The Evolution of Fairness under an Assortative Matching Rule in the Ultimatum Game *

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Abstract

This paper studies how a matching rule affects the evolution of fairness in the ultimatum game. Gale et al. [1995] show that under the random matching rule, a partially fair imperfect Nash equilibrium in which all proposers are fair but some responders are selfish is asymptotically stable in the limit as noise in learning vanishes if responders are noisier than proposers. This paper shows that, under an assortative matching rule, a mutually fair imperfect Nash equilibrium in which all proposers are fair and all responders are reciprocal is limit asymptotically stable as noise due to committed agents vanishes.

1 Introduction

Why would people behave in a fair manner while sacrificing their own monetary payoffs? In the ultimatum game, subgame perfection predicts that selfish individuals will make proposals to exploit almost all of the total surplus, and denotes those who accept these unfair offers as responders. In contrast to this prediction by standard game theory, many experimental data show that people tend to equally divide the total surplus (e.g. Binmore et al. [2002], Güth et al. [1982]). In the present study, this paradox is investigated within the framework of the evolutionary game theory by focusing on a matching rule.

Gale et al. [1995] study the replicator dynamics of the ultimatum mini game. There are two populations—proposers (population 1) and responders (population 2). The two populations are equal in size. In each period, an agent in one population matches with an agent in the other population at random. Each pair of agents plays the ultimatum (mini) game, as shown in Figure 1.

In the game, agent 1 proposes either a high (H) or low (L) offer. If she adopts strategy H, it is assumed that agent 2 (responder) always accepts it. If she adopts strategy L, the responder decides to either accept (Y) or reject (N) it.

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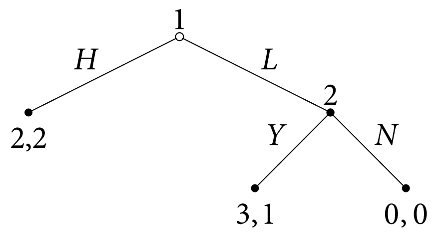

Figure 1: The ultimatum mini game.

We call a proposer selfish if she adopts strategy L and fair if she adopts H. Similarly, we call a responder selfish if he adopts strategy Y and reciprocal if he adopts N.[1] Let x_1 denote the proportion of selfish proposers in population 1 and x_2 denote the proportion of selfish responders in population 2. A state of the system is represented by a pair $x = (x_1, x_2) \in [0,1] \times [0,1]$. The average fitness of action k ($k = L, H, Y, N$) at state x is denoted by $f_k(x)$. Then, the average fitness of population 1 and population 2 is given by $\phi_1(x) = x_1 f_L(x) + (1 - x_1) f_H(x)$ and $\phi_2(x) = x_2 f_Y(x) + (1 - x_2) f_N(x)$, respectively.

The standard replicator dynamics (RD)[2] is described as follows:

$$\dot{x}_1 = g_1(x) = x_1(f_L(x) - \phi_1(x)) = x_1(1 - x_1)(f_L(x) - f_H(x)) \tag{1}$$

$$\dot{x}_2 = g_2(x) = x_2(f_Y(x) - \phi_2(x)) = x_2(1 - x_2)(f_Y(x) - f_N(x)). \tag{2}$$

In the ultimatum game, $f_L(x) = 3x_2$, $f_H(x) = 2$, $f_Y(x) = x_1 + 2(1 - x_1)$, and $f_N(x) = 2(1 - x_1)$. Therefore, the RD for the ultimatum game is described by $g_1(x) = x_1(1 - x_1)(3x_2 - 2)$ and $g_2(x) = x_2(1 - x_2)x_1$.

As illustrated by the phase diagram of RD for the ultimatum game (1)–(2) in Figure 2, Gale et al. [1995] show that the subgame perfect equilibrium and the imperfect Nash equilibria are locally (Liapunov) stable and that the subgame perfect equilibrium is the unique asymptotically stable point. Boundedly rational people may play wrong strategies by learning errors. To represent such an evolutionary drift, Gale et al. [1995] and Binmore and Samuelson [1999] introduce the following perturbed selection dynamics: For all $i = 1, 2$,

$$\dot{x}_i = g_i(x) + h_i(x).$$

They show that if the drift function h_1 of proposers is constant in the fitness difference between actions L and H and h_2 of responders is strictly decreasing in the fitness difference between actions Y and N (which implies that the noise level is higher for responders than for proposers near the imperfect

[1] Reciprocal is one of descriptions of this strategy. It can alternatively be called fair or spiteful.

[2] We consider only the (perturbed) replicator dynamics. However, this can be interpreted as an approximation to many learning models (Gale et al. [1995]). Section 5 constructs an imitation learning model leading to the replicator dynamics.

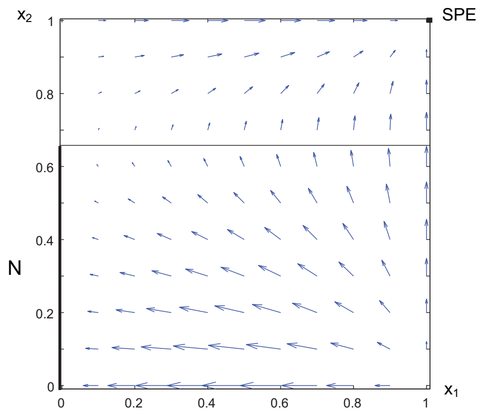

Figure 2: Phase diagram under the random matching rule. N is the set of imperfect Nash equilibria and SPE is the subgame perfect equilibrium.

equilibria)[3], there exists an asymptotically stable point that converges to an imperfect Nash equilibrium, resulting in the fair allocation as noise vanishes. This result implies that in the long run, people may behave in a fair manner as a consequence of noisy learning.

The asymptotically stable point, however, critically depends on the form of h (Binmore and Samuelson [1999]). For example, if h_2 is not sensitive to payoffs, then only the subgame perfect equilibrium is asymptotically stable. Thus, if we consider any other noisy learning, then under the replicator dynamics with the random matching rule, only the unfair behavior survives.

Unlike these previous approaches, this paper studies how a matching rule affects the evolution of fairness under the replicator dynamics. In particular, we analyze an *assortative matching rule* introduced by Becker [1973, 1974]. According to the assortative matching rule, pairing of similar types of individuals is more likely than they are paired under the random matching rule.

In contrast to the random matching rule, an interaction rate between individuals depends on their own actions under the assortative matching rule. This property leads to replicator dynamics with non-linear fitness functions (Taylor and Nowak [2006]). Taylor and Nowak [2006] introduce a generalized matching rule with non-uniform interaction rates for symmetric 2×2 strategic form games in a single population. They show that the non-uniform interaction rates generate interior equilibria even if one strategy dominates another. Bergstrom [2003] introduces another type of the assortative matching rule in the prisoners' dilemma game, and Taylor and Nowak [2006] and Bergstrom [2003] show that cooperation survives under the assortative matching rule in the prisoners' dilemma game.

Departing from the assortative matching in a single population, we introduce assortativity into interactions between two different populations. Following Gale et al. [1995], we assume that agent 1 is drawn from the population of proposers and that agent 2 is drawn from the population of responders.

[3] Gale et al. [1995] give a plausible explanation for this drift.

It is widely observed that bargaining in two-sided markets such as marriage, and labor markets, in which the two sides of overall population are clearly separate. Many interactions in these markets often have assortativity because of social signals such as region, education, skills, and conventions.[4] Section 5 constructs a model of partner choice resulting in assortative matching. Biologically, it is important to study assortative matching by size, color, or signals in conflicts between two different sexes and species (Marrow et al. [1996], Zu et al. [2008]).

The main result of this paper is that under a certain assortative matching rule, there exists a limit asymptotically stable point in which equal allocation may prevail. In particular, if the matching rule is completely assortative, there exist only two limit asymptotically stable states—the mutually fair equilibrium and the selfish equilibrium. The results provide evolutionary support for the fair allocation observed in many experiments in the ultimatum game.

The reminder of the paper is organized as follows. Section 2 defines an assortative matching rule and the selection dynamics of the ultimatum mini game. Section 3 presents the main results. Section 4 provides an example of an assortative matching rule. Section 5 discusses the results, and Section 6 concludes the paper.

2 The Model

We first extend the RD (1)–(2) for the ultimatum game in Figure 1 to the RD with a general matching rule. The state space of the RD is $[0,1]\times[0,1]$. We define a matching rule $\alpha = (p_1(x), p_2(x), q_1(x), q_2(x))$ on $[0,1]\times[0,1]$ between population 1 and population 2. Here, $p_1(x)$ is the probability that a selfish proposer (L) meets a selfish responder (Y), and $q_1(x)$ is the probability that a fair proposer (H) meets a reciprocal responder (N) at state x (Figure 3). Similarly, $p_2(x)$ is the probability that a selfish responder meets a selfish proposer, and $q_2(x)$ is the probability that a reciprocal responder meets a fair proposer at state x.

Definition 1. A quadruplet $\alpha = (p_1(x), p_2(x), q_1(x), q_2(x))$ is a matching rule on $[0,1]\times[0,1]$ if for all $i = 1,2$ and all $x \in [0,1]\times[0,1]$, $p_i(x)$ and $q_i(x)$ satisfy

$$x_1 p_1(x) = x_2 p_2(x), \tag{3}$$

$$(1-x_1)q_1(x) = (1-x_2)q_2(x), \tag{4}$$

$$x_1(1-p_1(x)) = (1-x_2)(1-q_2(x)), \tag{5}$$

$$(1-x_1)(1-q_1(x)) = x_2(1-p_2(x)). \tag{6}$$

Equations (3), (4), (5), and (6) are parity equations and imply that probability functions p_i and

[4] The literature on search theory (e.g. Atakan [2006], Shimer and Smith [2000]) investigates the assortative matching in two-sided market. Mendes et al. [2007] empirically find assortative matching between firms and workers, which they measure as correlation between a firm-specific productivity and the time-average share of high educated workers using data from Portugal.

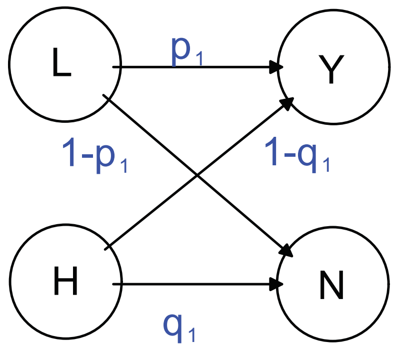

Figure 3: Matching probability in population 1: $p_1 = \Pr(L \text{ meets } Y)$, $q_1 = \Pr(H \text{ meets } N)$.

q_i are consistent as a matching rule. All agents can be paired as long as these equations are satisfied. Note that any one of four equations is derived from the others, and that if any one of p_1, p_2, q_1, and q_2 is given, then the other variables are uniquely determined by these equations. The random matching rule on $[0,1] \times [0,1]$ underlying the standard RD (1)–(2) assumes that $p_i(x) = q_i(x) = x_j$ for all $i, j = 1,2$ $(i \neq j)$ and all $x \in [0,1] \times [0,1]$, independently of the distribution in population i, x_i.

Under a matching rule $\alpha = (p_1(x), p_2(x), q_1(x), q_2(x))$ on $[0,1] \times [0,1]$, the average fitness of selfish and fair proposers is given by $f_L^\alpha(x) = 3p_1(x)$ and $f_H^\alpha(x) = 2$, respectively. Similarly, the average fitness of selfish and reciprocal responders is given by $f_Y^\alpha(x) = p_2(x) + 2(1 - p_2(x))$ and $f_N^\alpha(x) = 2q_2(x)$, respectively. Parallel to the RD (1)–(2), the *Replicator Dynamics with matching rule* α (RD$^\alpha$) of the ultimatum game is defined by the following system:

$$\dot{x}_1 = x_1(1 - x_1)\psi_1^\alpha(x) \tag{7}$$

$$\dot{x}_2 = x_2(1 - x_2)\psi_2^\alpha(x), \tag{8}$$

where $\psi_1^\alpha(x) = f_L^\alpha(x) - f_H^\alpha(x)$ and $\psi_2^\alpha(x) = f_Y^\alpha(x) - f_N^\alpha(x)$. Equations (7) and (8) are different from (1) and (2) only in that the average fitness f_k is replaced by f_k^α for every action $k = L, H, Y, N$.

Next, we introduce an assortative matching rule.

Definition 2. A matching rule $\alpha = (p_1(x), p_2(x), q_1(x), q_2(x))$ is *assortative* if for all $i, j = 1,2$ $(i \neq j)$, p_i and q_i satisfy the following conditions:

(i) $p_i(x)$ is monotonically non-increasing in x_i and monotonically non-decreasing in x_j, and $q_i(x)$ is monotonically non-increasing in x_j and monotonically non-decreasing in x_i.

(ii) $p_i(x) \geq x_j$ and $q_i(x) \geq 1 - x_j$ for all $x \in (0,1) \times (0,1)$.

(iii) p_i and q_i are Lipschitz continuous on the open set $(0,1) \times (0,1)$.[5]

[5] A function p on the product $(a,b) \times (c,d)$ is *Lipschitz continuous* if there exists a constant k such that $|p(x) - p(y)| < k(|x_1 - y_1| + |x_2 - y_2|)$ for any $x, y \in (a,b) \times (c,d)$.

The assortative matching rule has two characteristics. By condition (i), the increase in the frequency of selfish agents x_i causes the decrease in the probability $p_i(x)$. Under the random matching rule, $p_i(x)(= x_j)$ is independent of x_i. By condition (ii), the probability $p_i(x)$ that a selfish agent meets a selfish opponent is higher than that under the random matching rule for any interior state $x \in (0,1) \times (0,1)$. Condition (iii) is a technical property so that the RD$^\alpha$ has a unique solution path to the initial value problem within domain $(0,1) \times (0,1)$. Let $\mathbb{A}$ be the set of all assortative matching rules. We henceforth assume that a matching rule is assortative.

Section 5 shows that there exists an assortative matching rule $\alpha \in \mathbb{A}$ that cannot be extended to a Lipschitz continuous rule on the the closed set $[0,1] \times [0,1]$. To overcome the difficulty, we use a "perturbation" method, which is in the same spirit as the *trembling-hand* approach (Selten [1975]) in the literature on refinement. Roughly, we introduce a slightly perturbed version of RD$^\alpha$ (7)–(8), and analyze a limit point of asymptotically stable states of the perturbed system as perturbation vanishes.

Formally, we define a perturbed replicator dynamics on state space $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$ for the RD$^\alpha$ and noise level $\epsilon \in (0, 1/2)$ as follows.

Definition 3. A *Perturbed Replicator Dynamics of the RD$^\alpha$ with noise* ϵ (PRD$^{\alpha,\epsilon}$) is defined by

$$\dot{x}_1 = g_1^{\alpha,\epsilon}(x) = \frac{1}{1-2\epsilon}(x_1 - \epsilon)(1 - \epsilon - x_1)\psi_1^\alpha(x) \tag{9}$$

$$\dot{x}_2 = g_2^{\alpha,\epsilon}(x) = \frac{1}{1-2\epsilon}(x_2 - \epsilon)(1 - \epsilon - x_2)\psi_2^\alpha(x), \tag{10}$$

where $\psi_1^\alpha(x) = f_L^\alpha(x) - f_H^\alpha(x) = 3p_1(x) - 2$, $\psi_2^\alpha(x) = f_Y^\alpha(x) - f_N^\alpha(x) = 2 - p_2(x) - 2q_2(x)$, and $x \in [\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$. The value ϵ is a noise level of the PRD$^{\alpha,\epsilon}$.

Although this dynamics is not the standard replicator dynamics, it has the same properties—*regularity* and *monotonicity* (Binmore and Samuelson [1999]). In the PRD$^{\alpha,\epsilon}$ (9)–(10), the growth rate is continuous on state space $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$ (regularity) and the growth rate of a relatively low-payoff action is smaller than that of a relatively high-payoff action (monotonicity). Section 5 discusses an interpretation of this perturbed dynamics and constructs the PRD$^{\alpha,\epsilon}$ from a simple imitation learning model.

Finally, we define a limit rest point and a limit asymptotically stable point. Recall the standard stability concepts of a dynamical system (Vega-Redondo [2003]).

Definition 4. Let $\dot{x}_i = g_i(x)$, $i = 1, \ldots, k$, be a dynamical system in $\mathbb{R}^k$.

(1) A state $x^* \in \mathbb{R}^k$ is a *rest point* if $g_k(x^*) = 0$ for all k.

(2) A state $x^* \in \mathbb{R}^k$ is an *asymptotically stable point* if the following two conditions hold:

   (i) (Liapunov stability) Given any neighborhood U_1 of x^*, there exists some neighborhood U_2 of x^* such that for any path $x = x(t)$, $x(0) \in U_2$ implies $x(t) \in U_1$ for all $t > 0$.

(ii) There exists some neighborhood V of x^* such that for any path $x = x(t)$, $x(0) \in V$ implies $\lim_{t\to\infty} x(t) = x^*$.

We then define a notion of limit stability. A sequence of noise levels $\{\epsilon_n\}_{n=1}^{\infty}$ is *admissible* if ϵ_n converges to 0 as n goes to infinity. Since p_i and q_i are Lipschitz continuous on $(0,1) \times (0,1)$ for $i = 1,2$, the PRD$^{\alpha,\epsilon_n}$ is Lipschitz continuous on $[\epsilon_n, 1-\epsilon_n] \times [\epsilon_n, 1-\epsilon_n]$ for all ϵ_n.

Definition 5. (1) A state $x = (x_1, x_2) \in [0,1] \times [0,1]$ is a *limit rest point* of RD$^\alpha$ if for any admissible sequence $\{\epsilon_n\}_{n=1}^{\infty}$, there exist some sequence $\{x_n\}_{n=1}^{\infty}$ and $\bar{n}$ such that (i) $x_n = (x_{1n}, x_{2n}) \in [\epsilon_n, 1-\epsilon_n] \times [\epsilon_n, 1-\epsilon_n]$ is a rest point of the PRD$^{\alpha,\epsilon_n}$ for all $n > \bar{n}$, and (ii) x_n converges to x as n goes to infinity.

(2) A state $x = (x_1, x_2) \in [0,1] \times [0,1]$ is a *limit asymptotically stable point* of RD$^\alpha$ if for any admissible sequence $\{\epsilon_n\}_{n=1}^{\infty}$, there exist some sequence $\{x_n\}_{n=1}^{\infty}$ and $\bar{n}$ such that (i) $x_n = (x_{1n}, x_{2n}) \in [\epsilon_n, 1-\epsilon_n] \times [\epsilon_n, 1-\epsilon_n]$ is an asymptotically stable point of the PRD$^{\alpha,\epsilon_n}$ for all $n > \bar{n}$, and (ii) x_n converges to x as n goes to infinity.

For any limit rest point and any limit asymptotically stable point of RD$^\alpha$, there exist a rest point and an asymptotically stable point of PRD$^{\alpha,\epsilon}$ in ϵ-neighborhood of those states for any sufficiently small $\epsilon > 0$, respectively.

3 Results

We characterize a set of limit asymptotically stable states of RD$^\alpha$. Given a noise level $\epsilon \in (0, \frac{1}{2})$, let $\mathcal{R}(\alpha, \epsilon) = \{\, x \mid g_1^{\alpha,\epsilon}(x) = g_2^{\alpha,\epsilon}(x) = 0 \,\}$ be the set of all rest points of PRD$^{\alpha,\epsilon}$ (9)–(10) for an assortative matching rule $\alpha = (p_1, p_2, q_1, q_2) \in \mathbb{A}$, and $\mathcal{R}(\alpha)$ be the set of all limit rest points of RD$^\alpha$ as ϵ vanishes.

Proposition 1. $\mathcal{R} \equiv \bigcup_{\alpha\in\mathbb{A}} \mathcal{R}(\alpha) = \big\{\, (1,0), (1,1)\} \cup \{(0,c) \,\big|\, c \in [0,1]\} \cup \{(c, \frac{1}{2}) | c \in [0, \frac{3}{4}] \,\big\}$.

Proof. For all $i = 1,2$, $g_i^{\alpha,\epsilon}(x) = 0$ in (9)–(10) if and only if $x_i = \epsilon$, $x_i = 1-\epsilon$, or $\psi_i^\alpha(x) = 0$. Obviously, the four states (ϵ,ϵ), $(\epsilon, 1-\epsilon)$, $(1-\epsilon, \epsilon)$, and $(1-\epsilon, 1-\epsilon)$ are in $\mathcal{R}(\alpha,\epsilon)$ for all ϵ and all $\alpha \in \mathbb{A}$. Hence, $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ are in $\mathcal{R}(\alpha)$ for any α. We then examine the stability of other five cases in $\mathcal{R}(\alpha,\epsilon)$.

Case (i): $(x_1(\alpha,\epsilon), 1-\epsilon)$ with $\psi_1^\alpha(x_1, 1-\epsilon) = 0$. We show that there exists no solution $x_1(\alpha,\epsilon)$ of $\psi_1^\alpha(x_1, 1-\epsilon) = 0$ for any $\alpha \in \mathbb{A}$ if $\epsilon < \frac{1}{3}$. Suppose that $\psi_1^\alpha(x_1, 1-\epsilon) = 0$. Then, $p_1(x_1, 1-\epsilon) = \frac{2}{3}$. However, this contradicts condition (ii) in Definition 2 since $p_1(x_1, 1-\epsilon) = \frac{2}{3} < 1-\epsilon$ if $\epsilon < \frac{1}{3}$. Thus, there exists no $(x_1(\alpha,\epsilon), 1-\epsilon)$ with $\psi_1^\alpha(x_1, 1-\epsilon) = 0$.

Now, the following four cases remain; case (ii) $(x_1(\alpha,\epsilon), \epsilon)$ with $\psi_1^\alpha(x_1, \epsilon) = 0$, case (iii) $(\epsilon, x_2(\alpha,\epsilon))$ with $\psi_2^\alpha(\epsilon, x_2) = 0$, case (iv) $(1-\epsilon, x_2(\alpha,\epsilon))$ with $\psi_2^\alpha(1-\epsilon, x_2) = 0$, and case (v) $(x_1(\alpha,\epsilon), x_2(\alpha,\epsilon))$ with $\psi_1^\alpha(x_1, x_2) = \psi_2^\alpha(x_1, x_2) = 0$.

Case (ii): $(x_1(\alpha,\epsilon),\epsilon)$ with $\psi_1^\alpha(x_1,\epsilon)=0$. We first prove that a limit rest point of RD$^\alpha$ is $(0,0)$ for any α in this case. Consider a rule α under which there exists a solution x_1 of $\psi_1^\alpha(x_1,\epsilon)=0$ for any ϵ. Then, $p_1(x_1,\epsilon)=\frac{2}{3}$ since $\psi_1^\alpha(x_1,\epsilon)=0$. By (3) and $p_2\leq 1$,

$$x_1=\frac{\epsilon}{p_1(x_1,\epsilon)}p_2(x_1,\epsilon)\leq\frac{3}{2}\epsilon.$$

Thus, $(x_1(\alpha,\epsilon_n),\epsilon_n)$ converges to $(0,0)$ for any α as n goes to infinity.

We next construct an assortative matching rule α under which there exists a solution $x_1(\alpha,\epsilon)$ of $\psi_1^\alpha(x_1,\epsilon)=0$ for any ϵ. Consider an assortative matching rule $\alpha(\lambda)=(p_1,p_2,q_1,q_2)$ such that

$$p_1(x_1,x_2)=(1-\lambda)x_2+\lambda(\min\left\{\frac{x_2}{x_1},1\right\}) \tag{11}$$

for all $x\in[\epsilon,1-\epsilon]\times[\epsilon,1-\epsilon]$, where $\lambda\in(0,1]$. Take $\lambda=1$. Then, $x_1=\frac{3}{2}\epsilon\in[\epsilon,1-\epsilon]$ is a solution of $\psi_1^{\alpha(1)}(x_1,\epsilon)=3p_1(x)-2=0$.

Case (iii): $(1-\epsilon,x_2(\alpha,\epsilon))$ with $\psi_2^\alpha(1-\epsilon,x_2)=0$. Similarly to case (ii), we can show that under the rule $\alpha(1)$, there exists a solution $x_2(\alpha,\epsilon)$ of $\psi_2^{\alpha(1)}(1-\epsilon,x_2)=0$. Next, we show that $(1,1)$ is a limit rest point for any α. Substituting $p_2,q_1\leq 1$ and (4) into $\psi_2^\alpha(1-\epsilon,x_2)=0$, we obtain

$$\begin{aligned}2&=p_2(1-\epsilon,x_2)+2q_2(1-\epsilon,x_2)\\&=p_2(1-\epsilon,x_2)+2q_1(1-\epsilon,x_2)\frac{\epsilon}{1-x_2}\\&\leq 1+2\frac{\epsilon}{1-x_2}.\end{aligned}$$

Thus,

$$x_2(\alpha,\epsilon)\geq 1-2\epsilon. \tag{12}$$

Hence, the rest point $(1-\epsilon_n,x_2(\alpha,\epsilon_n))$ converges to $(1,1)$ for any α as n goes to infinity.

Case (iv): $(\epsilon,x_2(\alpha,\epsilon))$ with $\psi_2^\alpha(\epsilon,x_2)=0$. We first show that for any $c\in[0,1]$, there exists some $\lambda\in(0,1]$ such that under the assortative matching rule $\alpha(\lambda)$, $(0,c)$ is a limit rest point of RD$^{\alpha(\lambda)}$. Since $x_2\geq\epsilon$, (11) implies $p_1(\epsilon,x_2)=(1-\lambda)x_2+\lambda$ under the rule $\alpha(\lambda)$. Substituting (3) and (5) into $\psi_2^{\alpha(\lambda)}(\epsilon,x_2)=2-p_2(\epsilon,x_2)-2q_2(\epsilon,x_2)=0$ yields $\frac{2x_2}{1+x_2}=p_1(\epsilon,x_2)$. Thus, any solution $x_2(\alpha(\lambda),\epsilon)$ of $\psi_2^{\alpha(\lambda)}(\epsilon,x_2)=0$ satisfies $x_2=\frac{\lambda}{1-\lambda}$. When $\lambda\in[\frac{\epsilon}{1+\epsilon},\frac{1-\epsilon}{2-\epsilon}]$, $x_2\in[\epsilon,1-\epsilon]$. Then, there exists a solution (ϵ,x_2) of $\psi_2^{\alpha(\lambda)}(\epsilon,x_2)=0$ under the rule $\alpha(\lambda)$.

Let $\{\epsilon_n\}$ be any admissible sequence. For any $d\in[0,\frac{1}{2}]$, choose $\lambda_n\in[\frac{\epsilon_n}{1+\epsilon_n},\frac{1-\epsilon_n}{2-\epsilon_n}]$ such that $\{\lambda_n\}$ converges to d as n goes to infinity. Then, the solution $x_2(\alpha(\lambda_n),\epsilon_n)=\frac{\lambda_n}{1-\lambda_n}$ of $\psi_2^{\alpha(\lambda_n)}(\epsilon_n,x_2)=0$ converges to $\frac{d}{1-d}$ as n goes to infinity. Hence, $(0,\frac{d}{1-d})$ is a limit rest point. Since $\frac{d}{1-d}\in[0,1]$, there exists a rule $\alpha(\lambda)$ under which $(0,c)$ is a limit rest point for all $c\in[0,1]$. For any other rule $\alpha\in\mathbb{A}$, if there exists a limit rest point, then it is in $\{(0,c) \mid c \in [0,1]\}$ since $\{\epsilon_n\}$ converges to 0 as n goes to infinity.

Case (v): $(x_1(\alpha,\epsilon), x_2(\alpha,\epsilon))$ with $\psi_1^\alpha(x_1,x_2) = \psi_2^\alpha(x_1,x_2) = 0$. Since $\psi_1^\alpha(x_1,x_2) = 0$, $p_1(x_1,x_2) = \frac{2}{3}$. By (3) and (5), $p_2(x_1,x_2) = \frac{2x_1}{3x_2}$ and $q_2(x_1,x_2) = 1 - \frac{x_1}{3(1-x_2)}$. Hence, $\psi_2^\alpha(x_1,x_2) = 0$ implies $x_2(\alpha,\epsilon) = \frac{1}{2}$. Then, $x_1(\alpha,\epsilon) \leq \frac{3}{4}$ by $p_2 \leq 1$. Using the same procedure as in case (iv), for all $c \in [0, \frac{3}{4}]$, there exists a rule $\alpha(\lambda)$ with $\lambda \in [\frac{1}{3}, 1]$ under which $(c, \frac{1}{2})$ is a limit rest point as n goes to infinity. Then, any state $(c, \frac{1}{2})$ with $c \in [0, \frac{3}{4}]$ is a limit rest point.

By the cases (i)–(v), $\mathcal{R} = \Big\{ (1,0),(1,1)\Big\} \cup \Big\{(0,c) \;\Big|\; c \in [0,1]\Big\} \cup \Big\{(c,\frac{1}{2}) | c \in [0,\frac{3}{4}]\Big\}$. □ □

Since any state x with $x_1 = 0$ is in $\mathcal{R}$, the equal allocation is supported by a limit rest point of RD$^\alpha$ for some α. The following proposition shows that the equal allocation is asymptotically stable for some α. Let $\mathcal{A}(\alpha,\epsilon)$ be the set of all asymptotically stable states of PRD$^{\alpha,\epsilon}$, and $\mathcal{A}(\alpha)$ be the set of all limit asymptotically stable states of RD$^\alpha$.

Proposition 2. $\mathcal{A} \equiv \bigcup_{\alpha \in \mathbb{A}} \mathcal{A}(\alpha) = \{\ (1,1)\ \} \cup \{\ (0,c) \mid c \in [0,1/2)\ \}$.

Proof. Obviously, $\mathcal{A}(\alpha,\epsilon) \subset \mathcal{R}(\alpha,\epsilon)$. Therefore, we only examine the asymptotic stability of all rest points in $\mathcal{R}(\alpha,\epsilon)$ for any α and any ϵ. We first prove two claims.

Claim 1. *Both $(\epsilon, 1-\epsilon)$ and $(1-\epsilon,\epsilon)$ are not asymptotically stable states of PRD$^{\alpha,\epsilon}$ for any assortative matching rule α and any sufficiently small noise level ϵ.*

We show only that $(\epsilon,1-\epsilon) \notin \mathcal{A}(\alpha,\epsilon)$ for all sufficiently small ϵ and all $\alpha \in \mathbb{A}$. The Jacobian matrix of g at $(\epsilon, 1-\epsilon)$ is:

$$\begin{aligned}
&\frac{\partial g}{\partial x}(\epsilon,1-\epsilon)\\
&= \begin{pmatrix} (3p_1(\epsilon,1-\epsilon)-2) & 0 \\ 0 & -(2-p_2(\epsilon,1-\epsilon)-2q_2(\epsilon,1-\epsilon)) \end{pmatrix}\\
&= \begin{pmatrix} (3p_1(\epsilon,1-\epsilon)-2) & 0 \\ 0 & (\frac{2-\epsilon}{1-\epsilon}p_1(\epsilon,1-\epsilon)-2) \end{pmatrix}.
\end{aligned}$$

It is well-known that a rest point of the system is asymptotically stable if and only if the real parts of both eigenvalues of the Jacobian are negative (e.g. Arnold [2006]). Thus, $(\epsilon, 1-\epsilon)$ is asymptotically stable if and only if both eigenvalues $(3p_1(\epsilon,1-\epsilon)-2)$ and $(\frac{2-\epsilon}{1-\epsilon}p_1(\epsilon,1-\epsilon)-2)$ are negative. That is, $p_1(\epsilon,1-\epsilon) < \min\Big\{\frac{2}{3}, \frac{2(1-\epsilon)}{2-\epsilon}\Big\}$. However, since $p_1(\epsilon,1-\epsilon) \geq 1-\epsilon$ by (ii) in Definition 2, $p_1(\epsilon,1-\epsilon) > \frac{2}{3}$ for any $\epsilon < \frac{1}{3}$. We can show that $(1-\epsilon,\epsilon)$ is not asymptotically stable for any α and any sufficiently small ϵ by the same procedure.

Claim 2. *Any solution $x(\alpha,\epsilon)$ of $\psi_1^\alpha(x) = \psi_2^\alpha(x) = 0$ is a saddle point for any assortative matching rule α and any sufficient small noise level ϵ.*

Recall the case (v) in Proposition 1: $p_1(x_1, x_2) = \frac{2}{3}$ and $x_2 = \frac{1}{2}$ for any solution (x_1, x_2) of $\psi_1^\alpha(x_1, x_2) = \psi_2^\alpha(x_1, x_2) = 0$. The Jacobian matrix of g at (x_1, x_2) is

$$\frac{\partial g}{\partial x}(x_1, x_2) = \frac{1}{1-2\epsilon}\begin{pmatrix} (x_1-\epsilon)(1-\epsilon-x_1)(\frac{\partial \psi_1^\alpha}{\partial x_1}(x)) & (x_1-\epsilon)(1-\epsilon-x_1)(\frac{\partial \psi_1^\alpha}{\partial x_2}(x)) \\ (x_2-\epsilon)(1-\epsilon-x_2)(\frac{\partial \psi_2^\alpha}{\partial x_1}(x)) & (x_2-\epsilon)(1-\epsilon-x_2)(\frac{\partial \psi_2^\alpha}{\partial x_2}(x)) \end{pmatrix}$$
$$= \frac{1}{1-2\epsilon}\begin{pmatrix} (x_1-\epsilon)(1-\epsilon-x_1)(3\frac{\partial p_1}{\partial x_1}(x)) & (x_1-\epsilon)(1-\epsilon-x_1)(3\frac{\partial p_1}{\partial x_2}(x)) \\ (\frac{1}{2}-\epsilon)^2(-6x_1\frac{\partial p_1}{\partial x_1}(x)) & (\frac{1}{2}-\epsilon)^2(x_1(\frac{16}{3}-6\frac{\partial p_1}{\partial x_2}(x))) \end{pmatrix}.$$

The determinant of this matrix $(x_1-\epsilon)(1-\epsilon-x_1)(1-2\epsilon)(4x_1\frac{\partial p_1}{\partial x_1}(x)) \le 0$ since $\frac{\partial p_1}{\partial x_1}(x) \le 0$. This implies that one eigenvalue is non-negative and the other is non-positive. Therefore, $x(\alpha, \epsilon)$ is a saddle point for any α and any ϵ.

We next examine the asymptotic stability of (ϵ, ϵ), $(1-\epsilon, 1-\epsilon)$, and points in cases (ii)–(iv). First, consider the rest point (ϵ, ϵ) of PRD$^{\alpha,\epsilon}$. By (3) and (5), the Jacobian matrix of g at (ϵ, ϵ) is:

$$\frac{\partial g}{\partial x}(\epsilon, \epsilon) = \begin{pmatrix} (3p_1(\epsilon, \epsilon) - 2) & 0 \\ 0 & (2 - p_2(\epsilon, \epsilon) - 2q_2(\epsilon, \epsilon)) \end{pmatrix}$$
$$= \begin{pmatrix} (3p_1(\epsilon, \epsilon) - 2) & 0 \\ 0 & (\frac{2\epsilon}{1-\epsilon} - p_1(\epsilon, \epsilon)\frac{1+\epsilon}{1-\epsilon}). \end{pmatrix}$$

If a rule α satisfies $\frac{2\epsilon}{1+\epsilon} < p_1(\epsilon, \epsilon) < \frac{2}{3}$, then both eigenvalues are negative. Under the rule $\alpha(\frac{1}{10})$ (defined by (11)), these conditions are satisfied. Hence, there exists an assortative matching rule α under which (ϵ, ϵ) is an asymptotically stable point of PRD$^{\alpha,\epsilon}$ for any ϵ. For other rest points, $(1-\epsilon, 1-\epsilon)$ and points in cases (ii)–(iv), we can show that there exists a rule α under which each rest point is asymptotically stable for any ϵ by the same procedure.

For each state in $\mathcal{A}(\alpha, \epsilon)$, the conditions for rule α that it is an asymptotically stable point of PRD$^{\alpha,\epsilon}$ are given as follows:

(a) (ϵ, ϵ) is asymptotically stable if α satisfies

$$\frac{2\epsilon}{1+\epsilon} < p_1(\epsilon, \epsilon) < \frac{2}{3}. \tag{13}$$

(b) $(1-\epsilon, 1-\epsilon)$ is asymptotically stable if α satisfies

$$\frac{2}{3} < p_1(1-\epsilon, 1-\epsilon) < 1 - \frac{\epsilon}{2-\epsilon}. \tag{14}$$

(c) Case (ii): $(x_1(\alpha,\epsilon),\epsilon)$ with $\psi_1^\alpha(x_1,\epsilon) = 0$ is asymptotically stable if α satisfies

$$\begin{aligned} \frac{\partial p_1}{\partial x_1}(x_1(\alpha,\epsilon),\epsilon) < 0 \\ \frac{2\epsilon}{1+\epsilon} < p_1(x_1(\alpha,\epsilon),\epsilon) = \frac{2}{3}. \end{aligned} \tag{15}$$

(d) Case (iii): $(1-\epsilon, x_2(\alpha,\epsilon))$ with $\psi_2^\alpha(1-\epsilon,x_2) = 0$ is asymptotically stable if α satisfies

$$\begin{aligned} \frac{2}{3} < p_1(1-\epsilon, x_2(\alpha,\epsilon)) = 2\frac{x_2(\alpha,\epsilon)}{1+x_2(\alpha,\epsilon)} \\ \frac{\partial p_1}{\partial x_2}(1-\epsilon, x_2(\alpha,\epsilon)) > 2\frac{1-x_2(\alpha,\epsilon)}{1+x_2(\alpha,\epsilon)}. \end{aligned} \tag{16}$$

(e) Case (iv): $(\epsilon, x_2(\alpha,\epsilon))$ with $\psi_2^\alpha(\epsilon,x_2) = 0$ is asymptotically stable if α satisfies

$$\begin{aligned} \frac{2}{3} > p_1(\epsilon, x_2(\alpha,\epsilon)) = 2\frac{x_2(\alpha,\epsilon)}{1+x_2(\alpha,\epsilon)} \\ \frac{\partial p_1}{\partial x_2}(\epsilon, x_2(\alpha,\epsilon)) > 2\frac{1-x_2(\alpha,\epsilon)}{1+x_2(\alpha,\epsilon)}. \end{aligned} \tag{17}$$

States $(0,0)$ and $(1,1)$ are limit asymptotically stable since the asymptotically stable states (ϵ_n,ϵ_n) and $(1-\epsilon_n, 1-\epsilon_n)$ converge to $(0,0)$ and $(1,1)$, respectively, as n goes to infinity for any $\alpha \in \mathbb{A}$. Next, consider the the asymptotically stable states $(x_1(\alpha,\epsilon),\epsilon)$ and $(1-\epsilon, x_2(\alpha,\epsilon))$ such that $x_1(\alpha,\epsilon)$ and $x_2(\alpha,\epsilon)$ are solutions of $\psi_1^\alpha(x_1,\epsilon) = 0$ and $\psi_2^\alpha(1-\epsilon,x_2) = 0$ respectively (cases (ii) and (iii)). By Proposition 1, for any $\alpha \in \mathbb{A}$, $(x_1(\alpha,\epsilon_n),\epsilon_n)$ and $(1-\epsilon_n, x_2(\alpha,\epsilon_n))$ converge to $(0,0)$ and $(1,1)$, respectively, as n goes to infinity.

Finally, consider the asymptotically stable point $(\epsilon, x_2(\alpha,\epsilon))$ such that $x_2(\alpha,\epsilon)$ is a solution of $\psi_2^\alpha(\epsilon,x_2) = 0$ (case (iv)). By Proposition 1, for each state $(0,c)$ with $c \in [0,1]$, there exists a rule α under which rest point $(\epsilon_n, x_2(\alpha,\epsilon_n))$ converges to $(0,c)$ as n goes to infinity. By (3) and (5), $\psi_2^\alpha(\epsilon,z_2) = 0$ implies the following:

$$p_1(\epsilon, x_2(\alpha,\epsilon)) = 2\frac{x_2(\alpha,\epsilon)}{1+x_2(\alpha,\epsilon)}.$$

By (17), $(\epsilon, x_2(\alpha,\epsilon))$ is asymptotcally stable if $x_2(\alpha,\epsilon) < \frac{1}{2}$. Then, for any $(0,c)$ with $c \in [0,1/2)$, we can construct an assortative matching rule α such that $(0,c)$ is limit asymptotically stable.

Therefore, $\mathcal{A} = \{\,(1,1)\,\} \cup \{\,(0,c) \mid c \in [0,1/2)\,\}$. □ □

Proposition 2 shows that there is a limit asymptotically stable point in the set of imperfect Nash equilibria $\{(0,c)|c \in [0,1/2)\}$ for *some* assortative matching rules. We call each imperfect Nash equilibrium state $(0,c)$ with $c \in (0,1/2)$ an *partially fair equilibrium state* in which all proposers are fair but there exists a non-negligible quantity of selfish responders; the imperfect Nash equilibrium state $(0,0)$ the *mutually fair equilibrium state* in which all proposers are fair and all responders are reciprocal; and the subgame perfect equilibrium state $(1,1)$ the *selfish equilibrium state* in which all agents are selfish.

Proposition 3. *The mutually fair equilibrium state and the selfish equilibrium state are limit asymptotically stable states if assortative matching rule α satisfies $\frac{\partial p_1}{\partial x_1}(x_1(\alpha,\epsilon),\epsilon) < 0$ and $\frac{2\epsilon}{1+\epsilon} < p_1(\epsilon,\epsilon)$ for any $\epsilon \in (0,\frac{1}{2})$, where $x_1(\alpha,\epsilon)$ is a solution of $\psi_1^\alpha(x_1,\epsilon) = 3 - 2p_1(x_1,\epsilon) = 0$.*

Proof. When $p_1(\epsilon,\epsilon) < \frac{2}{3}$, (ϵ,ϵ) is asymptotically stable by $\frac{2\epsilon}{1+\epsilon} < p_1(\epsilon,\epsilon)$ and (13). When $p_1(\epsilon,\epsilon) \geq \frac{2}{3}$, we consider $(x_1(\alpha,\epsilon),\epsilon)$. Since $p_1(x_1(\alpha,\epsilon),\epsilon) = \frac{2}{3}$, it is asymptotically stable by $\frac{\partial p_1}{\partial x_1}(x_1(\alpha,\epsilon),\epsilon) < 0$ and (15). Thus, since $\lim_{n\to\infty}(\epsilon_n,\epsilon_n) = \lim_{n\to\infty}(x_1(\alpha,\epsilon_n),\epsilon_n) = (0,0)$, the mutually fair equilibrium state is limit asymptotically stable under any α that satisfies $\frac{\partial p_1}{\partial x_1}(y_1(\alpha,\epsilon),\epsilon) < 0$ and $\frac{2\epsilon}{1+\epsilon} < p_1(\epsilon,\epsilon)$.[6]

When $1-\frac{\epsilon}{2-\epsilon} > p_1(1-\epsilon,1-\epsilon)$, $(1-\epsilon,1-\epsilon)$ is asymptotically stable by (14) and $p_1(1-\epsilon,1-\epsilon) \geq 1-\epsilon > \frac{2}{3}$ for any $\epsilon \in (0,\frac{1}{3})$. When $1-\frac{\epsilon}{2-\epsilon} \leq p_1(1-\epsilon,1-\epsilon)$, we consider $(1-\epsilon,x_2(\alpha,\epsilon))$ with $\psi_2^\alpha(1-\epsilon,x_2) = 0$. By (ii) in Definition 2 and (12), we obtain $p_1(1-\epsilon,x_2(\alpha,\epsilon)) > x_2(\alpha,\epsilon) > \frac{2}{3}$ for any $\epsilon \in (0,\frac{1}{6})$. Furthermore, by (5) and (i) in Definition 2 ($\frac{\partial q_2}{\partial x_2} \geq 0$),

$$\frac{\partial p_1}{\partial x_2}(x) \geq \frac{1-p_1(x)}{1-x_2}.$$

Substituting (3) and (5) into $\psi_2^\alpha(1-\epsilon,x_2) = 0$, we obtain $\frac{1-p_1(1-\epsilon,x_2(\alpha,\epsilon))}{1-x_2(\alpha,\epsilon)} = \frac{1}{1+x_2(\alpha,\epsilon)}$. Combining these with (12) yields that for $\epsilon \in (0,\frac{1}{4})$,

$$\frac{\partial p_1}{\partial x_2}(1-\epsilon,x_2(\alpha,\epsilon)) \geq \frac{1}{1+x_2(\alpha,\epsilon)} > 2\frac{1-x_2(\alpha,\epsilon)}{1+x_2(\alpha,\epsilon)}.$$

Thus, $(1-\epsilon,x_2(\alpha,\epsilon))$ is an asymptotically stable point of $\mathrm{PRD}^{\alpha,\epsilon}$ for any $\epsilon \in (0,\frac{1}{6})$ by (16). Since $\lim_{n\to\infty}(1-\epsilon_n,1-\epsilon_n) = \lim_{n\to\infty}(1-\epsilon_n,x_2(\alpha,\epsilon_n)) = (1,1)$, the selfish equilibrium state is limit asymptotically stable under any assortative matching rule. □

Proposition 3 provides a sufficient condition of assortative matching rules for the existence of the limit asymptotically stable mutually fair equilibrium state. Since $\frac{2\epsilon}{1+\epsilon} \approx \epsilon$ (the difference converges to 0 as ϵ goes to 0) and any assortative matching rule satisfies $p_1(\epsilon,\epsilon) > \epsilon$, the mutually fair equilibrium state is a limit asymptotically stable point unless the matching rules has very low assortativity. However, the selfish equilibrium state is also a limit asymptotically stable point for any matching rule, and then either fair or selfish behavior survives depending on the initial state.

Intuitively, it is easy for reciprocal responders to encounter fair proposers compared to selfish responders by assortativity. Hence, if proposers are almost fair ($x_1 \approx 0$), then action N generates

[6] However, partially fair equilibrium states are not always limit asymptotically stable. A counter-example is given in Section 4.

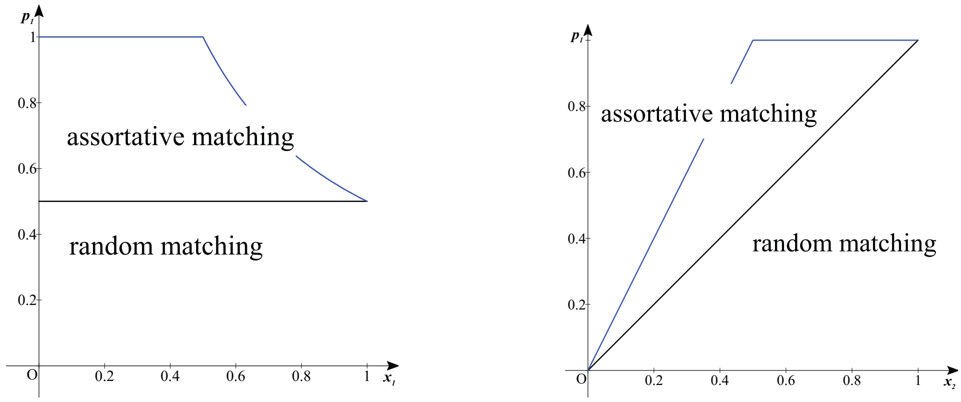

Figure 4: Probability $p_1 = \Pr(L \text{ meets } Y)$. The graph on the left is $p_1(x_1, 0.5)$. The graph on the right is $p_1(0.5, x_2)$.

higher average fitness than action Y. If proposers are almost selfish ($x_1 \approx 1$), in contrast, the average fitness is less for action N than action Y. Therefore, both the mutually fair equilibrium state and the selfish equilibrium state are limit asymptotically stable.

4 An Example

This section provides a completely assortative matching rule under which only the mutually fair equilibrium and the selfish equilibrium are limit asymptotically stable states.

Definition 6. A matching rule $\bar{\alpha} = (\bar{p}_1, \bar{p}_2, \bar{q}_1, \bar{q}_2)$ on $[0,1] \times [0,1]$ is *completely assortative* if the matching probability is defined as follows: In population 1,

$$\bar{p}_1 = \begin{cases} \frac{x_2}{x_1} & \text{if } x_1 > x_2 \\ 1 & \text{otherwise,} \end{cases} \qquad \bar{q}_1 = \begin{cases} \frac{1-x_2}{1-x_1} & \text{if } x_1 \leq x_2 \\ 1 & \text{otherwise.} \end{cases}$$

In population 2,

$$\bar{p}_2 = \begin{cases} \frac{x_1}{x_2} & \text{if } x_1 \leq x_2 \\ 1 & \text{otherwise,} \end{cases} \qquad \bar{q}_2 = \begin{cases} \frac{1-x_1}{1-x_2} & \text{if } x_1 > x_2 \\ 1 & \text{otherwise.} \end{cases}$$

The rule $\bar{\alpha}$ is equal to the rule $\alpha(1)$ defined by (11). Figure 4 shows the probability that each selfish proposer encounter a selfish responder under this assortative matching rule and under the random matching rule. It is evident that fair proposers are more likely to encounter reciprocal responders than selfish proposers. The completely assortative matching rule maximizes the number of pairs that consist of a fair proposer and a reciprocal responder.

Under $\bar{\alpha}$, the PRD$^{\bar{\alpha},\epsilon}$ is given as:

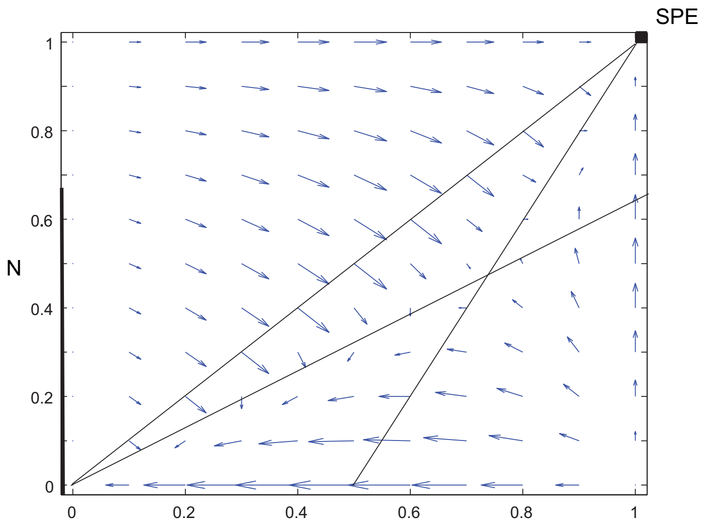

Figure 5: Phase diagram under the completely assortative matching rule. N is the set of imperfect Nash equilibria, and SPE is the subgame perfect equilibrium.

Case 1 : When $x_1 > x_2$,

$$\dot{x}_1 = g_1^{\bar{\alpha},\epsilon}(x_1, x_2) = \frac{1}{1-2\epsilon}(x_1 - \epsilon)(1 - \epsilon - x_1)(3\frac{x_2}{x_1} - 2) \tag{18}$$

$$\dot{x}_2 = g_2^{\bar{\alpha},\epsilon}(x_1, x_2) = \frac{1}{1-2\epsilon}(x_2 - \epsilon)(1 - \epsilon - x_2)(1 - 2\frac{1-x_1}{1-x_2}). \tag{19}$$

Case 2 : When $x_1 \le x_2$,

$$\dot{x}_1 = g_1^{\bar{\alpha},\epsilon}(x_1, x_2) = \frac{1}{1-2\epsilon}(x_1 - \epsilon)(1 - \epsilon - x_1)(3 - 2) \tag{20}$$

$$\dot{x}_2 = g_2^{\bar{\alpha},\epsilon}(x_1, x_2) = \frac{1}{1-2\epsilon}(x_2 - \epsilon)(1 - \epsilon - x_2)(\frac{x_1}{x_2} + 2 - 2\frac{x_1}{x_2} - 2). \tag{21}$$

Figure 5 illustrates the phase diagram for the nonlinear system (18)–(21).

Proposition 4. *Let $\bar{\mathcal{A}}(\epsilon) = \mathcal{A}(\bar{\alpha}, \epsilon)$ be the set of asymptotically stable states of the system (18)–(21) under a noise level ϵ. Then, $\bar{\mathcal{A}}(\epsilon) = \{\, (\frac{3}{2}\epsilon, \epsilon), (1-\epsilon, 1-2\epsilon) \,\}$.*

Proof. It is straightforward that the system (18)–(21) has the following set of rest points:

$$\mathcal{R}(\bar{\alpha}, \epsilon) = \{\, (\epsilon, \epsilon), (\epsilon, 1-\epsilon), (1-\epsilon, \epsilon), (1-\epsilon, 1-\epsilon) \,\} \\ \cup \left\{\, (\frac{3}{2}\epsilon, \epsilon), (1-\epsilon, 1-2\epsilon)(\frac{3}{4}, \frac{1}{2}) \,\right\}.$$

To prove the proposition, we examine the eigenvalues of the Jacobian matrix at each rest point. In the case of $(1-\epsilon, \epsilon)$, the Jacobian matrix at $(1-\epsilon, \epsilon)$ is

$$\frac{\partial g}{\partial x}(1-\epsilon, \epsilon) = \begin{pmatrix} 2 - \frac{3\epsilon}{1-\epsilon} & 0 \\ 0 & 1 - \frac{2\epsilon}{1-\epsilon} \end{pmatrix}.$$

Therefore, $(1-\epsilon, \epsilon)$ is not asymptotically stable for sufficiently small ϵ. Similarly, we can show that $(\epsilon, 1-\epsilon)$ is not asymptotically stable.

In the case of $(1-\epsilon, 1-\epsilon)$, it is not sufficient to consider the system in any one case since both cases of the system is clearly included in any neighborhood of $(1-\epsilon, 1-\epsilon)$. The Jacobian matrices at $(1-\epsilon, 1-\epsilon)$ in the cases 1 and 2 are the same and are given by

$$\frac{\partial g}{\partial x}(1-\epsilon, 1-\epsilon) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $(1-\epsilon, 1-\epsilon)$ is a saddle state for sufficiently small ϵ. We can show that $(3/4, 1/2)$, (ϵ, ϵ) are saddle states in the same manner.

Finally, we examine the case of $(\frac{3}{2}\epsilon, \epsilon)$. The Jacobian matrix at $(\frac{3}{2}\epsilon, \epsilon)$ is as follows:

$$\frac{\partial g}{\partial x}(\frac{3}{2}\epsilon, \epsilon) = \frac{1}{1-2\epsilon} \begin{pmatrix} -\frac{2}{3} + \frac{5}{3}\epsilon & 1 - \frac{5}{2}\epsilon \\ 0 & \frac{-1+2\epsilon}{1-\epsilon} \end{pmatrix}.$$

Thus, $(\frac{3}{2}\epsilon, \epsilon)$ is asymptotically stable for sufficiently small ϵ. We can show that $(1-\epsilon, 1-2\epsilon)$ is asymptotically stable in the same manner. □

Since $(\frac{3}{2}\epsilon_n, \epsilon_n)$ and $(1-\epsilon_n, 1-2\epsilon_n)$ converge to $(0, 0)$ and $(1, 1)$, respectively, as n goes to infinity for any admissible sequence $\{\epsilon_n\}$, Proposition 4 shows that only the mutually fair equilibrium state and the selfish equilibrium state are limit asymptotically stable states when the matching rule is completely assortative.

5 Discussion

Assortative Matching Rule By Definition 2, any assortative matching rule is Lipschitz continuous on the open set $(0, 1) \times (0, 1)$. We show that if the state space is the closed set $[0, 1] \times [0, 1]$, then there exists an assortative matching rule α that is discontinuous on $[0, 1] \times [0, 1]$. Consider the assortative matching rule $\alpha(\lambda)$ defined by (11). Then $p_1(x) = (1-\lambda)x_2 + \lambda \min\left\{\frac{x_2}{x_1}, 1\right\}$ for all $x \in (0, 1) \times (0, 1)$. Under $\alpha(\lambda)$, $\lim_{x_2 \to 0} p_1(x_1, x_2) = 0$ for all $x_1 \in (0, 1)$. Hence, $\lim_{x_1 \to 0} \lim_{x_2 \to 0} p_1(x_1, x_2) = 0$. However, since $p_1(x, x) = (1-\lambda)x + \lambda$ for all $x \in (0, 1) \times (0, 1)$, $\lim_{x \to 0} p_1(x, x) = \lambda > 0$. Therefore, the rule $\alpha(\lambda)$ is discontinuous at $(0, 0)$ if the state space is the closed set $[0, 1] \times [0, 1]$.

In words, suppose that almost all proposers are fair and a small proportion $\epsilon > 0$ of them are selfish, whereas all responders are reciprocal ($(x_1, x_2) = (\epsilon, 0)$). Then, under any matching rule, the probability that every selfish proposer meets a selfish responder is zero, independent of ϵ. However, if the population of responders also has a ϵ-proportion of selfish responders ($(x_1, x_2) = (\epsilon, \epsilon)$), then under an assortative matching rule, the probability that a selfish proposer meets a selfish responder is $\lambda > \epsilon$ for any $\epsilon > 0$. This implies that an assortative matching rule is discontinuous at $(0, 0)$. Therefore, we introduce the perturbed replicator dynamics on state space $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$.

Perturbed Replicator Dynamics We have studied the PRD$^{\alpha,\epsilon}$ (9)–(10). This perturbed dynamics is interpreted as follows. There are two types of agents—*committed* and *uncommitted* agents.[7]

Suppose that there is a small exogenous fraction (say 2ϵ) of committed agents in the two populations. They play one of two actions repeatedly as a dominant strategy. An evolutionary interpretation for commitments may be as follows. Committed agents are (genetically) programmed to play a prescribed action. They do not change their actions even when they receive a revision opportunity. We assume that each action k is played by ϵ of committed agents, $k = L, H, Y, N$.

The remaining $1-2\epsilon$ of agents are called uncommitted. The evolution of behavior of uncommitted agents is subject to the replicator dynamics. We normalize the population size of uncommitted agents to 1. Let $y_i \in [0, 1]$ be the normalized mass of selfish uncommitted agents. Thus, the mass of selfish agents $x_i^\epsilon = (1-2\epsilon)y_i + \epsilon$ for $i = 1, 2$. Note that $x_i^\epsilon \in [\epsilon, 1-\epsilon]$ for $i = 1, 2$. Fix $\alpha \in \mathbb{A}$. Then, parallel to the replicator dynamics (1)–(2), the replicator dynamics for uncommitted agents is given by the following system:

$$\dot{y}_1 = y_1(f_L^\alpha(x^\epsilon) - \phi_1^\alpha(x^\epsilon)) = y_1(1-y_1)(f_L(x^\epsilon) - f_H(x^\epsilon)) \tag{22}$$
$$\dot{y}_2 = y_2(f_L^\alpha(x^\epsilon) - \phi_1^\alpha(x^\epsilon)) = y_2(1-y_1)(f_Y(x^\epsilon) - f_N(x^\epsilon)), \tag{23}$$

where ϕ_i^α is the average fitness of uncommitted agents in population i ($i = 1, 2$) given by

$$\phi_1^\alpha(x^\epsilon) = y_1 f_L^\alpha(x^\epsilon) + (1-y_1) f_H^\alpha(x^\epsilon)$$
$$\phi_2^\alpha(x^\epsilon) = y_2 f_Y^\alpha(x^\epsilon) + (1-y_2) f_H^\alpha(x^\epsilon).$$

Therefore, since $x_i^\epsilon = (1-2\epsilon)y_i + \epsilon$, the dynamics for the total populations is given by

$$\dot{x}_1^\epsilon = (1-2\epsilon)\dot{y}_1 = \frac{1}{1-2\epsilon}(x_1^\epsilon - \epsilon)(1-\epsilon-x_1^\epsilon)(f_L^\alpha(x^\epsilon) - f_H^\alpha(x^\epsilon))$$
$$\dot{x}_2^\epsilon = (1-2\epsilon)\dot{y}_1 = \frac{1}{1-2\epsilon}(x_2^\epsilon - \epsilon)(1-\epsilon-x_2^\epsilon)(f_Y^\alpha(x^\epsilon) - f_N^\alpha(x^\epsilon)).$$

This dynamics is equal to the PRD$^{\alpha,\epsilon}$ (9)–(10).

The PRD$^{\alpha,\epsilon}$ (9)–(10) is Lipschitz continuous on state space $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$, and the growth rate of

[7] This interpretation is suggested by an anonymous referee.

any action has a finite limit as state x goes to the boundary (regularity). The growth of dying action is 0 when a state is on the boundary. Thus, there is no extinction of any actions at all time. Furthermore, $PRD^{\alpha,\epsilon}$ satisfies the monotonicity, which is violated in the perturbed replicator dynamics defined by Gale et al. [1995]. When a state is in interior, the growth rate of relatively high-payoff action in the $PRD^{\alpha,\epsilon}$ is smaller than that in the RD^{α}, but the signs are always the same. Thus, although the $PRD^{\alpha,\epsilon}$ is not standard, it is a qualitative invariant of the standard RD^{α}. Note that every regular and monotonic selection dynamics has the same set of asymptotically stable states (Cressman [1997], Samuelson and Zhang [1992]).

Imitation Learning In the above, we assume that the evolution of behavior of uncommitted agents is given by the replicator dynamics. The replicator dynamics can be interpreted as an approximation of some learning models (e.g. reinforcement learning (Bögers and Sarin [1997]) and imitation learning (Schlag [1998])). We derive the replicator dynamics for uncommitted agents from the following proportional imitation learning rule.

The imitation learning rule is given as follows. Consider the normalized mass of uncommitted agents in population 1, $y_1 \in [0,1]$. We assume that each uncommitted proposer only imitates uncommitted proposers. According to Gale et al. [1995], divide time into discrete periods of length Δt. In each period, each uncommitted proposer k independently receives an opportunity to learn with probability Δt. Let $a_k(t)$ be the action adopted by k and $g_k(t)$ be the payoff of k at time t.

The imitation rule is given as follows. Suppose that uncommitted proposer k with $a_k(t) = H$ receives an opportunity. This event occurs with probability $\Delta t(1 - y_1(t))$. After receiving the opportunity, k randomly samples another uncommitted proposer l. Then, k observes l's current action $a_l(t)$ and payoff $g_l(t)$. If $a_l(t) = L$ and $g_l(t) > g_k(t)$, then k imitate l's action (i.e. $a_k(t + \Delta t) = L$) with some positive probability that is proportional to the payoff difference $g_l(t) - g_k(t)$, otherwise, k does not imitate ($a_k(t + \Delta t) = H$). That is, if $g_l(t) > g_k(t)$, k switches her action at time $t + \Delta t$ to $a_k(t+\Delta t) = L$ with probability $y_1(t)\beta(g_l(t)-g_k(t))$, where β is a constant switching rate. Each k with $a_k(t) = L$ imitates by the same rule. Therefore, the average net increase of uncommitted proposers who adopt action L is given by

$$y_1(t + \Delta t) - y_1(t) = \Delta t y_1(t)(1 - y_1(t))\beta(f_L^{\alpha}(x^{\epsilon}(t)) - f_H^{\alpha}(x^{\epsilon}(t))),$$

where $f_L^{\alpha}(x^{\epsilon}(t))$ and $f_H^{\alpha}(x^{\epsilon}(t))$ are the average payoffs of uncommitted proposers who adopt actions L and H, respectively. Hence,

$$\frac{y_1(t + \Delta t) - y_1(t)}{\Delta t} = \beta y_1(t)(1 - y_1(t))(f_L^{\alpha}(x^{\epsilon}(t)) - f_H^{\alpha}(x^{\epsilon}(t))).$$

	(L,Y)	(H,N)
(L,Y)	2, 2	1, 1
(H,N)	1, 1	2, 2

Figure 6: Payoff matrix in the role game with 2 types.

	(L,Y)	(H,N)	(H,Y)	(L,N)
(L,Y)	2, 2	1, 1	2.5, 1.5	0.5, 1.5
(H,N)	1, 1	2, 2	2, 2	1, 1
(H,Y)	1.5, 2.5	2, 2	2, 2	1.5, 2.5
(L,N)	1.5, 0.5	1, 1	2.5, 1.5	0, 0

Figure 7: Payoff matrix in the role game with 4 types.

Taking the limit as Δt goes to 0,

$$\dot{y}_1 = \beta y_1(1-y_1)(f_L^\alpha(x^\epsilon) - f_H^\alpha(x^\epsilon)).$$

Since the same derivation holds true for population 2, the dynamics is equal to the replicator dynamics (22)–(23) when $\beta = 1$.

Comparison to Role Game Consider the following role game. All agents belong to the same single population. They play the role of proposer with probability $\frac{1}{2}$ and that of responder with probability $\frac{1}{2}$ after they are paired under some matching rule. We assume that selfish agents play L or Y and that fair agents play H or N, according to their roles. This role game with two types has a symmetric payoff matrix, as shown in Figure 6. Under the random matching rule, the replicator dynamics is given by $\dot{x} = x(1-x)(2x-1)$, where x is a frequency of selfish agents. Since $\frac{d\dot{x}}{dx} = -6x^2 + 6x - 1 < 0$ when $x = 0$, the state in which all agents adopt (H,N) is asymptotically stable. Similarly, the state in which all agents adopt (L,N) is also asymptotically stable. Hence, either selfish or fair behavior survives even under the random matching rule without noise, depending on the initial state.

Under the assortative matching rule, the above result holds according to Taylor and Nowak [2006]. That is, both the selfish equilibrium state (L,Y) and the fair equilibrium state (H,N) are asymptotically stable for any level of assortativity. Suppose that all agents are fair and a proportion ϵ of entrants invade the population. Let $p(x)$ be the probability that each selfish agent meets a selfish agent, and let $q(x)$ be the probability that each fair agent meets a fair agent when the frequency of selfish agents is x. If a proportion ϵ of mutants invade the population, then they play each role with equal probability. Then, $p(\epsilon) < q(\epsilon)$ by the parity equation $\epsilon(1-p(\epsilon)) = (1-\epsilon)(1-q(\epsilon))$. The average fitness of each mutant is $\frac{1}{2}(3p(\epsilon)) + \frac{1}{2}(p(\epsilon) + 2(1-p(\epsilon)))$, and the average fitness of each fair agent is $1 + q(\epsilon)$. Therefore, the selfish action never diffuse since

$$\frac{1}{2}(3p(\epsilon)) + \frac{1}{2}(p(\epsilon) + 2(1-p(\epsilon))) - (1+q(\epsilon)) = p(\epsilon) - q(\epsilon) < 0.$$

However, if the populations are separate and an entrant adopting Y invades, then this selfish responding action diffuses when the assortativity is low.[8]

Suppose that, in addition to selfish (L,Y) agents and fair (H,N) agents, there are (H,Y) agents

[8] Cressman [2006] studies relation between the asymptotically stability and the uninvadability in n-population games.

and (L, N) agents. This role game with four types is shown in Figure 7. In the standard replicator dynamics, the state in which all agents adopt (L, Y) is the unique asymptotically stable point since $((L, Y), (L, Y))$ is the unique strict Nash equilibrium and there is no interior Nash equilibrium. Thus, only the selfish behavior survives under the random matching rule.

Even if we introduce the assortative matching rule, any state in which all agents are $(H, \cdot)$ agents (agents who propose H) is not stable. For any state in which there are only (H, Y) agents and (H, N) agents, (H, Y) agents can diffuse under any matching rule. Further, once there are only (H, Y) agents, (L, Y) agents diffuse under any matching rule, since both the maximum payoff for (H, Y) agents and the minimum payoff for (L, Y) agents are 2. Therefore, any imperfect Nash equilibrium of the ultimatum game, in which equal allocation is achieved, is not asymptotically stable and the selfish subgame perfect equilibrium is asymptotically stable.

Partner Choice and Reputation Bergstrom [2003] studies a model in which all players can choose their partners according to some labels or signals as a foundation of assortative matching.[9] Suppose the following: each player's label represents his/her true action, search cost is negligible, and proposers can unilaterally propose an offer to any responder. Since each selfish proposer obtains a higher payoff if her partner is also selfish, the number of pairs consisting of a selfish proposer and a selfish responder is maximized (i.e. the completely assortative matching by labels) in the stable matching.

Suppose that labels are not accurate but based on a social reputation system (Nowak et al. [2000]).[10] Consider a reputation system in which every responder obtains a bad reputation if he accepts any unfair offer in a previous period and a good reputation otherwise. Each selfish proposer has an incentive to choose a responder who has a bad reputation in the reputation system since her unfair offer will be accepted. Therefore, the social reputation system prompts assortativity.

Inequity Aversion We focused only on an evolution of behavior of populations with assortativity in the ultimatum bargaining. Fehr and Schmidt [1999] and Bolton and Ockenfels [2000] point out that people's preferences are usually not self-interested but inequity averse. The preferences of inequity-averse subjects depend not only on their own monetary payoffs but also on fairness or equity. This notion can approximately justify the fair behavior surviving the evolutionary pressure with assortativity.

6 Concluding Remarks

We investigated the role of matching rules in the replicator dynamics in the ultimatum game. Gale et al. [1995] shows that if encounters are random, then the imperfect Nash equilibrium resulting in

[9] The assortative matching by partner choice is also studied in the field of search theory.

[10] Nowak and Sigmund [1998] and Ohtsuki and Iwasa [2004] study the social reputation system in the prisoners' dilemma game.

fair allocation is a limit asymptotically stable point as noise vanishes when the learning of responders is noisier than that of proposers.

Here, we considered the evolution of fair actions under an assortative matching rule introduced by Becker [1973, 1974]. This can be regarded as an alternative explanation of fair behavior. The assortative matching rule leads to the replicator dynamics with non-linear fitness functions, and thus expands the set of stable states. For the ultimatum game, under some assortative matching rules, the average fitness of reciprocal responders is higher than that of selfish responders when there is a large mass of fair proposers. Then, the mutually fair imperfect Nash equilibrium state is limit asymptotically stable as the noise due to committed agents vanishes. Therefore, the fair behavior may survive in the long run.

Our study has some limitations. First, the selfish equilibrium state is also asymptotically stable. The dynamic path depends on an initial state and an assortative matching rule. Second, it is not known if the same result holds for other types of perturbed dynamics. The ultimatum "mini" game is restricted. The analysis of general games is left for future work.

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