Another E-kKP→0–1KP revised a little bit

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We present a new transformation from a variant of the classical 0–1 knapsack problem (0–1KP) into the original 0–1KP, viz E-kKP→0–1KP, employing the collapsing knapsack problem (CKP), and also mention a new kKP→0–1KP, concretely kKP→E-kKP→CKP→0–1KP.

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1 Introduction

This piece shows a new transformation from E-kKP, which is a variant of the 0–1 knapsack problem (hereafter 0–1KP), back to the original and more simple 0–1KP. The 0–1KP is a classical and well-known combinatorial optimisation problem such that we pack a lot of given items of profit and weight, both of which are positive integers, into a knapsack of capacity c so that without the total weight of packed items exceeding the capacity c, the total profit of those is maximised—it goes without saying that an item is of weight $\leq c$ and we cannot pack items into the knapsack all together. The 0–1KP is formulated as, with $N = \{1, 2, \ldots, n\}$, $z^* = \max\{\sum_{j\in N} p_j x_j \mid \sum_{j\in N} w_j x_j \leq c, x_j \in \{0,1\}\}$ where p_j, w_j indicate profit and weight of item $j \in N$ respectively, and 0–1 variable x_j depicts the choice of item j as $x_j = 1$ (packed)/0 (unpacked). In particular, following, a word *solution* corresponds to items selected—that is, we call n-vector of $x = (x_j)_{j\in N}$ a solution according to the literature whilst in this piece we call $S \subseteq N$ a solution too, that is, we identify x with S as $x_j = 1 \Leftrightarrow j \in S$. In the light of this, the cardinality of solution x is $\sum_{j\in N} x_j$ ($= \sum_{j\in S} 1$, usually denoted as $|S|$). In addition, a solution fulfilling all constraints is said to be feasible. A solution which accomplishes z^* is naturally feasible, and we call the maximised z^* optimal value. For further details on 0–1KP and related, see Kellerer *et al* [6].

Adding to 0–1KP a constraint such that the number of packed items $\leq k$ leads to kKP, and more tightly, k even (i.e., $\sum_{j\in N} x_j = k$) leads to E-kKP. The next section presents a new transformation from E-kKP to 0–1KP. Taking advantage of the new, we also develop kKP→0–1KP.

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2 From E-kKP to 0–1KP via CKP

Although the transformation of E-kKP into 0–1KP has already been proposed in [2] we'll devise another one in this section. For the sake of brevity we hereafter assume $p_1 \geq p_2 \geq \cdots \geq p_n$ and $w_1 \leq w_2 \leq \cdots \leq w_n$ (neither provokes the re-ordering of items).

In actual fact, E-kKP is equivalent to the collapsing knapsack problem (CKP, for short) of

$$c(j) = \begin{cases} c, & j = k \\ 0, & j \in N \setminus \{k\} \end{cases}$$

where CKP is an extension of 0–1KP, having capacity not a constant but a functional over the number of packed items as $c(\sum_{j \in N} x_j)$. Therefore transforming the CKP with the method by Iida and Uno [5] produces a 0–1KP which is equivalent to given E-kKP as follows:

$$\begin{aligned}
&P = \max\left\{\sum_{j=1}^{n-1} p_j - \text{pmin} + 1, 0\right\},\ W = \max\left\{c - \sum_{j=1}^{k+1} w_j + 1, 0\right\},\\
&p'_j = \begin{cases} p_j + P, & 1 \leq j \leq n \\ (2n+1-j)P + p_n, & n < j \leq 2n, \end{cases}\\
&w'_j = \begin{cases} w_j + W, & 1 \leq j \leq n \\ (3n-1-j)W + c + 1, & n < j \leq 2n,\ j \neq n+k \\ (2n-1-k)W + 1, & j = n+k, \end{cases}\\
&c' = (2n-1)W + c + 1
\end{aligned} \tag{1}$$

where p'_j, w'_j and c' represent new profit and weight of item j and new capacity, respectively. The pmin in (1) is the profit of (given by) a feasible solution S' of CKP (also feasible in given E-kKP) as $\text{pmin} = \sum_{j \in S'} p_j$, provided $\sum_{j \in S'} w_j \leq c(|S'|)$. For example as in Kellerer *et al* [6, p. 272] we may adopt the lightest k items as S'. If not so (i.e., $\sum_{j \in S' = \{1,2,\ldots,k\}} w_j > c(|S'| = k) = c$) the given instance of E-kKP is unsolvable, including no feasible solution. Also if we're allowed to assume $k < n$, we have $P = \sum_{j=1}^{n-1} p_j -$ pmin $+1$. Indeed, Kellerer *et al* [6, p. 272] have assumed $2 \leq k < n$. In what follows, on (1) we call an item of index $j > n$ (new n items added) large item, and others (of index $j \in N$) small items.

Here we will briefly pick up the salient points of (1). For more details, see [5]. In short, the optimal value of 0–1KP (1) is attained by a combination of large item $n+k$ and k small items. Moreover, the k small items are our goal, that is, those convey optimal value to given E-kKP. First, focusing on weight, because $w'_{2n} + w'_{2n-1} \geq c' + 1$ even for $k = n$ or $k = n - 1$ (only w'_{n+k} does not contain c amongst weights of index $> n$) we cannot pack two large items together. In the case where we especially choice large item $n+k$, we can pack at most k small items, since

$$\sum_{j=1}^{k+1} w_j + (k+1)W \geq c + 1 + kW > c + kW = c' - w'_{n+k}.$$

If the number of small items is k even, after weight kW being subtracted, a solution whose weight $\leq c$—feasible in given E-kKP—only remains. As for another large item $j\ (\neq n+k)$, owing to remaining capacity $c' - w'_j = (j-n)W$ we can pack at most $j-n-1$ small items, which actually implies that such a solution does not achieve the optimal value of 0–1KP (1). This is because P is provided so that profit given by large item j and less than $j-n$ small items is beneath

$$p'_{n+k} + \text{pmin} + kP = (n+1)P + p_n + \text{pmin} \tag{2}$$

where profit (2) is given by a solution of 0–1KP (1) corresponding to a solution which gives pmin in given E-kKP (in the CKP of only $c(k) = c$, too). More precisely,

$$\begin{aligned} p'_j + \sum_{i=1}^{j-n-1} p_i + (j-n-1)P &\leq (n+1)P + p_n + \sum_{i=1}^{j-n-1} p_i - \sum_{i=1}^{n-1} p_i + \text{pmin} - 1 \\ &\leq (n+1)P + p_n + \text{pmin} - 1. \end{aligned}$$

Furthermore, we have no alternative but to pack a large item, since the total profit of all small items $\sum_j p_j + (n+1)P - P \leq (n+1)P + p_n + \text{pmin} - 1$, so it's also below (2). It should be pointed out that the last two arguments "$p'_j + \sum_{i=1}^{j-n-1} p_i + (j-n-1)P$ and $\sum_{j\in N} p_j + nP$ are both less than (2)" hold even for $P = 0$. Indeed, $P = 0$ in the definition of P (1) leads to $\sum_{j=1}^{n-1} p_j < \text{pmin}$, thereby us having $\sum_j p_j < p_n + \text{pmin}$.

We would here like to add that the latter part of kKP→E-kKP→0–1KP in [3] is a transformation proposed in [2]. Then, it will be possible to obtain another transformation of kKP→0–1KP by replacing the latter with (1); however, 0–1KP (1) has $2n$ items, twice as many as the previous, and moreover too large coefficients. For this reason, it seems not to be promising.[1] Still, in the remain-

[1]Regarding E-kKP→0–1KP proposed in [2], under the same assumptions in this piece to $\{p_j, w_j\}$ and with pmin (the profit of some feasible solution of given E-kKP), we set $P = W = \max\{\sum_{j=1}^{k-1} p_j - \text{pmin} + 1, c - \sum_{j=1}^{k+1} w_j + 1, 0\}$ and $(p'_j, w'_j) = (p_j + P, w_j + W)$, $c' = c + kW$ where the total number of items is still n (unchanged). Here we would like to note that this W makes the number of packed items at most k. In particular, when we set $W = c+1$ and $c' = c + kW$, unless

der of this section we will note it down a little.

Notice that $p'_{2n+2k} = P$, because the smallest profit p_{n+k} is 0 in converting, by (1), E-kKP being made up of given kKP plus k dummy items $(p_{n+j}, w_{n+j}) = (0, 0)$. Like this, the total number of items in resultant 0–1KP is $2(n+k)$ (all n appearing in (1) is replaced with $n+k$). In addition, we have $W = c - w_1 + 1 > 0$ (the total weight of k lightest items is 0 and $w_1 \leq c$). Here we should note that $W > 0$ since $W = 0$ might cause a trouble due to a dummy item of weight 0 added. For example, we saw that large item j can be combined with at most $j - n$ small items; nevertheless, an item of weight 0 would, in spite of the weight constraint, lead to unexpected result against the design of the reduction in [5].[2] On the other hand, because a dummy item added in kKP→E-kKP by Kellerer *et al* [6, p. 272] is not $(0, 0)$ but $(1, 1)$, which may be preferable with no possible trouble albeit larger coefficients (e.g., the capacity of resulting E-kKP is up to $k(c+1) - 1$ [1, p. 2]).

In the same way as [3] when we pack only one item of the largest profit, we obtain pmin $= p_1$—since $w_j \leq c$, a solution consisting of one item is always feasible in given kKP (A solution corresponding to the solution in the E-kKP of $n+k$ items shall contain $k-1$ dummy items). In this case, however, we have $P = \sum_{j=2}^{n+k-1} p_j + 1 = \sum_{j=2}^{n} p_j + 1$ pretty huge. The larger the pmin is, the smaller the P is. If we address E-kKP then we can make use of a solution being composed of k lightest items as mentioned previously [6, p. 272] but now is kKP and the solution is not always available—there may not exist a solution of cardinality k. To replace the solution giving pmin $= p_1$, anybody will think of a framework of greedy heuristic such that until exceeding capacity we iterate packing an item according to the sequence of items sorted in nonascending order of efficiency p_j/w_j or just profit. Needless to say, we stop the iteration straightaway when the cardinality reaches k (if necessary[3]).

$w_j < 0$, we cannot pack $k+1$ items of original weight plus W since $c' < (k+1)W$. If $W = 0$, from the definition of W we have $c < \sum_{j=1}^{k+1} w_j$. In this case, even sans W we cannot pack $k+1$ items, or more; thus, W is indeed needless. Conversely, P has us pack k items, or more. Then, $P = 0$ implies $\sum_{j=1}^{k-1} p_j <$ pmin, and in the same way as W, indeed P is needless in the case of $P = 0$.

[2]For simplicity let n' be $n+k$: the total number of items in E-kKP including k dummy items. On large item $j < n' + k$, we can pack $j - n'$ dummy items against remaining capacity $(j - n')W$ even when $W > 0$; nonetheless, because the total profit of $j - n'$ dummy items is 0, its profit $(n'+1)P < (n'+1)P +$ pmin, and does not become optimal where pmin > 0 follows from that a solution of one item is feasible in given kKP as will be mentioned afterwards. At the same time, $P = 0$ brings $\sum_{j=1}^{n'-1} p_j <$ pmin impossible, so we have $P > 0$.

[3]Notice that it's all over when we've packed the most profitable k items in applying to kKP the greedy heuristic along the nonascending order of profit.

3 Conclusions

We have hereby defined a new transformation from E-kKP to 0–1KP, yet unfortunately we cannot contend that the new is an alternative to that of [2]. Finally, we hope that this piece will become a start of something new.

References

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[4] Hiroshi Iida, Another E-kKP→0–1KP. pp. 1–4, 28 Feb 2016; `http://researchmap.jp/?action=cv_download_main&upload_id=103753` (An erratum: 'from E-kKP to 0–1KP' at line 8 in page 3 should be, of course, 'from kKP to 0–1KP').

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