Economic Review (Otaru University of Commerce),
Vol. 72, No. 2 & 3, 1-8, December, 2021.

Supplement to the paper "Maximization of some types of information for unidentified item response models with guessing parameters"

Haruhiko Ogasawara

This article supplements Ogasawara (2021).

Reference

Ogasawara, H. (2021). Maximization of some types of information for unidentified item response models with guessing parameters. *Psychometrika, 86* (2), 544-563, https://doi.org/10.1007/s11336-021-09763-4.

In the following, the number of distinct θ_j's among $\theta_j (j=1,...,N)$ is assumed to be sufficiently large with the largest one being *N*. As addressed in Ogasawara (2021), in the case of the 1PL-G model, k_2 is associated with the location indeterminacies of $a^*\theta_j^*$ and $a^*b_i^*$. Consequently, under $\bar{\theta} = \bar{\theta}^* = k_\theta$, k_2 can be set to 1. Define $\mathrm{var}\{\ln(e^{a\theta} + k_1)\}$ as the variance of $\ln\{\exp(a\theta_j) + k_1\}$ $(j=1,...,N)$. Let

$$\theta_{\min} \equiv \min\{\theta_j;\ j=1,...,N\} \ \text{ with } \ \text{inf-}k_1 \equiv -\exp(a\theta_{\min}). \tag{a.1}$$

Then, we have the following result.

Lemma 1. *In the case of the 1PL-G model*,

$$\lim_{k_1 \to \text{inf-}k_1 + 0} \mathrm{var}\{\ln(e^{a\theta} + k_1)\} = +\infty. \tag{a.2}$$

Proof. Let $K_j \equiv \exp(a\theta_j) + k_1$ $(j=1,..,N)$ and $K_{\min} \equiv \exp(a\theta_{\min}) + k_1$. Then,

$$\begin{aligned}\operatorname{var}\{\ln(e^{a\theta}+k_1)\} &= N^{-1}\sum_{j=1}^{N}\left[\ln\{\exp(a\theta_j)+k_1\}-\overline{\ln(e^{a\theta}+k_1)}\right]^2 \\ &= N^{-1}\sum_{j=1}^{N}\left(\ln K_j - N^{-1}\sum_{m=1}^{N}\ln K_m\right)^2 > N^{-1}\left(\ln K_{\min} - N^{-1}\sum_{m=1}^{N}\ln K_m\right)^2 \\ &= N^{-1}\left\{(1-N^{-1})\ln K_{\min} - N^{-1}\sum_{m=1\,(m\neq\min)}^{N}\ln K_m\right\}^2 .\end{aligned} \tag{a.3}$$

When $k_1 \to \text{inf-}k_1 + 0$, by definition $\ln K_{\min} \to -\infty$. Then, since $-N^{-1}\sum_{m=1\,(m\neq\min)}^{N}\ln K_m$ is finite, the last result in (a.3) goes to $+\infty$ Q.E.D.

A.1 The results under $a^* = [\operatorname{var}\{\ln(e^{a\theta}+k_1)\}]^{1/2}$

In this section the results under $a^* = [\operatorname{var}\{\ln(e^{a\theta}+k_1)\}]^{1/2}$ with $\bar{\theta} = \bar{\theta}^* = 0$ and $\operatorname{var}(\theta) = \operatorname{var}(\theta^*) = 1$ are shown.

Theorem 2. *Under* $a^* = [\operatorname{var}\{\ln(e^{a\theta}+k_1)\}]^{1/2}$ *in the 1PL-G model,*

$$\begin{aligned} &\lim_{k_1\to\text{inf-}k_1+0} a^* = +\infty, \quad \lim_{k_1\to\text{inf-}k_1+0} b_i^* = \frac{N^{1/2}}{N-1}, \\ &0 < \lim_{k_1\to\text{inf-}k_1+0} c_i^* = \frac{c_i\exp(ab_i)-\text{inf-}k_1}{\exp(ab_i)-\text{inf-}k_1} < 1, \\ &\lim_{k_1\to\text{inf-}k_1+0} \theta^*_{\min} = -N^{1/2} \quad \text{with} \quad \theta^*_{\min} = \ln\{\exp(a\theta_{\min})+k_1\} \\ &\lim_{k_1\to\text{inf-}k_1+0} \theta_j^* = \lim_{k_1\to\text{inf-}k_1+0} b_i^* = \frac{N^{1/2}}{N-1}\ (i=1,\dots,n;\ j=1,\dots,N;\ j\neq\min). \end{aligned} \tag{a.4}$$

Proof. $\lim_{k_1\to\text{inf-}k_1+0} a^* = +\infty$ is given by Lemma 1. For b_i^*, let $K_j^* = 1/K_j = 1/\{\exp(a\theta_j)+k_1\}$ $(j=1,\dots,N)$ and $K^*_{\min} = 1/K_{\min}$. Denote $\operatorname{var}\{\ln(e^{a\theta}+k_1)\} = \operatorname{var}[\ln\{1/(e^{a\theta}+k_1)\}]$ by $\operatorname{var}(\ln K^*)$. When $k_1 \to \text{inf-}k_1 + 0$, we find from Lemma 1 that the denominator of b_i^* in the first paragraph of Section 4 i.e., $\{\operatorname{var}(\ln K^*)\}^{1/2} \to +\infty$. On the other hand, for the numerator of b_i^*, when $k_1 \to \text{inf-}k_1 + 0$, using $\ln K_{\min} \to -\infty$ and $\ln K_{\min}^* \to +\infty$, we have

$$\begin{aligned}&\lim_{k_1 \to \text{inf-}k_1+0} [\ln\{\exp(ab_i) - k_1\} - \overline{\ln(e^{a\theta} + k_1)}]\\&= \ln\{\exp(ab_i) - \text{inf-}k_1\} - N^{-1} \sum_{j=1(j \neq \min)}^{N} \ln\{\exp(a\theta_j) + \text{inf-}k_1\}\\&\quad + N^{-1} \lim_{K_{\min}^* \to +\infty} \ln K_{\min}^*\\&= N^{-1} \lim_{K_{\min}^* \to +\infty} \ln K_{\min}^* = +\infty.\end{aligned} \tag{a.5}$$

Then,

$$\begin{aligned}&\lim_{K_{\min}^* \to +\infty} b_i^* = \lim_{K_{\min}^* \to +\infty} \frac{N^{-1} \ln K_{\min}^*}{\{\text{var}(\ln K^*)\}^{1/2}}\\&= \lim_{K_{\min}^* \to +\infty} N^{-1/2} \left\{ \sum_{j=1}^{N} \left(\frac{\ln K_j^*}{\ln K_{\min}^*} - N^{-1} \sum_{m=1}^{N} \frac{\ln K_m^*}{\ln K_{\min}^*} \right)^2 \right\}^{-1/2} = \frac{N^{-1/2}}{1 - N^{-1}} = \frac{N^{1/2}}{N-1}\end{aligned} \tag{a.6}$$

and the results for c_i^* are obvious $(i = 1, ..., n)$.

For $\theta_{\min}^* = \ln\{\exp(a\theta_{\min}) + k_1\}$, we have

$$\begin{aligned}&\lim_{k_1 \to \text{inf-}k_1+0} \theta_{\min}^* = \lim_{k_1 \to \text{inf-}k_1+0} \frac{\ln\{\exp(a\theta_{\min} + k_1)\} - \overline{\ln(e^{a\theta} + k_1)}}{[\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}}\\&= \lim_{K_{\min}^* \to +\infty} \frac{-\ln K_{\min}^* + \overline{\ln K^*}}{\{\text{var}(\ln K^*)\}^{1/2}} = -\frac{1 - N^{-1}}{N^{-1/2}(1 - N^{-1})} = -N^{1/2}.\end{aligned} \tag{a.7}$$

For $\theta_j^* (j = 1, ..., N; j \neq \min)$, as for b_i^*, we obtain

$$\begin{aligned}&\lim_{k_1 \to \text{inf-}k_1+0} \theta_j^* = \lim_{k_1 \to \text{inf-}k_1+0} \frac{\ln\{\exp(a\theta_j + k_1)\} - \overline{\ln(e^{a\theta} + k_1)}}{[\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}}\\&= \lim_{K_{\min}^* \to +\infty} \frac{N^{-1} \ln K_{\min}^*}{\{\text{var}(\ln K^*)\}^{1/2}} = \lim_{k_1 \to \text{inf-}k_1+0} b_i^* \ (i = 1, ..., n) = \frac{N^{1/2}}{N-1}.\ \text{Q.E.D.}\end{aligned} \tag{a.8}$$

It is easily confirmed that

$$\overline{\lim_{k_1 \to \text{inf-}k_1+0} \theta^*} \equiv N^{-1} \sum_{j=1}^{N} \lim_{k_1 \to \text{inf-}k_1+0} \theta_j^* = 0 . \quad \text{(a.9)}$$

However,

$$\text{var}\left(\lim_{k_1 \to \text{inf-}k_1+0} \theta^* \right) \equiv N^{-1} \sum_{j=1}^{N} \left(\lim_{k_1 \to \text{inf-}k_1+0} \theta_j^* - \overline{\lim_{k_1 \to \text{inf-}k_1+0} \theta^*} \right)^2 = \frac{N}{N-1} > 1 . \quad \text{(a.10)}$$

When $k_1 \to \text{inf-}k_1 + 0$, $\Psi^*_{i\min}$ $(\equiv \Psi^*_{ij} = 1/[1+\exp\{-a^*(\theta_j^* - b_i^*)\}]$ when $\theta_j^* = \theta^*_{\min} = \ln\{\exp(a\theta_{\min}) + k_1\}$) goes to zero, and consequently, $P_{i\min}$ $(\equiv P_{ij}$ when $\theta_j = \theta_{\min}$ or equivalently $\theta_j^* = \theta^*_{\min}$) goes to c_i^*. The last result holds only for $\theta^*_{\min}$ since $-a^*(\theta_j^* - b_i^*) = \ln\{\exp(ab_i) - k_1\} - \ln\{\exp(a\theta_j) + k_1\}$ is finite for θ_j $(j = 1,...,N; j \neq \min)$.

Lemma 2. *Under* $a^* = [\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}$ *in the 1PL-G model*,

$$\lim_{k_1 \to \text{inf-}k_1+0} \frac{\partial P_i^*}{\partial \theta^*}\Big|_{\theta^* = \theta^*_{\min}} \equiv \lim_{k_1 \to \text{inf-}k_1+0} \frac{\partial P_i^*}{\partial \theta^*_{\min}} = 0 \quad (i = 1,...,n) . \quad \text{(a.11)}$$

Proof. Recall that $K_j^* = 1/K_j = 1/\ln\{\exp(a\theta_j) + k_1\}$ $(j = 1,...,N)$ and $K^*_{\min} = 1/K_{\min}$. Then, as derived in Section 3 we have

$$\frac{\partial P_i^*}{\partial \theta^*_{\min}} = \frac{\{\text{var}(\ln K^*)\}^{1/2}\{\exp(a\theta_{\min}) + k_1\}(1 - P_{i\min})}{\exp(a\theta_{\min}) + \exp(ab_i)}$$

$$\equiv \frac{\{\text{var}(\ln K^*)\}^{1/2}}{K^*_{\min}} h_i \quad (i = 1,...,n), \quad \text{(a.12)}$$

where $h_i = (1 - P_{i\min}) / \{\exp(a\theta_{\min}) + \exp(ab_i)\}$ does not depend on k_1; and $\text{var}(\ln K^*) = \text{var}[\ln\{1/(e^{a\theta} + k_1)\}] = \text{var}\{\ln(e^{a\theta} + k_1)\}$.

When $k_1 \to \text{inf-}k_1 + 0$, we have $\ln K^*_{\min} \to +\infty$ and from Lemma 1 $\text{var}(\ln K^*) \to +\infty$. Using L'Hôpital's rule, we obtain

$$\lim_{k_1 \to \text{inf-}k_1+0} \frac{\partial P_i^*}{\partial \theta_{\min}^*} = \lim_{K_{\min}^* \to +\infty} \frac{\partial \{\text{var}(\ln K^*)\}^{1/2} / \partial K_{\min}^*}{\partial K_{\min}^* / \partial K_{\min}^*} h_i$$

$$= \frac{1}{2}\{\text{var}(\ln K^*)\}^{-1/2} 2 \lim_{K_{\min}^* \to +\infty} \left(N^{-1} \frac{\ln K_{\min}^*}{K_{\min}^*} - N^{-2} \sum_{j=1}^{N} \frac{\ln K_j^*}{K_{\min}^*} \right) h_i \qquad \text{(a.13)}$$

$$= 0 \quad (i = 1,...,n),$$

where $\lim_{K_{\min}^* \to +\infty} (\ln K_{\min}^*) / K_{\min}^* = 0$ is given again by L'Hôpital's rule. Q.E.D.

Then, we obtain the following main result.

Theorem 3. *Under* $a^* = [\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}$ *in the 1PL-G model*,

$$\lim_{k_1 \to \text{inf-}k_1+0} \sum_{i=1}^{n} I_{\text{F}i\min*} = \lim_{k_1 \to \text{inf-}k_1+0} I_{\text{S}\min*} = \lim_{k_1 \to \text{inf-}k_1+0} I_{\text{Q}\min*} = 0 \quad \text{and} \qquad \text{(a.14)}$$

$$\lim_{k_1 \to \text{inf-}k_1+0} I_{\text{F*}}^{+} = \lim_{k_1 \to \text{inf-}k_1+0} I_{\text{S*}}^{+} = \lim_{k_1 \to \text{inf-}k_1+0} I_{\text{Q*}}^{+} = +\infty, \qquad \text{(a.15)}$$

where $I_{\text{S}\min*} = I_{\text{S}j*}$ *when* $\theta_j = \theta_{\min}$ *with other similar expressions defined similarly.*

On the other hand, when $k_1 \to \text{sup-}k_1 - 0$, *all the values of* $I_{\text{F*}}^{+}, I_{\text{S*}}^{+}$ *and* $I_{\text{Q*}}^{+}$ *are finite and their unattained limiting values are given by* $k_1 = \text{sup-}k_1$ *in* $\partial P_i^* / \partial \theta_j^*$ $(i = 1,...,n; j = 1,...,N)$ *of the total informations, and* $c_{\text{sup-}k_1}^*$ $(\equiv c_i^*$ *when* $b_i = \min\{b_m; m = 1,...,n\}$) *goes to* $-\infty$.

Proof. The first set of limiting zero informations (see (a.14)) is given by Lemma 2. For the second set of their infinite limiting values (see (a.15)), when $k_1 \to \text{inf-}k_1 + 0$, it is found that

$$\frac{\partial P_i^*}{\partial \theta_j^*} = [\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2} \{\exp(a\theta_j) + k_1\} h_i$$
$$(i = 1,...,n; j = 1,...,N; j \neq \min) \qquad \text{(a.16)}$$

go to $+\infty$ since $\text{var}\{\ln(e^{a\theta} + k_1)\} \to +\infty$ and $\exp(a\theta_j) + k_1$ is finite as h_i, which gives the second set of infinite limiting informations.

The results when $k_1 \to \text{sup-}k_1 - 0$ are obviously derived since all the factors in $\partial P_i^* / \partial \theta_j^*$ are finite for this limiting case while

$c_i^* = \{c_i \exp(ab_i) - k_1\} / \{\exp(ab_i) - k_1\}$, when $c_i^* = c_{\text{sup-}k_1}^*$ goes to $-\infty$ since the numerator is negative and finite and the denominator approaches +0. Q.E.D.

A.2 The results under $a = a^* = k_3\ (> 0)$

Next, we consider the case of parametrization with $a = a^* = k_3\ (> 0)$, where $k_3 = 1$ is used without loss of generality. That is, ab_i and $a\theta_j$ are redefined as b_i and θ_j, respectively before transformation with $\bar{\theta} = N^{-1}\sum_{j=1}^{N}\theta_j = 0$ to remove the location indeterminacy. After transformation, using $a^* = 1$ we have

$$b_i^* = \ln\{\exp(b_i) - k_1\} - \overline{\ln(e^{\theta} + k_1)},\ \ c_i^* = \frac{c_i \exp(b_i) - k_1}{\exp(b_i) - k_1} \qquad \text{(a.17)}$$

$$\theta_j^* = \ln\{\exp(\theta_j) + k_1\} - \overline{\ln(e^{\theta} + k_1)} \ \text{ with } \ \bar{\theta}^* = N^{-1}\sum_{m=1}^{N}\theta_m^* = 0$$

$$(i = 1,...,n; j = 1,...,N).$$

We have two possible regions of k_1 as given in Section 2:

$$\text{inf-}k_1 = -\min\{\exp(\theta_j);\ j = 1,...,N\} < k_1 < \min\{\exp(b_i);\ i = 1,...,n\} = \text{sup-}k_1, \qquad \text{(a.18)}$$

$$\text{and} \ \ \text{inf-}k_1 < k_1 \le \min\{c_i \exp(b_i);\ i = 1,...,n\} = \text{max-}k_1 < \text{sup-}k_1. \qquad \text{(a.19)}$$

Define $\theta_{\min} = \min\{\theta_j; j = 1,...,N\}$ as before with similar expressions defined similarly. Then, we have the following results.

Theorem 4. *Under* $a = a^* = 1$ *and* $\bar{\theta} = \bar{\theta}^* = 0$ *in the 1PL-G model*,

$$\lim_{k_1 \to \text{inf-}k_1+0} b_i^* = +\infty,\ \ \lim_{k_1 \to \text{inf-}k_1+0} c_i^* = \frac{c_i \exp(b_i) - \text{inf-}k_1}{\exp(b_i) - \text{inf-}k_1}\ (< 1) \ \text{ is finite,}$$

$$\lim_{k_1 \to \text{sup-}k_1-0} c_{\text{sup-}k_1}^* = -\infty,\ \ \lim_{k_1 \to \text{sup-}k_1-0} c_{i\ (i \ne \text{sup-}k_1)}^* \ \text{ is finite,} \qquad \text{(a.20)}$$

$$\lim_{k_1 \to \text{inf-}k_1+0} \theta_{\min}^* = -\infty,\ \ \lim_{k_1 \to \text{inf-}k_1+0} \theta_{j(j \ne \min)}^* = +\infty \ \text{ with } \ \overline{\lim_{k_1 \to \text{inf-}k_1+0} \theta^*} = 0 \ \text{ and}$$

$$\text{var}\left(\lim_{k_1 \to \text{inf-}k_1+0} \theta^*\right) = +\infty \quad (i = 1,...,n; j = 1,...,N).$$

Proof. The results are given as in Lemma 1 and Theorem 2 with

$a = a^* = 1$ and $\bar{\theta} = \bar{\theta}^* = 0$. Q.E.D.

Lemma 3. *Under* $a = a^* = 1$ *and* $\bar{\theta} = \bar{\theta}^* = 0$ *in the 1PL-G model,*

$$\lim_{k_1 \to \text{inf-}k_1 + 0} \partial P_i^* / \partial \theta_{\min}^* = 0 \quad \textit{and} \quad \lim_{k_1 \to \text{inf-}k_1 + 0} \partial P_i^* / \partial \theta_j^* \qquad \text{(a.21)}$$

$(i = 1, ..., n;\ j = 1, ..., N; j \neq \min)$ *are positive and finite.*

Proof. The zero limiting value is given by $\lim_{k_1 \to \text{inf-}k_1 + 0} \partial P_i^* / \partial \theta_{\min}^* =$ $\lim_{k_1 \to \text{inf-}k_1 + 0} \frac{\{\exp(\theta_{\min}) + k_1\}(1 - P_{ij})}{\exp(\theta_{\min}) + \exp(b_i)} = 0$ since $\exp(\theta_{\min}) + k_1 \to +0$. On the other hand,

$$\lim_{k_1 \to \text{inf-}k_1 + 0} \frac{\partial P_i^*}{\partial \theta_j^*} = \lim_{k_1 \to \text{inf-}k_1 + 0} \frac{\{\exp(\theta_j) + k_1\}(1 - P_{ij})}{\exp(\theta_j) + \exp(b_i)}$$

$$= \frac{\{\exp(\theta_j) + \text{inf-}k_1\}(1 - P_{ij})}{\exp(\theta_j) + \exp(b_i)} (i = 1, ..., n;\ j = 1, ..., N; j \neq \min), \qquad \text{(a.22)}$$

which are obviously positive and finite by definition. Q.E.D.

Theorem 5. *Under* $a = a^* = 1$ *and* $\bar{\theta} = \bar{\theta}^* = 0$ *in the 1PL-G model,*

$$\lim_{k_1 \to \text{inf-}k_1 + 0} \sum_{i=1}^{n} I_{\text{F}i\min *} = \lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{S}\min *} = \lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{Q}\min *} = 0 \text{ ; and} \qquad \text{(a.23)}$$

$$\lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{F}*}^{+} = \sum_{j=1}^{N} \sum_{i=1}^{n} I_{\text{F}ij*}, \lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{S}*}^{+} = \sum_{j=1}^{N} I_{\text{S}j*} \text{ and } \lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{Q}*}^{+} = \sum_{j=1}^{N} I_{Qj*} \qquad \text{(a.24)}$$

are finite, where the right-hand side in each equation of (a.24) is defined to be given by $k_1 = \text{inf-}k_1$.

When $k_1 \to \text{sup-}k_1 - 0$, *all the values of* $I_{\text{F}*}^{+}, I_{\text{S}*}^{+}$ *and* $I_{\text{Q}*}^{+}$ *are finite and their unattained limiting values are given by* $k_1 = \text{sup-}k_1$ *in* $\partial P_i^* / \partial \theta_j^*$ $(i = 1, ..., n; j = 1, ..., N)$ *of the total informations, and* $c_{\text{sup-}k_1}^*$ $(\equiv c_i^*$ *when* $b_i = \min\{b_m;\ m = 1, ..., n\}$*) goes to* $-\infty$.

Proof. Using Lemma 3 and the definitions of the informations, (a.23) and (a.24) follow. The results when $k_1 \to \text{sup-}k_1 - 0$ are given as in Theorem 3. Q.E.D.

Recall that under $a^* = [\mathrm{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}$, $\mathrm{var}\left(\lim_{k_1 \to \text{inf-}k_1 + 0} \theta^*\right)$ is finite while $I^+_{\mathrm{F}*}$, $I^+_{\mathrm{S}*}$ and $I^+_{\mathrm{Q}*}$ go to $+\infty$ when $k_1 \to \text{inf-}k_1 + 0$. To the contrary, under $a = a^* = 1$, the opposite results with infinite $\mathrm{var}\left(\lim_{k_1 \to \text{inf-}k_1 + 0} \theta^*\right)$ and finite $I^+_{\mathrm{F}*}$, $I^+_{\mathrm{S}*}$ and $I^+_{\mathrm{Q}*}$ when $k_1 \to \text{inf-}k_1 + 0$ are obtained.

Theorem 6. *Under* $a = a^* = 1$ *and* $\bar{\theta} = \bar{\theta}^* = 0$ *in the 1PL-G model, using the possible region of (a.18) for* k_1, *the total informations* $I^+_{\mathrm{F}*}$, $I^+_{\mathrm{S}*}$ *and* $I^+_{\mathrm{Q}*}$ *have no maxima though their suprema are finite, which are given when* $k_1 = \sup\text{-}k_1$. *When the possible region of (a.19) for* k_1 *is used, the total informations have finite maxima, which are obtained by* $k_1 = \max\text{-}k_1$.

Proof. Since $\dfrac{\partial P_i^*}{\partial \theta_j^*} = \dfrac{\{\exp(\theta_j) + k_1\}(1 - P_{ij})}{\exp(\theta_j) + \exp(b_i)}$ $(i = 1, ..., n;\ j = 1, ..., N)$, the total informations are increasing functions of k_1, which gives the results depending on the domains of definition for k_1. Q.E.D.

Corollary 2. *Under the same condition as in Theorem 6 using* $\max\text{-}k_1$ *in (a.19) for* k_1, *when* $c_i = 0$ *for at least one item, the maxima of the informations are already attained before transformation*.

Proof. When $c_i = 0$ for an item, $\max\text{-}k_1$ $= \min\{c_m \exp(b_m);\ m = 1, ..., n\}$ becomes 0, which gives the required result. Q.E.D.

Corollary 2 shows a flexibility of the model with negative c_i^*. Even when $c_i = 0$ for all items, the informations can further be increased. Note that in this case the model before transformation is the usual 1-parameter logistic or Rasch model.