Economic Review (Otaru University of Commerce),
Vol. 52, No.1, 239-241, July, 2001.

A Supplimentary Note on the Joint Maximum Likelihood Estimation of the IRT Equating Coefficients and Abilities for the Common-Examinee Design

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This note was initially written as an appendix of Ogasawara (20001) to give some explanation about the joint maximum likelihood estimation of the IRT equating coefficients and ability parameters for the common-examinee design. However, the appendix was not included in the article due to space limitation.

Let the probability of a correct or an incorrect response in Tests 1 and 2 be described as:

$$P_1(x_{1ij}|\theta_i, a_{1j}, b_{1j}) \equiv \frac{\exp\{-Da_{1j}(\theta_i - b_{1j})(1-x_{1ij})\}}{1+\exp\{-Da_{1j}(\theta_i - b_{1j})\}}, \tag{A1}$$

$(i = 1,\ldots,N; j = 1,\ldots,p)$,

$$P_2(x_{2ij}|\theta_i, A, B, a_{2j}, b_{2j}) \equiv \frac{\exp\{-D(a_{2j}/A)(\theta_i - Ab_{2j} - B)(1-x_{2ij})\}}{1+\exp\{-D(a_{2j}/A)(\theta_i - Ab_{2j} - B)\}}, \tag{A2}$$

$(i = 1,\ldots,N; j = 1,\ldots,q)$,

where the notations are defined similarly to the case of the MML estimation except that the θ_i, $(i = 1,\ldots,N)$ are unknown parameters in the JML estimation (for simplicity the same notation is used here).

Then, the likelihood to be maximized is

$$L = \prod_{i=1}^{N}[\{\prod_{j=1}^{p} P_1(x_{1ij}|\cdot)\}\{\prod_{j=1}^{q} P_2(x_{2ij}|\cdot)\}]. \tag{A3}$$

The following method of maximizing L is similar to the usual JML estimation of item and ability parameters (see e.g., Lord, 1980, 12.2). The estimation proceeds in two steps in each iteration. Let l = ln$L$.In the first step A and B are renewed by treating θ_i, (i = 1,...,N) as fixed values:

$$\begin{pmatrix} \hat{A}^{(k+1)} \\ \hat{B}^{(k+1)} \end{pmatrix} = \begin{pmatrix} \hat{A}^{(k)} \\ \hat{B}^{(k)} \end{pmatrix} + \left[E\left(\frac{-\partial^2 l}{\partial(A,B)'\partial(A,B)} \right)_{\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \hat{A}^{(k)} \\ \hat{B}^{(k)} \end{pmatrix}} \right]^{-1} \frac{\partial l}{\partial(A,B)'}_{\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \hat{A}^{(k)} \\ \hat{B}^{(k)} \end{pmatrix}} \tag{A4}$$

where the superscript (k) indicates the value in the k -th iteration and

$$\frac{\partial l}{\partial(A,B)'} = \sum_{i=1}^{N}\sum_{j=1}^{q}(x_{2ij} - P_2(x_{2ij}=1|\cdot))Da_{2j}\begin{pmatrix} -(\theta_i - B)/A^2 \\ -1/A \end{pmatrix}, \tag{A5}$$

$$E\left(\frac{-\partial^2 l}{\partial(A,B)'\partial(A,B)}\right) = \sum_{i=1}^{N}\sum_{j=1}^{q} P_2(x_{2ij}=1|\cdot)(1-P_2(x_{2ij}=1|\cdot)) \tag{A6}$$

$$\times D^2 a_{2j}^2 \begin{pmatrix} (\theta_i - B)^2/A^4 & (\theta_i - B)/A^3 \\ (\theta_i - B)/A^3 & 1/A^2 \end{pmatrix}.$$

In the second step, θ_i, (i = 1,...,N) are revised by regarding A and B as fixed parameters in the following way:

$$\hat{\theta}_i^{(k+1)} = \hat{\theta}_i^{(k)} + \left\{ \frac{\partial l}{\partial \theta_i} \Big/ E\left(\frac{-\partial^2 l}{\partial \theta_i^2} \right) \right\}_{\theta_i = \hat{\theta}_i^{(k)}}, (i = 1, \ldots, N), \tag{A7}$$

where

$$\frac{\partial l}{\partial \theta_i} = \sum_{j=1}^{p}(x_{1ij} - P_1(x_{1ij}=1|\cdot))Da_{1j} + \sum_{j=1}^{q}(x_{2ij} - P_2(x_{2ij}=1|\cdot))Da_{2j}/A,$$

$$E\left(\frac{-\partial^2 l}{\partial \theta_i^2}\right) = \sum_{j=1}^{p} P_1(x_{1ij}=1|\cdot)(1-P_1(x_{1ij}=1|\cdot))D^2 a_{1j}^2 \tag{A8}$$

$$+ \sum_{j=1}^{q} P_2(x_{2ij}=1|\cdot)(1-P_2(x_{2ij}=1|\cdot))D^2 a_{2j}^2/A^2.$$

References

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