Three kinds of kKP→ □ →0-1KP: a survey

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This piece presents three sorts of transformation, all reducing kKP to the classical 0-1 knapsack problem (0-1KP) where kKP is a variant of 0-1KP with additional constraint such that the number of packed items is k or less. Every transformation is not direct but via another problem □ as in the title viz rubber knapsack, collapsing knapsack, or E-kKP. Such a transformation makes both possible to solve kKP as 0-1KP and not to devise a tailored method for kKP. Anyway it shall be better that candidates for solving kKP augment.

keywords: combinatorial optimisation, knapsack problem, cardinality constraint

1 Introduction

We argue about a transformation from kKP, which is a variant of the 0-1 knapsack problem (hereafter 0-1KP), back to the original and more simple 0-1KP. The 0-1KP is a classical and well-known combinatorial optimisation problem such that we pack given items of profit and weight (both are positive integers) into a knapsack of capacity c so that the total profit of packed items is maximised without the total weight of those exceeding the

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c—needless to say there is no item of weight $> c$ and a case that the total weight of all items $\leq c$ is ruled out. The 0-1KP can be formulated as, with $N := \{1, 2, \ldots, n\}$, $z^* :=$ $\max\{\sum_{j \in N} p_j x_j \mid \sum_{j \in N} w_j x_j \leq c, x_j \in \{0, 1\}\}$ where p_j, w_j indicate profit and weight of item $j \in N$ respectively, and 0-1 variable x_j indicates the choice of item j as $x_j = 1$ (packed)/0 (otherwise). In particular, following, a word *solution* corresponds to the selection of items—that is, we call n-vector of $x := (x_j)_{j \in N}$ a solution according to the literature while in this piece we call $S \subseteq N$ a solution too, that is, we identify x with S as $x_j = 1 \Leftrightarrow j \in S$. By this, the cardinality of x means $\sum_{j \in N} x_j$. Also, a solution fulfilling all constraints is said to be feasible. A solution which gives z^* is of course feasible, and we call the maximised z^* optimal value. For more details on 0-1KP and related, see Kellerer *et al* [7].

Adding to 0-1KP a constraint such that the number of packed items $\leq k$ leads to kKP, and more tightly, k even (i.e., $\sum_{j \in N} x_j = k$) leads to E-kKP. In this piece we argue about a transformation from kKP to 0-1KP. All three transformations in the next section are not direct but via another problem.

2 □ := rubber | CKP | E-kKP

First, we consider a transformation via rubber knapsack (which is mentioned afterwards). This transformation is a modification of E-kKP→kKP. Kellerer *et al* [7, p. 273] proposed a transformation of E-kKP→kKP such that $P := \sum_j p_j$ and $W := \sum_j w_j$ are added to each item's profit and weight respectively, and new capacity $c' :=$ $c + kW$. In fact, the W is redundant. It was provided so that under the subset-sum case (i.e., $p_j = w_j$ for all $j \in N$), the condition is kept in resulting kKP. Indeed, Caprara *et al* [1] employ P only (due to P we shall pack k or more items). However the W makes the transformation the one from E-kKP to not kKP but 0-1KP. This is because we have $c' < (k+1)W$ owing to $\sum_j w_j > c$ also in E-kKP (and in kKP too) as so in 0-1KP (if $\sum_j w_j \leq c$ then we can pack k most profitable items and an instance of E-kKP given becomes trivial), we cannot pack more than k items without the constraint $\sum_j x_j \leq k$ of kKP, that is, the transformation is to not kKP but 0-1KP. Namely, by merely solving resulting problem as 0-1KP, we can find a solution (a subset of N) that maximises $\sum_{j\in N} (p_j + P) x_j$ and is of cardinality k.

Note that excluding P from the E-kKP→0-1KP does not produce kKP→0-1KP though impossible to pack more than k items certainly. This is because it may happen that on a solution of cardinality $k' < k$, slack $(k - k')W$ makes the solution feasible in resulting 0-1KP (i.e., $\leq c'$) against of total weight $> c$ in the original (for a concrete example, see footnote no. 1 of [4]). To help this defect, we remove the slack by using expanding knapsack problem—a knapsack expands like rubber according to the number of packed items [7, p. 416]; following for the sake of brevity we call the problem rubber knapsack. More precisely, rubber knapsack's capacity is not a constant but a function over the number of packed items $\sum_{j\in N} x_j$ as $c(\sum_j x_j)$. Then we replace the constant $c' = c + kW$ with

$$c(j) = c + \begin{cases} jW, & 1 \leq j < k \\ kW, & k \leq j \leq n. \end{cases} \tag{1}$$

Iida and Uno [6] proposed two transformations from CKP (collapsing knapsack problem, which we will mention afterwards) to 0-1KP where the former does not use a property that $c(\cdot)$ is monotonically nonascending on CKP; thus, we can transform rubber knapsack (1) to 0-1KP by the former. Although resultant 0-1KP obtained is terrible, we will herein note it down. Before writing it, we would like to add that we can reduce the W.

Specifically, a requirement is $\sum_{j=1}^{k+1} w_j + (k+1)W > c + kW$ under $w_1 \le w_2 \le \cdots \le w_n$ [2, 3], it's better to set $W := c - \sum_{j=1}^{k+1} w_j + 1$. If this $W \le 0$, the total weight of the lightest $k+1$ items is $> c$; thus, we may solve given kKP without any transformation. Consequently we have a solution of cardinality $\le k$ naturally.

We assume $p_1 \ge p_2 \ge \cdots \ge p_n$ and $w_1 \le w_2 \le \cdots \le w_n$ (this doesn't provoke the sorting of items). For simplicity, we also assume $W \ge 0$. Moreover for the sake of $\max_{i \ne j}\{c(i) + c(j)\} = 2(c + kW)$ we assume $k < n$. Indeed $k = n$ is non sense, and according to Kellerer *et al* [7, p. 272] we assume $2 \le k < n$ (but considering $\sum_j w_j > c$, $k = n - 1$ is still meaningless). In what follows, on resultant 0-1KP obtained, (p'_j, w'_j) indicates the two properties of new items and c' does new capacity.

$$W = c - \sum_{j=1}^{k+1} w_j + 1, \quad A = \sum_{j=3}^{k+1} w_j,$$

$$w'_j = \begin{cases} w_j + c - w_1 - w_2 + 1, & 1 \le j \le n \\ (3n - 1 - j)A + c + (2k + n - j)W + 1, & n < j < n + k \\ (3n - 1 - j)A + c + kW + 1, & n + k \le j \le 2n, \end{cases}$$

$$c' = (2n - 1)A + 2(c + kW) + 1,$$

$$C = \sum_{j=2}^{n-1} p_j + 1,$$

$$p'_j = \begin{cases} p_j + C, & 1 \le j \le n \\ (2n + 1 - j)C + p_n, & n < j \le 2n \end{cases}$$

where A should be $c - W - w_1 - w_2 + 1$ according to [6], yet the definition of W makes it so. In addition, as a feasible solution of rubber knapsack, we adopt one being made up of only one item of the largest profit. Moreover an assumption $2 \le k < n$ leads to $n \ge 3$ and $A > 0$. Furthermore because of

$W + A > 0$ we have $W + A = c - w_1 - w_2 + 1 > 0$, and it follows that $w'_j > 0$ $(j \leq n)$.

Second, we show another transformation which is also related to a capacity function $c\,(\sum_j x_j)$. As mentioned in footnote no. 2 of [5] we can assume kKP as a collapsing knapsack problem (CKP). In the CKP, as its name indicates, the knapsack will collapse according to the number of packed items as $c(1) \geq c(2) \geq \cdots \geq c(n)$. Therefore, a CKP of

$$c(j) = \begin{cases} c, & 1 \leq j \leq k \\ 0, & j > k \end{cases} \tag{2}$$

is identical to kKP; then, we shall gain 0-1KP by transforming CKP (2) with a method proposed by Iida and Uno [6]. Assuming $w_1 \leq w_2 \leq \cdots \leq w_n$, the total number of items in resulting 0-1KP is $n + k' := \min\{k, \max\{\ell \mid \sum_{j=1}^{\ell} w_j \leq c\}\}$. Further we assume $p_1 \geq p_2 \geq \cdots \geq p_n$. Then, pmin appearing afterwards indicates profit given by some feasible solution of CKP. For example, as so in rubber knapsack, when we adopt a solution including only one item of the largest profit, pmin $= p_1$ (although hidden, it's same on (3) which will appear afterwards). To summarise, assuming $c(1) = c(2) = c$ (i.e., $k \geq 2$) we have the following 0-1KP by transforming CKP of (2) with the 2nd method proposed by Iida and Uno [6], which takes advantage of the monotonicity of $c(\cdot)$:

$$W = \max\{c - w_1 - w_2 + 1, 0\},$$

$$w'_j = \begin{cases} w_j + W, & 1 \leq j \leq n \\ (2k' + n - 1 - j)\,W + c + 1, & n < j \leq n + k', \end{cases}$$

$$c' = (2k' - 1)\,W + 2c + 1,$$

$$P = \max\left\{\sum_{j=1}^{k'-1} p_j - \text{pmin} + 1, 0\right\},$$

$$p'_j = \begin{cases} p_j + P, & 1 \le j \le n \\ (2n+1-j)P + \sum_{i=k'}^{n} p_i, & n < j \le n + k'. \end{cases}$$

Now we have seen two transformations, both of which are complicated. To our knowledge, the one in [4] is the simplest, which is kKP→E-kKP→0-1KP. Third, we cite coefficients of 0-1KP obtained by the transformation as follows:

$$P = W = \max\left\{\sum_{j=2}^{k-1} p_j, c - \min_{1 \le j \le n} w_j\right\} + 1,$$

$$(p'_j, w'_j) = \begin{cases} (p_j + P, w_j + W), & j \in N \\ (P, W), & 1 \le j - n \le k, \end{cases} \tag{3}$$

$$c' = c + kW$$

where we assume $p_1 \ge p_2 \ge \cdots \ge p_n$ in kKP given. In addition $k \ge 2$, and if $k = 2$ then $\sum_{j=2}^{k-1} p_j = 0$. By extra k items of index $j > n$ provided, each solution of cardinality $< k$ in given kKP can become a solution of cardinality k even in E-kKP. In other words, these k dummy items produce one-to-one correspondence of feasible solutions between kKP and E-kKP (for example, an empty set in kKP is feasible while a solution of $x_j = 1, \forall j > n$ in E-kKP corresponds to the empty set). Why does solving 0-1KP (3) lead to solving E-kKP equivalent to given kKP? Roughly speaking, because of $W > c - \min_{1 \le j \le n} w_j$ by the definifion of W, the total weight of the lightest $k+1$ items $\min_{1 \le j \le n} w_j + (k+1)W > c + kW = c'$; thus, we can pack at most k items. In addition, because of $P > \sum_{j=2}^{k-1} p_j$ by the definition of P, we have

$$\sum_{j=1}^{k-1} p_j + (k-1)P < \sum_{j=1}^{k-1} p_j + kP - \sum_{j=2}^{k-1} p_j = p_1 + kP. \tag{4}$$

Thus we must pack k or more items. In consequence we shall consider a solution of cardinality k even only. Let $z^* + kP$ be an optimal (maximised) value in 0-1KP (3). Then, for its optimality, we can contend that z^* is maximised in original kKP (a solution obtained by discarding $j > n$ (if exist) from the one which gives $z^* + kP$ gives z^* in original kKP).

In comparison with the one via CKP, the total number of items in (3) (i.e., $n + k$) may greater than $n + k'$ whereas the capacity $c + kW$ is, in the case of $W = c - \min_{1 \le j \le n} w_j + 1$, almost the half of $(2k' - 1)W + 2c + 1$ (so as to keep subset-sum case like E-kKP→kKP (0-1KP) proposed by Kellerer *et al* [7, p. 273], P and W in (3) are defined [2]).

To conduct a further comparison with the third (via E-kKP), on the same framework of kKP→E-kKP→0-1KP, we will build it by pure elements kKP→E-kKP and E-kKP→0-1KP both proposed by Kellerer *et al* [7, pp. 272-3] (as indicated at the start of Section 2, although described E-kKP→kKP but in fact →0-1KP) and note down the coefficients of 0-1KP obtained by the built. Notice that new capacity by the first kKP→E-kKP is $k(c+1) - 1$ according to the claim in [2] (next E-kKP→0-1KP does $+kW$):

$$P = k\left(1 + \sum_{j\in N} p_j\right), \quad W = k\left(1 + \sum_{j\in N} w_j\right),$$

$$(p'_j, w'_j) = \begin{cases} (kp_j + P, kw_j + W), & j \in N \\ (1 + P, 1 + W), & 1 \le j - n \le k, \end{cases}$$

$$c' = k(c + W + 1) - 1.$$

It should be pointed out that under $\sum_j w_j > c$ assumed previously, we have $c' < k\sum_j w_j + kW + k = (k+1)W$. Although simple, it's in no doubt that these coefficients are bigger than (3).

3 Conclusions

Until now we have seen three transformations from kKP to 0-1KP. Finally we would like to note three points for further study.

- On the transformation of E-kKP→0-1KP, as seen in (4), as an upper bound of profit gained among solutions of cardinality $< k$, we adopt the total profit of the most profitable k-1 items [2,3]. Then, if the total weight of a solution which gives the upper bound is $> c'$, we can improve the bound. Is there alternative upper bound? In addition, as a solution which gives pmin in the transformation via rubber or E-kKP, one consisting of only one item of the largest profit is ordinary.

- We have focused on weight as to the transformation via rubber or CKP. Does focusing on profit and reformulating kKP bring something new? For example, a solution of cardinality $> k$ shall have penalty, or conversely, a solution of cardinality $\leq k$ shall have extra large profit.

- Does there exist another □, that is, a problem which can be inserted between kKP and 0-1KP? It is well known that bounded knapsack problem (BKP, the available number of each item is determined beforehand in the integer knapsack problem) can be reduced to 0-1KP [7, Subsect 7.1.1]; but, it seems kKP has no connection with BKP. Also, is there a direct transformation from kKP to 0-1KP?

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