Augmenting a $(k-1)$-Vertex-Connected Multigraph to an ℓ-Edge-Connected and k-Vertex-Connected Multigraph *

Toshimasa Ishii[†], Hiroshi Nagamochi[††] and Toshihide Ibaraki[†††]

[†]Department of Information and Computer Sciences,
Toyohashi University of Technology,
Toyohashi 441-8580, Japan.
ishii@ics.tut.ac.jp

[††]Department of Applied Mathematics and Physics,
Graduate School of Informatics,
Kyoto University
Kyoto 606-8501, Japan.
nag@amp.i.kyoto-u.ac.jp

[†††]Department of Informatics, School of Science and Technology,
Kwansei Gakuin University
Sanda 669-1337, Japan.
ibaraki@ksc.kwansei.ac.jp

Abstract For two integers $k, \ell > 0$ and an undirected multigraph $G = (V, E)$, we consider the problem of augmenting G by the smallest number of new edges to obtain an ℓ-edge-connected and k-vertex-connected multigraph. In this paper, we show that a $(k-1)$-vertex-connected multigraph G can be made ℓ-edge-connected and k-vertex-connected by adding at most $\max\{\ell+1, 2k-4\}$ surplus edges over the optimum in $O(\min\{k, \sqrt{n}\}kn^3 + n^4)$ time, where $n = |V|$.

Corresponding Author: Toshimasa Ishii Phone: +81-532-44-6767 Fax: +81-532-44-6757

Keywords: undirected multigraph, edge-connectivity, vertex-connectivity, graph augmentation, polynomial time approximation algorithm, deterministic algorithm

1 Introduction

Let $G = (V, E)$ stand for an undirected multigraph with a set V of *vertices* and a set E of *edges*. For a nonnegative integer ℓ, a multigraph G is called *ℓ-edge-connected* if the deletion of any $\ell-1$ or fewer edges leaves a connected multigraph. For a nonnegative integer k, a multigraph G is called *k-vertex-connected* if $|V| \geq k+1$ and the deletion of any $k-1$ or fewer vertices leaves a connected multigraph. Given a multigraph $G = (V, E)$ and two positive integers ℓ and k, we consider the problem of augmenting G by the smallest number of new edges to obtain an ℓ-edge-connected and k-vertex-connected multigraph (, which we call an (ℓ, k)-connected multigraph). We call this problem the *edge- and vertex-connectivity augmentation problem*, denoted by EVAP(ℓ, k)

*A preliminary version of this paper was presented at 7th Annual European Symposium on Algorithms, Prague, July 1999.

(where $\ell \geq k$ is assumed since a k-vertex-connected graph is k-edge-connected). The problem of augmenting a graph by adding the smallest number of new edges to meet edge-connectivity or vertex-connectivity requirements has been extensively studied as an important subject in the network design problem, the data security problem [16], the graph drawing problem [15] and others, and many efficient algorithms have been developed so far.

In the case of $k = 1$, EVAP$(\ell, 1)$ is called the *edge-connectivity augmentation problem.* Watanabe and Nakamura [24] showed that the edge-connectivity augmentation problem can be solved in polynomial time for an arbitrary ℓ, and the complexity has been improved [5, 6, 22]. In the case of $\ell = k$, EVAP(k, k) is called the *vertex-connectivity augmentation problem.* It is still open whether the vertex-connectivity augmentation problem is polynomially solvable for an arbitrary k, even if an initial graph G is $(k-1)$-vertex-connected. For small k, polynomial time algorithms are known (see [3, 9] for $k = 2$, [8, 25, 26] for $k = 3$ and [7] for $k = 4$). For an arbitrary $k \geq 5$, Jordán presented a polynomial time approximation algorithm [13, 14] such that the gap between the number of new edges added by the algorithm and the optimal value is at most $(k-2)/2$. It was shown that EVAP$(\ell, 2)$ is polynomially solvable [10] and that EVAP$(\ell, 3)$ can be solved in polynomial time, under the assumption that ℓ is a fixed constant [11, 12]. However, for arbitrary ℓ and k with $\ell > k \geq 4$, it is not known that there is a polynomial time approximation algorithm such that the gap is small, say, $O(\ell)$, even if an initial graph is $(k-1)$-vertex-connected.

In this paper, we consider EVAP(ℓ, k) for arbitrary ℓ and k with $\ell \geq k \geq 4$. We first present a lower bound on the number of edges that is necessary to make a given graph G (ℓ, k)-connected, and then show that the lower bound plus $\max\{\ell+1, 2k-4\}$ edges suffices if G is $(k-1)$-vertex-connected. The task of constructing such set of new edges can be done in $O(\min\{k, \sqrt{n}\}kn^3+n^4)$ time.

The paper is organized as follows. In Section 2, we state our main result that EVAP(ℓ, k) is approximable within the absolute error $\max\{\ell + 1, 2k - 4\}$ for a $(k-1)$-vertex-connected graph, after introducing some basic notations and deriving two lower bounds on the optimal value of the problem. In Section 3, we describe a basic idea on our approximation algorithm, called EV-AUG, for EVAP(ℓ, k). In Section 4, we describe that the first lower bound can be computed in polynomial time. We show in Section 5 some previously known and newly derived edge-splitting operations, while stating several properties of k-vertex-connected graphs in Section 5.2. In Section 6, we give algorithm EV-AUG and prove its correctness and time complexity. In Section 7, we state some concluding remarks.

2 Main Theorem

Let $G = (V, E)$ stand for an undirected multigraph with a set V of *vertices* and a set E of *edges*, where we denote $|V|$ by n and the number of pairs of vertices which are adjacent in G by m. An edge with end vertices u and v is denoted by (u, v). In $G = (V, E)$, its vertex set V and edge set E may be denoted by $V(G)$ and $E(G)$, respectively. A singleton set $\{x\}$ may be simply written as x, and " $\subset$ " implies proper inclusion while " $\subseteq$ " means " $\subset$ " or " $=$ ". A *partition* $X_1, \ldots, X_t$ of the vertex set V means a family of nonempty disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset V' of V.

For a subset $V' \subseteq V$ (resp., $E' \subseteq E$) in G, $G[V']$ (resp., $G[E']$) denotes the subgraph induced by V' (resp., $G[E'] = (V, E')$). For $V' \subset V$ (resp., $E' \subseteq E$), we denote subgraph $G[V - V']$ (resp., $G[E - E']$) by $G - V'$ (resp., $G - E'$). For $E' \subset E$, we denote by $V[E']$ the set of all end vertices of edges in E'. For an edge set E' with $E' \cap E = \emptyset$, we denote the augmented graph $G = (V, E \cup E')$ by $G + E'$.

For two disjoint subsets of vertices X, $Y \subset V$, we denote by $E_G(X, Y)$ the set of edges $e = (x, y)$ such that $x \in X$ and $y \in Y$, and also denote $|E_G(X, Y)|$ by $c_G(X, Y)$. In particular, $E_G(u, v)$ is the set of edges with end vertices u and v.

A *cut* is defined as a subset X of V with $\emptyset \neq X \neq V$, and the *size* of a cut X is defined by $c_G(X, V - X)$, which may also be written as $c_G(X)$. A cut with the minimum size is called a (*global*) *minimum cut*, and its size, denoted by $\lambda(G)$, is called the *edge-connectivity* of G. The *local edge-connectivity* $\lambda_G(x, y)$ for two vertices $x, y \in V$ is defined to be the minimum size of a cut in G that separates x and y (i.e., $|\{x, y\} \cap X| = 1$). Equivalently, $\lambda_G(x, y)$ is the maximum number of edge-disjoint paths between x and y by Menger's theorem.

For a subset X of V, a vertex $v \in V - X$ is called a *neighbor* of X if it is adjacent to some vertex $u \in X$, and the set of all neighbors of X is denoted by $\Gamma_G(X)$. A maximal connected subgraph G' in a graph G is called a *component* of G (for notational convenience, a component H may be represented by its vertex set $X = V(H)$). Let us denote the number of components in G by $p(G)$. A *disconnecting set* of G is defined as a subset S of V such that $p(G - S) > p(G)$ holds and no $S' \subset S$ has this property. The *local vertex-connectivity* $\kappa_G(x, y)$ for two vertices $x, y \in V$ is defined to be the number of internally-disjoint paths between x and y in G. Again by Menger's theorem, $\kappa_G(x, y)$ for nonadjacent vertices x and y is equal to the minimum size of a disconnecting set S that disconnects x and y (i.e., x and y is contained in distinct components in $G - S$). A connected multigraph G with $|V(G)| \geq 3$ always has a disconnecting set unless every two vertices are adjacent. For a connected G, a disconnecting set of the minimum size is called a *minimum disconnecting set*, and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G; we define $\kappa(G) = 0$ if G is not connected, and $\kappa(G) = n - 1$ if G is a complete graph K_n. For a nonnegative integer ℓ (resp., k), we call a multigraph G *ℓ-edge-connected* (resp., *k-vertex-connected*) if $\lambda(G) \geq \ell$ (resp., $\kappa(G) \geq k$).

A subset $T \subset V$ is called *tight* if $\Gamma_G(T)$ is a minimum disconnecting set in G. A tight set D is called *minimal* if no proper subset D' of D is tight (hence a minimal tight set D induces a connected subgraph $G[D]$). We denote the family of all minimal tight sets in G by $\mathcal{D}(G)$, and the maximum number of pairwise disjoint minimal tight sets in G by $t(G)$ (hence $|\mathcal{D}(G)| \geq t(G)$).

For an initial graph G and given integers $\ell, k \geq 0$, let $opt_{\ell,k}(G)$ denote the optimal value of the EVAP(ℓ, k) in G, i.e., the minimum size $|E'|$ of a set E' of new edges such that $G + E'$ is an (ℓ, k)-connected graph.

Several algorithms have been developed for EVAP(ℓ, k). These algorithms use the following lower bound on $opt_{\ell,k}(G)$. We define

$$\alpha_{\ell,k}(G) = \max\left\{\sum_{i=1}^{q_1}(\ell - c_G(X_i)) + \sum_{i=q_1+1}^{q_2}(k - |\Gamma_G(X_i)|)\right\}, \tag{2.1}$$

where the maximum is taken over all subpartitions $\mathcal{X} = \{X_1, \ldots, X_{q_1}, X_{q_1+1}, \ldots, X_{q_2}\}$ of V such that $V - X_i - \Gamma_G(X_i) \neq \emptyset,\ i = q_1 + 1, \ldots, q_2$. We also define

$$\beta_k(G) = \max_{\substack{\text{all vertex subsets} \\ S \text{ with } |S| = k-1}} \left\{p(G - S)\right\}. \tag{2.2}$$

Let

$$\gamma_{\ell,k}(G) = \max\{\lceil \alpha_{\ell,k}(G)/2 \rceil, \beta_k(G) - 1\}.$$

We easily observe that $\gamma_{\ell,k}(G)$ is a lower bound on $opt_{\ell,k}(G)$ (see [10] for the proof). In the case of $k = 1$ (the edge-connectivity augmentation problem), it is known that $opt_{\ell,1}(G) = \gamma_{\ell,1}(G)$ for an arbitrary $\ell \geq 1$ and EVAP$(\ell, 1)$ is polynomially solvable [24]. In the case of $\ell = k$ (the vertex-connectivity augmentation problem), Eswaran and Tarjan [3] proved that EVAP(2,2) can be solved by finding a set of $\gamma_{2,2}(G)$ edges. Watanabe and Nakamura [25] stated the same result for EVAP(3,3). Thus $\gamma_{k,k}(G)$ is indeed the optimal value for $k = 2, 3$, while it is known that $\gamma_{k,k}(G)$ can be smaller than $opt_{k,k}(G)$ for arbitrary $k \geq 4$. It is reported in [7] that EVAP(4,4) can be solved in polynomial time for an arbitrary initial graph G, by using $\gamma_{4,4}(G)$ and another lower bound. For $k \geq 5$, Jordán proved [13, 14] that EVAP(k, k) for a graph G with $\kappa(G) = k-1$ can be solved by an approximation algorithm which finds a solution with absolute error at most $(k - 2)/2$, using a slightly different lower bound. In the case of $\ell > k > 0$, Ishii et al. showed that EVAP($\ell, 2$) can be solved in polynomial time by adding $\gamma_{\ell,2}(G)$ new edges for an arbitrary ℓ [10]. However $\gamma_{\ell,k}(G)$ can be smaller than the optimal value for an arbitrary $\ell > k \geq 3$. Ishii et al. showed that EVAP(ℓ,3) can be solved in polynomial time for a fixed ℓ [11, 12].

In this paper, we use the lower bound $\gamma_{\ell,k}(G) = \max\{\lceil \alpha_{\ell,k}(G)/2 \rceil, \beta_k(G) - 1\}$ and show the next result.

Theorem 2.1 *Let G be a $(k-1)$-vertex-connected multigraph G with n vertices and $k \geq 4$. Then, for any integer $\ell \geq k$,*

$$\gamma_{\ell,k}(G) \leq opt_{\ell,k}(G) \leq \gamma_{\ell,k}(G) + \max\{\ell + 1, 2k - 4\}$$

holds and a feasible solution F of EVAP(ℓ, k) *with $\gamma_{\ell,k}(G) \leq |F| \leq \gamma_{\ell,k}(G) + \max\{\ell+1, 2k-4\}$ can be found in $O(\min\{k, \sqrt{n}\}kn^3 + n^4)$ time.* □

3 Outline of Algorithm

3.1 s-basal (ℓ, k)-connectivity

A graph H with a designated vertex $s \in V(H)$, where $H - s$ is denoted by G, is called *s-basally (ℓ, k)-connected* if G is $(k-1)$-vertex-connected and

$$c_H(X) \geq \ell \qquad \text{for all cuts } X \text{ with } \emptyset \neq X \subset V(G), \qquad (3.1)$$

$$c_H(s, D) \geq 1 \qquad \text{for all minimal tight sets } D \in \mathcal{D}(G). \qquad (3.2)$$

Given a graph H with a designated vertex s and vertices $u, v \in \Gamma_H(s)$ (possibly $u = v$), an *edge-splitting* operation is defined as replacing two edges (s, u) and (s, v) by a new edge (u, v). We say that $H' = (H - \{(s, u), (s, v)\}) \cup \{(u, v)\}$ is obtained from H by *splitting* a pair of edges (s, u) and (s, v) (or by splitting (s, u) and (s, v)). A sequence of splittings is *complete* if the resulting graph H' does not have any neighbor of s. Conversely, we say that H' is obtained from H by *hooking up* an edge $(u, v) \in E(H - s)$ at s, if we construct H' by replacing an edge (u, v) with two edges (s, u) and (s, v) in H.

In particular, we say that a pair of edges (s, u) and (s, v) is *ℓ-splittable* (resp., *k-splittable*) in H satisfying (3.1) (resp., (3.2)) if the graph $(H - \{(s, u), (s, v)\}) + \{(u, v)\}$ resulting from splitting (s, u) and (s, v) also satisfies (3.1) (resp., (3.2)).

The following theorem about ℓ-splittable splittings is known.

Theorem 3.1 [5, 18] *Let $H = (V \cup \{s\}, E)$ be a multigraph with a designated vertex $s \notin V$ and even $c_H(s)$ satisfying* (3.1)*, and $\ell \geq 2$ be an integer. Then for each $u \in \Gamma_H(s)$ there is a vertex $v \in \Gamma_H(s)$ such that $\{(s, u), (s, v)\}$ is ℓ-splittable.* □

3.2 Basic idea on our algorithm

Before showing our algorithm, we give a basic idea on our algorithm, called EV-AUG, which augments a given $(k-1)$-vertex-connected graph G to (ℓ, k)-connectivity by adding a set E^* of new edges with $|E^*| \leq opt_{\ell,k}(G) + \max\{\ell+1, 2k-4\}$. The algorithm EV-AUG is based on Frank's algorithm [5] for augmenting the edge-connectivity, and Cheriyan and Thurimella's algorithm [2] for augmenting the vertex-connectivity by one, which is a variant of Jordán's algorithm [13] with edge-splitting operations.

Let us review these algorithms with related splitting theorems. The following algorithm EC-AUG is an outline of Frank's algorithm which delivers optimal solutions.

Algorithm EC-AUG
Input: An undirected multigraph $G = (V, E)$ and an integer $\ell \geq 2$.
Output: A set E^* of new edges with $|E^*| = opt_{\ell,1}(G)$ such that $\lambda(G + E^*) \geq \ell$.

Step 1: By [5, Lemma 4.2], we can add to G a new vertex s and a set F_1 of new edges with $|F_1| = \alpha_{\ell,1}(G)$ such that the resulting graph $H = (V \cup \{s\}, E \cup F_1)$ satisfies (3.1). If $|F_1|$ is odd, then add an arbitrary one edge to F_1.
Step 2: By Theorem 3.1, we can repeat splitting of a pair of two edges (s, u) and (s, v) which is ℓ-splittable until the vertex s is isolated. □

Note that the solution E^* obtained from algorithm EC-AUG is optimal, because we have $|E^*| = \lceil \alpha_{\ell,1}(G)/2 \rceil$ and the graph $G + E^*$ satisfies (3.1), which means $\lambda(G + E^*) \geq \ell$.

The following algorithm VC-AUG is an outline of Cheriyan and Thurimella's algorithm [2], which delivers at most $k - 3$ over the optimal.

Algorithm VC-AUG
Input: An undirected graph $G = (V, E)$ with $\kappa(G) \geq k - 1$ and an integer $k \geq 2$.
Output: A set E^* of new edges with $|E^*| - opt_{k,k}(G) \leq k - 3$ such that $\kappa(G + E^*) \geq k$.
Step 1: (i) If $\alpha_{k,k}(G) = t(G) \leq 2k - 3$, then by [13, Lemma 3.5], we can find a solution E^* with $|E^*| - opt_{k,k}(G) \leq k - 3$.
(ii) If $\alpha_{k,k}(G) = t(G) \geq 2k - 2$, then by [13] we can add to G a new vertex s and a set F_1 of new edges with $|F_1| = t(G) = \alpha_{k,k}(G)$ such that the resulting graph $H = (V \cup \{s\}, E \cup F_1)$ satisfies (3.2). Let $H' := H$ and $G' := G$.
Step 2: While $\alpha_{k,k}(G') = t(G') \geq 2k - 2$, then by [2, Algorithm 3] we can repeat the following procedure (a) or (b).

(a) If $\beta_k(G') - 1 \geq \lceil t(G')/2 \rceil$, then we can find a solution E^* with $|E^*| = \beta_k(G) - 1$ by [13, Theorem 2.4].

(b) If $\beta_k(G') - 1 < \lceil t(G')/2 \rceil$, then we have one of the following two cases (b-1) or (b-2) by [2, Lemmas 5.6, 5.7, and 5.8].
(b-1) We can split a k-splittable pair $\{(s, u), (s, v)\}$ such that the resulting graph $H'' = (H' - \{(s, u), (s, v)\}) \cup \{(u, v)\}$ satisfies $t(H'' - s) = t(G') - 2$. After letting $E' := E' \cup \{(u, v)\}$, $H' := H''$, and $G' := G' + \{(u, v)\}$, we go to Step 2.
(b-2) We have $t(G') = 2k - 2$ and G' can be made k-vertex-connected by adding a set E'' of at most $2k - 4$ new edges. Output $E^* = E' \cup E''$.

Step 3: We have $t(G') \leq 2k - 3$. Then G' can be made k-vertex-connected by adding a set E'' of at most $2k - 4$ new edges by the following Lemma 3.1 which is easily derived from [13, 19]. Output $E^* = E' \cup E''$. □

Lemma 3.1 *Let $G = (V, E)$ be a graph with $\kappa(G) = k - 1$ and T be an arbitrary inclusion minimal cover of all tight sets of G.*
(i) If $|T| \geq k + 1$ holds, then $|T| = t(G)$.
(ii) Let E' be the edge set of the complete graph on T. Then $G + E'$ is k-vertex-connected. Moreover, if $F \subseteq E'$ is an arbitrary inclusion minimal edge set such that $G + F$ is k-vertex-connected, then F is a forest, and thus $|F| \leq |T| - 1$. □

Note that the solution E^* obtained in Steps 2 or 3 satisfies $|E^*| = \beta_k(G) - 1$ or $|E^*| \leq \lceil \alpha_{k,k}(G)/2 \rceil + 2k - 4$.

Let us return to the outline of algorithm EV-AUG. Intuitively, it is an extension of these two algorithms EC-AUG and VC-AUG. In algorithm EV-AUG, first, there are two possible cases (I) $t(G) \leq \max\{\ell + 2, 2k - 3\}$ and (II) $t(G) \geq \max\{\ell + 3, 2k - 2\}$.

The cases of $t(G) \leq \max\{\ell + 2, 2k - 3\}$ are not difficult to deal with. Then we can make G k-vertex-connected by a set E' of new edges with $|E'| \leq \max\{\ell + 1, 2k - 4\}$ by Lemma 3.1. According to algorithm EC-AUG, G can be made ℓ-edge-connected by adding a set E'' of new edges with $|E''| = \lceil \alpha_{\ell,1}(G)/2 \rceil$. Since $|E' \cup E''| \leq \lceil \alpha_{\ell,1}(G)/2 \rceil + \max\{\ell + 1, 2k - 4\} \leq opt_{\ell,k}(G) + \max\{\ell + 1, 2k - 4\}$ holds, we can obtain a required solution.

We consider the cases of $t(G) \geq \max\{\ell + 3, 2k - 2\}$. Similarly to the first steps of algorithms EC-AUG and VC-AUG, we first add to G a new vertex s and a set F_1 of new edges with

$|F_1| = \alpha_{\ell,k}(G)$ such that the resulting graph $H = (V \cup \{s\}, E \cup F_1)$ is s-basally (ℓ, k)-connected. In Section 4, we will prove the following lemma.

Lemma 3.2 *Assume that $t(G) \geq \max\{\ell+2, 2k-2\}$. Given a $(k-1)$-vertex-connected graph $G = (V, E)$ with integers $\ell \geq k \geq 4$, $\alpha_{\ell,k}(G)$ can be computed in $O(n^2 m + n^3 \log n)$ time.* □

The next step is intuitively a combination of the second steps of algorithms EC-AUG and VC-AUG. While the current graph H' satisfies $t(H'-s) \geq \max\{\ell+3, 2k-2\}$, we repeat finding an ℓ-splittable and k-splittable pair $\{(s,u),(s,v)\}$ of two edges in H' such that the graph $H'' = (H' - \{(s,u),(s,v)\}) \cup \{(u,v)\}$ obtained by splitting (s,u) and (s,v) satisfies $\max\{\beta_k(H''-s)-1, \lceil t(H''-s)/2\rceil\} = \max\{\beta_k(H'-s)-1, \lceil t(H'-s)/2\rceil\} - 1$. In particular, in the cases where the initial graph G satisfies $\beta_k(G) - 1 > \lceil \alpha_{\ell,k}(G)/2\rceil$, we may need to hook up and resplit some edges which has been split before then, or replace some edges in F_1 so that a solution with cardinality $\beta_k(G)-1$ can be obtained. For these splitting operations, we will extend in Section 5 splitting theorems, which will be the basis of algorithms EC-AUG and VC-AUG.

In consequence of this second step, we obtain a solution E^* with $|E^*| = \beta_k(G) - 1$ or the resulting graph H' satisfies the s-basally (ℓ,k)-connectivity and $t(H'-s) \leq \max\{\ell+2, 2k-3\}$. In the latter case, we can the graph $G' = H' - s$ k-vertex-connected by adding a set E' of new edges with $|E'| \leq \max\{\ell+1, 2k-4\}$ by Lemma 3.1, similarly to Step 3 of algorithm VC-AUG. Since the graph H' satisfies (3.1), there is a complete ℓ-splittable splitting in H' by Theorem 3.1. Let E'' be the set of split edges by such complete splitting. Then $E^* = (E(G') - E) \cup E' \cup E''$ is a required solution, since $|(E(G') - E) \cup E''| = \lceil \alpha_{\ell,k}(G)/2\rceil$ holds.

In Section 6, we give a clearer description of algorithm EV-AUG, after showing the proof of Lemma 3.2 in Section 4 and splitting theorems in Section 5.

4 Proof of Lemma 3.2

A k-vertex-connected graph has the following property.

Lemma 4.1 [13] *If $t(G) \geq \kappa(G)+1$, then any two minimal tight sets $X, Y \in \mathcal{D}(G)$ are pairwise disjoint.* □

In the case of $\kappa(G) = k-1$, if $t(G) \geq \max\{\ell+2, 2k-3\}$, then we have $t(G) \geq k+2$ (by $\ell \geq k$), and it follows from Lemma 4.1 that every two minimal tight sets in $\mathcal{D}(G)$ are pairwise disjoint.

We prove Lemma 3.2 by using the following lemma which is given in [12].

Lemma 4.2 [12] *For a multigraph $G = (V, E)$, an integer ℓ and a subpartition $\mathcal{Y} = \{Y_1, \ldots, Y_r\}$ of V, let $H = (V \cup \{s\}, E \cup F)$ be a multigraph obtained by adding a new vertex s and a set F of new edges between s and V such that*

$$c_H(X) \geq \ell \quad \textit{for all cuts } X \in V$$
$$c_H(s, Y) \geq 1 \quad \textit{for all subsets } Y \in \mathcal{Y}.$$

Such an F with the minimum cardinality and a subpartition $\mathcal{X}$ with $\Gamma_H(s) \subseteq \cup_{X \in \mathcal{X}} X$ such that each $X \in \mathcal{X}$ satisfies (a) $c_H(X) = \ell$ or (b) $c_H(s, X) = 1$ and $X \in \mathcal{Y}$ can be found in $O(n^2 m + n^3 \log n)$ time.

Proof: The property of the lemma has been in [12] in the case where $\mathcal{Y}$ is a set $\mathcal{D}(G)$ of minimal tight sets in a 2-vertex-connected graph G (in which minimal tight sets are always pairwise disjoint). However, the property remains valid for a general subpartition $\mathcal{Y}$, since the proof in [12] relies on only the pairwise disjointness of $\mathcal{D}(G)$. □

Proof of Lemma 3.2: Here we construct an s-basally (ℓ,k)-connected graph $H_1 = (V \cup \{s\}, E \cup F_1)$ by adding to $G = (V, E)$ a new vertex s and a set F_1 of new edges between

s and V so that $|F_1|$ is the minimum. As seen in the above, every two minimal tight sets in $\mathcal{D}(G)$ are pairwise disjoint. Hence by applying Lemma 4.2, we can obtain such F_1 and a subpartition $\mathcal{X}$ with $\Gamma_{H_1}(s) \subseteq \cup_{X\in\mathcal{X}} X$ such that each $X \in \mathcal{X}$ satisfies (a) $c_{H_1}(X) = \ell$ or (b) $c_{H_1}(s, X) = 1$ and $X \in \mathcal{D}(G)$. Note that every $X \in \mathcal{X}$ satisfying (a) satisfies $c_{H_1}(s, X) = \ell - c_G(X)$ and every $X \in \mathcal{X}$ satisfying (b) satisfies $c_{H_1}(s, X) = k - |\Gamma_G(X)|$ $(= 1)$. Hence we have $|F_1| = \sum_{i=1}^{p_1}(\ell - c_G(X_i)) + \sum_{i=p_1+1}^{p_2}(k - |\Gamma_G(X_i)|) \leq \alpha_{\ell,k}(G)$ for $\mathcal{X} = \{X_1, \ldots, X_{p_2}\}$ from the maximality of $\alpha_{\ell,k}(G)$. Moreover, $|F_1| \geq \alpha_{\ell,k}(G)$ holds, since otherwise H_1 violates (3.1) or (3.2). Therefore we see $|F_1| = \alpha_{\ell,k}(G)$.

Finally, we analyze the complexity of finding F_1. We first claim that in the case of $t(G) \geq \max\{\ell + 2, 2k - 3\}$, all sets in $\mathcal{D}(G)$ can be computed in $O(\min\{k - 1, \sqrt{n}\}mn)$ time by using n times standard network flow techniques [4]. First we construct $H = (V \cup \{s\}, E \cup F)$ by adding a new vertex s to G and connecting s and each vertex in V by one edge. Clearly, H satisfies (3.2). Then for each edge $(s, v) \in F$, we check whether the removal of (s, v) preserves (3.2) or not by computing a network flow between s and v. If the removal of (s, v) preserves (3.2), then redenote $H := H - \{(s, v)\}$, and otherwise we can find a minimal tight set containing v. It is not hard to see from Lemma 4.1 that if $t(G) \geq \max\{\ell + 2, 2k - 3\}$, then the family of minimal tight sets obtained by this procedure turns out to be $\mathcal{D}(G)$. Moreover, it can be found by n times the network flow computation.

Given $\mathcal{D}(G)$, the edge set F_1 can be computed in $O(n^2m + n^3 \log n)$ time by Lemma 4.2. □

5 Edge-splitting Operations

In this section, we show several splitting theorems by extending splitting theorems on which algorithms EC-AUG and VC-AUG in the previous section are based. We first give theorems about preserving the ℓ-edge-connectivity in Section 5.1. Next we give theorems about preserving the k-vertex-connectivity in Section 5.3 after stating several properties of k-vertex-connected graphs in Section 5.2. Finally, Section 5.4 gives theorems about preserving the (ℓ, k)-connectivity.

In Sections 5.1, 5.3, and 5.4, let $\ell \geq k \geq 2$ be an integer and $H = (V \cup \{s\}, E)$ denote a multigraph with $s \notin V$ such that $G = V - s$ satisfies $\kappa(G - s) = k - 1$, if no confusion occurs.

5.1 Preserving ℓ-edge-connectivity

Here we give several theorems about ℓ-splittable splittings, in addition to Theorem 3.1. A set X *intersects* another set Y if none of subsets $X \cap Y$, $X - Y$ and $Y - X$ is empty. We say that a set X *crosses* another set Y if they intersect each other and in addition $V - (X \cup Y) \neq \emptyset$ holds. Now the following properties hold for two crossing cuts X and Y in $G = (V, E)$:

$$\begin{array}{l} c_G(X) + c_G(Y) = c_G(X \cap Y) + c_G(X \cup Y) + 2c_G(X - Y, Y - X), \\ c_G(X) + c_G(Y) = c_G(X - Y) + c_G(Y - X) + 2c_G(X \cap Y, V - (X \cup Y)). \end{array} \tag{5.1}$$

Let $H = (V \cup \{s\}, E)$ satisfy (3.1). We call a cut $X \subset V$ *dangerous* if $c_H(X) \leq \ell + 1$ holds. Note that $\{(s, u), (s, v)\}$ is not ℓ-splittable if and only if there is a dangerous cut $X \subset V$ with $\{u, v\} \subseteq X$.

For a fixed vertex $u \in \Gamma_H(s)$, dangerous cuts containing u satisfies the following lemma and corollary. In the subsequent arguments, these properties will be used for seeking an edge $e \in E_H(s)$ such that $\{(s, u), e\}$ is ℓ-splittable. In particular, Corollary 5.1 says that the number of such e is at least $c_H(s) - (\ell + 2 - k)$.

Lemma 5.1 [5] *Let $H = (V \cup \{s\}, E)$ satisfy (3.1), and $\ell \geq 2$ be an integer. Let $u \in \Gamma_H(s)$. Then there are at most two maximal dangerous cuts X containing u; no cut $X' \supset X$ is dangerous. In particular, if there are exactly two maximal dangerous cuts X_1 and X_2 with $u \in X_1 \cap X_2$, then $c_H(X_1 \cup X_2) = \ell + 2$, $c_H(X_1 \cap X_2) = \ell$, $c_H(X_1 - X_2) = c_H(X_2 - X_1) = \ell$, and $c_H(s, X_1 \cap X_2) = 1$ hold.* □

Corollary 5.1 *Let $H = (V \cup \{s\}, E)$ satisfy the assumption of Lemma 5.1 and $\lambda(H - s) \geq k$. For each vertex $u \in \Gamma_H(s)$, $|\{u\} \cup \{v \in \Gamma_H(s) | \{(s,u),(s,v)\}$ is not ℓ-splittable$\}| \leq |\{(s,u)\} \cup \{e \in E_H(s) |\ \{(s,u), e\}$ is not ℓ-splittable$\}| \leq \ell + 2 - k$.* □

Moreover, the following lemma gives a sufficient condition for an ℓ-splittable pair $\{(s,x),(s,y)\}$, $x \neq u \neq y$ for some fixed $u \in \Gamma_H(s)$. For example, in the case where every ℓ-splittable pair containing (s,u) is not k-splittable, this lemma may help us to find another candidate.

Lemma 5.2 *Let $H = (V \cup \{s\}, E)$ satisfy (3.1), and $\ell \geq 2$ be an integer. Let $u \in \Gamma_H(s)$ and $N \subseteq \Gamma_H(s) - u$ denote the set of all vertices $x \in \Gamma_H(s)$ such that $\{(s,u),(s,x)\}$ is not ℓ-splittable. Assume that $c_H(s,N) \geq 2$ and that $\{(s,y),(s,x_i)\}$, $i = 1,2$ is not ℓ-splittable for some $(s,y) \in E_H(s, V - N - \{u\})$ and some set $\{(s,x_1),(s,x_2)\} \subseteq E_H(s,N)$. Then $N = \{x_1, x_2\}$ and $c_H(s,N) = 2$. Moreover, if $(s,z) \in E_H(s, V - N - \{u,y\})$ exists, then $\{(s,z),(s,x_1)\}$ or $\{(s,z),(s,x_2)\}$ is ℓ-splittable.*

Proof: By Theorem 3.1, $\Gamma_H(s) - u - N \neq \emptyset$ holds; an edge (s,y) of the statement exists. Let $Y_i \subset V$ denote a dangerous cut with $\{x_i, y\} \subseteq Y_i$ for $i = 1,2$. Note that $u \notin Y_i$ holds for $i = 1,2$ from $y \notin N$. Recall that the number of maximal dangerous cuts containing u is at most two from Lemma 5.1. There are the following two possible cases (1) and (2).

(1) There is a dangerous cut $X_1 \subset V$ with $\{u, x_1, x_2\} \subseteq X_1$: From $u \in X_1 - Y_i$, $y \in Y_i - X_1$, $x_i \in Y_i \cap X_1$ and $s \in V - (X_1 \cup Y_i)$, X_1 and Y_i cross each other in H for $i = 1,2$. From (5.1) and $c_H(s, Y_1 \cap X_1) \geq 1$, we have $\ell + 1 + \ell + 1 \geq c_H(X_1) + c_H(Y_1) = c_H(X_1 - Y_1) + c_H(Y_1 - X_1) + 2c_H(s, Y_1 \cap X_1) \geq \ell + \ell + 2$. It follows that $c_H(s, X_1 \cap Y_1) = 1$ and $c_H(X_1 - Y_1) = \ell$. By $c_H(s, X_1 \cap Y_1) = 1$, we have $x_2 \in X_1 - Y_1$. From $u \in (X_1 - Y_1) - Y_2$, $y \in Y_2 - (X_1 - Y_1)$, $x_2 \in Y_2 \cap (X_1 - Y_1)$ and $s \in V - (X_1 - Y_1) - Y_2$, $X_1 - Y_1$ and Y_2 cross each other in H. From (5.1) and $c_H(s, Y_2 \cap (X_1 - Y_1)) \geq 1$, we have $\ell + \ell + 1 \geq c_H(X_1 - Y_1) + c_H(Y_2) = c_H((X_1 - Y_1) - Y_2) + c_H(Y_2 - (X_1 - Y_1)) + 2c_H(s, Y_2 \cap (X_1 - Y_1)) \geq \ell + \ell + 2$, a contradiction.

(2) H has no dangerous cut $X \subset V$ with $\{u, x_1, x_2\} \subseteq X$; there are two distinct maximal dangerous cuts $X_1, X_2 \subset V$ such that $\{u, x_1\} \subseteq X_1$, $\{u, x_2\} \subseteq X_2$, $x_2 \notin X_1$ and $x_1 \notin X_2$ hold. Note that $N \subseteq X_1 \cup X_2$ holds from Lemma 5.1. Since X_1 and X_2 cross each other in H, we have $c_H(X_1 - X_2) = c_H(X_2 - X_1) = \ell$ and $c_H(s, X_1 \cap X_2) = 1$. As seen in (1), X_1 and Y_1 cross each other and $c_H(s, X_1 \cap Y_1) = 1$ holds. Moreover, $X_1 - X_2$ and Y_1 cannot cross each other in H from $c_H(X_1 - X_2) = \ell$, $c_H(Y_1) \leq \ell + 1$ and (5.1). It follows that $X_1 - X_2 \subset Y_1$ holds, from which $X_1 - X_2 \subseteq X_1 \cap Y_1$ and $c_H(s, X_1 - X_2) = 1$. Similarly $c_H(s, X_2 - X_1) = 1$ holds. From $c_H(s, X_1 \cap X_2) = 1$ and $u \in X_1 \cap X_2 - N$, it follows that $c_H(s,N) = c_H(s, X_1 - X_2) + c_H(s, X_2 - X_1) = 2$.

Moreover, we consider the case where an edge (s,z) of the statement exists. Now by reversing the roles of (s,u) and (s,y), we can observe that $Y_1 \neq Y_2$ and $E_H(s, Y_1 \cup Y_2) = \{(s,y),(s,x_1),(s,x_2)\}$. For proving the last statement, assume by contradiction that $\{(s,z), (s,x_i)\}$, $i = 1,2$ is not ℓ-splittable. Let Z_i denote a maximal dangerous cut containing $\{z, x_i\}$. Note that by $z \notin N$, we can apply the above arguments about (s,y) to (s,z). Hence we can observe that $E_H(s, Z_1 \cup Z_2) = \{(s,z),(s,x_1),(s,x_2)\}$. Now we have $c_H(X_1) + c_H(Y_1) + c_H(Z_1) \geq c_H(X_1 - Y_1 - Z_1) + c_H(Y_1 - Z_1 - X_1) + c_H(Z_1 - X_1 - Y_1) + c_H(X_1 \cap Y_1 \cap Z_1) + 2c_H(s, X_1 \cap Y_1 \cap Z_1)$. Since each of X_1, Y_1, and Z_1 is dangerous, it follows that the left side of this inequality is at most $3(\ell + 1)$. Note that and $(s, x_1) \in E_H(s, X_1 \cap Y_1 \cap Z_1)$, $u \in X_1 - Y_1 - Z_1$, $y \in Y_1 - Z_1 - X_1$, and $z \in Z_1 - X_1 - Y_1$. It follows from (3.1) that the right side is at least $4\ell + 2$. These contradict $\ell \geq 2$. □

Before closing this subsection, we give other operations while preserving ℓ-edge-connectivity. Given a multigraph $H = (V \cup \{s\}, E)$ and a vertex $u \in \Gamma_H(s)$, a *shifting* operation is defined as replacing the edge (s,u) to one new edge (s,v) for $v \in V$. We say that the resulting graph $H' = (H - \{(s,u)\}) \cup \{(s,v)\}$ is obtained from H by *shifting* (s,u) to (s,v). For an edge $e = (s,v)$ with $v \in V$ in H satisfying (3.1), if $H - e$ violates (3.1), then there is a cut $X \subset V$ such that

X satisfies $c_H(X) = \ell$ and $v \in X$ and all cuts $X' \subset X$ with $v \in X'$ satisfy $c_H(X') > \ell$. We say that such X is λ-*critical* with respect to $v \in \Gamma_H(s)$. It is not difficult to observe from (5.1) that for a vertex $v \in \Gamma_H(s)$, a cut X which is λ-critical with respect to v is unique. The following proposition implies that as an method for isolating s in H, we can use not only edge-splitting operations but also shifting operations.

Proposition 5.1 *Let $H = (V \cup \{s\}, E)$ satisfy* (3.1). *Let X be the λ-critical cut with respect to some $v \in \Gamma_H(s)$ if any; $X = V$ otherwise. Then if $|X| \geq 2$ holds, then for any $v' \in X$, the shifted multigraph $(H - (s, v)) + (s, v')$ also satisfies* (3.1). □

Proof: Otherwise H has a cut $Y \subset V$ with $v \in Y$ and $v' \notin Y$ with $c_H(Y) = \ell$. Note that X and Y cross each other in H from $Y - X \neq \emptyset$ by the minimality of X. Hence from (5.1) and $v \in X \cap Y$, we have $2\ell = c_H(X) + c_H(Y) = c_H(X - Y) + c_H(Y - X) + 2c_H(s, X \cap Y) \geq 2\ell + 2$, a contradiction. □

5.2 Structure of k-Vertex-Connected Graphs

We in this section review some properties of a k-vertex-connected graph, which will be a basis for deriving edge-splitting operations while preserving the vertex-connectivity.

For a vertex set S in G, we call the components in $G - S$ *the S-components*, and denote the family of all S-components by $\mathcal{C}(G - S)$. Note that the vertex set S is a disconnecting set in a connected G if and only if $|\mathcal{C}(G - S)| \geq 2$. Clearly, for a minimum disconnecting set S, the union of some (but not all) S-components is also tight.

Two crossing tight sets satisfies the following property.

Lemma 5.3 [13] *Given two tight sets X and Y with $X \cap Y \neq \emptyset$ and $|V - (X \cup Y)| \geq \kappa(G)$ in a graph $G = (V, E)$, $X \cap Y$ is tight. Moreover, if $|V - (X \cup Y)| \geq \kappa(G) + 1$ holds, then $X \cup Y$ is tight.* □

The following lemma says that in the case of $\beta_k(G) - 1 \geq \lceil t(G)/2 \rceil \geq \kappa(G)$, the number of minimum disconnecting sets S with $p(G - S) = \beta_k(G)$ is at most two. Moreover, this indicates that if the number of S is two, then $t(G) = 2(\beta_k(G) - 1)$.

Lemma 5.4 [13] *Let $G = (V, E)$ be a multigraph with $\beta_k(G) - 1 \geq \lceil t(G)/2 \rceil \geq \kappa(G)$. Then G has at most two minimum disconnecting sets S with $p(G - S) = \beta_k(G)$. If G has exactly two disconnecting sets S_1 and S_2 with $p(G - S_1) = p(G - S_2) = \beta_k(G)$, then the following (i) and (ii) hold:*
(i) $(\beta_k(G) - 1)$ S_1-components (resp., S_2-components) are contained in some S_2-component T_2 (resp., S_1-component T_1),
(ii) every minimal tight set $D \in \mathcal{D}(G)$ is contained in some $T \in \mathcal{C}(G - S_1) \cup \mathcal{C}(G - S_2) - \{T_1, T_2\}$ and D is the unique minimal tight set contained in such T. □

We call a disconnecting set S a *shredder* if $|\mathcal{C}(G - S)| \geq 3$. Let S_1 and S_2 be two disconnecting sets. We say that S_2 *meshes* S_1 if $S_2 \cap T_1 \neq \emptyset$ and $S_2 \cap T_2 \neq \emptyset$ hold for at least two sets $T_1, T_2 \in \mathcal{C}(G - S_1)$. The following Lemma 5.5 implies that not only T_1, T_2 but also any $T \in \mathcal{C}(G - S_1)$ satisfies $S_2 \cap T \neq \emptyset$. The following Lemma 5.6 implies that if S_1 is a shredder, then all S_1-components $T \in \mathcal{C}(G - S_1)$ but one satisfies $T \subset S_2$ or all S_2-components $T \in \mathcal{C}(G - S_2)$ but one satisfies $T \subset S_1$.

Lemma 5.5 [2] *If S_1 and S_2 are (not necessarily minimum) disconnecting sets of a graph G such that S_1 meshes S_2, then every component in $\mathcal{C}(G - S_1)$ (resp., in $\mathcal{C}(G - S_2)$) has a vertex in S_2 (resp., in S_1). Hence, the meshing relation on pairs of disconnecting sets is symmetric.* □

Lemma 5.6 [2] *Let S be a shredder with $|S| = \kappa(G)$ in a connected graph $G = (V, E)$. If there is a minimum disconnecting set S_1 that meshes with S, then $\mathcal{C}(G - S)$ or $\mathcal{C}(G - S_1)$ contains a component T satisfying $V - (S \cup S_1) \subseteq T$.* □

A tight set T is called a *superleaf*, if T contains exactly one minimal tight set in $\mathcal{D}(G)$ and no $T' \supset T$ satisfies this property. In the case of $t(G) \geq \kappa(G)+3$, a superleaf enjoys the following property.

Lemma 5.7 [13, Claim I(a)] *Let $G=(V,E)$ be a connected multigraph with $t(G) \geq \kappa(G)+3$. Then the following holds.*
(1) For every set in $\mathcal{D}(G)$, as well as every superleaf, the induced subgraph is connected.
(2) For each set $D_i \in \mathcal{D}(G)$, there is a unique superleaf Q_i containing it.
(3) Every two superleaves are pairwise disjoint. Hence, a superleaf is disjoint from all other sets in $\mathcal{D}(G)$, except for the set in $\mathcal{D}(G)$ contained in it. □

Given a shredder S, the following lemma shows a sufficient condition for an S-component to be a superleaf.

Lemma 5.8 *Let S be a shredder with $|S| = \kappa(G)$ and $D \in \mathcal{D}(G)$ be a minimal tight set in a connected graph $G=(V,E)$. If D is contained in a set $T \in \mathcal{C}(G-S)$ and no set in $\mathcal{D}(G)$ other than D is contained in T, then T is the superleaf with $T \supseteq D$.*

Proof: Otherwise there is a superleaf Q with $Q \supseteq D$ such that $Q \supset T$ holds from the definition of superleaves. Let $k = \kappa(G)$ and $S' = \Gamma_G(Q)$. Lemma 5.5 implies that S does not mesh S', since T does not contain a vertex in S' by $Q \supset T$. Hence there is exactly one set $T_1 \in \mathcal{C}(G-S)$ with $(T_1 \cup S) \supset S'$ (note $S \neq S'$ from $T \subset Q$). Now G has another set $T_2 \in \mathcal{C}(G-S)$ with $T \neq T_2 \neq T_1$, since S is a shredder. It can be seen from the k-vertex-connectivity of G that there are k internally-disjoint paths between every two vertices $u \in T$ and $v \in T_2$ in $G[S \cup T \cup T_2]$. Moreover, from $S' \cap T_1 \neq \emptyset$ it follows that at least one of those paths survives in $G - S'$. Therefore, every vertex in $T \cup T_2$ is contained in the same S'-component, i.e., $Q \supset T \cup T_2$ holds. Since the tight set T_2 contains a set $D_2 (\neq D) \in \mathcal{D}(G)$, Q contains at least two sets in $\mathcal{D}(G)$. This contradicts the definition of superleaves. □

Given a shredder S, the following lemma shows a sufficient condition for which every superleaf is disjoint with S.

Lemma 5.9 [13, Lemma 2.2(c)] *Let S be a shredder with $|S| = \kappa(G)$ in a connected graph $G=(V,E)$. If $p(G-S) \geq \kappa(G)+1$ holds, then every superleaf Q in G satisfies $Q \cap S = \emptyset$.* □

The following lemma shows a sufficient condition for a new edge $(x,y) \notin E$ to satisfy $t(G+(x,y)) \leq t(G)-1$.

Lemma 5.10 *Suppose that $t(G) \geq \kappa(G)+2$, let $D \in \mathcal{D}(G)$ be a minimal tight set in G, and Q be the superleaf containing D. Then for any two vertices $x \in D$ and $y \in V-(Q \cup \Gamma_G(Q))$, we have $t(G+(x,y)) \leq t(G)-1$.*

Proof: Let $k = \kappa(G)$. Every two sets in $\mathcal{D}(G)$ are pairwise disjoint in G and $t(G) = |\mathcal{D}(G)|$ holds from $t(G) \geq k+1$ and Lemma 4.1. Assume by contradiction that the lemma does not hold; $|\mathcal{D}(G+(x,y))| = t(G) \geq k+2$ holds. Then D is not a tight set in $G+(x,y)$, and there is a set $X \in \mathcal{D}(G+(x,y))$ with $X \cap D \neq \emptyset$.

We first claim that each minimal tight set $D' \in \mathcal{D}(G+(x,y)) - \{X\}$ is minimal tight also in G; $\mathcal{D}(G+(x,y)) = (\mathcal{D}(G)-\{D\}) \cup \{X\}$. Assume by contradiction that there exists a minimal tight set $D_1 \in (\mathcal{D}(G)-\{D\}) - \mathcal{D}(G+(x,y))$ in G. Now note that each $D' \in \mathcal{D}(G+(x,y)) - \{X\}$ is minimal tight also in G. It follows from Lemma 4.1 and $t(G) \geq k+1$ that if $D' \cap D'' \neq \emptyset$ holds for some $D'' \in \mathcal{D}(G)$, then $D' = D''$ or $D' \cap D'' = \emptyset$. Hence, we can observe that D_1 is disjoint with any $D' \in \mathcal{D}(G+(x,y)) - \{X\}$ and $D_1 \cap X \neq \emptyset$ holds. Moreover, similarly, we can observe that D is disjoint with any $D' \in \mathcal{D}(G+(x,y)) - \{X\}$. It follows that $(\mathcal{D}(G+(x,y)) - \{X\}) \cup \{D, D_1\}$ is a collection of $t(G)+1$ tight sets in G. This is a contradiction.

This claim implies that $|V - (X \cup Q)| \geq |\mathcal{D}(G) - \{D\}| \geq k+1$. Since X is tight also in G, it follows from Lemma 5.3 and $X \cap Q \neq \emptyset$ that $X \cup Q$ is a tight set in G. This contradicts the maximality of Q. □

The following lemma for augmenting the vertex-connectivity is given in [2, Lemma 5.8]. This gives a sufficient condition for which G can be made k-vertex-connected optimally.

Lemma 5.11 [2, Lemma 5.8] *Let $G = (V, E)$ be a $(k-1)$-vertex-connected $(k \geq 2)$ with $t(G) \geq \max\{2k-2, k+2\}$ and $\beta_k(G) \leq \lceil t(G)/2 \rceil$. Suppose that G has a shredder S with $|S| = k-1$ and every S-component contains exactly one minimal tight set. Then the minimum number of edges required to make G k-vertex-connected is $2k-4$ if $G = K_{k-1,k-1}$, and $k-1+\lceil (k-1)/2 \rceil - 1$ otherwise. Moreover, such set of edges can be found in $O(n)$ time if all minimal tight sets in G have been found.* □

5.3 Preserving the k-Vertex-Connectivity

Let $H = (V \cup \{s\}, E)$ satisfy (3.2). Note that we have $|V| \geq k+1$ by $\kappa(G) = k-1$ (recall $G = H - s$). Also note that if $\{(s,u),(s,v)\}$ is not k-splittable, then the resulting graph $H' = (H - \{(s,u),(s,v)\}) \cup \{(u,v)\}$ has the following (i) and (ii):
(i) a disconnecting set $S \subset V$ with $|S| = k-1$ and $p(H' - S) = 2$, and
(ii) an S-component $T \in \mathcal{C}(H' - S)$ with $T \subset V$, $\{u,v\} \cap T \neq \emptyset$, $\{u,v\} \subseteq T \cup S$ and $E_{H'}(s,T) = \emptyset$.

The following theorem indicates that given a superleaf Q^* in G whose neighbor is not a shredder in G and an edge $(s,x^*) \in E_H(s,Q^*)$, there are some number of k-splittable pairs $\{(s,v),(s,x^*)\}$, $v \in \Gamma_H(s)$.

Theorem 5.2 *Let $H = (V \cup \{s\}, E)$ be a multigraph with $G = H - s$, and $k \geq 2$ be an integer such that $\kappa(G) = k-1$. Assume that $t(G) \geq k+2$ holds and H satisfies (3.2). Let Q^* be a superleaf in G such that $\Gamma_G(Q^*)$ is not a shredder in G and $x^* \in \Gamma_H(s) \cap D^*$ be a vertex where the set $D^* \in \mathcal{D}(G)$ with $D^* \subseteq Q^*$. Then there are at least $t(G) - k$ superleaves $Q (\neq Q^*)$ in G such that, for any set $D \in \mathcal{D}(G)$ with $D \subseteq Q$, $\{(s,x^*),(s,v)\}$ is k-splittable for all $v \in \Gamma_H(s) \cap D$. In particular, the graph H' resulting from splitting (s,x^*) and (s,v) satisfies $t(H'-s) = t(G) - 2$ if $c_H(s) = t(G)$ holds.*

Proof: Let $\mathcal{A}$ denote the family of all superleaves in G, where every two sets in $\mathcal{A}$ are pairwise disjoint from Lemma 5.7. For a set $Q_i \in \mathcal{A} - Q^*$, assume that $\{(s,x^*),(s,v_i)\}$ is not k-splittable for $v_i \in \Gamma_H(s) \cap D_i$, where D_i is the set in $\mathcal{D}(G)$ with $D_i \subseteq Q_i$. Let $H_1 = (H - \{(s,x^*),(s,v_i)\}) + (x^*, v_i)$ be the graph obtained from H by splitting (s,x^*) and (s,v_i). Let $T_i \subset V$ be a tight set in G with $c_{H_1}(s,T_i) = 0$ such that no $T' \supset T_i$ satisfies this property. Now there are the following three possibilities:

(a) $\{x^*, v_i\} \subseteq T_i$,
(b) $x^* \in T_i$ and $v_i \in \Gamma_G(T_i)$, or
(c) $v_i \in T_i$ and $x^* \in \Gamma_G(T_i)$ hold.

We first claim that $T_i \cap Q_j = \emptyset$ holds for every set $Q_j \in \mathcal{A} - \{Q^*, Q_i\}$. $T_i - Q_j \neq \emptyset$ and $Q_j - T_i \neq \emptyset$ clearly hold. For proving this claim, it suffices to show that the set T_i cannot cross any $Q_j \in \mathcal{A} - \{Q^*, Q_i\}$ in G. Assume by contradiction that some Q_j crosses T_i. Now $|V - Q_j - T_i| \geq |\mathcal{A}| - 3 \geq k-1$ holds. From Lemma 5.3, it follows that $Q_j \cap T_i$ is tight in G and hence we have $D_j \subseteq T_i$ from the definition of superleaves, where D_j is the set in $\mathcal{D}(G)$ contained in Q_j. This contradicts $c_{H_1}(s,T_i) = 0$ and $c_{H_1}(s,D_j) \geq 1$.

We next claim that $Q_i \cap \Gamma_G(Q^*) \neq \emptyset$ in both cases of (b) and (c). For this, it suffices to show that $T_i \subseteq Q^*$ (resp., $T_i \subseteq Q_i$) holds in the case of (b) (resp., (c)) (note $\Gamma_G(T_i) = \Gamma_{H_1}(T_i)$). Now we have $Q^* \cap T_i \neq \emptyset$ (resp., $Q_i \cap T_i \neq \emptyset$) by $x^* \in Q^* \cap T_i$ (resp., $v_i \in Q_i \cap T_i$). Moreover, $T_i \supset Q^*$ (resp., $T_i \supset Q_i$) would contradict the maximality of Q^* (resp., Q_i). We finally show that in the case of (b) (resp., (c)), T_i and Q^* (resp., Q_i) do not cross each other in H. Assume by contradiction that T_i and Q^* (resp., Q_i) cross each other in G. From the above first claim, we have $|V - Q^* - T_i| \geq |\mathcal{A}| - 2 \geq k$. (resp., $|V - Q_i - T_i| \geq k$). From this and Lemma 5.3, it follows that $Q^* \cup T_i$ (resp., $Q_i \cup T_i$) is tight in G. From the maximality of Q^* (resp., Q_i), we can observe that $Q^* \cup T_i$ (resp., $Q_i \cup T_i$) is not a superleaf in G and hence contains both of D^* and D_i. This contradicts the assumption of the case (b) (resp., (c)).

In the case of (a), we claim that $T_i \cap \Gamma_G(Q^*) \neq \emptyset$ holds. Assume by contradiction that $T_i \cap \Gamma_G(Q^*) = \emptyset$. Then there is a partition $\{T', T''\}$ of T_i with $c_G(T', T'') = 0$, $T' \subseteq Q^*$, and $T'' \subseteq V - Q^* - \Gamma_G(Q^*)$. This follows because two vertices $x^* \in T_i$ and $v_i \in T_i$ are disconnected by $\Gamma_G(Q^*)$ in G. From $|\Gamma_G(T_i)| = k - 1$, $|\Gamma_G(T')| \geq k - 1$ and $|\Gamma_G(T'')| \geq k - 1$, it follows that $\Gamma_G(Q^*) = \Gamma_G(T') = \Gamma_G(T'')(= \Gamma_G(T_i))$ (note $\Gamma_G(T_i) = \Gamma_G(T') \cup \Gamma_G(T'')$). This and $V - T_i - \Gamma_G(T_i) \neq \emptyset$ imply that $\Gamma_G(Q^*)$ is a shredder in G, a contradiction.

From above, if $\{(s, x^*), (s, v_i)\}$ is not k-splittable for $v_i \in \Gamma_H(s) \cap D_i$, then we have

$$T_i \cap \Gamma_G(Q^*) \neq \emptyset \text{ (in the case of (a)), or } Q_i \cap \Gamma_G(Q^*) \neq \emptyset \text{ (in the cases of (b) or (c)),}$$

where D_i, Q_i and T_i denote sets defined in the above. Then we claim that the number of such sets $Q_i \in \mathcal{A} - Q^*$ is at most $k - 1$, which proves this theorem.

Assume by contradiction that this claim does not hold. From $|\Gamma_G(Q^*)| = k - 1$, there is a vertex $v \in \Gamma_G(Q^*)$ with $v \in (Q_j \cup T_j) \cap (Q_h \cup T_h)$ and $j \neq h$. This implies $v \in T_j \cap T_h$, since every two superleaves are pairwise disjoint and each T_i satisfies $T_i \cap Q_{i'} = \emptyset$ for every set $Q_{i'} \in \mathcal{A} - \{Q^*, Q_i\}$. Moreover, note that both T_j and T_h satisfy the case of (a). This follows since otherwise $T_j \subseteq Q_j$, $T_j \subseteq Q^*$, $T_h \subseteq Q_h$ or $T_h \subseteq Q^*$ hold and $T_j \cap T_h \cap \Gamma_G(Q^*) \neq \emptyset$ cannot hold by $T_j \cap Q_h = \emptyset = T_h \cap Q_j$. From $v_j \in T_j - T_h$, $v_h \in T_h - T_j$, $x^* \in T_j \cap T_h$ and $|V - T_j - T_h| \geq |\mathcal{A}| - 3 \geq k - 1 > 0$, T_j and T_h cross each other in G (note $T_j \cap D_h = \emptyset = T_h \cap D_j$). From Lemma 5.3 and $|V - T_j - T_h| \geq |\mathcal{A}| - 3 \geq k - 1$, it follows that $T_j \cap T_h$ is tight in G. Then $(T_j \cap T_h) - Q^* \neq \emptyset$ implies that $T_j \cap T_h$ contains Q^* or that $T_j \cap T_h$ crosses Q^* in G. We can observe that in both cases, it would contradict the maximality of Q^*. This follows since in the latter case $(T_j \cap T_h) \cup Q^*$ is tight in G from Lemma 5.3 and $|V - (T_j \cap T_h) - Q^*| \geq |\mathcal{A}| - 1 \geq k$.

Finally, we see that $t(H' - s) = t(G) - 2$ holds for the resulting graph H' from splitting a k-splittable pair, in the case of $c_H(s) = t(G)$; every edge $e \in E_H(s)$ satisfies $c_H(s, D) = 1$ for a minimal tight set $D \in \mathcal{D}(G)$. □

The following lemma is a slight generalization of [2, Lemma 5.7] in which a graph H is assumed to satisfy (3.2) but removal of any edge incident to s from H violates this property. For a superleaf Q_1 in G whose neighbor is a shredder in G, the following lemma gives a sufficient condition for a k-splittable pair of two edges in $E_H(s, V - \Gamma_G(Q_1))$.

Lemma 5.12 *Let $H = (V \cup \{s\}, E)$ be a multigraph with $G = H - s$, and $k \geq 2$ be an integer such that $\kappa(G) = k - 1$. Assume that $t(G) \geq \max\{2k - 2, k + 2\}$ holds and H satisfies* (3.2)*. Let $Q_1 \subset V$ be an arbitrary superleaf such that $S^* = \Gamma_G(Q_1)$ is a shredder in G. If G has a set $T^* \in \mathcal{C}(G - S^*) - \{Q_1\}$ with $c_H(s, T^*) \geq 2$, then $\{(s, x), (s, y)\}$ is k-splittable for any pair $\{x, y\}$ such that $x \in \Gamma_H(s) \cap Q_1$ and $y \in \Gamma_H(s) \cap T^*$.*

Proof: Assume that for two vertices $x \in \Gamma_H(s) \cap Q_1$ and $y \in \Gamma_H(s) \cap T^*$, $\{(s, x), (s, y)\}$ is not k-splittable. Then the resulting graph $H' = (H - \{(s, x), (s, y)\}) \cup \{(x, y)\}$ has a disconnecting set $S' \subset V$ with $|S'| = k - 1$ and $p(H' - S') = 2$, and a set $T' \in \mathcal{C}(H' - S')$ with $T' \subset V$, $\{x, y\} \subseteq T' \cup S'$, $\{x, y\} \cap T' \neq \emptyset$ and $c_{H'}(s, T') = 0$. Clearly, $S^* \neq S'$ holds by $x \notin T^* \cup S^*$. Now we have $S' \cap T^* \neq \emptyset$ because $\{x, y\} \subseteq T' \cup S'$, $c_{H'}(s, T') = 0$ and $c_H(s, T^*) \geq 2$ imply $(V - T') \cap T^* \neq \emptyset$ (note that $G[T^*]$ is connected).

We consider the case where S^* does not mesh S'. Then Lemma 5.5 and $S^* \neq S'(|S^*| = |S'|)$ imply that there is exactly one set $T_1 \in \mathcal{C}(G - S^*)$ satisfying $S' \cap T_1 \neq \emptyset$. We have $T_1 = T^*$ by $S' \cap T^* \neq \emptyset$. Let T_2 be another S^*-component in G with $Q_1 \neq T_2 \neq T^*$ (T_2 exists since S^* is a shredder in G). Note $x \in T'$ from $S' \cap Q_1 = \emptyset$. We now claim that $T_2 \subset T'$ holds. It can be seen from the $(k - 1)$-vertex-connectivity of G that there are $(k - 1)$ internally-disjoint paths between every two vertices $u \in Q_1$ and $v \in T_2$ in $G[Q_1 \cup S^* \cup T_2]$. Moreover, from $S' \cap T^* \neq \emptyset$, it follows that at least one path of those survives in $G - S'$, which proves $T_2 \subset T'$. By this claim, $c_{H'}(s, T') \geq c_{H'}(s, T_2) = c_H(s, T_2) \geq 1$ holds, contradicting $c_{H'}(s, T') = 0$.

Therefore S^* meshes S'. Then Lemma 5.5 says that every set in $\mathcal{C}(G - S^*)$ contains a vertex in S'. Hence, every set T in $\mathcal{C}(G - S^*)$ satisfies $|(T \cup S^*) \cap S'| \leq k - 3$ since S^* is a shredder in G. We claim $|S^* \cap T'| \geq 2$. Let z be a vertex in $\{x, y\} \cap T'$ and T_0 be the set in $\mathcal{C}(G - S^*)$ with $z \in T_0$ (note $T_0 = Q_1$ or $T_0 = T^*$). Let $S^* = \{v_1, \ldots, v_{k-1}\}$. It can be seen from the $(k-1)$-vertex-connectivity of G that $G[T_0 \cup S^*]$ has $(k-1)$ internally-disjoint paths $P_1, \ldots, P_{k-1}$ such that P_i connects two vertices z and $v_i \in S^*$. From $|(T_0 \cup S^*) \cap S'| \leq k - 3$, it follows that at least two of those paths survive in $G - S'$, proving the claim.

Lemma 5.6 says that there are two possibilities:
(I) $V - (S^* \cup S') \subseteq T_3$ holds for a set $T_3 \in \mathcal{C}(G - S')$, and
(II) $V - (S^* \cup S') \subseteq T_3$ holds for a set $T_3 \in \mathcal{C}(G - S^*)$.
We first consider the case of (I). Then $T_3 = T'$ holds from $\{x, y\} \cap T' \neq \emptyset$. Let D_i, $i = 1, \ldots, t(G)$, denote all sets in $\mathcal{D}(G)$ (note that every two sets in $\mathcal{D}(G)$ are pairwise disjoint by Lemma 4.1). For every set D_i with $\{x, y\} \cap D_i = \emptyset$, we have $\Gamma_{H'}(s) \cap D_i \subset V - T' = (S^* \cup S') - T'$ by $c_{H'}(s, T') = 0$. Now $|\{v \in (S^* \cup S') - T' \mid c_{H'}(s, v) \geq 1\}| \leq 2k - 4$ holds from $|S^* \cup S'| \leq 2k - 2$ and $|S^* \cap T'| \geq 2$. Hence we have $|\{D_i \mid \{x, y\} \cap D_i = \emptyset\}| = 2k - 4$, $\{x, y\} \cap S' = \emptyset$ and $x \in D_1 \subseteq Q_1$, since $t(G) \geq 2k - 2$ holds and at most two D_i satisfy $D_i \cap \{x, y\} \neq \emptyset$. Moreover, each single vertex in $(S^* \cup S') - T'$ gives rise to a set in $\mathcal{D}(G)$ by Lemma 5.3. Hence $Q_1 \cap S' \neq \emptyset$ implies that a vertex $w \in Q_1 \cap S'$ is a set $\{w\} = D_j \in \mathcal{D}(G) - \{D_1\}$, contradicting the definition of superleaves.

In the case of (II), $x \notin S'$ and $T_3 = Q_1$ hold or $y \notin S'$ and $T_3 = T^*$ hold. We first consider the case where $x \notin S'$ and $T_3 = Q_1$ hold. Then $T^* \subseteq S'$ holds. Now we have $|\{v \in (S^* \cup S') - T' \mid c_H(s, v) \geq 1\}| \leq 2k-4$ from $|S^* \cup S'| \leq 2k-2$ and $|S^* \cap T'| \geq 2$. This and $\Gamma_H(s) \cap (Q_1 - S') = \{x\}$ mean $t(G) \leq 2k - 3$, contradicting $t(G) \geq 2k - 2$. We finally consider the case where $y \notin S'$ and $T_3 = T^*$ hold. Then $Q_1 \subseteq S'$ holds. Now $|\{v \in (S^* \cup S') - T' \mid c_H(s, v) \geq 1\}| \leq 2k - 4$ holds from $|S^* \cup S'| \leq 2k - 2$ and $|S^* \cap T'| \geq 2$. This and $\Gamma_H(s) \cap (T^* - S') = \{y\}$ mean $t(G) \leq 2k - 3$, contradicting $t(G) \geq 2k - 2$. □

5.4 Preserving the (ℓ, k)-Connectivity

In an s-basally (ℓ, k)-connected graph $H = (V \cup \{s\}, E)$, we consider a splitting pair of two edges incident to s which is ℓ-splittable and k-splittable, i.e., which preserves the s-basal (ℓ, k)-connectivity of H.

Let $\mathcal{H}$ denote the family of all graphs $H^* = (V \cup \{s\}, E \cup E^*)$ with a designated vertex $s \notin V$ satisfying the following (i)–(iv):

(i) H^* is an s-basally (ℓ, k)-connected graph with integers $\ell \geq k \geq 2$,
(ii) $E^* \subseteq E_{H^*}(s)$ holds,
(iii) $G = H^* - s$ satisfies $\kappa(G) = k - 1$, and
(iv) $H = (V \cup \{s\}, E)$ satisfies (3.2) (H does not necessarily satisfy (3.1)).

Then we derive some conditions that admit a pair of two edges in $E_H(s)$ which is ℓ-splittable and k-splittable in H^*. In this paper, we need only the case of $E^* = \emptyset$ to prove the correctness of Step II of algorithm EV-AUG described in Section 6. We, however, handle a more general case where E^* may not be empty, because the resulting edge-splitting results can be used to wide applications.

In this subsection, we introduce the following three key properties about ℓ-splittable and k-splittable pairs. The following first theorem gives a sufficient condition for a pair $\{(s, x), (s, y)\}$ with $\beta_k(G + (x, y)) = \beta_k(G) - 1$, in the case of $\beta_k(G) - 1 \geq \lceil t(G)/2 \rceil$.

Theorem 5.3 *Let $H^* = (V \cup \{s\}, E \cup E^*)$ satisfy property $H^* \in \mathcal{H}$, $t(G) \geq \max\{\ell + 2, 2k - 2\}$, and $\beta_k(G) - 1 \geq \lceil t(G)/2 \rceil$. Let S^* be a shredder in G with $|S^*| = k - 1$ and $p(G - S^*) =$*

$\beta_k(G)$. *Then if G has an S^*-component $T \in \mathcal{C}(G - S^*)$ with $c_H(s,T) \geq 2$, then there is a pair $\{(s,x),(s,y)\} \subseteq E_H(s)$ such that $\{(s,x),(s,y)\}$ is ℓ-splittable and k-splittable in H^* and we have $\beta_k(G + (x,y)) = \beta_k(G) - 1$.*

Proof: Note that a k-splittable pair in H is k-splittable also in H^*. In the following, we show that there is a pair of two edges in $E_H(s)$ which is ℓ-splittable in H^* and k-splittable in H.

By $t(G) \geq \ell + 2 \geq \kappa(G) + 3$ and Lemma 4.1, every two minimal tight sets are disjoint (hence $t(G) = |\mathcal{D}(G)|$). By $p(G - S^*) \geq \kappa(G) + 1$, Lemma 5.9 tells that

$$\text{every superleaf } Q \text{ in } G \text{ satisfies } Q \cap S^* = \emptyset. \tag{5.2}$$

Since $2p(G - S^*) - 2 \geq t(G)$ holds from $\beta_k(G) - 1 \geq t(G)/2$ and $p(G - S^*) = \beta_k(G)$, there are at least two S^*-components $Q \in \mathcal{C}(G - S^*)$ such that each Q contains exactly one minimal tight set. By $t(G) \geq \kappa(G) + 3$ and Lemma 5.8, such Q is a superleaf in G. Let $Q_1 \in \mathcal{C}(G - S^*)$ be such superleaf in G such that $c_H(s, Q_1)$ is the minimum and $x_1 \in \Gamma_H(s) \cap Q_1$. Let N_1 be the set of all vertices $v \in \Gamma_H(s) \cap (V - S^*)$ such that $\{(s,x_1),(s,v)\}$ is ℓ-splittable in H^* and v is contained in some minimal tight set $D \in \mathcal{D}(G)$. Now $t(G) \geq \max\{\ell+2, 2k-2\}$ and (5.2) imply $|\Gamma_H(s) \cap (V - S^*)| \geq \max\{\ell + 2, 2k-2\}$. From Corollary 5.1, it follows that we have

$$|N_1| \geq \max\{\ell+2, 2k-2\} - \{\ell + 2 - (k-1)\} = \max\{k-1, 3k - \ell - 5\} \geq 1. \tag{5.3}$$

Moreover, we can observe that

$$\text{if } |N_1| = 1, \text{ then } k = 2 \text{ and } c_H(s, V - N_1 - S^*) = \ell + 2 - (k-1). \tag{5.4}$$

There are the following two possible cases (1) and (2):
(1) G has another disconnecting set $S' \neq S^*$ satisfying $p(G - S') = \beta_k(G)$, and
(2) G has exactly one disconnecting set S^* satisfying $p(G - S^*) = \beta_k(G)$.

(1) Lemma 5.4 says that G has exactly one disconnecting set $S' \neq S^*$ satisfying $p(G - S') = \beta_k(G)$, and there is a set $T_1 \in \mathcal{C}(G - S^*)$ (resp., $T_1' \in \mathcal{C}(G - S')$) which contains $(\beta_k(G) - 1)$ S'-components (resp., S^*-components). Moreover, again from Lemma 5.4, every minimal tight set $D \in \mathcal{D}(G)$ is contained in either $T \in \mathcal{C}(G - S^*) - T_1$ or $T \in \mathcal{C}(G - S') - T_1'$. Note that $Q_1 \neq T_1$ holds, because Q_1 contains exactly one minimal tight set in G from the definition of superleaves. Since S^* is a shredder and $\beta_k(G) - 1 \geq 2$ holds, Lemma 5.12 implies that

$$\begin{array}{l} \{(s,x),(s,y)\} \text{ is } k\text{-splittable in } H \text{ for any pair } x \in T \text{ and } y \in T' \\ \text{with } T \in \mathcal{C}(G - S^*) - \{T_1\} \text{ and } T' \in \mathcal{C}(G - S') - \{T_1'\}. \end{array} \tag{5.5}$$

Hence if there is a pair $\{(s,x),(s,y)\} \subseteq E_H(s)$ which is ℓ-splittable in H^* for x in some $T \in \mathcal{C}(G - S^*) - \{T_1\}$ and y in some $T' \in \mathcal{C}(G - S') - \{T_1'\}$, then the pair is desirable.

We consider the case where this does not hold; no pair $\{(s,x),(s,y)\}$ is ℓ-splittable in H^* for any pair of two vertices $x \in T$ and $y \in T'$ with $T \in \mathcal{C}(G - S^*) - \{T_1\}$ and $T' \in \mathcal{C}(G - S') - \{T_1'\}$. Lemma 5.2 and $|N_1| \geq 1$ imply that the number of S^*-components other than T_1 is at most two. Now $\beta_k(G) - 1 \geq 2$, and hence it follows that $\beta_k(G) - 1 = 2$. From the assumption, we have $4 = 2(\beta_k(G) - 1) \geq t(G) \geq \max\{\ell + 2, 2k-2\}$. By $\ell \geq k \geq 2$, it follows that $\ell = k = 2$. Hence, a k-splittable pair means an ℓ-splittable pair, which contradicts (5.5).

(2) Let $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) denote the family of all S^*-components $T \in \mathcal{C}(G - S^*)$ with $c_H(s,T) = 1$ (resp., $c_H(s,T) \geq 2$). By the assumption and the choice of Q_1, we have $\mathcal{T}_2 - \{Q_1\} \neq \emptyset$. Let (s, y_1) and (s, y_2) be two distinct edges in $E_H(s)$ for $\{y_1, y_2\} \subseteq T'$ for $T' \in \mathcal{T}_2 - \{Q_1\}$. If $\{(s,x_1),(s,y_1)\}$ or $\{(s,x_1),(s,y_2)\}$ is ℓ-splittable in H^*, then the pair is k-splittable (from Lemma 5.12), and desirable.

We consider the case where this does not hold; $y_1, y_2 \in V - N_1 - S^*$. Let $z_1, z_2 \in N_1$ ($z_1 = z_2$ may hold), and T_i denote the S^*-components containing z_i, $i = 1, 2$ (note that $N_1 \neq \emptyset$ by (5.3)). Assume $T_1 \neq T' \neq T_2$, since if $T_i = T' \in \mathcal{T}_2 - \{Q_1\}$, then $\{(s,x_1),(s,z_i)\}$ is a desirable pair. Moreover, assume that $\{(s,y'),(s,z')\}$ is not ℓ-splittable in H^* for any pair $y' \in \{y_1, y_2\}$ and

$z' \in \{z_1, z_2\}$ (otherwise the pair $\{(s, y'), (s, z')\}$ is desirable). Hence, by Lemma 5.2, we can observe that $E_H(s, V - N_1 - S^*) = \{(s, x_1), (s, y_1), (s, y_2)\}$. Moreover, if $z_1 \neq z_2$ holds, then this assumption contradicts the last statement of Lemma 5.2.

We finally consider the case of $|N_1| = 1$. From (5.4) and $E_H(s, V - N_1 - S^*) = \{(s, x_1), (s, y_1), (s, y_2)\}$, it follows that $k = 2$ and $3 = c_H(s, V - N_1 - S^*) = \ell + 2 - (k - 1)$. Hence $\ell = k$ holds, and a k-splittable pair means an ℓ-splittable pair. This contradicts that $\{(s, x_1), (s, y_1)\}$ is not ℓ-splittable.

Finally, we claim that $\beta_k(G + (u, v)) = \beta_k(G) - 1$ holds in $H^* - \{(s, u), (s, v)\} + \{(u, v)\}$ for a pair of edges $\{(s, u), (s, v)\}$ chosen in the above (1) and (2). This follows since the edge (u, v) connects two distinct S-components for a disconnecting set S in G that satisfies $p(G - S) = \beta_k(G)$. □

The following lemma indicates a possibility that even if there is no desirable splitting pair, then shifting or hooking up operations help us to find a new desirable pair. This also deals with the case of $\beta_k(G) - 1 \geq \lceil t(G)/2 \rceil$.

Lemma 5.13 *Let $H^* = (V \cup \{s\}, E \cup E^*)$ satisfy $H^* \in \mathcal{H}$, $t(G) \geq \max\{\ell + 2, 2k - 2\}$, and $\beta_k(G) - 1 \geq \lceil t(G)/2 \rceil$. Let S^* be a shredder in G with $|S^*| = k - 1$ and $p(G - S^*) = \beta_k(G)$. Assume that every S^*-component $T \in \mathcal{C}(G - S^*)$ satisfies $c_H(s, T) = 1$. In each case of the following (1)–(4), the corresponding graph H_1 can be obtained. Moreover, H_1 has an ℓ-splittable and k-splittable pair of two edges in $E_{H_1}(s) - E^*$ such that the resulting graph H_2 satisfies $\beta_k(H_2 - s) \leq \beta_k(G) - 1$.*
(1) Assume that there is an edge $e = (s, v)$ with $v \in S^*$. Let H_1 be a graph obtained by shifting (s, v) with a new edge (s, v') with $v' \in V - S^*$, while preserving (3.1).
(2) Assume there is a split edge $e = (u, v) \in E(G)$ with $\kappa(G - e) = k - 1$, $u \in S^*$ and $v \in T_1 \in \mathcal{C}(G - S^*)$. Let H_1 be the graph obtained by hooking up the edge (u, v).
(3) Assume that there is a split edge $e = (u, v) \in E(G)$ with $\kappa(G - e) = k - 1$ and $u, v \in T_1 \in \mathcal{C}(G - S^*)$ such that $p(G - S^*) = p((G - e) - S^*)$ holds. Let H_1 be the graph obtained by hooking up the edge (u, v).
(4) There is a split edge $e = (u, v) \in E(G)$ with $\kappa(G - e) = k - 1$ and $u, v \in S^*$. In the graph obtained by hooking up the edge (u, v) in H^*, let H_1 be the graph obtained by shifting (s, v) with a new edge (s, v') with $v' \in V - S^*$, while preserving (3.1).

Proof: Every set $T \in \mathcal{C}(G - S^*)$ contains exactly one vertex in $\Gamma_H(s)$. From Lemma 5.8 and $t(G) \geq k + 2 \geq \kappa(G) + 3$, it follows that such T is a superleaf in G. We have (5.2) by $p(G - S^*) \geq \kappa(G) + 1$ and Lemma 5.9. It follows that $t(G) = \beta_k(G) = |\mathcal{C}(G - S^*)|$. Now from $p(G - S^*) \geq \ell + 1$, we have $c_G(X) \geq p(G - S^*)|X| \geq \ell + 1$ for every cut $X \subseteq S^*$. For proving the lemma, we show that in each of (1)–(4), we can obtain a multigraph H_1 in which we can apply Theorem 5.3 to find a desirable pair in $E_{H_1}(s) - E^*$. Note that edges newly obtained by hooking up some split edge or shifting some edge can be used for an edge-splitting in H_1.

(1) Let X be a λ-critical cut with respect to v in H^* if exists, $X = V$ otherwise. Now G has a set $T_1 \in \mathcal{C}(G - S^*)$ satisfying $T_1 \cap X \neq \emptyset$, since every cut $Y \subseteq S^*$ satisfies $c_G(Y) \geq \ell + 1$. Let $H_1 := (H - e) + e'$ be the graph resulting from shifting e to an edge $e' = (s, v_1)$ with $v_1 \in X \cap T_1$. Then H_1 satisfies (3.1) and (3.2) by Proposition 5.1 and by (5.2), respectively. Thus $c_{H_1}(s, T_1) = 2$ holds.

(2),(3) Let $H_1 := (H - e) + \{(s, u), (s, v)\}$ be the graph resulting from hooking up e. H_1 also satisfies $\beta_k(G) - 1 = \beta_k(G - (u, v)) - 1 \geq \lceil t(G - (u, v))/2 \rceil$, because $t(G) = \beta_k(G) \geq \max\{k + 2, 2k - 2\}$ and $k \geq 2$ hold. Thus $c_{H_1}(s, T_1) \geq 2$ holds.

(4) Let $H'' = (H - e) + \{(s, u), (s, v)\}$ be the graph resulting from hooking up e. $\beta_k(G) - 1 = \beta_k(G - (u, v)) - 1 \geq \lceil t(G - (u, v))/2 \rceil$ also holds, because $t(G) = \beta_k(G) \geq \max\{k + 2, 2k - 2\}$ and $k \geq 2$ hold. Then H'' satisfies the statement (1), which proves the lemma. □

The following lemma deals with the case of $\beta_k(G) \leq \lceil t(G)/2 \rceil$. This shows that if there exists no desirable pair, then we can attain the k-vertex-connectivity by adding newly at most $2k - 4$

edges.

Lemma 5.14 *Let $H^* = (V \cup \{s\}, E \cup E^*)$ satisfy $H^* \in \mathcal{H}$, $t(G) \geq \max\{\ell+3, 2k-2\}$, and $\beta_k(G) \leq \lceil t(G)/2 \rceil$. Then one of the following holds:*
(a) there is a pair $\{(s,x),(s,y)\} \subseteq E_H(s)$ such that $\{(s,x),(s,y)\}$ is ℓ-splittable and k-splittable in H^* and $t(G+(x,y)) \leq t(G)-1$ holds,
(b) $t(G) = 2k-2$ holds and G can be made k-vertex-connected by adding at most $2k-4$ new edges.

Proof: Every two superleaves in G are pairwise disjoint by Lemma 5.7. Let Q_1 be a superleaf in G, choose $x_1 \in D_1 \cap \Gamma_H(s)$ with $Q_1 \supseteq D_1 \in \mathcal{D}(G)$, and let $S = \Gamma_G(Q_1)$. There are two cases to be considered: (1) S is a shredder in G, (2) otherwise.

(1) S is a shredder in G. If every set $T \in \mathcal{C}(G-S)$ contains exactly one minimal tight set in G, then Lemma 5.11 says that $t(G) = 2k-2$ holds and G can be made k-vertex-connected by adding at most $2k-4$ new edges, implying (b).

We consider the case where G has a set $T \in \mathcal{C}(G-S)$ containing at least two minimal tight sets in G. Let $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) denote the family of S-components $T \in \mathcal{C}(G-S) - \{Q_1\}$ which contains exactly one minimal tight set (resp., at least two minimal tight sets). Let $\mathcal{D}_1$ (resp., $\mathcal{D}_2$) denote the family of sets in $\mathcal{D}(G)$ contained in some set $T \in \mathcal{T}_1$ (resp., $\mathcal{T}_2$). Assume that there is no edge (s,y) with $y \in (V - Q_1 - S) \cap D$ for $D \in \mathcal{D}(G)$ such that $\{(s,x_1),(s,y)\}$ is ℓ-splittable in H^* and k-splittable. Let Y denote the set of all vertices $v \in \Gamma_H(s)$ such that $\{(s,x_1),(s,v)\}$ is not ℓ-splittable in H^*. Then from Corollary 5.1, we can observe that there exists a cut $X \subset V$ with $Y \cup \{x_1\} \subseteq X$ and $c_H(s,X) \leq \ell + 2 - (k-1)$. Now Lemma 5.12 says that $\{(s,x_1),(s,v)\}$ is k-splittable for every vertex $v \in \Gamma_H(s) \cap D$ with $D \in \mathcal{D}_2$. Hence it follows that X contains all $v \in \Gamma_H(s) \cap D$ with $D \in \mathcal{D}_2$ and that every $D \in \mathcal{D}(G)$ with $D \cap \Gamma_H(s) - X \neq \emptyset$ belongs to $\mathcal{D}_1$.

We here claim that there are at least two minimal tight sets $D_1, D_2 \in \mathcal{D}_1$. Note that any dangerous cut Y' induces a connected graph. This follows since if a partition $\{Y_1, Y_2\}$ of Y' satisfies $c_H(Y_1, Y_2) = 0$, then $c_H(Y') = c_H(Y_1) + c_H(Y_2) \geq 2\ell \geq \ell + 2$ (by (3.1) and $\ell \geq 2$), a contradiction. Hence, it follows that $X \cap S \neq \emptyset$. From this and $c_H(s,X) \leq \ell+2-(k-1)$ and $|S| = k-1$, we can observe that $c_H(s, X \cup S) \leq \ell + 2 - (k-1) + (k-1) - 1 = \ell + 1$. Then $t(G) \geq \ell + 3$ proves the claim.

Let $y_i \in \Gamma_H(s) \cap D_i$, $i = 1,2$. Lemma 5.2 implies that a pair $\{(s,y'),(s,x')\}$ is λ-splittable for some $y' \in \{y_1, y_2\}$ and some $x' \in D \cap \Gamma_H(s)$ with $D \in \mathcal{D}_2$. From the definition of $\mathcal{D}_j$, two vertices y' and x' are contained in distinct S-components. From Lemma 5.12, the pair is k-splittable and a desirable pair.

(2) S is not a shredder in G. Let $\mathcal{A}_1$ denote the family of superleaves $Q \neq Q_1$ in G such that $\{(s,x_1),(s,v)\}$ is k-splittable for some $v \in \Gamma_H(s) \cap D$ with $Q \supseteq D \in \mathcal{D}(G)$. Theorem 5.2 says that $|\mathcal{A}_1| \geq t(G) - k \geq \max\{\ell - k + 3, k-2\}$. From Corollary 5.1, the number of edges (s,v') such that $\{(s,x_1),(s,v')\}$ is not ℓ-splittable in H^* is at most $\ell + 1 - (k-1) = \ell - k + 2$. Then there is a k-splittable pair $\{(s,x_1),(s,v)\}$ which is ℓ-splittable in H^* and k-splittable for some $v \in \Gamma_H(s) \cap D$ with $D \in \mathcal{D}(G)$.

Moreover, $t(H'-s) \leq t(G)-1$ follows from the choice of two edges incident to s, Lemma 5.10 in the case of (1) and Theorem 5.2 in the case of (2). □

6 Augmentation Algorithm

Based on Sections 4 and 5, we give algorithm EV-AUG which makes a given $(k-1)$-connected graph (ℓ,k)-connected by adding at most $\gamma_{\ell,k}(G) + \max\{\ell+1, 2k-4\}$ new edges. Define $\gamma'_k(G) = \max\{\beta_k(G) - 1, \lceil t(G)/2 \rceil\}$.

Algorithm EV-AUG

Input: An undirected multigraph $G = (V,E)$ with $|V| \geq k+1$, $\kappa(G) = k-1$, and an integer $\ell \geq k \geq 4$.

Output: A set of new edges E^* with $|E^*| \le opt_{\ell,k}(G) + \max\{\ell+1, 2k-4\}$ such that $G^* = G + E^*$ satisfies $\lambda(G^*) \ge \ell$ and $\kappa(G^*) \ge k$.

Initialization: Set E^*, E_1^*, E_2^*, E_3^*, E', and E'' to be an empty, $p = \max\{\ell + 1, 2k - 4\}$.

Step I: If $t(G) \le p+1$ holds, then compute a set E_1^* of new edges with $|E_1^*| = \lceil \alpha_{\ell,1}(G)/2 \rceil$ and $\lambda(G + E_1^*) \ge \ell$ by using algorithm EC-AUG. Compute a set E_2^* of new edges with $|E_2^*| \le p$ and $\kappa(G + E_2^*) \ge k$ by using Lemma 3.1. Halt after outputting $E^* = E_1^* \cup E_2^*$.

If $t(G) \ge p+2$ holds, add to G a new vertex s and a set F_1 of new edges with $|F_1| = \alpha_{\ell,k}(G)$ such that $H = (V \cup \{s\}, E \cup F_1)$ is s-basally (ℓ, k)-connected. If $c_H(s)$ is odd, then add an arbitrary one edge to F_1. After setting $H' := H$ and $G' := G$, go to Step II.

Step II: While $t(G') \ge p + 2$ do:

(i) If there exists an ℓ-splittable and κ-splittable pair $\{(s, u_1), (s, u_2)\}$ of two edges in $\gamma'_k(G' + (u_1, u_2)) = \gamma'_k(G') - 1$, then set $E_3^* := E_3^* \cup \{(u_1, u_2)\}$, $F_1 := F_1 - \{(s, u_1), (s, u_2)\}$, $H' := (H' - \{(s, u_1), (s, u_2)\}) \cup \{(u_1, u_2)\}$, and $G' := H' - s$.

/** If (i) does not hold and $\beta_k(G') - 1 \ge \lceil t(G')/2 \rceil$ holds, then G' has exactly one disconnecting set S^* such that each S^*-component $T \in \mathcal{C}(G' - S^*)$ satisfies $c_{H'}(s, T) = 1$. Moreover, S^* is a shredder by $\beta(G') - 1 \ge \lceil t(G')/2 \rceil \ge k - 1 \ge 3$.**/

(ii) Else if $\beta_k(G') - 1 \ge \lceil t(G')/2 \rceil$ holds, execute the following procedure (a)–(d).

(a) If $E_{H'}(s, S^*) \ne \emptyset$, then shift an edge $(s, v) \in E_{H'}(s, S^*)$ to (s, v'), $v' \in V - S^*$ while preserving s-basally (ℓ, k)-connectivity, set $F_1 := (F_1 - \{(s, v)\}) \cup \{(s, v')\}$, $H' := (H' - \{(s, v)\}) \cup \{(s, v')\}$, and $G' := H' - s$, and go to the above (i).

(b) If there exists $(u, v) \in E_3^*$ with $u \in S^*$ and $v \in V - S^*$, then hook up the edge (u, v), set $E_3^* := E_3^* - \{(u, v)\}$, $F_1 := F_1 \cup \{(s, u), (s, v)\}$, and $H' := (H' - \{(u, v)\}) \cup \{(s, u), (s, v)\}$, and $G' := H' - s$, and go to the above (i).

(c) If there exists $(u, v) \in E_3^*$ with $u, v \in V - S^*$ and $p(G' - S^*) = p((G - (u, v)) - S^*)$, then hook up the edge (u, v), set $E_3^* := E_3^* - \{(u, v)\}$, $F_1 := F_1 \cup \{(s, u), (s, v)\}$, and $H' := (H' - \{(u, v)\}) \cup \{(s, u), (s, v)\}$, and $G' := H' - s$, and go to the above (i).

(d) If there exists $(u, v) \in E_3^*$ with $u, v \in S^*$, then hook up the edge (u, v), set $E_3^* := E_3^* - \{(u, v)\}$, $F_1 := F_1 \cup \{(s, u), (s, v)\}$, and $H' := (H' - \{(u, v)\}) \cup \{(s, u), (s, v)\}$, and $G' := H' - s$, and go to the above (a).

(iii) Else if $\beta_k(G') - 1 < \lceil t(G')/2 \rceil$ holds and the addition of a new edge set E' with $|E'| \le 2k - 4$ can make G' k-vertex-connected, then add E' to E_3^* and go to Step III.

(iv) Else then find a set E' of new edges with $|E'| = \beta_k(G) - 1 - |E_3^*|$ such that $G' + E'$ is (ℓ, k)-connected. Halt after outputting $E^* := E_3^* \cup E'$ as an optimal solution.

Step III: Find a set E' of new edges with $\kappa(G' + E') \ge k$ and $|E'| \le p$ if $\kappa(G') = k - 1$. Find a set E'' of new edges with $\lambda(G' + E'') \ge \ell$ and $|E''| \le |F_1'|/2$. Output $E^* = E_3^* \cup E' \cup E''$.
□

6.1 Correctness of algorithm EV-AUG

Here we prove the correctness of algorithm EV-AUG. In the cases of $t(G) \le p + 1$, we see that $G + E^*$ is (ℓ, k)-connected and $|E^*| \le \lceil \alpha_{\ell,k}(G)/2 \rceil + p$ holds from Lemma 3.1 and algorithm EC-AUG.

Analogously with this, we can prove that if Step II works correctly, then the edge set E^* obtained in Step III is a required solution. Since $G + E^*$ is clearly (ℓ, k)-connected, we prove that the absolute error is at most p. Immediately before Step III, the current graph H' is s-basally (ℓ, k)-connected and $|F_1|$ is even. If we reach Step III through the case (iii) in Step II, then $G' = H' - s$ satisfies $|E_3^*| \le \lceil (\alpha_{\ell,k}(G) - |F_1|)/2 \rceil + 2k - 4$ and $\kappa(G') \ge k$, which implies $E' = \emptyset$. Otherwise we have $\kappa(G') = k-1$, $|E_3^*| = \lceil (\alpha_{\ell,k}(G) - |F_1|)/2 \rceil$, and $t(G') \le p+1$, which implies $|E'| \le p$. Therefore, in both cases, we have $|E^*| \le \lceil \alpha_{\ell,k}(G)/2 \rceil + p$.

In the rest of this section, we show the correctness of Step II. For this, we prove that for the current graph H', one of four cases (i)–(iv) always holds. Through this step, H' always satisfies $t(H'-s) \ge p+2$ and belongs to $\mathcal{H}$ which is defined in Section 5.4. In the cases of $\beta_k(G') \le \lceil t(G')/2 \rceil$, Lemma 5.14 says that one of two cases (i) and (iii) holds.

We consider the cases of $\beta_k(G') - 1 \ge \lceil t(G')/2 \rceil$. Let S^* be a shredder in G' with $p(G' - S^*) = \beta_k(G') - 1$ (note $\beta_k(G') - 1 \ge \lceil t(G')/2 \rceil \ge k-1 \ge 3$). If G' has an S^*-component $T \in \mathcal{C}(G' - S^*)$ with $c_{H'}(s,T) \ge 2$, then the case (i) holds by Theorem 5.3.

We consider the case where each S^*-component $T \in \mathcal{C}(G' - S^*)$ satisfies $c_{H'}(s,T) = 1$. Then if there exists an edge $e \in E_3^* \cup F_1$ such that $e \in E_{H'}(s, S^*)$ holds or $e \in E(G[T \cup S^*])$ with $p(G' - S^*) = p((G'-e) - S^*)$ for some $T \in \mathcal{C}(G' - S^*)$, then the case (ii) holds by Lemma 5.13. Note that $\gamma'_k(H''_2 - s) = \gamma'_k(G') - 1$ holds, since $\beta_k(G') = t(G')$ and $k \ge 4$ imply $\beta_k(G') - 1 > \lceil t(G')/2 \rceil$.

If H' has no such edge e, then we claim that the case (iv) holds. Then we have $E_{H'}(s, S^*) = \emptyset$ and every edge $e \in E_3^*$ satisfies $V[e] \subseteq V - S^*$ and $p(G' - S^*) = p((G'-e) - S^*) - 1$. Now according to Theorem 3.1, we can execute a complete ℓ-splittable splitting at s since H' is s-basally (ℓ,k)-connected and $|F_1|$ is even. Let E_4^* be the set of all edges added by the complete splitting and $G'' = G' + E_4^*$ (note $\lambda(G'') \ge \ell$). Since $c_{H'}(s,T) = 1$ holds for each S^*-component $T \in \mathcal{C}(G' - S^*)$, every edge $e \in E_3^* \cup E_4^*$ satisfies $p(G'' - S^*) = p((G''-e) - S^*) - 1$. Hence $p(G - S^*) = p(G'' - S^*) + |E_3^*| + |E_4^*|$ holds. Let $\mathcal{C}(G'' - S^*) = \{T_1, T_2, \ldots, T_b\}$, where $b = p(G'' - S^*)$.

Claim 6.1 *$\kappa(G'' + E_5^*) \ge k$ holds for $E_5^* = \{(x_i, x_{i+1}) \mid i = 1, \ldots, b-1\}$, where $x_i \in T_i \cap \Gamma_{H'}(s)$.*

Proof: Otherwise $G'' + E_5^*$ has a disconnecting set $S_1 \ne S^*$ with $|S_1| = k-1$. Then S_1 does not mesh S^* in G' by Lemma 5.5 and $p(G' - S^*) \ge k$. So $G'' + E_5^*$ has a tight set $T' \subset T_1$ with $\Gamma_{H'}(s) \cap T' = \emptyset$ for some $T_1 \in \mathcal{C}(G' - S^*)$. This contradicts that H' satisfies (3.2). □

Note that $|E_3^*| + |E_4^*| + |E_5^*| = |E_3^*| + |E_4^*| + p(G'' - S^*) - 1 = p(G - S^*) - 1 \le \beta_k(G) - 1$. This implies that $E_3^* \cup E_4^* \cup E_5^*$ is an optimal solution, since $|E_3^*| + |E_4^*| + |E_5^*|$ attains a lower bound $\beta_k(G) - 1$.

Consequently, the correctness of algorithm is proven. □

6.2 Complexity

We analyze the complexity of algorithm EV-AUG. First the family $\mathcal{D}(G)$ of all minimal tight sets and superleaves in G can be computed in $O(\min\{k, \sqrt{n}\}mkn)$ time by using the standard network flow computation [4] $O(kn)$ times.

In Step I, if $t(G) \le p+1$ holds, then it can be implemented to run in $O(\min\{k, \sqrt{n}\}kn^3)$ time as follows. It is known in [1, 22] that E_1^* can be computed in $O(n(m + n\log n)\log n)$ time. E_2^* can be computed in $O(\min\{k, \sqrt{n}\}kn^3)$ time by applying Phase 5 of Jordán's algorithm in [13], on a sparsified spanning subgraph of G with $O(kn)$ edges, where such sparsification takes $O(m + n\log n)$ time [20, 21]. If $t(G) \ge p+2$ holds, then the edge set F_1 can be computed in $O(n^2 m + n^3 \log n)$ time as described in Section 4.

Similarly to the cases of $t(G) \le p+1$ in Step I, Step III can be executed in the same time.

Finally, we show that Step II is executed in $O(\min\{k, \sqrt{n}\}k^2n^2 + n^4)$ time as follows. The number of iterations of Step II is at most n by $\gamma'_k(G) \le n$. In the case (iv), the complexity depends on that of computing a complete ℓ-splittable splitting, which can be done in $O(n(m + n\log n)\log n)$ time. In the case (iii), the edge set E_3^* can be found in $O(n)$ time from Lemma 5.11, since Lemma 5.14 says that G' in this case satisfies the sufficient conditions of Lemma 5.11.

We analyze the complexity in the case (i) and (ii). Let $b := \max\{p(G' - \Gamma_{G'}(Q)) \mid Q$ is a superleaf in $G'\}$ in the current graph G'. We can see that $b - 1 \ge \lceil t(G')/2 \rceil$ holds if and only if

$\beta_k(G') - 1 \geq \lceil t(G')/2 \rceil$ holds. Actually, if $b - 1 \geq \lceil t(G')/2 \rceil$ holds, then $\beta_k(G') - 1 \geq \lceil t(G')/2 \rceil$ clearly holds. In this case, moreover, $b = \beta_k(G')$ holds, since in the case of $\beta_k(G') - 1 \geq \lceil t(G')/2 \rceil$, there is a superleaf Q in G' satisfying $p(G' - \Gamma_{G'}(Q)) = \beta_k(G')$, as observed in the first part of the proof of Theorem 5.3. Hence if $b - 1 < \lceil t(G')/2 \rceil$ holds, then $\beta_k(G') - 1 \geq \lceil t(G')/2 \rceil$ cannot hold. Thus in the case of $\beta_k(G') - 1 \geq \lceil t(G')/2 \rceil$, we can easily obtain the value of $\beta_k(G')$ and the disconnecting sets S^* satisfying $p(G' - S^*) = \beta_k(G')$. Furthermore, after splitting two edges edges (s, x_1) and (s, x_2), the update of $\mathcal{D}(G')$ takes $O(\min\{k, \sqrt{n}\}m)$ time. This follows since all minimal tight sets not containing x_1 or x_2 remain minimal tight and at most two times of network flow computation between x_i and $y \in D \in \mathcal{D}(G')$ with $\{x_1, x_2\} \cap D = \emptyset$ for $i = 1, 2$ suffice.

Hence it suffices to show that we can find an ℓ-splittable and k-splittable pair of two edges incident to s according to Step II in $O(\min\{k, \sqrt{n}\}k^2 n + n^3)$ time. We consider the time complexity of finding a k-splittable pair. Except the case of applying Theorem 5.2, for an edge $e_1 \in E_{H'}(s)$, we can find in $O(n)$ time the set of edges $e_2 \in E_{H'}(s)$ such that $\{e_1, e_2\}$ is k-splittable, from $|\Gamma_{H'}(s)| \leq n$ and Lemma 5.12. In the case of applying Theorem 5.2, we can find a k-splittable pair of two edges incident to s in $O(\min\{k, \sqrt{n}\}k^2 n)$ time by using standard network flow techniques [4] at most k times, since the number of finding non-k-splittable pairs is at most $k - 1$.

With respect to ℓ-splittable pairs, if a pair $\{(s, u_1), (s, u_2)\}$ of two edges in $E_{H'}(s)$ is not ℓ-splittable, then a maximal dangerous cut $X \subset V$ with $\{u_1, u_2\} \subseteq X$ can be found in $O(n^3)$ time. This can be done by computing a maximum flow between two vertices u_1 and s in $(H' - \{(s, u_1), (s, u_2)\}) + (u_1, u_2)$ [17]. Since Lemma 5.1 says that the number of maximal dangerous cuts is at most two, the set of edges $e' = (s, u)$ such that $\{(s, u_1), e'\}$ is ℓ-splittable can be found in $O(n^3)$ time. In the cases where edge-shifting operations are necessary, a λ-critical cut $X \subset V$ can be found in $O(n(m + n \log n))$ time [22]. Thus each iteration can be executed in $O(\min\{k, \sqrt{n}\}k^2 n + n^3)$ time, since for a given edge $e_1 = (s, u_1)$, whether there is an edge $e_2 = (s, u_2)$ such that $\{e_1, e_2\}$ is ℓ-splittable and k-splittable can be computed in $O(\min\{k, \sqrt{n}\}k^2 n + n^3)$ time. Consequently, each iteration of the case (i) and (ii) in Step II takes $O(\min\{k, \sqrt{n}\}k^2 n + n^3)$ time.

Summarizing the argument given so far, Theorem 2.1 is now established. □

7 Concluding Remarks

In this paper, we gave a polynomial time algorithm for augmenting a given $(k-1)$-vertex-connected multigraph G to an ℓ-edge-connected and k-vertex-connected graph by adding at most $\max\{\ell + 1, 2k - 4\}$ surplus edges over the optimum for $k \geq 4$. However, if $\ell = k \geq 4$, there is an algorithm [13, 14] that produces at most $(k-2)/2$ surplus edges over the optimum. Therefore, it is a future work to close the gap between this and our bound.

Acknowledgments

We are grateful to the anonymous referees for careful reading and suggestions that significantly improved the presentation. This research was partially supported by the Scientific Grant-in-Aid from Ministry of Education, Science, Sports and Culture of Japan, and the subsidy from the Inamori Foundation.

References

[1] J. Bang-Jensen, H. N. Gabow, T. Jordán and Z. Szigeti, *Edge-connectivity augmentation with partition constraints*, SIAM J. Discrete Math., Vol.12, 1999, pp. 160–207.

[2] J. Cheriyan and R. Thurimella, *Fast algorithms for k-shredders and k-node connectivity augmentation*, J. Algorithms, Vol.33, 1999, pp. 15–50.

[3] K. P. Eswaran and R. E. Tarjan, *Augmentation problems*, SIAM J. Comput., Vol.5, 1976, pp. 653–665.

[4] S. Even and R. E. Tarjan, *Network flow and testing graph connectivity*, SIAM J. Comput., Vol.4, 1975, pp. 507–518.

[5] A. Frank, *Augmenting graphs to meet edge-connectivity requirements*, SIAM J. Discrete Math., Vol.5, 1992, pp. 25–53.

[6] H. N. Gabow, *Applications of a poset representation to edge connectivity and graph rigidity*, Proc. 32nd IEEE Symp. Found. Comp. Sci., 1991, pp. 812–821.

[7] T. Hsu, *Undirected vertex-connectivity structure and smallest four-vertex-connectivity augmentation*, Lecture Notes in Comput. Sci., 1004, Springer-Verlag, Algorithms and Computation (Proc. ISAAC '95), 1995, pp. 274–283.

[8] T. Hsu and V. Ramachandran, *A linear time algorithm for triconnectivity augmentation*, Proc. 32nd IEEE Symp. Found. Comp. Sci., 1991, pp. 548–559.

[9] T. Hsu and V. Ramachandran, *Finding a smallest augmentation to biconnect a graph*, SIAM J. Comput., Vol.22, 1993, pp. 889–912.

[10] T. Ishii, H. Nagamochi and T. Ibaraki, *Augmenting edge-connectivity and vertex-connectivity simultaneously*, Lecture Notes in Comput. Sci., 1350, Springer-Verlag, Algorithms and Computation (Proc. ISAAC '97), 1997, pp. 102–111.

[11] T. Ishii, H. Nagamochi and T. Ibaraki, *k-edge and 3-vertex connectivity augmentation in an arbitrary multigraph*, Lecture Notes in Comput. Sci., 1533, Springer-Verlag, Algorithms and Computation (Proc. ISAAC '98), 1998, pp. 159–168.

[12] T. Ishii, H. Nagamochi and T. Ibaraki, *Optimal augmentation of a 2-vertex-connected multigraph to a k-edge-connected and 3-vertex-connected multigraph*, J. Combinatorial Optimization, Vol.4, 2000, pp. 35–78.

[13] T. Jordán, *On the optimal vertex-connectivity augmentation*, J. Combin. Theory Ser. B, Vol.63, 1995, pp. 8–20.

[14] T. Jordán, *A note on the vertex-connectivity augmentation problem*, J. Combin. Theory Ser. B, Vol.71, 1997, pp. 294–301.

[15] G. Kant, *Algorithms for drawing planar graphs*, PhD thesis, Dept. of Computer Science, Utrecht University, the Netherlands, 1993.

[16] M. -Y. Kao, *Data security equals graph connectivity*, SIAM J. Discrete Math., Vol.9, 1996, pp. 87–100.

[17] A. V. Karzanov, *Determining the maximal flow in a network by the method of preflows*, Soviet Math. Doklady, Vol.15, 1974, pp. 434–437.

[18] L. Lovász, Combinatorial Problems and Exercises, North-Holland, 1979.

[19] W. Mader, *Ecken vom Grad n in minimalen n-fach zusammenhängenden Graphen*, Arch. Math. Vol.23, 1972, pp. 219–224.

[20] H. Nagamochi and T. Ibaraki, *A linear-time algorithm for finding a sparse k-connected spanning subgraph of a k-connected graph*, Algorithmica, Vol.7, 1992, pp. 583–596.

[21] H. Nagamochi and T. Ibaraki, *Computing edge-connectivity of multigraphs and capacitated graphs*, SIAM J. Discrete Math., Vol. 5, 1992, pp. 54–66.

[22] H. Nagamochi and T. Ibaraki, *Deterministic* $\tilde{O}(nm)$ *time edge-splitting in undirected graphs*, J. Combinatorial Optimization, Vol.1, 1997, pp. 5–46.

[23] H. Nagamochi, S. Nakamura and T. Ibaraki, *A simplified* $\tilde{O}(nm)$ *time edge-splitting algorithm in undirected graphs*, Algorithmica, Vol.26, 2000, pp. 50–67.

[24] T. Watanabe and A. Nakamura, *Edge-connectivity augmentation problems*, J. Comput. System Sci., Vol.35, 1987, pp. 96–144.

[25] T. Watanabe and A. Nakamura, *A smallest augmentation to 3-connect a graph*, Discrete Appl. Math., Vol.28, 1990, pp. 183–186.

[26] T. Watanabe and A. Nakamura, *A minimum 3-connectivity augmentation of a graph*, J. Comput. System Sci., Vol.46, 1993, pp. 91–128.