1\theta_{01}}{\{\theta_{01}(1-\theta_{01})\}^2}(1-2\theta_{01}) & 0 \\ 0 & \frac{-n_2}{\theta_{02}(1-\theta_{02})} - \frac{m_{21} - n_2\theta_{02}}{\{\theta_{02}(1-\theta_{02})\}^2}(1-2\theta_{02}) \end{bmatrix},$$

$$\mathrm{E}_\theta\left(\frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0'} \right) = -n^{-1}\begin{bmatrix} \frac{n_1}{\theta_{01}(1-\theta_{01})} & 0 \\ 0 & \frac{n_2}{\theta_{02}(1-\theta_{02})} \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{c_1}{\theta_{01}(1-\theta_{01})} & 0 \\ 0 & \frac{c_2}{\theta_{02}(1-\theta_{02})} \end{bmatrix}$$

$$= -\mathbf{I}_0 \quad (c_1 = n_1/n = O(1), c_2 = n_2/n = O(1)),$$

$$\mathbf{M} \equiv \frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0 '} - \mathrm{E}_\theta \left(\frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0 '} \right)$$

$$= -n^{-1} \begin{bmatrix} \frac{m_{11} - n_1\theta_{01}}{\{\theta_{01}(1-\theta_{01})\}^2}(1-2\theta_{01}) & 0 \\ 0 & \frac{m_{21} - n_2\theta_{02}}{\{\theta_{02}(1-\theta_{02})\}^2}(1-2\theta_{02}) \end{bmatrix},$$

$$\mathrm{E}_\theta \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0 '} \right) = n^{-1} \begin{pmatrix} \frac{c_1}{\theta_{01}(1-\theta_{01})} & 0 \\ 0 & \frac{c_2}{\theta_{02}(1-\theta_{02})} \end{pmatrix} = n^{-1}\mathbf{I}_0,$$

$$\mathrm{E}_\theta \left((\mathbf{M})_{11} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0 '} \right) = n^{-1} \left(-\frac{c_1}{\{\theta_{01}(1-\theta_{01})\}^2}(1-2\theta_{01}),\ 0 \right),$$

$$\mathrm{E}_\theta \left((\mathbf{M})_{22} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0 '} \right) = n^{-1} \left(0,\ -\frac{c_2}{\{\theta_{02}(1-\theta_{02})\}^2}(1-2\theta_{02}) \right),$$

$$\mathrm{E}_\theta \left((\mathbf{M})_{12} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0 '} \right) = \mathrm{E}_\theta \left((\mathbf{M})_{21} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0 '} \right) = \mathbf{0}',$$

$$\boldsymbol{\Lambda}_0^{*-1} = \begin{pmatrix} -\mathbf{I}_0 & \mathbf{H}_0 \\ \mathbf{H}_0 ' & 0 \end{pmatrix}^{-1}.$$

Since $\theta_{01} = \theta_{02}$, define $\theta_0 \equiv (\boldsymbol{\theta}_0)_1 = (\boldsymbol{\theta}_0)_2 = \theta_{01} = \theta_{02}$. Then, using

$$\mathbf{I}_0 = \frac{1}{\theta_0(1-\theta_0)} \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \text{ and } \mathbf{I}_0^{-1} = \theta_0(1-\theta_0) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix},$$

$$\boldsymbol{\Lambda}_0^{*-1}=\begin{bmatrix} -\mathbf{I}_0^{-1}+\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\left\{(1,-1)\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\right\}^{-1}(1,-1)\mathbf{I}_0^{-1} & \text{sym.} \\ \left\{(1,-1)\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\right\}^{-1}(1,-1)\mathbf{I}_0^{-1} & \left\{(1,-1)\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\right\}^{-1} \end{bmatrix}$$

$$=\begin{bmatrix} \theta_0(1-\theta_0)\left\{-\begin{pmatrix}c_1^{-1} & 0\\ 0 & c_2^{-1}\end{pmatrix}+c_1c_2\begin{pmatrix}c_1^{-1}\\-c_2^{-1}\end{pmatrix}(c_1^{-1},-c_2^{-1})\right\} & \text{sym.} \\ c_1c_2(c_1^{-1},-c_2^{-1}) & \dfrac{c_1c_2}{\theta_0(1-\theta_0)} \end{bmatrix},$$

$$=\begin{bmatrix} \theta_0(1-\theta_0)\begin{pmatrix}-\dfrac{1-c_2}{c_1} & -1\\ -1 & -\dfrac{1-c_1}{c_2}\end{pmatrix} & \begin{pmatrix}c_2\\-c_1\end{pmatrix} \\ (c_2,-c_1) & \dfrac{c_1c_2}{\theta_0(1-\theta_0)} \end{bmatrix}$$

$$=\begin{bmatrix} -\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} & \begin{pmatrix}c_2\\-c_1\end{pmatrix} \\ (c_2,-c_1) & \dfrac{c_1c_2}{\theta_0(1-\theta_0)} \end{bmatrix},$$

$$\mathbf{q}_0^*=\left(\frac{k}{\theta_{01}}-\frac{k}{1-\theta_{01}},\frac{k}{\theta_{02}}-\frac{k}{1-\theta_{02}}\right)'=k\left(\frac{1-2\theta_{01}}{\theta_{01}(1-\theta_{01})},\frac{1-2\theta_{02}}{\theta_{02}(1-\theta_{02})}\right)'$$

$$=\frac{k(1-2\theta_0)}{\theta_0(1-\theta_0)}(1,1)'.$$

$$\boldsymbol{\Lambda}_{\mathrm{W}}^{(1)}\mathbf{I}_0^{(1)}=\boldsymbol{\Lambda}_{\mathrm{ML}}^{(1)}\mathbf{I}_0^{(1)}=-\boldsymbol{\Lambda}_0^{(\cdot 1)}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}=-\begin{bmatrix}-\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ (c_2,-c_1)\end{bmatrix}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0},$$

where $\boldsymbol{\Lambda}_{\mathrm{W}}^{(1)}=\boldsymbol{\Lambda}_{\mathrm{ML}}^{(1)}=-\boldsymbol{\Lambda}_0^{(\cdot 1)}=-(\boldsymbol{\Lambda}_0^{(11)}\ \boldsymbol{\Lambda}_0^{(21)}{}')'$, $\mathbf{I}_0^{(1)}=\partial\bar{l}/\partial\boldsymbol{\theta}_0$.

$$-n^{-1}\begin{pmatrix}\mathbf{\Lambda}_0^{(11)}\\ \mathbf{\Lambda}_0^{(21)}\end{pmatrix}\mathbf{q}_0^{*}=n^{-1}\begin{bmatrix}\theta_0(1-\theta_0)\begin{pmatrix}1&1\\1&1\end{pmatrix}\\ -(c_2,-c_1)\end{bmatrix}\begin{pmatrix}1\\1\end{pmatrix}\frac{k(1-2\theta_0)}{\theta_0(1-\theta_0)}$$

$$=n^{-1}\begin{bmatrix}2\\2\\(c_1-c_2)/\{\theta_0(1-\theta_0)\}\end{bmatrix}k(1-2\theta_0),$$

where $\mathbf{\Lambda}_0^{(11)}\mathbf{q}_0^{*}=-\begin{pmatrix}1\\1\end{pmatrix}2k(1-2\theta_0)$.

For nonzero elements of $\mathbf{J}_0^{(3)}$,

$$(\mathbf{J}_0^{(3)})_{(\theta_1,\theta_1\theta_1)}$$

$$=n^{-1}\left\{\frac{n_1 2(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2}-(m_{11}-n_1\theta_0)\left(-\frac{2(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}-\frac{2}{\{\theta_0(1-\theta_0)\}^2}\right)\right\}$$

$$=2n^{-1}\left\{\frac{n_1(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2}+(m_{11}-n_1\theta_0)\left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}+\frac{1}{\{\theta_0(1-\theta_0)\}^2}\right)\right\},$$

$$\mathbf{E}_\theta\{(\mathbf{J}_0^{(3)})_{(\theta_1,\theta_1\theta_1)}\}=2c_1\frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2},$$

$$\{\mathbf{J}_0^{(3)}-\mathbf{E}_\theta(\mathbf{J}_0^{(3)})\}_{(\theta_1,\theta_1\theta_1)}$$

$$=2n^{-1}(m_{11}-n_1\theta_0)\left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}+\frac{1}{\{\theta_0(1-\theta_0)\}^2}\right).$$

Similarly,

$$\mathbf{E}_\theta\{(\mathbf{J}_0^{(3)})_{(\theta_2,\theta_2\theta_2)}\}=2c_2\frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2},$$

$$\{\mathbf{J}_0^{(3)}-\mathbf{E}_\theta(\mathbf{J}_0^{(3)})\}_{(\theta_2,\theta_2\theta_2)}$$

$$=2n^{-1}(m_{21}-n_2\theta_0)\left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}+\frac{1}{\{\theta_0(1-\theta_0)\}^2}\right).$$

Nonzero elements of $\mathbf{J}_0^{(4)}$ are

$$\mathbf{E}_\theta\{(\mathbf{J}_0^{(4)})_{(\theta_1,\theta_1\theta_1\theta_1)}\} = 2c_1\left(-\frac{3(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} - \frac{3}{\{\theta_0(1-\theta_0)\}^2}\right)$$

$$= -6c_1\left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} + \frac{1}{\{\theta_0(1-\theta_0)\}^2}\right),$$

$$\mathbf{E}_\theta\{(\mathbf{J}_0^{(4)})_{(\theta_2,\theta_2\theta_2\theta_2)}\} = -6c_2\left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} + \frac{1}{\{\theta_0(1-\theta_0)\}^2}\right).$$

$$\mathbf{\Lambda}_{\mathrm{W}}^{(2)}\mathbf{l}_0^{(2)} = \mathbf{\Lambda}_{\mathrm{ML}}^{(2)}\mathbf{l}_0^{(2)} = (\mathbf{\Lambda}_{\mathrm{ML}}^{(2-1)}\ \mathbf{\Lambda}_{\mathrm{ML}}^{(2-2)})\left\{\mathrm{v}(\mathbf{M})'\otimes\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}, \left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)^{<2>}\right\}',$$

where

$$(\mathbf{\Lambda}_{\mathrm{ML}}^{(2-1)})_{\bullet(ijk^*)} = \sum\nolimits_{(ij)}^{2}(\mathbf{\Lambda}_0^{(\bullet 1)})_{\bullet i}\frac{2-\delta_{ij}}{2}(\mathbf{\Lambda}_0^{(11)})_{jk^*}$$

$$= \sum_{(ij)}^{2}\begin{bmatrix} -\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \\ (c_2, -c_1)\end{bmatrix}_{\bullet i}\frac{2-\delta_{ij}}{2}\{-\theta_0(1-\theta_0)\}$$

$$= \begin{bmatrix} \{\theta_0(1-\theta_0)\}^2(2-\delta_{ij}) \\ -\sum\nolimits_{(ij)}^{2}\theta_0(1-\theta_0)(c_2,-c_1)_i(2-\delta_{ij})/2\end{bmatrix}\quad (1 \le i \le j \le 2; k^* = 1,2),$$

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(2-2)})_{\bullet(ij)} = -\frac{1}{2}\boldsymbol{\Lambda}_0^{*-1}\begin{pmatrix}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\\ \mathbf{O}\end{pmatrix}\{(\boldsymbol{\Lambda}_0^{(11)})_{\bullet i}\otimes(\boldsymbol{\Lambda}_0^{(11)})_{\bullet j}\}$$

$$= -\frac{1}{2}\boldsymbol{\Lambda}_0^{(\bullet 1)}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\{(\boldsymbol{\Lambda}_0^{(11)})_{\bullet i}\otimes(\boldsymbol{\Lambda}_0^{(11)})_{\bullet j}\}$$

$$= -\frac{1}{2}\begin{bmatrix}-\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ (c_2, -c_1)\end{bmatrix}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\{\theta_0(1-\theta_0)\}^2$$

$$= -\begin{bmatrix}-\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ (c_2, -c_1)\end{bmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}(1-2\theta_0) = \begin{pmatrix}1\\1\\0\end{pmatrix}\theta_0(1-\theta_0)(1-2\theta_0)$$

$(i, j = 1,\ldots,q).$

$$\boldsymbol{\Lambda}_{\mathrm{W}}^{(3)}\mathbf{l}_0^{(3)} = \boldsymbol{\Lambda}_{\mathrm{ML}}^{(3)}\mathbf{l}_0^{(3)} + \boldsymbol{\Lambda}_{\mathrm{W}}^{(\Delta)}n^{-1}\mathbf{l}_0^{(\mathrm{W})},$$

$$\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3)}\mathbf{l}_0^{(3)}$$

$$= (\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-1)}\ \boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-2)}\ \boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-3)}\ \boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-4)})\left\{\mathrm{v}(\mathbf{M})'^{<2>}\otimes\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}, \mathrm{v}(\mathbf{M})'\otimes\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)^{<2>},\right.$$

$$\left.\mathrm{vec}((\mathbf{J}_0^{(3)} - \mathrm{E}_{\mathrm{T}}(\mathbf{J}_0^{(3)}))'\otimes\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)^{<2>}, \left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)^{<3>}\right\}',$$

$$\boldsymbol{\Lambda}_{\mathrm{W}}^{(\Delta)} n^{-1} \mathbf{l}_0^{(\mathrm{W})} = (\boldsymbol{\Lambda}_{\mathrm{W}}^{(\Delta-1)}\ \boldsymbol{\Lambda}_{\mathrm{W}}^{(\Delta-2)}) n^{-1} \left\{ \mathrm{v}(\mathbf{M})', \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right\}',$$

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-1)})_{\bullet(ijk^*l^*m)} = -\sum_{(ij)}^{2} \sum_{(k^*l^*)}^{2} (\boldsymbol{\Lambda}_0^{(\bullet 1)})_{\bullet i} \frac{2-\delta_{ij}}{2} (\boldsymbol{\Lambda}_0^{(11)})_{jk^*} \frac{2-\delta_{k^*l^*}}{2} (\boldsymbol{\Lambda}_0^{(11)})_{l^*m}$$

$$= -\sum_{(ij)}^{2} \sum_{(k^*l^*)}^{2} \begin{bmatrix} -\theta_0(1-\theta_0)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{bmatrix}_{\bullet i} \frac{(2-\delta_{ij})(2-\delta_{k^*l^*})}{4} \{\theta_0(1-\theta_0)\}^2$$

$$= \begin{bmatrix} \{\theta_0(1-\theta_0)\}^3 (2-\delta_{ij}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -\sum_{(ij)}^{2} (c_2, -c_1)_i \{(2-\delta_{ij})/2\} \{\theta_0(1-\theta_0)\}^2 \end{bmatrix} (2-\delta_{k^*l^*})$$

$$(1 \le i \le j \le 2; 1 \le k^* \le l^* \le 2; m = 1, 2),$$

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-2)})_{\bullet(ijk^*l^*)}$$

$$= \sum_{(ij)}^{2} (\boldsymbol{\Lambda}_0^{(\bullet 1)})_{\bullet i} \frac{2-\delta_{ij}}{2} \left(\frac{1}{2} \boldsymbol{\Lambda}_0^{(11)} \mathrm{E}_\theta(\mathbf{J}_0^{(3)}) \{ (\boldsymbol{\Lambda}_0^{(11)})_{\bullet k^*} \otimes (\boldsymbol{\Lambda}_0^{(11)})_{\bullet l^*} \} \right)_j$$

$$+ \boldsymbol{\Lambda}_0^{(\bullet 1)} \mathrm{E}_\theta(\mathbf{J}_0^{(3)}) \left[(\boldsymbol{\Lambda}_0^{(11)})_{\bullet k^*} \otimes \sum_{(ij)}^{2} \left\{ (\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} \frac{2-\delta_{ij}}{2} (\boldsymbol{\Lambda}_0^{(11)})_{jl^*} \right\} \right]$$

$$= \sum_{(ij)}^{2} (\boldsymbol{\Lambda}_0^{(\bullet 1)})_{\bullet i} \frac{2-\delta_{ij}}{2} (\boldsymbol{\Lambda}_0^{(11)})_{j\bullet} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (1-2\theta_0)$$

$$- \boldsymbol{\Lambda}_0^{(\bullet 1)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} 2\theta_0(1-\theta_0)(1-2\theta_0)(2-\delta_{ij})$$

$$= \sum_{(ij)}^{2} \begin{pmatrix} -\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \\ (c_2, -c_1) \end{pmatrix}_{\bullet i} \frac{2-\delta_{ij}}{2}(-1)\theta_0(1-\theta_0)(1-2\theta_0)$$

$$- \begin{pmatrix} -\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \\ (c_2, -c_1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} 2\theta_0(1-\theta_0)(1-2\theta_0)(2-\delta_{ij})$$

$$= \begin{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} 3(2-\delta_{ij})\{\theta_0(1-\theta_0)\}^2(1-2\theta_0) \\ -\sum_{(ij)}^{2}(c_2, -c_1)_i \frac{2-\delta_{ij}}{2}\theta_0(1-\theta_0)(1-2\theta_0) \end{pmatrix}$$

$(1 \le i \le j \le 2; k^*, l^* = 1, 2)$,

$$(\mathbf{\Lambda}_{\mathrm{ML}}^{(3-3)})_{\bullet(ijk^*l^*m)} = -\frac{1}{2}(\mathbf{\Lambda}_0^{(\bullet 1)})_{\bullet i}(\mathbf{\Lambda}_0^{(11)})_{jl^*}(\mathbf{\Lambda}_0^{(11)})_{k^*m}$$

$$= -\frac{1}{2}\begin{pmatrix} -\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \\ (c_2, -c_1) \end{pmatrix}_{\bullet i} \{\theta_0(1-\theta_0)\}^2$$

$$= \frac{1}{2}\begin{pmatrix} \theta_0(1-\theta_0)\begin{pmatrix}1\\1\end{pmatrix} \\ -(c_2, -c_1) \end{pmatrix}_{\bullet i} \{\theta_0(1-\theta_0)\}^2$$

$(i, j, k^*, l^*, m = 1, 2)$.

$(\mathbf{\Lambda}_{\mathrm{ML}}^{(3-4)})_{\bullet(ijk^*)}$

$$= -\frac{1}{2}\mathbf{\Lambda}_0^{(\bullet 1)}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\left[(\mathbf{\Lambda}_0^{(11)})_{\bullet i} \otimes \left\{\mathbf{\Lambda}_0^{(11)}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\{(\mathbf{\Lambda}_0^{(11)})_{\bullet j} \otimes \mathbf{\Lambda}_0^{(11)})_{\bullet k}\}\right\}\right]$$

$$+\frac{1}{6}\mathbf{\Lambda}_0^{(\bullet 1)}\mathrm{E}_\theta(\mathbf{J}_0^{(4)})\{(\mathbf{\Lambda}_0^{(11)})_{\bullet i} \otimes (\mathbf{\Lambda}_0^{(11)})_{\bullet j} \otimes \mathbf{\Lambda}_0^{(11)})_{\bullet k^*}\},$$

where the first term is

$$-\frac{1}{2}\boldsymbol{\Lambda}_0^{(\cdot 1)}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\left[(\boldsymbol{\Lambda}_0^{(11)})_{\cdot i}\otimes\left\{\boldsymbol{\Lambda}_0^{(11)}\begin{pmatrix}c_1\\c_2\end{pmatrix}2(1-2\theta_0)\right\}\right]$$

$$=-\frac{1}{2}\boldsymbol{\Lambda}_0^{(\cdot 1)}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\left\{\theta_0(1-\theta_0)\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}\theta_0(1-\theta_0)2(1-2\theta_0)\right\}$$

$$=-\frac{1}{2}\boldsymbol{\Lambda}_0^{(\cdot 1)}4\begin{pmatrix}c_1\\c_2\end{pmatrix}(1-2\theta_0)^2$$

$$=2\begin{pmatrix}1\\1\\0\end{pmatrix}\theta_0(1-\theta_0)(1-2\theta_0)^2$$

and the second term is

$$\frac{1}{6}\boldsymbol{\Lambda}_0^{(\cdot 1)}6\begin{pmatrix}c_1\\c_2\end{pmatrix}\{(1-2\theta_0)^2+\theta_0(1-\theta_0)\}$$

$$=-\begin{pmatrix}1\\1\\0\end{pmatrix}[\theta_0(1-\theta_0)(1-2\theta_0)^2+\{\theta_0(1-\theta_0)\}^2].$$

Then,

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(3-4)})_{\cdot(ijk*)}=\begin{pmatrix}1\\1\\0\end{pmatrix}\left[\theta_0(1-\theta_0)(1-2\theta_0)^2-\{\theta_0(1-\theta_0)\}^2\right].$$

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(\Delta-1)})_{\cdot(ij)}=\sum_{(ij)}^{2}(\boldsymbol{\Lambda}_0^{(\cdot 1)})_{\cdot i}\frac{2-\delta_{ij}}{2}(\boldsymbol{\Lambda}_0^{(11)}\mathbf{q}_0^*)_j$$

$$=\sum_{(ij)}^{2}(\boldsymbol{\Lambda}_0^{(\cdot 1)})_{\cdot i}\frac{2-\delta_{ij}}{2}\{-2k(1-2\theta_0)\}$$

$$=-\sum_{(ij)}^{2}\begin{pmatrix}-\theta_0(1-\theta_0)\begin{pmatrix}1&1\\1&1\end{pmatrix}\\(c_2,-c_1)\end{pmatrix}_{\cdot i}(2-\delta_{ij})k(1-2\theta_0)$$

$$=\begin{pmatrix}\theta_0(1-\theta_0)2(2-\delta_{ij})\\-\sum_{(ij)}^{2}(c_2,-c_1)_i(2-\delta_{ij})\end{pmatrix}k(1-2\theta_0)$$

$(1\le i\le j\le 2)$,

where $\mathbf{q}_0^* = \frac{k(1-2\theta_0)}{\theta_0(1-\theta_0)}(1,1)'$, $\boldsymbol{\Lambda}_0^{(\bullet 1)} = \begin{pmatrix} -\theta_0(1-\theta_0)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \\ (c_2, -c_1) \end{pmatrix}$ and consequently,

$\boldsymbol{\Lambda}_0^{(11)}\mathbf{q}_0^* = -\begin{pmatrix}1\\1\end{pmatrix}2k(1-2\theta_0)$ is used.

$$(\boldsymbol{\Lambda}_{\mathrm{ML}}^{(\Delta-2)})_{\bullet i} = -\boldsymbol{\Lambda}_0^{(\bullet 1)}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\{(\boldsymbol{\Lambda}_0^{(11)})_{\bullet i} \otimes (\boldsymbol{\Lambda}_0^{(11)}\mathbf{q}_0^*)\} + \boldsymbol{\Lambda}_0^{(\bullet 1)}\frac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0{}'}(\boldsymbol{\Lambda}_0^{(11)})_{\bullet i}$$

$$= -\boldsymbol{\Lambda}_0^{(\bullet 1)}\mathrm{E}_\theta(\mathbf{J}_0^{(3)})\left\{\left(-\theta_0(1-\theta_0)\begin{pmatrix}1\\1\end{pmatrix}\right) \otimes \left(-\begin{pmatrix}1\\1\end{pmatrix}2k(1-2\theta_0)\right)\right\}$$

$$+\boldsymbol{\Lambda}_0^{(\bullet 1)}\left[-\frac{2}{\theta_0(1-\theta_0)} - \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2}\right]k\mathbf{I}_{(2)}(\boldsymbol{\Lambda}_0^{(11)})_{\bullet i}$$

$$= -\boldsymbol{\Lambda}_0^{(\bullet 1)}\begin{pmatrix}c_1\\c_2\end{pmatrix}\frac{4k(1-2\theta_0)^2}{\theta_0(1-\theta_0)}$$

$$+\boldsymbol{\Lambda}_0^{(\bullet 1)}\left[\frac{2}{\theta_0(1-\theta_0)} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2}\right]k\theta_0(1-\theta_0)\begin{pmatrix}1\\1\end{pmatrix}$$

$$= 4k(1-2\theta_0)^2\begin{pmatrix}1\\1\\0\end{pmatrix} + \left[\frac{2}{\theta_0(1-\theta_0)} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2}\right]$$

$$\times k\theta_0(1-\theta_0)\begin{pmatrix}-\theta_0(1-\theta_0)2\\ -\theta_0(1-\theta_0)2\\ c_2-c_1\end{pmatrix}$$

$$= k\begin{bmatrix}\{-4\theta_0(1-\theta_0)+2(1-2\theta_0)^2\}\begin{pmatrix}1\\1\end{pmatrix} \\ \left\{2+\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}(c_2-c_1)\end{bmatrix} \quad (i=1,2).$$

3.2 Studentized estimators

3.2.1 $t_{\mathrm{W}\theta}$ ($t_{\mathrm{W}\theta_1}$ and $t_{\mathrm{W}\theta_2}$)

Using θ_{01} and θ_{02} rather than θ_0 for differentiation,

$$\mathbf{I}_0^{*-1} = \{-\mathrm{E}_\theta(\boldsymbol{\Lambda}_0^*)\}^{-1} = \begin{pmatrix} \mathbf{I}_0 & -\mathbf{H}_0 \\ -\mathbf{H}_0' & 0 \end{pmatrix}^{-1}$$

$$= \begin{bmatrix} \mathbf{I}_0^{-1} - \mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\left\{(1,-1)\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\right\}^{-1}(1,-1)\mathbf{I}_0^{-1} & \text{Sym.} \\ -\left\{(1,-1)\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\right\}^{-1}(1,-1)\mathbf{I}_0^{-1} & -\left\{(1,-1)\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\right\}^{-1} \end{bmatrix},$$

where $\mathbf{I}_0^{-1} = \begin{pmatrix} \dfrac{c_1}{\theta_{01}(1-\theta_{01})} & 0 \\ 0 & \dfrac{c_2}{\theta_{02}(1-\theta_{02})} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{\theta_{01}(1-\theta_{01})}{c_1} & 0 \\ 0 & \dfrac{\theta_{02}(1-\theta_{02})}{c_2} \end{pmatrix}.$

Then, noting $\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix} \theta_{01}(1-\theta_{01})/c_1 \\ -\theta_{02}(1-\theta_{02})/c_2 \end{pmatrix}$ and

$(1,-1)\mathbf{I}_0^{-1}\begin{pmatrix}1\\-1\end{pmatrix} = \dfrac{\theta_{01}(1-\theta_{01})}{c_1} + \dfrac{\theta_{02}(1-\theta_{02})}{c_2}$, it follows that

$\mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}' & d \end{pmatrix}$, where

$$\mathbf{A} = \begin{pmatrix} \dfrac{\theta_{01}(1-\theta_{01})}{c_1} & 0 \\ 0 & \dfrac{\theta_{02}(1-\theta_{02})}{c_2} \end{pmatrix} - \left(\frac{\theta_{01}(1-\theta_{01})}{c_1} + \frac{\theta_{02}(1-\theta_{02})}{c_2}\right)^{-1}$$

$$\times \begin{pmatrix} \dfrac{\theta_{01}(1-\theta_{01})}{c_1} \\ -\dfrac{\theta_{02}(1-\theta_{02})}{c_2} \end{pmatrix} \left(\frac{\theta_{01}(1-\theta_{01})}{c_1}, -\frac{\theta_{02}(1-\theta_{02})}{c_2}\right),$$

$$\mathbf{b} = -\left(\frac{\theta_{01}(1-\theta_{01})}{c_1} + \frac{\theta_{02}(1-\theta_{02})}{c_2}\right)^{-1} \left(\frac{\theta_{01}(1-\theta_{01})}{c_1}, -\frac{\theta_{02}(1-\theta_{02})}{c_2}\right)',$$

and $d = -\left(\frac{\theta_{01}(1-\theta_{01})}{c_1} + \frac{\theta_{02}(1-\theta_{02})}{c_2} \right)^{-1}$.

Since direct differentiation of $\mathbf{I}_0^{*-1}$ with respect to θ_{0k} is tedious, we use the formula of $\partial \mathbf{I}_0^{*-1} / \partial \theta_{0k^*} = -\mathbf{I}_0^{*-1}(\partial \mathbf{I}_0^* / \partial \theta_{0k^*})\mathbf{I}_0^{*-1}$, where

$$\mathbf{I}_0^* = \begin{pmatrix} \mathbf{I}_0 & -\mathbf{H}_0 \\ -\mathbf{H}_0' & 0 \end{pmatrix} = \begin{pmatrix} \frac{c_1}{\theta_{01}(1-\theta_{01})} & 0 & -1 \\ 0 & \frac{c_2}{\theta_{02}(1-\theta_{02})} & 1 \\ -1 & 1 & 0 \end{pmatrix}$$ though

$$\mathbf{I}_0^{*-1} = \begin{pmatrix} \theta_0(1-\theta_0)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & -\begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \\ -(c_2, -c_1) & -\frac{c_1 c_2}{\theta_0(1-\theta_0)} \end{pmatrix}$$ after evaluation using $\theta_{01} = \theta_{02}$

looks simple.

$$(\mathbf{i}_{\theta_1}^{(1)})_j = \frac{1}{2}\left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{11} (i_0^{(11)})^{-3/2}$$

$$= \frac{1}{2}\{\theta_0(1-\theta_0)\}^2 \frac{-c_j(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2}\{\theta_0(1-\theta_0)\}^{-3/2}$$

$$= -\frac{c_j}{2}(1-2\theta_0)\{\theta_0(1-\theta_0)\}^{-3/2}$$

$(j = 1, 2)$,

$$(\mathbf{i}_{\theta_2}^{(1)})_j = \frac{1}{2}\left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{22} (i_0^{(11)})^{-3/2} = -\frac{c_j}{2}(1-2\theta_0)\{\theta_0(1-\theta_0)\}^{-3/2}$$

$(j = 1, 2)$.

That is, $\mathbf{i}_{\theta_1}^{(1)} = \mathbf{i}_{\theta_2}^{(1)}$. Note that

$$\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\right)_{11}=\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\right)_{22}=-c_j(1-2\theta_0)\ \ (j=1,2).$$

In $t_{\mathrm{W}\theta}\equiv n^{1/2}(\hat{i}_{\mathrm{W}}^{\theta\theta})^{-1/2}(\hat{\theta}_{\mathrm{W}}-\theta_0)$, $\hat{i}_{\mathrm{W}}^{\theta\theta}$ is the first or second diagonal element of $\mathbf{I}_0^{*-1}$ with $\boldsymbol{\theta}_0$ replaced by $\hat{\boldsymbol{\theta}}_{\mathrm{W}}$ and becomes $\hat{\theta}_{\mathrm{W}}(1-\hat{\theta}_{\mathrm{W}})$. Then, $t_{\mathrm{W}\theta}=n^{1/2}\{\hat{\theta}_{\mathrm{W}}(1-\hat{\theta}_{\mathrm{W}})\}^{-1/2}(\hat{\theta}_{\mathrm{W}}-\theta_0)$, where note that $t_{\mathrm{W}\theta}$ is defined using $n^{1/2}$ rather than $n_{\mathrm{W}}^{1/2}$. That is, $\hat{\theta}_{\mathrm{W}}$ and $t_{\mathrm{W}\theta}$ reduce to those of a single group proportion with size n and pseudocounts $4k$ in total.

$$(\mathbf{i}_{\theta_1}^{(2)})_{(jk^*)}$$

$$=\left(\frac{1}{4}\mathbf{I}_0^{*-1}\frac{\partial^2\mathbf{I}_0^*}{\partial\theta_{0j}\partial\theta_{0k^*}}\mathbf{I}_0^{*-1}-\frac{1}{2}\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0k^*}}\mathbf{I}_0^{*-1}\right)_{11}\{\theta_0(1-\theta_0)\}^{-3/2}$$

$$+\frac{3}{8}\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\right)_{11}\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0k^*}}\mathbf{I}_0^{*-1}\right)_{11}\{\theta_0(1-\theta_0)\}^{-5/2}$$

$$=\left[\delta_{jk^*}\frac{c_j}{2}\left\{1+\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}-\frac{1}{2}c_jc_{k^*}\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right]\theta_0(1-\theta_0)\}^{-3/2}$$

$$+\frac{3}{8}c_jc_{k^*}(1-2\theta_0)^2\{\theta_0(1-\theta_0)\}^{-5/2}$$

$(j,k^*=1,2)$,

where $\dfrac{\partial\mathbf{I}_0^*}{\partial\theta_{01}}=\begin{pmatrix}-\dfrac{c_1(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $\dfrac{\partial\mathbf{I}_0^*}{\partial\theta_{02}}=\begin{pmatrix}0 & 0 & 0\\ 0 & -\dfrac{c_1(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} & 0\\ 0 & 0 & 0\end{pmatrix}$

and non-zero second derivatives are

$$\frac{\partial^2 \mathbf{I}_0^*}{(\partial\theta_{01})^2} = \begin{bmatrix} 2c_1\left(\frac{1}{\{\theta_0(1-\theta_0)\}^2}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\frac{\partial^2 \mathbf{I}_0^*}{(\partial\theta_{02})^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2c_2\left(\frac{1}{\{\theta_0(1-\theta_0)\}^2}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$i_0^{11} = i_0^{22} = \theta_0(1-\theta_0)$, $(\mathbf{i}_{\theta_1}^{(2)})_{(jk^*)} = (\mathbf{i}_{\theta_2}^{(2)})_{(jk^*)}$.

Incidentally,

$$\sum_{j=1}^{2}\sum_{k^*=1}^{2}(\mathbf{i}_{\theta_1}^{(2)})_{(j,k^*)} = \sum_{j=1}^{2}\sum_{k^*=1}^{2}(\mathbf{i}_{\theta_2}^{(2)})_{(j,k^*)}$$

$$= \left[\frac{c_1+c_2}{2}\left\{1+\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\} - \frac{1}{2}(c_1+c_2)(c_1+c_2)\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right]\{\theta_0(1-\theta_0)\}^{-3/2}$$

$$+\frac{3}{8}(c_1+c_2)(c_1+c_2)(1-2\theta_0)^2\{\theta_0(1-\theta_0)\}^{-5/2}$$

$$= \frac{1}{2}\{\theta_0(1-\theta_0)\}^{-3/2} + \frac{3}{8}(1-2\theta_0)\{\theta_0(1-\theta_0)\}^{-5/2}$$

$$= i_0^{(2)},$$

which is a scalar in the case of a single group.

3.2.2 $t_{\mathrm{W}\eta}$

Expand $(-\hat{i}_{\mathrm{W}}^{\eta\eta})^{-1/2}$ about $(-i_0^{\eta\eta})^{-1/2}\left(=\left\{\frac{c_1c_2}{\theta_0(1-\theta_0)}\right\}^{-1/2}\right)$,

where $\mathrm{avar}(n^{-1}\hat{\eta}_{\mathrm{W}}) = n^{-1}(-\mathbf{I}_0^{*-1})_{\eta\eta} = n^{-1}\frac{c_1c_2}{\theta_0(1-\theta_0)}$.

$$(-\hat{i}_{\mathrm{W}}^{\eta\eta})^{-1/2}=(-i_0^{\eta\eta})^{-1/2}-\frac{1}{2}\sum_{j=1}^{2}\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\right)_{\eta\eta}(-i_0^{\eta\eta})^{-3/2}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)_j$$

$$+\sum_{j=1}^{2}\sum_{k^*=1}^{2}\left\{\left(-\frac{1}{4}\mathbf{I}_0^{*-1}\frac{\partial^2\mathbf{I}_0^*}{\partial\theta_{0j}\partial\theta_{0k^*}}\mathbf{I}_0^{*-1}+\frac{1}{2}\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0k^*}}\mathbf{I}_0^{*-1}\right)_{\eta\eta}(-i_0^{\eta\eta})^{-3/2}\right.$$

$$\left.+\frac{3}{8}\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\right)_{\eta\eta}\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0k^*}}\mathbf{I}_0^{*-1}\right)_{\eta\eta}(-i_0^{\eta\eta})^{-5/2}\right\}$$

$$\times(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)_j(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)_{k^*}+O_p(n^{-3/2})$$

$$\equiv(-i_0^{\eta\eta})^{-1/2}+\mathbf{i}_\eta^{(1)}{}'(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)+\mathbf{i}_\eta^{(2)}{}'(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<2>}+O_p(n^{-3/2}),$$

where

$$\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{01}}\mathbf{I}_0^{*-1}\right)_{\eta\eta}=c_2\frac{-c_1(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2}c_2=-c_1c_2^2\frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2},$$

$$\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{02}}\mathbf{I}_0^{*-1}\right)_{\eta\eta}=-c_1^2c_2\frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2},$$

consequently,

$$(\mathbf{i}_\eta^{(1)})_j=-\frac{1}{2}\left(\mathbf{I}_0^{*-1}\frac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\right)_{\eta\eta}(-i_0^{\eta\eta})^{-3/2}$$

$$=\frac{1}{2}\frac{(c_1c_2)^2}{c_j}\frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2}\left\{\frac{c_1c_2}{\theta_0(1-\theta_0)}\right\}^{-3/2}$$

$$=\frac{1}{2}\frac{(c_1c_2)^{1/2}}{c_j}\frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^{1/2}}\quad(j=1,2),$$

where $\left(\mathbf{I}_0^{*-1}\dfrac{\partial\mathbf{I}_0^*}{\partial\theta_{0j}}\mathbf{I}_0^{*-1}\right)_{\eta\eta}=-\dfrac{(c_1c_2)^2}{c_j}\dfrac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2}.$

$$
\begin{aligned}
(\mathbf{i}_\eta^{(2)})_{(jk^*)} &= \left[-\frac{\delta_{jk^*}}{2} \frac{(c_1 c_2)^2}{c_j} \left(\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right) \right. \\
&\quad \left. + \frac{1}{2}(-1)^{1-\delta_{jk^*}} (c_1 c_2)^2 \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right] \left\{ \frac{c_1 c_2}{\theta_0(1-\theta_0)} \right\}^{-3/2} \\
&\quad + \frac{3}{8} \frac{c_1^4 c_2^4}{c_j c_{k^*}} \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^4} \left\{ \frac{c_1 c_2}{\theta_0(1-\theta_0)} \right\}^{-5/2} \\
&= -\frac{\delta_{jk^*}}{2} \frac{(c_1 c_2)^{1/2}}{c_j} \left(\frac{1}{\{\theta_0(1-\theta_0)\}^{1/2}} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^{3/2}} \right) \\
&\quad + \frac{1}{2}(-1)^{1-\delta_{jk^*}} (c_1 c_2)^{1/2} (1-2\theta_0)^2 \{\theta_0(1-\theta_0)\}^{-3/2} \\
&\quad + \frac{3}{8} \frac{(c_1 c_2)^{3/2}}{c_j c_{k^*}} (1-2\theta_0)^2 \{\theta_0(1-\theta_0)\}^{-3/2}.
\end{aligned}
$$

Note that while $t_{\mathrm{W}\theta} \equiv n^{1/2}(\hat{i}_{\mathrm{W}}^{\theta\theta})^{-1/2}(\hat{\theta}_{\mathrm{W}} - \theta_0)$, $t_{\mathrm{W}\eta} \equiv n^{-1/2}(-\hat{i}_{\mathrm{W}}^{\eta\eta})^{-1/2}\hat{\eta}_{\mathrm{W}}$. Alternatively, $t_{\mathrm{W}\eta} = n^{1/2}(-\hat{i}_{\mathrm{W}}^{\eta\eta})^{-1/2}(n^{-1}\hat{\eta}_{\mathrm{W}})$, where

$$
n^{-1}\hat{\eta}_{\mathrm{W}} = \frac{n_{\mathrm{W}}}{n} \frac{c_{\mathrm{W}1} c_{\mathrm{W}2} (\hat{p}_{\mathrm{W}2} - \hat{p}_{\mathrm{W}1})}{\hat{\theta}_{\mathrm{W}}(1-\hat{\theta}_{\mathrm{W}})} = O_p(n^{-1/2}).
$$

4. Results using special properties of Example 2.1 (Example 1.2)

4.1 Partial derivatives of $\hat{\eta}$ with respect to $\hat{p}_1$ and $\hat{p}_2$

Note that $\hat{\theta} = c_1 \hat{p}_1 + c_2 \hat{p}_2$ and $n^{-1}\hat{\eta} = \dfrac{c_1 c_2 (\hat{p}_2 - \hat{p}_1)}{\hat{\theta}(1-\hat{\theta})}$, then

$$
\begin{aligned}
n^{-1} \frac{\partial \hat{\eta}}{\partial \hat{\mathbf{p}}} \Big|_{\hat{\mathbf{p}}=\boldsymbol{\theta}_0} &= c_1 c_2 \left\{ \frac{(-1,1)'}{\hat{\theta}(1-\hat{\theta})} - \frac{\hat{p}_2 - \hat{p}_1}{\{\hat{\theta}(1-\hat{\theta})\}^2} (1-2\hat{\theta})(c_1, c_2)' \right\}_{\hat{\mathbf{p}}=\boldsymbol{\theta}_0} \\
&= c_1 c_2 \frac{(-1,1)'}{\theta_0(1-\theta_0)},
\end{aligned}
$$

$$n^{-1}\frac{\partial^2\hat{\eta}}{(\partial\hat{\mathbf{p}})^{<2>}}\Big|_{\hat{\mathbf{p}}=\boldsymbol{\theta}_0}=c_1c_2\left\{-\frac{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}}{\{\hat{\theta}(1-\hat{\theta})\}^2}(1-2\hat{\theta})\right.$$

$$\left.+2\frac{\hat{p}_2-\hat{p}_1}{\{\hat{\theta}(1-\hat{\theta})\}^2}\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}+\frac{2(\hat{p}_2-\hat{p}_1)}{\{\hat{\theta}(1-\hat{\theta})\}^3}(1-2\hat{\theta})^2\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}\right\}_{\hat{\mathbf{p}}=\boldsymbol{\theta}_0}$$

$$=-\frac{c_1c_2(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\right\}$$

$$=-\frac{c_1c_2(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2}(-2c_1,c_1-c_2,c_1-c_2,2c_2)',$$

$$n^{-1}\frac{\partial^3\hat{\eta}}{(\partial\hat{\mathbf{p}})^{<3>}}\Big|_{\hat{\mathbf{p}}=\boldsymbol{\theta}_0}=2c_1c_2\left[\frac{1}{\{\theta_0(1-\theta_0)\}^2}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\right]$$

$$\times\left\{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}+\begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\right\}$$

$$=2c_1c_2\left[\frac{1}{\{\theta_0(1-\theta_0)\}^2}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\right]$$

$$\times\{(-2c_1^2,-2c_1c_2,c_1^2-c_1c_2,c_1c_2-c_2^2,c_1^2-c_1c_2,c_1c_2-c_2^2,2c_1c_2,2c_2^2)'$$

$$+(-c_1^2,c_1^2,-c_1c_2,c_1c_2,-c_1c_2,c_1c_2,-c_2^2,c_2^2)'\}$$

$$=2c_1c_2\left[\frac{1}{\{\theta_0(1-\theta_0)\}^2}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\right](-3c_1^2,c_1^2-2c_1c_2,c_1^2-2c_1c_2,$$

$$2c_1c_2-c_2^2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,2c_1c_2-c_2^2,3c_2^2)'$$

where

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}\otimes\begin{pmatrix}-1\\1\end{pmatrix}=(c_1^2,c_1c_2,c_1c_2,c_2^2)'\otimes(-1,1)'$$

$$=(-c_1^2,c_1^2,-c_1c_2,c_1c_2,-c_1c_2,c_1c_2,-c_2^2,c_2^2)'.$$

4.2 Asymptotic cumulants of $\hat{\theta}$ and $\hat{\eta}$ by ML

Since $\hat{\theta} = c_1\hat{p}_1 + c_2\hat{p}_2$ is a usual sample proportion of size *n*,

$$\kappa_1(\hat{\theta} - \theta_0) = 0,$$

$$\kappa_2(\hat{\theta}) = n^{-1}\theta_0(1-\theta_0),$$

$$\kappa_3(\hat{\theta}) = n^{-2}\theta_0(1-\theta_0)(1-2\theta_0),$$

$$\kappa_4(\hat{\theta}) = n^{-3}\theta_0(1-\theta_0)\{1-6\theta_0(1-\theta_0)\}.$$

On the other hand, since

$$n^{-1}\hat{\eta} = \frac{c_1c_2(\hat{p}_2 - \hat{p}_1)}{\hat{\theta}(1-\hat{\theta})} = \frac{c_1c_2(\hat{p}_2 - \hat{p}_1)}{(c_1\hat{p}_1 + c_2\hat{p}_2)(1 - c_1\hat{p}_1 - c_2\hat{p}_2)},$$

we use

$$\kappa_1(\hat{\mathbf{p}}) = 0,$$

$$\kappa_2(\hat{\mathbf{p}}) = n^{-1}\theta_0(1-\theta_0)\,\mathrm{vec}\ \mathrm{diag}(c_1^{-1}, c_2^{-1}),$$

$$\kappa_3(\hat{\mathbf{p}}) = n^{-2}\theta_0(1-\theta_0)(1-2\theta_0)(c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})',$$

$$\kappa_4(\hat{\mathbf{p}}) = n^{-3}\theta_0(1-\theta_0)\{1-6\theta_0(1-\theta_0)\}(c_1^{-3}, \mathbf{0}'_{(14)}, c_2^{-3})'$$

$$(n\ \mathrm{cov}(\hat{\mathbf{p}}) = \theta_0(1-\theta_0)\,\mathrm{diag}(c_1^{-1}, c_2^{-1})).$$

$$\kappa_1(n^{-1}\hat{\eta}) = \frac{1}{2}\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\mathrm{E}_\theta\{(\hat{\mathbf{p}} - \mathbf{p})\}^{<2>} + O(n^{-2)})$$

$$= \frac{1}{2}\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}} n^{-1}\theta_0(1-\theta_0)\,\mathrm{vec}\ \mathrm{diag}(c_1^{-1}, c_2^{-1}) + O(n^{-2)})$$

$$= -n^{-1}\frac{c_1c_2(1-2\theta_0)}{2\theta_0(1-\theta_0)}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix} + \begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\right\}\mathrm{vec}\ \mathrm{diag}(c_1^{-1}, c_2^{-1})$$

$$+O(n^{-2})$$

$$= -n^{-1}\frac{c_1c_2(1-2\theta_0)}{\theta_0(1-\theta_0)}(-1+1) + O(n^{-2})$$

$$= O(n^{-2})\ \ (\alpha_{\eta 1} = 0).$$

$$\kappa_2(n^{-1}\hat{\eta}) = \frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} n^{-1}\theta_0(1-\theta_0)\,\mathrm{diag}(c_1^{-1}, c_2^{-1})\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}}$$

$$+\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\kappa_3(\hat{\mathbf{p}}, \hat{\mathbf{p}}'^{<2>})\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p})^{<2>}} + \frac{1}{4}\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\kappa_2(\hat{\mathbf{p}}^{<2>}, \hat{\mathbf{p}}'^{<2>})\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p})^{<2>}}$$

$$-n^{-2}\alpha_{\eta 1}^2 + \frac{1}{3}\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\kappa_2(\hat{\mathbf{p}}, \hat{\mathbf{p}}'^{<3>})\frac{n^{-1}\partial^3\eta_0}{(\partial\mathbf{p})^{<3>}} + O(n^{-3})$$

$$= n^{-1}c_1c_2\frac{(-1,\ 1)}{\theta_0(1-\theta_0)}\theta_0(1-\theta_0)\,\mathrm{diag}(c_1^{-1}, c_2^{-1})c_1c_2\frac{(-1,\ 1)'}{\theta_0(1-\theta_0)}$$

$$-n^{-2}\left[c_1c_2\frac{(-1,\ 1)}{\theta_0(1-\theta_0)}\otimes\left\{\frac{c_1c_2(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\right\}'\right\}\right]$$

$$\times\theta_0(1-\theta_0)(1-2\theta_0)(c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})'$$

$$+\frac{n^{-2}}{2}\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\{n\,\mathrm{cov}(\hat{\mathbf{p}})\}^{<2>}\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p})^{<2>}}$$

$$+n^{-2}\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}[n\,\mathrm{cov}(\hat{\mathbf{p}})\otimes\mathrm{vec}'\{n\,\mathrm{cov}(\hat{\mathbf{p}})\}]\frac{n^{-1}\partial^3\eta_0}{(\partial\mathbf{p})^{<3>}} + O(n^{-3})$$

$$= n^{-1}\frac{c_1c_2}{\theta_0(1-\theta_0)} - n^{-2}2(c_1c_2)^2\frac{1}{c_1c_2}\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2}$$

$$+\frac{n^{-2}}{2}\frac{(c_1c_2)^2(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^4}\{\theta_0(1-\theta_0)\}^2(-2c_1, c_1-c_2, c_1-c_2, 2c_2)$$

$$\times\mathrm{diag}\left(\frac{1}{c_1^2}, \frac{1}{c_1c_2}, \frac{1}{c_1c_2}, \frac{1}{c_2^2}\right)(-2c_1, c_1-c_2, c_1-c_2, 2c_2)'$$

$$+n^{-2}2\frac{(c_1c_2)^2}{\theta_0(1-\theta_0)}(-1,1)\left\{\begin{pmatrix}c_1^{-1} & 0\\ 0 & c_2^{-1}\end{pmatrix}\otimes(c_1^{-1}, 0, 0, c_2^{-1})\right\}$$

$$\times\left[\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\right]\{\theta_0(1-\theta_0)\}^2$$

$$\times(-3c_1^2, c_1^2-2c_1c_2, c_1^2-2c_1c_2, 2c_1c_2-c_2^2, c_1^2-2c_1c_2, 2c_1c_2-c_2^2,$$

$$2c_1c_2-c_2^2, 3c_2^2)' + O(n^{-3})$$

$$
\begin{aligned}
&= n^{-1}\frac{c_1c_2}{\theta_0(1-\theta_0)} - n^{-2}2c_1c_2\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \\
&\quad + \frac{n^{-2}}{2}\frac{(c_1c_2)^2(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2}\left\{4+\frac{2(c_1-c_2)^2}{c_1c_2}+4\right\} \\
&\quad + n^{-2}2(c_1c_2)^2\left[\frac{1}{\theta_0(1-\theta_0)}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2}\right] \\
&\qquad \times(-1,1)(-3+2-c_1^{-1}c_2, c_1c_2^{-1}-2+3)' + O(n^{-3}) \\
&= n^{-1}\frac{c_1c_2}{\theta_0(1-\theta_0)} - n^{-2}2c_1c_2\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \\
&\quad + n^{-2}\{4(c_1c_2)^2+c_1c_2(c_1-c_2)^2\}\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \\
&\quad + n^{-2}2(c_1c_2)^2\left[\frac{1}{\theta_0(1-\theta_0)}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2}\right](2+c_1^{-1}c_2+c_1c_2^{-1}) + O(n^{-3}) \\
&= n^{-1}\frac{c_1c_2}{\theta_0(1-\theta_0)} + n^{-2}\frac{c_1c_2}{\theta_0(1-\theta_0)}\left\{2+\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\} + O(n^{-3}) \\
&\equiv n^{-1}\alpha_{\eta 2} + n^{-2}\alpha_{\eta\Delta 2} + O(n^{-3}),
\end{aligned}
$$

where $4(c_1c_2)^2 + c_1c_2(c_1-c_2)^2 = c_1c_2$ and $2+c_1^{-1}c_2+c_1c_2^{-1} = 1/(c_1c_2)$ are used.

$$
\begin{aligned}
\kappa_3(n^{-1}\hat{\eta}) &= n^{-2}\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<3>} n^2\kappa_3(\hat{\mathbf{p}}) \\
&\quad + n^{-2}\frac{3}{2}\left\{\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\otimes\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<2>}\right\}\sum^{(3)}\{\mathrm{vec}\; n\mathrm{cov}(\hat{\mathbf{p}})\}^{<2>} \\
&\quad - n^{-2}\frac{3}{2}\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\mathrm{vec}\; n\mathrm{cov}(\hat{\mathbf{p}})\alpha_{\eta 2} + O(n^{-3})
\end{aligned}
$$

$$= n^{-2}\frac{(c_1c_2)^3}{\{\theta_0(1-\theta_0)\}^3}(-1,1)^{<3>}\theta_0(1-\theta_0)(1-2\theta_0)(c_1^{-2},\mathbf{0}'_{(6)},c_2^{-2})'$$

$$+n^{-2}3\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\,n\,\mathrm{cov}(\hat{\mathbf{p}})\frac{n^{-1}\partial^2\eta_0}{\partial\mathbf{p}\,\partial\mathbf{p}'}n\,\mathrm{cov}(\hat{\mathbf{p}})\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}}+O(n^{-3})$$

$$= n^{-2}\frac{c_1c_2(c_1-c_2)}{\{\theta_0(1-\theta_0)\}^2}(1-2\theta_0)-n^{-2}3\frac{(c_1c_2)^3}{\{\theta_0(1-\theta_0)\}^2}(1-2\theta_0)$$

$$\times(-1,1)\begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1}\end{pmatrix}\begin{pmatrix} -2c_1 & c_1-c_2 \\ c_1-c_2 & 2c_2\end{pmatrix}\begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1}\end{pmatrix}\begin{pmatrix} -1 \\ 1\end{pmatrix}+O(n^{-3})$$

$$= n^{-2}\frac{c_1c_2(c_1-c_2)}{\{\theta_0(1-\theta_0)\}^2}(1-2\theta_0)-n^{-2}3\frac{(c_1c_2)^3}{\{\theta_0(1-\theta_0)\}^2}(1-2\theta_0)$$

$$\times(-c_1^{-1},c_2^{-1})\begin{pmatrix} -2c_1 & c_1-c_2 \\ c_1-c_2 & 2c_2\end{pmatrix}\begin{pmatrix} -c_1^{-1} \\ c_2^{-1}\end{pmatrix}+O(n^{-3})$$

$$= n^{-2}\frac{c_1c_2(c_1-c_2)}{\{\theta_0(1-\theta_0)\}^2}(1-2\theta_0)-n^{-2}3\frac{(c_1c_2)^3}{\{\theta_0(1-\theta_0)\}^2}(1-2\theta_0)$$

$$\times\left\{-2c_1^{-1}-2\frac{c_1-c_2}{c_1c_2}+2c_2^{-1}\right\}+O(n^{-3})$$

$$= n^{-2}c_1c_2(c_1-c_2)\frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2}+O(n^{-3})$$

$$\equiv n^{-2}\beta_{\eta 3}+O(n^{-3}),$$

$$\kappa_4(n^{-1}\hat{\eta}) = n^{-3}\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<4>} n^3\kappa_4(\hat{\mathbf{p}})$$

$$+2n^{-3}\left\{\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<3>}\otimes\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\right\}\sum^{(10)}\{\mathrm{vec}\, n\,\mathrm{cov}(\hat{\mathbf{p}})\otimes n^2\kappa_3(\hat{\mathbf{p}})\}$$

$$+n^{-3}\left\{\frac{3}{2}\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<2>}\otimes\left(\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\right)^{<2>}+\frac{2}{3}\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<3>}\otimes\frac{n^{-1}\partial^3\eta_0}{(\partial\mathbf{p}')^{<3>}}\right\}$$

$$\times\sum^{(15)}\{\mathrm{vec}\, n\,\mathrm{cov}(\hat{\mathbf{p}})\}^{<3>} - 6n^{-3}\alpha_{\eta 2}\alpha_{\eta\Delta 2} + O(n^{-4}),$$

where the first term is

$$n^{-3}\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<4>} n^3\kappa_4(\hat{\mathbf{p}})$$

$$= n^{-3}\frac{(c_1c_2)^4}{\{\theta_0(1-\theta_0)\}^4}(-1,1)^{<4>}\theta_0(1-\theta_0)\{1-6\theta_0(1-\theta_0)\}(c_1^{-3},\mathbf{0}'_{(14)},c_2^{-3})'$$

$$= n^{-3}(c_1c_2^4 + c_1^4c_2)\frac{1-6\theta_0(1-\theta_0)}{\{\theta_0(1-\theta_0)\}^3}$$

$$= n^{-3}c_1c_2(1-3c_1c_2)\frac{1}{\{\theta_0(1-\theta_0)\}^2}\left\{\frac{1}{\theta_0(1-\theta_0)}-6\right\},$$

the second term is

$$2n^{-3}\left\{\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<3>}\otimes\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\right\}\sum^{(10)}\{\text{vec}\, n\text{cov}(\hat{\mathbf{p}})\otimes n^2\kappa_3(\hat{\mathbf{p}})\}$$

$$=-n^{-3}2\frac{(c_1c_2)^4}{\{\theta_0(1-\theta_0)\}^5}(1-2\theta_0)\{(-1,1)^{<3>}\otimes(-2c_1,c_1-c_2,c_1-c_2,2c_2)\}$$

$$\times\{\theta_0(1-\theta_0)\}^2(1-2\theta_0)\sum^{(10)}(c_1^{-1},0,0,c_2^{-1})'\otimes(c_1^{-2},\mathbf{0}'_{(6)},c_2^{-2})'$$

$$=-n^{-3}2\frac{(c_1c_2)^4}{\{\theta_0(1-\theta_0)\}^3}(1-2\theta_0)^2\{(-1,1)^{<3>}\otimes(-2c_1,c_1-c_2,c_1-c_2,2c_2)\}$$

$$\times\sum^{(10)}(c_1^{-1},0,0,c_2^{-1})'\otimes(c_1^{-2},\mathbf{0}'_{(6)},c_2^{-2})'$$

and the third term is

$$n^{-3}\left\{\frac{3}{2}\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<2>}\otimes\left(\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{<2>}}\right)^{<2>}+\frac{2}{3}\left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'}\right)^{<3>}\otimes\frac{n^{-1}\partial^3\eta_0}{(\partial\mathbf{p}')^{<3>}}\right\}$$

$$\times\sum^{(15)}\{\text{vec}\, n\text{cov}(\hat{\mathbf{p}})\}^{<3>}$$

$$=n^{-3}\left[\frac{3}{2}\frac{(c_1c_2)^4}{\{\theta_0(1-\theta_0)\}^6}(1-2\theta_0)^2\{(-1,1)^{<2>}\otimes(-2c_1,c_1-c_2,c_1-c_2,2c_2)^{<2>}\}\right.$$

$$+\frac{2}{3}\frac{(c_1c_2)^4}{\{\theta_0(1-\theta_0)\}^3}2\left[\frac{1}{\{\theta_0(1-\theta_0)\}^2}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\right]$$

$$\times\{(-1,1)^{<3>}\otimes(-3c_1^2,c_1^2-2c_1c_2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,c_1^2-2c_1c_2,$$

$$\left.2c_1c_2-c_2^2,2c_1c_2-c_2^2,3c_2^2)'\}\{\theta_0(1-\theta_0)\}^3\sum^{(15)}(c_1^{-1},0,0,c_2^{-1})^{<3>}\right].$$

Then,

$$\kappa_4(n^{-1}\hat{\eta}) = n^{-3}\Bigg[_{\text{(A)}} \frac{c_1c_2(1-3c_1c_2)}{\{\theta_0(1-\theta_0)\}^2}\left\{\frac{1}{\theta_0(1-\theta_0)}-6\right\}$$

$$-2\frac{(c_1c_2)^4}{\{\theta_0(1-\theta_0)\}^3}(1-2\theta_0)^2\{(-1,1)^{<3>}\otimes(-2c_1,c_1-c_2,c_1-c_2,2c_2)\}$$

$$\times\sum^{(10)}(c_1^{-1},0,0,c_2^{-1})'\otimes(c_1^{-2},\mathbf{0}'_{(6)},c_2^{-2})'$$

$$+(c_1c_2)^4\Bigg[_{\text{(B)}} \frac{3}{2}\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\{(-1,1)^{<2>}\otimes(-2c_1,c_1-c_2,c_1-c_2,2c_2)^{<2>}\}$$

$$+\frac{4}{3}\left[\frac{1}{\{\theta_0(1-\theta_0)\}^2}+\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3}\right]$$

$$\times\{(-1,1)^{<3>}\otimes(-3c_1^2,c_1^2-2c_1c_2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,c_1^2-2c_1c_2,$$

$$2c_1c_2-c_2^2,2c_1c_2-c_2^2,3c_2^2)'\}\Bigg]_{\text{(B)}}\sum^{(15)}(c_1^{-1},0,0,c_2^{-1})^{<3>}-6\alpha_{\eta 2}\alpha_{\eta\Delta 2}\Bigg]_{\text{(A)}}+O(n^{-4})$$

$$\equiv n^{-3}\alpha_{\eta 4}+O(n^{-4}),$$

4.3 Asymptotic cumulants of t_θ and t_η by maximum likelihood

$t_\theta = \dfrac{n^{1/2}(\hat{\theta}-\theta_0)}{\{\hat{\theta}(1-\hat{\theta})\}^{1/2}}$, where $\hat{\theta}$ is the MLE. The asymptotic cumulants of t_θ reduce to those of the studentized sample proportion (see e.g., Ogasawara, 2012, p.12):

$p \equiv \theta_0,\ q \equiv 1-\theta_0,$

$\kappa_1(t_\theta) = -n^{-1/2}\dfrac{(pq)^{-1/2}}{2}(1-2p) + O(n^{-3/2}) \equiv n^{-1/2}\alpha_{\theta 1}^{(t)} + O(n^{-2/3}),$

$\kappa_2(t_\theta) = 1 + n^{-1}\left\{\dfrac{7}{4}(1-2p)^2(pq)^{-1} + 3\right\} + O(n^{-2}) \equiv 1 + n^{-1}\alpha_{\theta\Delta 2}^{(t)} + O(n^{-2})$

$(\alpha_{\theta 2}^{(t)} = 1),$

$\kappa_3(t_\theta) = -n^{-1/2}2(pq)^{-1/2}(1-2p) + O(n^{-3/2}) \equiv n^{-1/2}\alpha_{\theta 3}^{(t)} + O(n^{-2/3}),$

$\kappa_4(t_\theta) = n^{-1}\{(pq)^{-1} + 9(1-2p)^2(pq)^{-1} + 6\} + O(n^{-3/2}) \equiv n^{-1}\alpha_{\theta 4}^{(t)} + O(n^{-2}),$

$n\,\mathrm{acov}(\hat{\theta}, \hat{\alpha}_{\theta 1}^{(t)}) = \dfrac{1}{4}(1-2p)^2(pq)^{-1/2} + (pq)^{1/2},$

$n\,\mathrm{acov}(\hat{\theta}, \hat{\alpha}_{\theta 3}^{(t)}) = (1-2p)^2(pq)^{-1/2} + 4(pq)^{1/2}.$

$$t_\eta = \frac{n^{1/2}(n^{-1}\hat{\eta})}{(-\hat{i}^{\eta\eta})^{1/2}} = \frac{n^{1/2}c_1c_2(\hat{p}_2-\hat{p}_1)/\{\hat{\theta}(1-\hat{\theta})\}}{[c_1c_2/\{\hat{\theta}(1-\hat{\theta})\}]^{1/2}}$$

$$= \frac{n^{1/2}(c_1c_2)^{1/2}(\hat{p}_2-\hat{p}_1)}{\{\hat{\theta}(1-\hat{\theta})\}^{1/2}} = \frac{\hat{p}_2-\hat{p}_1}{\{\hat{\theta}(1-\hat{\theta})/(nc_1c_2)\}^{1/2}},$$

where $\hat{\eta} = n\dfrac{c_1c_2(\hat{p}_2-\hat{p}_1)}{\hat{\theta}(1-\hat{\theta})}$ is the MLE and $\hat{i}^{\eta\eta} = -\dfrac{c_1c_2}{\hat{\theta}(1-\hat{\theta})}$.

Noting that $\dfrac{\hat{\theta}(1-\hat{\theta})}{nc_1c_2} = \left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)\hat{\theta}(1-\hat{\theta})$ and

$\mathrm{var}(\hat{p}_1-\hat{p}_2) = \left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)\theta_0(1-\theta_0)$, we find that t_η is the studentized

$\hat{p}_2-\hat{p}_1$.

$$\frac{\partial(\hat{p}_2-\hat{p}_1)}{\partial\hat{\mathbf{p}}}=(-1,1)',\ \frac{\partial\hat{\theta}}{\partial\hat{\mathbf{p}}}=\frac{\partial(c_1\hat{p}_1+c_2\hat{p}_2)}{\partial\hat{\mathbf{p}}}=(c_1,c_2)',$$

$$\frac{\partial t_\eta}{\partial\hat{\mathbf{p}}}=n^{1/2}(c_1c_2)^{1/2}\left[\frac{(-1,1)'}{\{\hat{\theta}(1-\hat{\theta})\}^{1/2}}-\frac{\hat{p}_2-\hat{p}_1}{2\{\hat{\theta}(1-\hat{\theta})\}^{3/2}}(1-2\hat{\theta})(c_1,c_2)\right]',$$

$$\frac{\partial^2 t_\eta}{(\partial\hat{\mathbf{p}})^{<2>}}=n^{1/2}(c_1c_2)^{1/2}\left[-\frac{1-2\hat{\theta}}{2\{\hat{\theta}(1-\hat{\theta})\}^{3/2}}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\right\}\right.$$

$$\left.+(\hat{p}_2-\hat{p}_1)\left[\frac{1}{\{\hat{\theta}(1-\hat{\theta})\}^{3/2}}+\frac{3}{4}\frac{(1-2\hat{\theta})^2}{\{\hat{\theta}(1-\hat{\theta})\}^{5/2}}\right]\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}\right],$$

$$\frac{\partial^3 t_\eta}{(\partial\mathbf{p})^{<3>}}=n^{1/2}(c_1c_2)^{1/2}\left[\frac{1}{\{\theta_0(1-\theta_0)\}^{3/2}}+\frac{3}{4}\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^{5/2}}\right]$$

$$\times\left\{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}+\begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\right\}.$$

$$\kappa_1(t_\eta)=\frac{1}{2}\frac{\partial^2 t_\eta}{(\partial\mathbf{p}')^{<2>}}\mathrm{E}_\theta\{(\hat{\mathbf{p}}-\mathbf{p})^{<2>}\}+O(n^{-3/2})$$

$$=\frac{1}{2}\frac{\partial^2 t_\eta}{(\partial\mathbf{p}')^{<2>}}n^{-1}\theta_0(1-\theta_0)\,\text{vec diag}(c_1^{-1},c_2^{-1})+O(n^{-3/2})$$

$$=-\frac{n^{-1/2}(c_1c_2)^{1/2}(1-2\theta_0)}{4\{\theta_0(1-\theta_0)\}^{1/2}}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\right\}'\text{vec diag}(c_1^{-1},c_2^{-1})$$

$$+O(n^{-3/2})$$

$$=-\frac{n^{-1/2}(c_1c_2)^{1/2}(1-2\theta_0)}{2\{\theta_0(1-\theta_0)\}^{1/2}}(-1+1)+O(n^{-3/2})$$

$$=O(n^{-3/2})\quad(\alpha_{\eta1}^{(t)}=0),$$

$$\kappa_2(t_\eta)=\frac{\partial t_\eta}{\partial \mathbf{p}'}n^{-1}\theta_0(1-\theta_0)\,\mathrm{diag}(c_1^{-1},c_2^{-1})\frac{\partial t_\eta}{\partial \mathbf{p}}$$

$$+\frac{\partial t_\eta}{\partial \mathbf{p}'}\kappa_3(\hat{\mathbf{p}},\hat{\mathbf{p}}'^{<2>})\frac{\partial^2 t_\eta}{(\partial \mathbf{p})^{<2>}}+\frac{1}{4}\frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{<2>}}\kappa_2(\hat{\mathbf{p}}^{<2>},\hat{\mathbf{p}}'^{<2>})\frac{\partial^2 t_\eta}{(\partial \mathbf{p})^{<2>}}$$

$$+\frac{1}{3}\frac{\partial t_\eta}{\partial \mathbf{p}'}\kappa_2(\hat{\mathbf{p}},\hat{\mathbf{p}}'^{<3>})\frac{\partial^3 t_\eta}{(\partial \mathbf{p})^{<3>}}-n^{-1}(\alpha_{\eta 1}^{(t)})^2+O(n^{-2}),$$

where the first term is

$$\frac{c_1c_2}{\theta_0(1-\theta_0)}\theta_0(1-\theta_0)(-1,1)\begin{pmatrix}c_1^{-1} & 0\\ 0 & c_2^{-1}\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix}=1,$$

the second term is

$$-n^{-1}\frac{c_1c_2(1-2\theta_0)}{2\{\theta_0(1-\theta_0)\}^2}\theta_0(1-\theta_0)(1-2\theta_0)$$

$$\times\left[\begin{pmatrix}-1\\ 1\end{pmatrix}\otimes\left\{\begin{pmatrix}-1\\ 1\end{pmatrix}\otimes\begin{pmatrix}c_1\\ c_2\end{pmatrix}+\begin{pmatrix}c_1\\ c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\ 1\end{pmatrix}\right\}\right]'(c_1^{-2},\mathbf{0}'_{(6)},c_2^{-2})'$$

$$=-n^{-1}\frac{c_1c_2(1-2\theta_0)^2}{\theta_0(1-\theta_0)}(c_1^{-1}+c_2^{-1})=-n^{-1}\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)},$$

the third term is

$$n^{-1}\frac{1}{2}\frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{<2>}}\{n\,\mathrm{cov}(\hat{\mathbf{p}})\}^{<2>}\frac{\partial^2 t_\eta}{(\partial \mathbf{p})^{<2>}}$$

$$=n^{-1}\frac{c_1c_2(1-2\theta_0)^2}{8\{\theta_0(1-\theta_0)\}^3}\{\theta_0(1-\theta_0)\}^2(-2c_1,c_1-c_2,c_1-c_2,2c_2)$$

$$\times\mathrm{diag}(c_1^{-2},c_1^{-1}c_2^{-1},c_2^{-1}c_1^{-1},c_2^{-2})(-2c_1,c_1-c_2,c_1-c_2,2c_2)'$$

$$=n^{-1}\frac{c_1c_2(1-2\theta_0)^2}{8\theta_0(1-\theta_0)}\left(4+2\frac{(c_1-c_2)^2}{c_1c_2}+4\right)$$

$$=n^{-1}\frac{(1-2\theta_0)^2}{4\theta_0(1-\theta_0)}$$

and the fourth term is

$$n^{-1}c_1c_2\frac{1}{\{\theta_0(1-\theta_0)\}^{1/2}}\left[\frac{1}{\{\theta_0(1-\theta_0)\}^{3/2}}+\frac{3}{4}\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^{5/2}}\right]$$

$$\times(-1,1)[n\,\mathrm{cov}(\hat{\mathbf{p}})\otimes\mathrm{vec}'\{n\,\mathrm{cov}(\hat{\mathbf{p}})\}]$$

$$\times\left\{\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}+\begin{pmatrix}c_1\\c_2\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}^{<2>}\otimes\begin{pmatrix}-1\\1\end{pmatrix}\right\}$$

$$=n^{-1}c_1c_2\left\{1+\frac{3}{4}\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}(-1,1)\left\{\begin{pmatrix}c_1^{-1}&0\\0&c_2^{-1}\end{pmatrix}\otimes(c_1^{-1},0,0,c_2^{-1})\right\}$$

$$\times(-3c_1^2,c_1^2-2c_1c_2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,$$

$$2c_1c_2-c_2^2,3c_2^2)'$$

$$=n^{-1}c_1c_2\left\{1+\frac{3}{4}\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}(-1,1)(-3+2-c_1^{-1}c_2,c_1c_2^{-1}-2+3)'$$

$$=n^{-1}c_1c_2\left\{1+\frac{3}{4}\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}(2+c_1^{-1}c_2+c_1c_2^{-1})$$

$$=n^{-1}\left\{1+\frac{3}{4}\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}.$$

Consequently,

$$\kappa_2(t_\eta)=1+n^{-1}\left\{1+\left(-1+\frac{1}{4}+\frac{3}{4}\right)\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}+O(n^{-2})$$

$$=1+n^{-1}+O(n^{-2})$$

$$=1+n^{-1}\alpha_{\eta\Delta 2}^{(t)}+O(n^{-2})\quad(\alpha_{\eta 2}^{(t)}=1).$$

$$\kappa_3(t_\eta)=\left(\frac{\partial t_\eta}{\partial\mathbf{p}'}\right)^{<3>}\kappa_3(\hat{\mathbf{p}})+\frac{3}{2}\left\{\frac{\partial^2 t_\eta}{(\partial\mathbf{p}')^{<2>}}\otimes\left(\frac{\partial t_\eta}{\partial\mathbf{p}'}\right)^{<2>}\right\}\sum^{(3)}\{\mathrm{vec}\,\mathrm{cov}(\hat{\mathbf{p}})\}^{<2>}$$

$$-n^{-1/2}3\alpha_{\eta 1}^{(t)}+O(n^{-3/2}),$$

where the first term is

$$n^{-1/2}\frac{(c_1c_2)^{3/2}}{\{\theta_0(1-\theta_0)\}^{3/2}}\theta_0(1-\theta_0)(1-2\theta_0)(-1,1)^{<3>}(c_1^{-2},\mathbf{0}'_{(6)},c_2^{-2})'$$

$$=n^{-1/2}\frac{(c_1c_2)^{3/2}(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^{1/2}}(-c_1^{-2}+c_2^{-2})$$

$$=n^{-1/2}\frac{(c_1-c_2)(1-2\theta_0)}{\{c_1c_2\theta_0(1-\theta_0)\}^{1/2}}$$

the sum of the second and third (actually 0) terms is

$$3\frac{\partial t_\eta}{\partial \mathbf{p}'}\mathrm{cov}(\hat{\mathbf{p}})\frac{\partial^2 t_\eta}{\partial \mathbf{p}\partial \mathbf{p}'}\mathrm{cov}(\hat{\mathbf{p}})\frac{\partial t_\eta}{\partial \mathbf{p}}$$

$$=-3n^{-1/2}\frac{(c_1c_2)^{3/2}}{\theta_0(1-\theta_0)}\{\theta_0(1-\theta_0)\}^2\frac{1-2\theta_0}{2\{\theta_0(1-\theta_0)\}^{3/2}}$$

$$\times(-1,1)\begin{pmatrix}c_1^{-1} & 0\\ 0 & c_2^{-1}\end{pmatrix}\left\{\begin{pmatrix}-1\\ 1\end{pmatrix}(c_1,c_2)+\begin{pmatrix}c_1\\ c_2\end{pmatrix}(-1,1)\right\}\begin{pmatrix}c_1^{-1} & 0\\ 0 & c_2^{-1}\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix}$$

$$=-n^{-1/2}\frac{3}{2}\frac{(c_1c_2)^{3/2}(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^{1/2}}(-c_1^{-1},c_2^{-1})\begin{pmatrix}-2c_1 & c_1-c_2\\ c_1-c_2 & 2c_2\end{pmatrix}\begin{pmatrix}-c_1^{-1}\\ c_2^{-1}\end{pmatrix}$$

$$=0,$$

where the quadratic form on the left-hand side of the last equation is 0.
Then,

$$\kappa_3(t_\eta)=n^{-1/2}\frac{(c_1-c_2)(1-2\theta_0)}{\{c_1c_2\theta_0(1-\theta_0)\}^{1/2}}+O(n^{-3/2})$$

$$\equiv n^{-1/2}\alpha_{\eta 3}^{(t)}+O(n^{-3/2}).$$

$$\kappa_4(t_\eta) = \left(\frac{\partial t_\eta}{\partial \mathbf{p}'}\right)^{<4>} \kappa_4(\hat{\mathbf{p}}) + 2\left\{\left(\frac{\partial t_\eta}{\partial \mathbf{p}'}\right)^{<3>} \otimes \frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{<2>}}\right\}$$

$$\times \sum^{(10)} \{\text{vec cov}(\hat{\mathbf{p}}) \otimes \kappa_3(\hat{\mathbf{p}})\}$$

$$+\left\{\frac{3}{2}\left(\frac{\partial t_\eta}{\partial \mathbf{p}'}\right)^{<2>} \otimes \left(\frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{<2>}}\right)^{<2>} + \frac{2}{3}\left(\frac{\partial t_\eta}{\partial \mathbf{p}'}\right)^{<3>} \otimes \frac{\partial^3 t_\eta}{(\partial \mathbf{p}')^{<3>}}\right\}$$

$$\times \sum^{(15)} \{\text{vec cov}(\hat{\mathbf{p}})\}^{<3>} - n^{-1} 6\alpha_{\eta\Delta 2}^{(t)} + O(n^{-3}),$$

where the first term is

$$n^{-1} \frac{(c_1 c_2)^2}{\{\theta_0(1-\theta_0)\}^2} \theta_0(1-\theta_0)\{1-6\theta_0(1-\theta_0)\}(-1,1)^{<4>}(c_1^{-3}, \mathbf{0}'_{(14)}, c_2^{-3})'$$

$$= n^{-1}\left(\frac{c_2^2}{c_1} + \frac{c_1^2}{c_2}\right)\left\{\frac{1}{\theta_0(1-\theta_0)} - 6\right\}.$$

Since $\frac{c_2^2}{c_1} + \frac{c_1^2}{c_2} = \frac{c_1^3 + c_2^3}{c_1 c_2} = \frac{1-3c_1c_2(c_1+c_2)}{c_1c_2} = \frac{1}{c_1c_2} - 3$, the first term becomes

$$n^{-1}\left(\frac{1}{c_1c_2} - 3\right)\left\{\frac{1}{\theta_0(1-\theta_0)} - 6\right\}.$$

The second term is

$$-n^{-1} 2 \frac{(c_1c_2)^2}{\{\theta_0(1-\theta_0)\}^{3/2}} \frac{1-2\theta_0}{2\{\theta_0(1-\theta_0)\}^{3/2}} \{\theta_0(1-\theta_0)\}^2(1-2\theta_0)$$

$$\times\{(-1,1)^{<3>} \otimes (-2c_1, c_1-c_2, c_1-c_2, 2c_2)\}\sum^{(10)}(c_1^{-1},0,0,c_2^{-1})' \otimes (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})'$$

and the third term is

$$n^{-1}\left[\frac{3}{2}\frac{(c_1c_2)^2}{\{\theta_0(1-\theta_0)\}^4}\frac{(1-2\theta_0)^2}{4}\{(-1,1)^{<2>}\otimes(-2c_1,c_1-c_2,c_1-c_2,2c_2)^{<2>}\}\right.$$

$$+\frac{2}{3}\frac{(c_1c_2)^2}{\{\theta_0(1-\theta_0)\}^{3/2}}\left[\frac{1}{\{\theta_0(1-\theta_0)\}^{3/2}}+\frac{3}{4}\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^{5/2}}\right]$$

$$\times(-1,1)^{<3>}\otimes(-3c_1^2,c_1^2-2c_1c_2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,$$

$$\left.2c_1c_2-c_2^2,3c_2^2)\right]\{\theta_0(1-\theta_0)\}^3\sum^{(15)}(c_1^{-1},0,0,c_2^{-1})'^{<3>}.$$

Then,

$$\kappa_4(t_\eta)=n^{-1}\underset{(\mathrm{A})}{\Bigg[}\left(\frac{1}{c_1c_2}-3\right)\left(\frac{1}{\theta_0(1-\theta_0)}-6\right)$$

$$-(c_1c_2)^2\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\{(-1,1)^{<3>}\otimes(-2c_1,c_1-c_2,c_1-c_2,2c_2)\}$$

$$\times\sum^{(10)}(c_1^{-1},0,0,c_2^{-1})'\otimes(c_1^{-2},\mathbf{0}'_{(6)},c_2^{-2})$$

$$\underset{(\mathrm{B})}{\Bigg[}\frac{3}{8}\frac{(c_1c_2)^2}{\theta_0(1-\theta_0)}(1-2\theta_0)^2\{(-1,1)^{<2>}\otimes(-2c_1,c_1-c_2,c_1-c_2,2c_2)^{<2>}\}$$

$$+\frac{2}{3}(c_1c_2)^2\left\{1+\frac{3}{4}\frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}$$

$$\times(-1,1)^{<3>}\otimes(-3c_1^2,c_1^2-2c_1c_2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,c_1^2-2c_1c_2,2c_1c_2-c_2^2,$$

$$2c_1c_2-c_2^2,3c_2^2)\underset{(\mathrm{B})}{\Bigg]}\sum^{(15)}(c_1^{-1},0,0,c_2^{-1})'^{<3>}-6\alpha_{\eta\Delta 2}^{(t)}\underset{(\mathrm{A})}{\Bigg]}+O(n^{-2})$$

$$\equiv n^{-1}\alpha_{\eta 4}^{(t)}+O(n^{-2}).$$

$n\,\mathrm{acov}(n^{-1}\hat{\eta},\hat{\alpha}_{\eta 1}^{(t)})=0$ (recall that $\alpha_{\eta 1}^{(t)}=0$),

$$n\,\mathrm{acov}(n^{-1}\hat{\eta},\hat{\alpha}_{\eta 3}^{(t)})=(c_1c_2)^{1/2}(c_1-c_2)n\,\mathrm{acov}\left[\frac{\hat{p}_2-\hat{p}_1}{\hat{\theta}(1-\hat{\theta})},\frac{1-2\hat{\theta}}{\{\hat{\theta}(1-\hat{\theta})\}^{1/2}}\right]$$

$$=\frac{(c_1c_2)^{1/2}}{\theta_0(1-\theta_0)}(c_1-c_2)\left[-\frac{2}{\{\theta_0(1-\theta_0)\}^{1/2}}-\frac{(1-2\theta_0)^2}{2\{\theta_0(1-\theta_0)\}^{3/2}}\right]$$

$$\times\theta_0(1-\theta_0)(-1,1)\begin{pmatrix}c_1^{-1} & 0\\ 0 & c_2^{-1}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

$$=0.$$

4.4 Asymptotic cumulants of $\hat{\theta}_{\mathrm{W}}$ and $\hat{\eta}_{\mathrm{W}}$ by the weighted score method

$$\hat{\theta}_{\mathrm{W}}=\frac{m_1+2k}{n+4k}=\frac{\hat{\theta}+n^{-1}2k}{1+n^{-1}4k}=\hat{\theta}\left(1-\frac{n^{-1}4k}{1+n^{-1}4k}\right)+\frac{n^{-1}2k}{1+n^{-1}4k}$$

$$=\hat{\theta}-n^{-1}4k\hat{\theta}+n^{-1}2k+O_p(n^{-2})=\hat{\theta}+n^{-1}2k(1-2\hat{\theta})+O_p(n^{-2}).$$

In the following, only the asymptotic cumulants different from those by ML are shown.

$$\kappa_1(\hat{\theta}_{\mathrm{W}}-\theta_0)=n^{-1}\alpha_{\theta 1}+n^{-1}2k(1-2\theta_0)+O(n^{-2})$$

$$=n^{-1}2k(1-2\theta_0)+O(n^{-2})$$

$$\equiv n^{-1}\alpha_{\mathrm{W}\theta 1}+O(n^{-2})\ \text{(recall that } \alpha_{\theta 1}=0\text{)},$$

$$\kappa_2(\hat{\theta}_{\mathrm{W}})=n^{-1}\alpha_{\theta 2}+n^{-2}(\alpha_{\theta\Delta 2}-8k\alpha_{\theta 2})+O(n^{-2})$$

$$\equiv n^{-1}\alpha_{\theta 2}+n^{-2}\alpha_{\mathrm{W}\theta\Delta 2}+O(n^{-3})\ \ (\alpha_{\theta 2}=\alpha_{\mathrm{W}\theta 2};\ \alpha_{\mathrm{W}\theta\Delta 2}\le\alpha_{\theta\Delta 2}).$$

$$\hat{\eta}_{\mathrm{W}}=\frac{n_{\mathrm{W}}c_{\mathrm{W}1}c_{\mathrm{W}2}(\hat{p}_{\mathrm{W}2}-\hat{p}_{\mathrm{W}1})}{\hat{\theta}_{\mathrm{W}}(1-\hat{\theta}_{\mathrm{W}})},\ \text{where}$$

$$n_{\mathrm{W}}\equiv n+4k,c_{\mathrm{W}1}=\frac{n_1+2k}{n+4k},c_{\mathrm{W}2}=\frac{n_2+2k}{n+4k},$$

$$\hat{p}_{\mathrm{W}1}=\frac{m_{11}+k}{n_1+2k},\ \hat{p}_{\mathrm{W}2}\equiv\frac{m_{21}+k}{n_2+2k},$$

$$c_{\mathrm{W}1}=\frac{c_1+n^{-1}2k}{1+n^{-1}4k}=c_1+n^{-1}2k(1-2c_1)+O(n^{-2}),$$

$$c_{\mathrm{W}2}=c_2+n^{-1}2k(1-2c_2)+O(n^{-2}),$$

$$\hat{p}_{\mathrm{W}1}=\frac{\hat{p}_1+n_1^{-1}k}{1+n_1^{-1}2k}=\frac{\hat{p}_1+n^{-1}c_1^{-1}k}{1+n^{-1}c_1^{-1}2k}=\hat{p}_1+n^{-1}c_1^{-1}k(1-2\hat{p}_1)+O_p(n^{-2}),$$

$$\hat{p}_{\mathrm{W}2}=\hat{p}_2+n^{-1}c_2^{-1}k(1-2\hat{p}_2)+O_p(n^{-2}),$$

$$\hat{p}_{\mathrm{W}2}-\hat{p}_{\mathrm{W}1}=\hat{p}_2-\hat{p}_1+n^{-1}k\{c_2^{-1}-c_1^{-1}-2(c_2^{-1}\hat{p}_2-c_1^{-1}\hat{p}_1)\}+O_p(n^{-2}),$$

$$\{\hat{\theta}_{\mathrm{W}}(1-\hat{\theta}_{\mathrm{W}})\}^{-1}=\{\hat{\theta}(1-\hat{\theta})\}^{-1}-\{\hat{\theta}(1-\hat{\theta})\}^{-2}(1-2\hat{\theta})(\hat{\theta}_{\mathrm{W}}-\hat{\theta})+O_p(n^{-2})$$

$$=\{\hat{\theta}(1-\hat{\theta})\}^{-1}-n^{-1}\{\hat{\theta}(1-\hat{\theta})\}^{-2}2k(1-2\hat{\theta})^2+O_p(n^{-2}).$$

From the above results,

$$n^{-1}\hat{\eta}_{\mathrm{W}}=n^{-1}(n+4k)\{c_1+n^{-1}2k(1-2c_1)\}\{c_2+n^{-1}2k(1-2c_2)\}$$

$$\times[\hat{p}_2-\hat{p}_1+n^{-1}k\{c_2^{-1}-c_1^{-1}-2(c_2^{-1}\hat{p}_2-c_1^{-1}\hat{p}_1)\}]$$

$$\times\left[\frac{1}{\hat{\theta}(1-\hat{\theta})}-n^{-1}\frac{2k(1-2\hat{\theta})^2}{\{\hat{\theta}(1-\hat{\theta})\}^2}\right]+O_p(n^{-2})$$

$$=\frac{c_1c_2(\hat{p}_2-\hat{p}_1)}{\hat{\theta}(1-\hat{\theta})}$$

$$+n^{-1}\left[\left\{4k+2kc_1^{-1}(1-2c_1)+2kc_2^{-1}(1-2c_2)-\frac{2k(1-2\hat{\theta})^2}{\hat{\theta}(1-\hat{\theta})}\right\}\right.$$

$$\left.\times\frac{c_1c_2(\hat{p}_2-\hat{p}_1)}{\hat{\theta}(1-\hat{\theta})}+k\{c_2^{-1}-c_1^{-1}-2(c_2^{-1}\hat{p}_2-c_1^{-1}\hat{p}_1)\}\frac{c_1c_2}{\hat{\theta}(1-\hat{\theta})}\right]+O_p(n^{-2})$$

$$=n^{-1}\hat{\eta}+n^{-1}\left[\left\{-4k+\frac{2k}{c_1c_2}-\frac{2k(1-2\hat{\theta})^2}{\hat{\theta}(1-\hat{\theta})}\right\}n^{-1}\hat{\eta}\right.$$

$$\left.+k\{c_1-c_2-2(c_1\hat{p}_2-c_2\hat{p}_1)\}\frac{1}{\hat{\theta}(1-\hat{\theta})}\right]+O_p(n^{-2}).$$

Then,

$$\kappa_1(n^{-1}\hat{\eta}_{\mathrm{W}} - n^{-1}\eta_0) = n^{-1}\left\{\alpha_{\eta 1} + k\frac{(c_1 - c_2)(1-2\theta_0)}{\theta_0(1-\theta_0)}\right\} + O(n^{-2})$$

$$= n^{-1}k\frac{(c_1 - c_2)(1-2\theta_0)}{\theta_0(1-\theta_0)} + O(n^{-2})$$

$$\equiv n^{-1}\alpha_{\mathrm{W}\eta 1} + O(n^{-2})$$

(recall that $\eta_0 = 0$ and $\alpha_{\eta 1} = 0$),

$$\kappa_2(n^{-1}\hat{\eta}_{\mathrm{W}}) = n^{-1}\alpha_{\eta 2} + n^{-2}\left[\alpha_{\eta\Delta 2} + 4k\left\{-2 + \frac{1}{c_1c_2} - \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}\alpha_{\eta 2}\right.$$

$$-2k\frac{(c_1 - c_2)(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2}\theta_0(1-\theta_0)(c_1, c_2)\begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1}\end{pmatrix}\frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}}$$

$$\left. -4k\frac{\partial(c_1p_2 - c_2p_1)}{\partial \mathbf{p}}\begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1}\end{pmatrix}\frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}}\right] + O(n^{-3})$$

$$= n^{-1}\alpha_{\eta 2} + n^{-2}\underset{\text{(A)}}{\Big[}\alpha_{\eta\Delta 2} + 4k\underset{\text{(B)}}{\Big[}\left\{-2 + \frac{1}{c_1c_2} - \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)}\right\}\alpha_{\eta 2}$$

$$- \frac{1-2c_1c_2}{\theta_0(1-\theta_0)}\underset{\text{(B)}}{\Big]}\underset{\text{(A)}}{\Big]} + O(n^{-3})$$

$$\equiv n^{-1}\alpha_{\eta 2} + n^{-2}\alpha_{\mathrm{W}\eta\Delta 2} + O(n^{-3})\ (\alpha_{\mathrm{W}\eta 2} = \alpha_{\eta 2}),$$

where $(c_1, c_2)\begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1}\end{pmatrix}\frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}} = 0$ and

$$\frac{\partial(c_1p_2 - c_2p_1)}{\partial \mathbf{p}}\begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1}\end{pmatrix}\frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}} = (-c_2, c_1)\begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1}\end{pmatrix}\begin{pmatrix} -1 \\ 1\end{pmatrix}\frac{c_1c_2}{\theta_0(1-\theta_0)}$$

$$= \left(\frac{c_2}{c_1} + \frac{c_1}{c_2}\right)\frac{c_1c_2}{\theta_0(1-\theta_0)} = \frac{1-2c_1c_2}{\theta_0(1-\theta_0)}$$

are used with $\dfrac{\partial n^{-1}\eta_0}{\partial \mathbf{p}} = \dfrac{c_1c_2}{\theta_0(1-\theta_0)}\begin{pmatrix} -1 \\ 1\end{pmatrix}$.

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