Minimum Augmentation of Edge-Connectivity with Monotone Requirements in Undirected Graphs ⋆

Toshimasa Ishii [a,*],

[a] *Department of Information and Management Science, Otaru University of Commerce, Hokkaido 047-8501 JAPAN*

Abstract

For a finite ground set V, we call a set-function $r : 2^V \to Z^+$ monotone, if $r(X') \geq r(X)$ holds for each $X' \subseteq X \subseteq V$, where Z^+ is the set of nonnegative integers. Given an undirected multigraph $G = (V, E)$ and a monotone requirement function $r : 2^V \to Z^+$, we consider the problem of augmenting G by a smallest number of new edges so that the resulting graph G' satisfies $d_{G'}(X) \geq r(X)$ for each $\emptyset \neq X \subset V$, where $d_G(X)$ denotes the degree of a vertex set X in G. This problem includes the edge-connectivity augmentation problem, and in general, it is NP-hard even if a polynomial time oracle for r is available. In this paper, we show that the problem can be solved in $O(n^4(m+n\log n+q))$ time, under the assumption that each $\emptyset \neq X \subset V$ satisfies $r(X) \geq 2$ whenever $r(X) > 0$, where $n = |V|$, $m = |\{\{u, v\} \mid (u, v) \in E\}|$, and q is the time required to compute $r(X)$ for each $X \subseteq V$.

Key words: undirected graph, connectivity augmentation problem, monotone requirement, polynomial time deterministic algorithm

1 Introduction

In a communication network, graph connectivity is a fundamental measure of its robustness. The connectivity augmentation problems have been extensively

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* Corresponding author.
Email address: ishii@res.otaru-uc.ac.jp (Toshimasa Ishii).

studied as an important subject in the network design problem [5] and so on, and many efficient algorithms have been developed so far (see [3,9] for surveys).

Let $G = (V, E)$ be an undirected multigraph and $d_G(X)$ be the number of edges between X and $V - X$ in G. A graph $G = (V, E)$ is k-edge-connected if every set $\emptyset \neq X \subset V$ satisfies $d_G(X) \geq k$. We consider the following problem of augmenting a given graph to meet the required edge-connectivity (RECAP): given a graph $G = (V, E)$ and a nonnegative integer set-function $r : 2^V \to Z^+$ where Z^+ denotes the set of nonnegative integers, add a smallest number of new edges F so that the augmented graph $G + F = (V, E \cup F)$ satisfies $d_{G+F}(X) \geq r(X)$ for every $\emptyset \neq X \subset V$. This formulation includes the *edge-connectivity augmentation problem* (*ECAP*), the *local edge-connectivity augmentation problem* (*LECAP*), the *node-to-area edge-connectivity augmentation problem* (*NAECAP*), and so on.

Let us briefly survey the developments in the edge-connectivity augmentation problems. ECAP is equivalent to RECAP in the case where every $\emptyset \neq X \subset V$ satisfies $r(X) = k$ for a given integer $k \in Z^+$. Watanabe and Nakamura [13] showed that it is polynomially solvable. The fastest known algorithm for it achieves complexity $O(mn+n^2 \log n)$ due to Nagamochi [10,11], where $n = |V|$ and $m = |\{\{u, v\} \mid u, v \in V\}|$.

In LECAP, we are given a local edge-connectivity requirement function $r'(u, v) \in Z^+$ on the set of pairs of vertices u and v, and hence the function r in RECAP is regarded as $r(X) = \max\{r'(u, v) \mid u \in X, v \in V - X\}$. Clearly, LECAP includes ECAP as a special case. Frank [2] showed that it is polynomially solvable. The fastest known algorithm, proposed by Gabow [4], runs in $O(n^2 m \log (n^2/m))$ time.

In NAECAP, we are given a family $\mathcal{W}$ of specified vertex subsets called *areas* and a requirement function $r'(W)$ on the family of areas $W \in \mathcal{W}$, and asked to augment G so that the edge-connectivity between each pair of $W \in \mathcal{W}$ and $v \in V - W$ becomes at least $r'(W)$; in the augmented graph G', every set $\emptyset \neq X \subset V$ is required to satisfy $d_{G'}(X) \geq r'(W)$ for each area $W \in \mathcal{W}$ with $W \cap X = \emptyset$ or $W \subseteq X$. Hence, the function r in RECAP is regarded as $r(X) = \max\{r'(W) \mid W \cap X = \emptyset, \text{ or } W \subseteq X\}$. NAECAP is also an extension of ECAP, because if $r'(W) = k$ holds for each area $W \in \mathcal{W}$ and some area $W' \in \mathcal{W}$ satisfies $|W'| = 1$, then the function r satisfies $r(X) = k$. Miwa and Ito [8] showed that even if $r'(W) = 1$ holds for every area $W \in \mathcal{W}$, NAECAP is NP-hard. On the other hand, Ishii and Hagiwara [6] showed that the case where $r'(W) \geq 2$ for every area $W \in \mathcal{W}$ can be solved in $O(n^3|\mathcal{W}|(m+n \log n))$ time.

More generally, RECAP can be extended to a problem of *covering* a given nonnegative integer set-function $p : 2^V \to Z^+$ by a smallest number of graph edges, where we say that an edge set F covers p if $d_{(V,F)}(X) \geq p(X)$ for every $X \subseteq V$. The p in RECAP is regarded as $p(X) = \max\{0, \max\{r(X), r(V - X)\} - d_G(X)\}$ (note that the degree of each set $\emptyset \neq X \subset V$ needs to be augmented up to $\max\{r(X), r(V - X)\}$ since G is undirected). Benczúr and Frank [1] showed that if p is a *symmetric supermodular* set-function, then such a problem of covering p can be solved in polynomial time, where $p : 2^V \to Z^+$ is symmetric if $p(X) = p(V - X)$ for every $X \subseteq V$, and p is (crossing) supermodular if $p(\emptyset) = 0$ and

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \tag{1.1}$$

for every $X, Y \subseteq V$ with $p(X) > 0$, $p(Y) > 0$ and $X \cap Y \neq \emptyset \neq V - (X \cup Y)$. Since $-d_G$ is symmetric supermodular, ECAP is a special case of this problem.

On the other hand, the functions p defined in LECAP and NAECAP are not symmetric supermodular, but symmetric *skew-supermodular*, as observed in [2] and [6], respectively, where $p : 2^V \to Z^+$ is skew-supermodular if $p(\emptyset) = 0$, and at least one of (1.1) and

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X) \tag{1.2}$$

holds for every $X, Y \subseteq V$ with $p(X) > 0$ and $p(Y) > 0$. Note that the problem of covering symmetric skew-supermodular functions is NP-hard since so is NAECAP. Recently, Nutov [12] proved that this problem is APX-hard and 7/4-approximable in polynomial time under the assumption that a polynomial time oracle for $\min_{X \subseteq V}\{\sum_{v \in X} g(v) + d_{(V,F)}(X) - p(X)\}$ is available, where $g : V \to Z^+$ is a function on V and F denotes a set of edges on V (note that such an oracle for a supermodular function p is always available as pointed in [1]). Some other problems as the *element-connectivity augmentation problem* (*ELCAP*) are also included in this problem as its special case, and ELCAP was shown to be NP-hard even if $r \in \{0, 2\}$ [7,12]. It remains a challenging question which type of the problem of covering symmetric skew-supermodular functions is polynomially solvable or not.

In this paper, we consider the *edge-connectivity augmentation problem with monotone requirements* (*MECAP*), which is RECAP with a *monotone* function r, where $r : 2^V \to Z^+$ is monotone if $r(X') \geq r(X)$ holds for every two sets $X', X \subseteq V$ with $\emptyset \neq X' \subseteq X$. NAECAP with $\mathcal{W}$ and $r' : \mathcal{W} \to Z^+$ is equivalent to MECAP with r'', where $r''(X) = \max\{r'(W) \mid W \cap X = \emptyset\}$ for each $\emptyset \neq X \subset V$. Indeed, the function r'' is monotone and the function r in NAECAP satisfies $r(X) = \max\{r''(X), r''(V - X)\}$. On the other hand, MECAP with r is equivalent to NAECAP with $\mathcal{W} = \{W \subset V \mid r(V - W) > 0\}$ and $r'(W) = r(V - W)$, $W \in \mathcal{W}$. Indeed, for each $\emptyset \neq X \subset V$, we have $\max\{r'(W) \mid W \cap X = \emptyset, W \in \mathcal{W}\} = r(X)$ by the monotonicity of r. In this sense, we may say that MECAP is a reformulation of NAECAP. It follows that the function p defined in MECAP is symmetric skew-supermodular and MECAP is NP-hard in general. However, the method of applying Ishii and Hagiwara's algorithm [6] to NAECAP with $\mathcal{W} = \{W \subset V \mid r(V - W) > 0\}$ and $r'(W) = r(V - W)$, $W \in \mathcal{W}$ is not a polynomial time one for MECAP, because their algorithm depends on the number of areas and $|\{W \subset V \mid r(V - W) > 0\}|$ may be exponential in n and m. In this paper, we propose an algorithm for solving MECAP in $O(n^4(m + n \log n + q))$ time, under the assumption that each $\emptyset \neq X \subset V$ satisfies $r(X) \geq 2$ whenever $r(X) > 0$, where q is the time required to compute $r(X)$ for each $X \subseteq V$; this gives rise to a polynomial time algorithm under the assumption that q is polynomial in the input size of the problem. In NAECAP with $\mathcal{W}$ and r', we have $r(X) = \max\{r'(W) \mid W \cap X = \emptyset\}$, and hence $r(X)$ can be computed in $O(|X| + \sum_{W \in \mathcal{W}} |W|)$ time; our algorithm is a polynomial time one also for NAECAP under the assumption that $r'(W) \geq 2$ holds for each $W \in \mathcal{W}$. Moreover, its time complexity improves Ishii and Hagiwara's one [6] in some case; e.g., in the case of $n = o(|\mathcal{W}|)$ and $\sum_{W \in \mathcal{W}} |W| = O(m + n \log n)$.

The paper is organized as follows. In Section 2, we define MECAP, after introducing some basic notations. In Section 3, we derive lower bounds on the optimal value to MECAP, and state our main result that MECAP is polynomially solvable under the assumption that $r(X) \geq 2$ holds for every $X \subseteq V$ whenever $r(X) > 0$. In Section 4, we introduce the so-called edge-splitting operation, and give an algorithm for solving MECAP, based on these lower bounds and the edge-splitting operation. In Section 5, we prove the correctness of the algorithm. In Section 6, we give concluding remarks.

2 Problem Definition

Let $G = (V, E)$ stand for an undirected graph with a set V of *vertices* and a set E of *edges*. An edge with end vertices u and v is denoted by (u, v). We denote $|V|$ by n (or by $n(G)$) and $|\{\{u, v\} | (u, v) \in E\}|$ by m (or by $m(G)$). A singleton set $\{x\}$ may be simply written as x, and " $\subset$ " implies proper inclusion while " $\subseteq$ " means " $\subset$ " or " $=$ ". In $G = (V, E)$, its vertex set V and edge set E may be denoted by $V(G)$ and $E(G)$, respectively. A maximal connected subgraph G' in a graph G is called a *component* of G (for notational convenience, a component H may be represented by its vertex set $X = V(H)$). For a subset $V' \subseteq V$ in G, the subgraph induced by V' is denoted by $G[V']$ or $G - (V - V')$. For an edge set E' with $E' \cap E = \emptyset$, we denote the augmented graph $(V, E \cup E')$ by $G + E'$. For an edge set E', we denote by $V[E']$ the set of all end vertices of edges in E'.

For two disjoint subsets $X, Y \subset V$ of vertices, we denote by $E_G(X, Y)$ the set of edges $e = (x, y)$ such that $x \in X$ and $y \in Y$, and also denote $|E_G(X, Y)|$ by $d_G(X, Y)$. In particular, $d_G(X, V - X)$ may be written as $d_G(X)$. Moreover, we define $d_G(\emptyset) = d_G(V) = 0$. For two sets $X, Y \subseteq V$ in a graph $G = (V, E)$, we say that X and Y *intersect* each other in G if none of $X \cap Y$, $X - Y$, $Y - X$ is empty. For a graph $G = (V, E)$, every two sets $X, Y \subset V$ satisfy the following equalities.

$$
\begin{aligned}
d_G(X) + d_G(Y) &= d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X - Y, Y - X). && (2.1)\\
d_G(X) + d_G(Y) &= d_G(X - Y) + d_G(Y - X) \\
&\qquad +2d_G(X \cap Y, V - (X \cup Y)). && (2.2)
\end{aligned}
$$

Given a ground set V, a set-function $r : 2^V \to Z^+$ is called *monotone* if $r(X') \geq r(X)$ holds for each set X, X' with $\emptyset \neq X' \subseteq X \subseteq V$. In this paper, we consider the following connectivity augmentation problem with monotone requirements.

Problem 1 (Edge-connectivity augmentation problem with monotone requirements, MECAP)
Input: *An undirected graph* $G = (V, E)$ *and a monotone function* $r : 2^V \to Z^+$.
Output: *A set* E^* *of new edges with the minimum cardinality such that each set* $\emptyset \neq X \subset V$ *satisfies* $d_{G+E^*}(X) \geq r(X)$. $\square$

We call a set $X \subseteq V$ *r-maximal* if $r(X) > 0$ and $r(X') = 0$ holds for each set $X' \supset X$. Let $\mathcal{R}$ denote the family of r-maximal subsets of V. A set $\emptyset \neq X \subset V$ is called *proper* if $X \subseteq M$ or $V - X \subseteq M$ for some $M \in \mathcal{R}$. Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the family of proper sets X such that X (resp. $V - X$) is contained in some r-maximal set (note that some proper set may belong to both of $\mathcal{A}$ and $\mathcal{B}$). Also notice that if X is in $\mathcal{A}$, then $X' \subseteq X$ is also in $\mathcal{A}$; if X is in $\mathcal{B}$, then X' with $X \subset X' \subset V$ is also in $\mathcal{B}$. From the symmetry of d_G, a set F of edges is feasible to MECAP if and only if all proper sets X satisfy $d_{G+F}(X) \geq R(X)$, where $R(X) = \max\{r(X), r(V - X)\}$. For a set-function $p' : 2^V \to Z^+$, we say that an edge set E' *covers* p' if $d_{(V,E')}(X) \geq p'(X)$ for each set $X \subseteq V$. We remark that a set E' of edges is feasible to MECAP if and only if E' covers p, where

$$
\begin{aligned}
&p(X) = \max\{0, R(X) - d_G(X)\} \text{ for every set } \emptyset \neq X \subset V,\\
&\text{and } p(\emptyset) = p(V) = 0.
\end{aligned}
$$

As mentioned in Section 1, p is symmetric skew-supermodular. We here give its proof for completing the paper.

Lemma 2 *Let* $r : 2^V \to Z^+$ *be a monotone set-function on* V. *Then* p *is symmetric skew-supermodular.* $\square$

Let $\mathcal{A}^*$ (resp. $\mathcal{B}^*$) denote the family of proper sets X in $\mathcal{A}$ (resp. $\mathcal{B}$) with $r(X) \geq r(V-X)$ (resp. $r(X) \leq r(V-X)$). Note that each proper set belongs to $\mathcal{A}^*$ or $\mathcal{B}^*$ and that $X \in \mathcal{A}^*$ if and only if $V-X \in \mathcal{B}^*$. By the monotonicity of r, it is not difficult to see the following properties.

Lemma 3 *Let $r : 2^V \to Z^+$ be a monotone set-function on V and X be a proper subset in $G=(V,E)$.*
(i) *If $X \in \mathcal{A}^*$, then any set $\emptyset \neq X' \subseteq X$ belongs to $\mathcal{A}$ and $R(X') \geq r(X') \geq R(X)$.*
(ii) *If $X \in \mathcal{B}^*$, then any set $V \supset X' \supseteq X$ belongs to $\mathcal{B}$ and $R(X') \geq r(V-X') \geq R(X)$.* $\square$

PROOF of Lemma 2: Clearly, p is symmetric by the symmetry of d_G and R. Since d_G satisfies both of (2.1) and (2.2), it suffices to show that R is skew-supermodular. For this, we show that every two intersecting proper subsets X, Y of V with $p(X), p(Y) > 0$ satisfy the followings (note that the cases of $X \subseteq Y$ or $X \cap Y = \emptyset$ clearly satisfy (1.1) or (1.2)):

$$\begin{aligned}&\text{If } (a)\ X, Y \in \mathcal{A}^*,\ (b)\ X, Y \in \mathcal{B}^*, \text{ or } (c)\ X \in \mathcal{A}^*,\ Y \in \mathcal{B}^*, \text{ and}\\ &V = X \cup Y, \text{ then } R(X) + R(Y) \leq R(X-Y) + R(Y-X).\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\text{If } X \in \mathcal{A}^*,\ Y \in \mathcal{B}^*,\ V \neq X \cup Y, \text{ then}\\ &R(X) + R(Y) \leq R(X \cap Y) + R(X \cup Y).\end{aligned} \tag{2.4}$$

In the case of (a) (resp. (b)), Lemma 3(i) implies that $R(X-Y) \geq r(X-Y) \geq R(X)$ and $R(Y-X) \geq r(Y-X) \geq R(Y)$ (resp. $R(Y-X) \geq r(Y-X) \geq R(V-X) = R(X)$ and $R(X-Y) \geq r(X-Y) \geq R(V-Y) = R(Y)$ from $V-X, V-Y \in \mathcal{A}^*$), implying (2.3). In the case of (c), $R(X-Y) = R(V-Y) = R(Y)$ and $R(Y-X) = R(V-X) = R(X)$ imply (2.3). In the remaining case, we have $R(X \cap Y) \geq R(X)$ (resp. $R(X \cup Y) \geq R(Y)$) by Lemma 3(i) (resp. by Lemma 3(ii) and $V \neq X \cup Y$), which implies (2.4). $\square$

3 Lower Bound on the Optimal Value

For a graph G and a fixed function $r : 2^V \to Z^+$, let $opt(G,r)$ denote the optimal value to MECAP in G, i.e., the minimum size $|E^*|$ of a set E^* of new edges which covers p. In this section, we derive lower bounds on $opt(G,r)$ to MECAP with G and r.

A family $\mathcal{X} = \{X_1, \ldots, X_t\}$ of nonempty vertex sets in $G=(V,E)$ is called a *subpartition of* V, if every two sets $X_i, X_j \in \mathcal{X}$ satisfy $X_i \cap X_j = \emptyset$. If X is proper, then it is necessary to add at least $p(X)$ edges between X and $V-X$. Let

$$\alpha(G,r) = \max_{\mathcal{X}}\left\{\sum_{X\in\mathcal{X}} p(X)\right\}, \tag{3.1}$$

where the maximization is taken over all subpartitions of V. Then any feasible solution to MECAP with G and r must contain an edge which joins two vertices from a set X with $p(X) > 0$ and the set $V-X$. Therefore we see the following property.

Remark 4 *$opt(G,r) \geq \lceil \alpha(G,r)/2 \rceil$ holds.* □

We remark that there is an instance with $opt(G,r) > \lceil \alpha(G,r)/2 \rceil$. Figure 1 gives an instance where $\mathcal{R} = \{M_1, M_2, M_3\}$ and all proper sets X satisfies $R(X) = 2$. Each set $\{v_i\}$, $i = 1,2,3,4,5$ is proper, $p(v_i) = 2 - d_G(v_i) = 1$ for $i = 1,2,3,5$ and $p(v_4) = 2 - d_G(v_4) = 2$. It is not hard to see that in (3.1) the maximum is achieved for the subpartition $\{\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_5\}\}$ and $\lceil \alpha(G,r)/2 \rceil = 3$. In order to obtain a feasible solution of three edges, we must add $E' = \{(v_1,v_2),(v_3,v_4),(v_4,v_5)\}$ or $E' = \{(v_1,v_4),(v_2,v_4),(v_3,v_5)\}$ without loss of generality. In both cases, E' is infeasible because the proper set X satisfies $d_{G+E'}(X) = 1$ for $X = M_1 - \{v_4, v_5\}$ in the former case and $X = M_1 - \{v_5\}$ in the latter case. We will show that all such instances can be

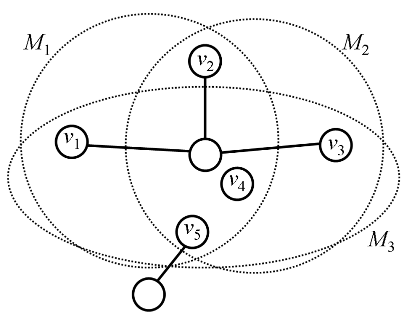

Fig. 1. Illustration of a graph G with $opt(G,r) > \lceil \frac{\alpha(G,r)}{2} \rceil$.

completely characterized.

Definition 5 *We say that a graph G has* property (P) *if there is a subpartition $\mathcal{X}$ of V with $\sum_{X\in\mathcal{X}} p(X) = \alpha(G,r)$ satisfying the following conditions $(P1)$–$(P3)$:*

$(P1)$ $\alpha(G,\, r)$ is even.

$(P2)$ There is a set $X^* \in \mathcal{X}$ with $p(X^*) = 1$.

$(P3)$ Let $\mathcal{X}_1$ denote the family of proper sets $X \in \mathcal{X}$ with $d_G(X) = 0$ and $p(X) = 2$. For each $X \in \mathcal{X} - \mathcal{X}_1 - \{X^*\}$, there is a set $Y_X \in \mathcal{B}^*$ such that the

following (i)–(iv) hold: (i) $X \cup X^* \subseteq Y_X$*, (ii)* $V - Y_X - (\cup_{X'' \in \mathcal{X}_1} X'') \neq \emptyset$*, (iii)* $\sum_{X' \in \mathcal{X}, X' \subset Y_X} p(X') \leq p(Y_X) + 1$*, and (iv) every set* $X' \in \mathcal{X}$ *satisfies* $X' \subset Y_X$ *or* $X' \cap Y_X = \emptyset$*.* $\square$

Note that G in Figure 1 has property (P) because $\alpha(G, r) = 6$ holds and the subpartition $\mathcal{X} = \{X^* = \{v_5\}, X_1 = \{v_1\}, X_2 = \{v_2\}, X_3 = \{v_3\}, X_4 = \{v_4\}\}$ of V satisfies $\mathcal{X}_1 = \{X_4\}$, $Y_{X_1} = (V - M_2) \cup \{v_5\}$, $Y_{X_2} = (V - M_3) \cup \{v_5\}$, and $Y_{X_3} = (V - M_1) \cup \{v_5\}$.

Lemma 6 *If G has property* (P)*, then* $opt(G, r) \geq \lceil \alpha(G, r)/2 \rceil + 1$*.*

PROOF. Assume by contradiction that G has property (P) and there is an edge set E^* with $|E^*| = \alpha(G, r)/2$ such that E^* covers p (note that $\alpha(G, r)$ is even by the property (P1)). Let $\mathcal{X} = \{X_1, \ldots, X_t\}$ denote a subpartition of V satisfying $\sum_{X \in \mathcal{X}} p(X) = \alpha(G, r)$, $p(X) > 0$ for each $X \in \mathcal{X}$, and the above (P2) and (P3). Since $|E^*| = \alpha(G, r)/2$ holds, each set $X \in \mathcal{X}$ satisfies $d_{G'}(X) = p(X)$, where $G' = (V, E^*)$. Therefore, any edge $(x, x') \in E^*$ satisfies $x \in X$ and $x' \in X'$ for some two sets $X, X' \in \mathcal{X}$ with $X \neq X'$. Hence $\sum_{v \in X''} d_{G'}(v) = d_{G'}(X'')$ for $X'' \in \mathcal{X}$. From this, there exists a set $X_1 \in \mathcal{X} - \{X^*\}$ with $E_{G'}(X^*, X_1) \neq \emptyset$. Now note that $\mathcal{X} - \mathcal{X}_1 - \{X^*\} \neq \emptyset$ holds since otherwise $\alpha(G, r) = 2|\mathcal{X}_1| + 1$ by the properties (P2) and (P3), contradicting that $\alpha(G, r)$ is even.

Assume that $X_1 \in \mathcal{X} - \mathcal{X}_1$ holds. Since G satisfies property (P), there is a set $Y_{X_1} \in \mathcal{B}^*$ which satisfies (P3), and hence $\sum_{v \in Y_{X_1}} d_{G'}(v) = \sum_{X' \in \mathcal{X}, X' \subset Y_{X_1}} d_{G'}(X') = \sum_{X' \in \mathcal{X}, X' \subset Y_{X_1}} p(X') \leq p(Y_{X_1}) + 1$. Since $G'[Y_{X_1}]$ contains one edge in $E_{G'}(X_1, X^*)$, the proper set Y_{X_1} satisfies $d_{G'}(Y_{X_1}) \leq (\sum_{v \in Y_{X_1}} d_{G'}(v)) - 2 \leq p(Y_{X_1}) - 1$, which contradicts that E^* covers p.

Assume that $X_1 \in \mathcal{X}_1$. From the properties (P2) and (P3), we have $d_{G'}(X^* \cup X_1) = 1$, and this implies that there exists an edge $e \in E^*$ connecting X_1 and some set in $\mathcal{X} - \{X^*, X_1\}$. Let $\mathcal{X}'_1 = \{X^*, X_1, X_2, \ldots, X_{t'}, X_{t'+1}\}$ be the family of sets in $\mathcal{X}$ such that we have $X_i \in \mathcal{X}_1$ for each $i = 1, 2, \ldots, t'$ and $X_{t'+1} \in \mathcal{X} - \mathcal{X}_1$ and $E_{G'}(X_i, X_{i+1}) \neq \emptyset$ for each $i = 1, \ldots, t'$ (note that such $X_{t'+1}$ exists by $\mathcal{X} - \mathcal{X}_1 - \{X^*\} \neq \emptyset$). Note that such $\mathcal{X}'_1$ is determined uniquely by

$$d_{G'}(X^*) = 1 \text{ and } d_{G'}(X) = 2 \text{ for each } X \in \mathcal{X}_1. \tag{3.2}$$

From the definition of property (P), there is a set $Y_{X_{t'+1}} \in \mathcal{B}^*$ satisfying (P3) for $X_{t'+1}$. Let $Y_{t'+1} = Y_{X_{t'+1}} \cup (\cup_{X \in \mathcal{X}'_1} X)$. Since we have $Y_{t'+1} \supseteq Y_{X_{t'+1}} \in \mathcal{B}^*$ and $V - Y_{X_{t'+1}} - (\cup_{X \in \mathcal{X}_1} X) \neq \emptyset$ (by the property (P3)), Lemma 3(ii) implies that $Y_{t'+1}$ is also proper and $R(Y_{t'+1}) \geq R(Y_{X_{t'+1}})$. Note that $d_G(Y_{t'+1}) = d_G(Y_{X_{t'+1}})$

by $d_G(X) = 0$ for each $X \in \mathcal{X}_1$. It follows that $p(Y_{t'+1}) \geq p(Y_{X_{t'+1}})$. Thus, we have

$$\sum_{v \in Y_{t'+1}} d_{G'}(v) \leq (p(Y_{t'+1}) + 1) + 2t' \tag{3.3}$$

by $\sum_{v \in Y_{X_{t'+1}}} d_{G'}(v) \leq p(Y_{X_{t'+1}}) + 1$, (3.2), and $p(Y_{t'+1}) \geq p(Y_{X_{t'+1}})$. Also by (3.2), we can observe that each edge in E^* incident to $(\cup_{X \in \mathcal{X}'_1 - \{X_{t'+1}\}} X)$ is contained in $E(G'[Y_{t'+1}])$; $E(G'[Y_{t'+1}])$ contains at least $t'+1$ edges in E^*. From (3.3) and this, we have $d_{G'}(Y_{t'+1}) \leq (p(Y_{t'+1}) + 1) + 2t' - 2(t'+1) = p(Y_{t'+1}) - 1$. Thus this contradicts that E^* covers p. $\Box$

In this paper, we prove that MECAP enjoys the following min-max theorem.

Theorem 7 *Let $G = (V, E)$ be an undirected graph and $r : 2^V \to Z^+$ be a monotone set-function on V such that $r(X) \geq 2$ holds whenever $r(X) > 0$. Then, for MECAP, $opt(G, r) = \lceil \alpha(G, r)/2 \rceil$ holds if G does not have property (P), and $opt(G, r) = \lceil \alpha(G, r)/2 \rceil + 1$ holds otherwise. Moreover, a solution E^* with $|E^*| = opt(G, r)$ can be obtained in $O(n^4(m + n \log n + q))$ time.* $\Box$

4 Edge-Splittings and Algorithm

4.1 Extensions

We adapt the so-called "edge-splitting" method for solving MECAP, which is known to be useful for solving connectivity augmentation problems [2]. In the edge-splitting method, after creating a new vertex s outside of G and adding new edges between s and G, we find an appropriate edge set to be added to G by splitting off a pair of edges incident to s in the extended graph. Given a graph $G = (V, E)$ and a function $r : 2^V \to Z^+$ on V, a graph $H = (V \cup \{s\}, E \cup F)$ obtained from G by adding a new vertex s and a set F of new edges connecting s and V is called a *p-extension of G* if

$$\text{all sets } X \subseteq V \text{ satisfy } d_H(s, X) \geq p(X). \tag{4.1}$$

In particular, a p-extension $H = (V \cup \{s\}, E \cup F)$ of G is called *critical* if $(V \cup \{s\}, E \cup F')$ violates (4.1) for any $F' \subset F$. In [2,12], it was shown that if p is symmetric skew-supermodular, then any critical p-extension $H = (V \cup \{s\}, E \cup F)$ of G satisfies $|F| = \alpha(G, r)$. From this and Lemma 2, we have the following theorem.

Theorem 8 *Let $G = (V, E)$ be a graph and $r : 2^V \to Z^+$ be a monotone function on V. Any critical p-extension $H = (V \cup \{s\}, E \cup F)$ of G satisfies $|F| = \alpha(G, r)$.* $\square$

4.2 Edge-splitting theorems

For a graph $H = (V \cup \{s\}, E)$ and a designated vertex $s \notin V$, an operation called *edge-splitting* (*at* s) is defined as deleting two edges $(s, u), (s, v) \in E$ and adding one new edge (u, v). That is, the graph $H' = (V \cup \{s\}, (E - \{(s, u), (s, v)\}) \cup \{(u, v)\})$ is obtained from such edge-splitting operation. Then we say that H' is obtained from H by *splitting* a pair of edges (s, u) and (s, v) (or by splitting (s, u) and (s, v)). A sequence of splittings is *complete* if the resulting graph H' does not have any neighbor of s.

Given a p-extension $H = (V \cup \{s\}, E \cup F)$ of $G = (V, E)$, a pair $\{(s, u), (s, v)\}$ is called *admissible* if the graph H' obtained from H by splitting (s, u) and (s, v) is also a p^{uv}-extension of $H' - s = G + \{(u, v)\}$, where $p^{uv}(X) = \max\{0, p(X) - 1\}$ for each set X with $|\{u, v\} \cap X| = 1$ and $p^{uv}(X) = p(X)$ otherwise. Notice that given a graph G, if there is a complete admissible splitting at s in its critical p-extension $H = (V \cup \{s\}, E \cup F)$, then the set E' of split edges is an optimal solution of MECAP to G and r. Indeed, in $H' = (V \cup \{s\}, E \cup E')$, $d_{H'}(s) = 0$ holds, and every set $\emptyset \neq X \subset V$ satisfies $0 = d_{H'}(s, X) \geq \max\{0, R(X) - d_{G+E'}(X)\}$, implying that E' is feasible to MECAP. Moreover, Theorem 8 implies that $|E'| = |F|/2 = \lceil \alpha(G, r)/2 \rceil$, which is a lower bound on $opt(G, r)$ by Remark 4. However, as indicated by Lemma 6, any critical p-extension of G with property (P) does not have a complete admissible splitting. If

$$\text{every set } X \subseteq V \text{ satisfies } r(X) \geq 2 \text{ whenever } r(X) > 0, \tag{4.2}$$

then we can characterize a graph with property (P) as follows.

Definition 9 *A p-extension $H = (V \cup \{s\}, E \cup F)$ of G has* property (P^*) *if H is a critical p-extension of G satisfying the following $(P1^*) - (P4^*)$:*

$(P1^*)$ $d_H(s)$ is even.
$(P2^*)$ G has exactly one component $C^* \subseteq V$ with $d_H(s, C^*) = 1$.
$(P3^*)$ For the edge (s, u^*) with $\{(s, u^*)\} = E_H(s, C^*)$, u^* is contained in a proper set $X \subseteq C^*$ with $d_H(s, X) = p(X)$.
$(P4^*)$ Let $\mathcal{C}_1$ be the family of all components C of G such that $d_H(C) = d_H(s, C) = 2$ and C is proper. For any edge $e \in E_H(s, V - \cup_{C \in \mathcal{C}_1} C)$, $\{(s, u^*), e\}$ is not admissible in H. $\square$

Theorem 10 *Let $G = (V, E)$ be a graph and $r : 2^V \to Z^+$ be a monotone* *function satisfying* (4.2). *Then, G has property* (P) *if and only if its critical p-extension has property* (P^*). $\square$

Moreover, the following properties hold about admissible splittings.

Theorem 11 *Let $r : 2^V \to Z^+$ be a monotone function on V satisfying* (4.2) *and $H = (V \cup \{s\}, E \cup F)$ be a critical p-extension of G. Then the following* (i) *and* (ii) *hold:*
(i) *Some graph H' obtained from H by adding at most one extra edge to G and some one extra edge incident to s to make the degree of s even* (*if necessary*) *has a complete admissible splitting at s.*
(ii) *If H does not have property* (P^*), *then H has a complete admissible splitting at s after replacing at most one edge incident to s with some new edge incident to s, and adding some one extra edge incident to s to make the degree of s even* (*if necessary*). $\square$

We give proofs of these two theorems in Section 5. Note that Lemma 6, Theorem 8, and Theorem 11(ii) prove the necessity of Theorem 10. Indeed, if a critical p-extension H of G does not have property (P^*), then by a complete admissible splitting according to Theorem 11(ii), we can obtain a feasible solution E' to MECAP with G and r such that $|E'| = \lceil d_H(s)/2 \rceil = \lceil \alpha(G, r)/2 \rceil$ (by Theorem 8), from which and Lemma 6 it follows that G does not have property (P). Let us discuss its consequences. Based on these two theorems, we give the following algorithm which delivers an optimal solution to MECAP with G and r satisfying (4.2).

Algorithm M-AUG

Input: A graph $G = (V, E)$ and a monotone function $r : 2^V \to Z^+$ on V satisfying (4.2).

Output: A set E^* of new edges with $|E^*| = opt(G, r)$ which covers p.

Step 1: Find a critical p-extension $H = (V \cup \{s\}, E \cup F)$ of G.

Step 2: If H does not have property (P*), then find a complete admissible splitting at s after replacing some one edge incident to s and adding some one edge between s and V to make the degree of s even according to Theorem 11(ii). Otherwise, after adding some edge to G according to Theorem 11(i), find a complete admissible splitting at s. Output the set E^* of all edges added to G as an optimal solution. $\square$

The details for Step 2 and the analysis of the time complexity of the algorithm will be given in Section 5. We here only observe that the set E^* obtained by the algorithm is optimal. If H does not have property (P*), then as observed above, we have $|E^*| = \lceil \alpha(G,r)/2 \rceil$, which is equal to a lower bound on $opt(G,r)$ by Remark 4. If H have property (P*), then $|E^*| = \lceil \alpha(G,r)/2 \rceil + 1$. Theorem 10 and Lemma 6 imply that also in this case, $|E^*|$ is equal to a lower bound on $opt(G,r)$.

5 Correctness of algorithm M-AUG

In this section, we will prove the correctness of algorithm M-AUG, give a detailed description of Step 2, and analyze the time complexity of algorithm M-AUG. For proving the correctness of the algorithm, it suffices to prove Theorems 10 and 11, as observed in the paragraph after the description of algorithm M-AUG in the previous section. We will show these two theorems in the following manner. After showing several preparatory properties about admissible splittings, we give a constructive proof of Theorem 11 in Section 5.1, which also proves the necessity of Theorem 10 as observed in the paragraph immediately after Theorem 11. In Section 5.2, we prove the sufficiency of Theorem 10, i.e., we give a proof that if a p-extension of G satisfies property (P*), then G has property (P). In Section 5.3, we give a detailed description of Step 2 of algorithm M-AUG, according to the constructive proof of Theorem 11, and finally analyze the time complexity of the algorithm.

Through this section, for a p-extension H of $G = (V, E)$, let $\mathcal{C}_1$ be the family of all components C of G such that $d_H(C) = d_H(s, C) = 2$ and C is proper, and $V_1 = \cup_{C \in \mathcal{C}_1} C$. Let $\mathcal{C}_2$ be the family of all components C of G such that $C \notin \mathcal{C}_1$ and $d_H(s, C) > 0$, and $V_2 = \cup_{C \in \mathcal{C}_2} C$.

We first show preparatory properties for proving the theorems. For seeking admissible pairs, we need to analyze situations where some splitting fails. For a p-extension $H = (V \cup \{s\}, E \cup F)$ of $G = (V, E)$, a pair $\{(s, u), (s, v)\} \subseteq F$ of two edges is not admissible if there is a proper set $Y \subset V$ with $\{u, v\} \subseteq Y$ and $d_H(s, Y) - p(Y) \leq 1$ (note that the graph H' obtained from H by splitting (s, u) and (s, v) satisfies $d_{H'}(s, Y) = d_H(s, Y) - 2 \leq p(Y) - 1 = p^{uv}(Y) - 1$). Also note that $d_H(s, Y) \geq 2$ implies that $p(Y) \geq d_H(s, Y) - 1 > 0$. Such set Y is called a *dangerous set*. Conversely, a pair $\{(s, u), (s, v)\}$ is not admissible only if there is a dangerous set $Y \subset V$ with $\{u, v\} \subseteq Y$.

As a corollary of Lemma 3, we can observe that the following property holds.

Corollary 12 *Let $r : 2^V \to Z^+$ be a monotone set-function on V and X, Y be proper subsets of V with $p(X), p(Y) > 0$.*
(i) If (a) $X, Y \in \mathcal{A}^*$, (b) $X, Y \in \mathcal{B}^*$, or (c) $X \in \mathcal{A}^*$, $Y \in \mathcal{B}^*$, and $V = X \cup Y$, then $p(X) + p(Y) \leq p(X - Y) + p(Y - X) - 2d_G(X \cap Y, V - (X \cup Y))$. In particular, in the cases of (a) or (b), if the equality holds, then $R(X - Y) =$

$r(X-Y)$ and $R(Y-X) = r(Y-X)$.
(ii) In all other cases, $p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y)$. $\square$

From the symmetry of p, we can observe that all neighbors of s in H cannot be included in one dangerous set.

Lemma 13 *Let $p : 2^V \to Z^+$ be a symmetric function and $H = (V \cup \{s\}, E \cup F)$ be a p-extension of $G = (V, E)$. If $Y \subset V$ is dangerous, then $d_H(s, V-Y) \ge d_H(s, Y) - 1 > 0$.*

PROOF. Since Y is dangerous and $p(Y) = p(V-Y)$, we have $d_H(s, Y) \le p(Y)+1 = p(V-Y)+1 \le d_H(s, V-Y)+1$. From the definition of dangerous sets, it follows that $d_H(s, Y) \ge 2$. $\square$

The next two lemmas show properties for proper sets Y with $d_H(s, Y) - p(Y) \le 1$ and $p(Y) > 0$ (note that Y is not necessarily dangerous). We will be often referred to the next Lemma 14 in the subsequent arguments, when we observe that a dangerous set of $\mathcal{A}^*$ induces a connected graph, or that a dangerous set which does not induce a connected graph belongs to $\mathcal{B}^*$.

Lemma 14 *Let $r : 2^V \to Z^+$ be a monotone function and $H = (V \cup \{s\}, E \cup F)$ be a p-extension of $G = (V, E)$. For every set $Y \subset V$ of $\mathcal{A}^*$ with $d_H(s, Y) - p(Y) \le 1$, $R(Y) \ge 2$, and $p(Y) > 0$, any set $\emptyset \ne Y' \subset Y$ satisfies $d_G(Y', Y - Y') \ge R(Y) - \lfloor \frac{d_H(Y)}{2} \rfloor$ (≥ 1).*

PROOF. By $p(Y) > 0$, $d_H(Y) = d_H(s, Y) + d_G(Y) \le R(Y)+1$. By Lemma 3 (i) and $Y \in \mathcal{A}^*$, we have $d_H(Y') = d_H(s, Y') + d_G(Y') \ge R(Y') \ge R(Y)$. Similarly, $Y - Y'$ satisfies this property. Hence, we have $d_G(Y', Y - Y') = \frac{1}{2}(d_H(Y') + d_H(Y - Y')) - \frac{d_H(Y)}{2} \ge R(Y) - \frac{d_H(Y)}{2} > 0$ by $R(Y) \ge 2$. $\square$

The next lemma is often used under a situation where two crossing dangerous cuts Y_1, Y_2 satisfy $d_H(s, Y_1 \cap Y_2) > 0$. We call a set $Y \subset V$ with $d_H(s, Y) = p(Y) > 0$ *tight* (note that each tight set Y with $d_H(s, Y) \ge 2$ is dangerous).

Lemma 15 *Let $r : 2^V \to Z^+$ be a monotone function and $H = (V \cup \{s\}, E \cup F)$ be a p-extension of $G = (V, E)$. Let Y_1 and Y_2 be two sets with $d_H(s, Y_i) - p(Y_i) \le 1$ and $p(Y_i) > 0$ for $i = 1, 2$, and $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) > 0$ such that Y_1 and Y_2 satisfy (i) $Y_1, Y_2 \in \mathcal{A}^*$ or (ii) $Y_1, Y_2 \in \mathcal{B}^*$. If Y_1 and Y_2 cross each other in H, then the following (a) – (d) hold:*
(a) $Y_1 - Y_2, Y_2 - Y_1 \in \mathcal{A}^*$.
(b) $d_H(s, Y_i) = p(Y_i) + 1$ for $i = 1, 2$.
(c) $d_H(s, Y_j - Y_k) = p(Y_j - Y_k)$ for $\{j, k\} = \{1, 2\}$. In particular, if $d_H(s, Y_j -$

$Y_k) > 0$, $Y_j - Y_k$ *is tight.*
(d) $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) = 1$.

PROOF. In the case of (i) (resp. (ii)), $Y_1 - Y_2, Y_2 - Y_1 \in \mathcal{A}$ hold by Lemma 3(i) (resp. by $Y_2 - Y_1 \subseteq V - Y_1 \in \mathcal{A}^*$, $Y_1 - Y_2 \subseteq V - Y_2 \in \mathcal{A}^*$, and Lemma 3(i)). In both cases, Corollary 12 implies that $2 \geq d_H(s, Y_1) - p(Y_1) + d_H(s, Y_2) - p(Y_2)$ $\geq d_H(s, Y_1 - Y_2) - p(Y_1 - Y_2) + d_H(s, Y_2 - Y_1) - p(Y_2 - Y_1) + 2d_G(Y_1 \cap Y_2, V - (Y_1 \cup Y_2)) + 2d_H(s, Y_1 \cap Y_2)$. Now we have $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) = d_G(Y_1 \cap Y_2, V - (Y_1 \cup Y_2)) + d_H(s, Y_1 \cap Y_2) \geq 1$ and $d_H(s, Y_j - Y_k) \geq p(Y_j - Y_k)$ for $\{j, k\} = \{1, 2\}$ by (4.1). It follows that every inequality turns out to be an equality. Hence, $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) = 1$, $d_H(s, Y_i) - p(Y_i) = 1$ for $i = 1, 2$, $d_H(s, Y_1 - Y_2) = p(Y_1 - Y_2)$, and $d_H(s, Y_2 - Y_1) = p(Y_2 - Y_1)$. Moreover, Corollary 12(i) indicates that $r(Y_1 - Y_2) = R(Y_1 - Y_2)$, and $r(Y_2 - Y_1) = R(Y_2 - Y_1)$. Hence, $Y_1 - Y_2, Y_2 - Y_1 \in \mathcal{A}^*$. $\square$

5.1 Proof of Theorem 11

We first define a new operation called *hooking up*, which is a reverse operation of edge-splittings. We say that H' is obtained from H by *hooking up* an edge $(u, v) \in E(H - s)$ at s, if we construct H' by replacing an edge (u, v) with two edges (s, u) and (s, v) in H.

For proving Theorem 11, it suffices to show the following Theorem 16 and Lemma 17.

Theorem 16 *Let* $r : 2^V \to Z^+$ *be a monotone function satisfying* (4.2) *and* $H = (V \cup \{s\}, E \cup F)$ *be a critical* p*-extension of* $G = (V, E)$*. Assume that there is no admissible pair in* H*. Then the following* (i) *or* (ii) *holds:*
(i) $d_H(s) = 3$*. After adding one edge incident to* s*, there is a complete admissible splitting.*
(ii) $d_H(s) = 4$ *and* G *has exactly two components* C_1 *and* C_2 *such that* (a) $d_H(s, C_1) = 3$ *and* $d_H(s, C_2) = 1$, (b) *every set* $\emptyset \neq X \subseteq C_1$ *satisfies* $d_H(X) \geq 2$*, and* (c) *every set* $\emptyset \neq X \subseteq C_1$ *with* $d_H(X) = 2$ *is a proper set of* $\mathcal{A}$*.* $\square$

Lemma 17 *Let* H *and* r *satisfy the assumption of Theorem 16 and* $d_H(s) = 4$*, and* C_1 *and* C_2 *be components in Theorem 16. Then for every edge* $e = (u, v)$ *in* $G[V - C_1]$ *(if exists), the graph* H' *obtained from* H *by hooking up the edge* e *has an admissible pair* $\{e_1, e_2\}$ *with* $e_1 \in E_{H'}(s, C_1) = E_H(s, C_1)$ *and* $e_2 \in E_{H'}(s, V - C_1)$*.* $\square$

Before showing these theorem and lemma, we give a proof of Theorem 11 as its consequences.

PROOF of Theorem 11: (i) Let H_1 denote the graph from H by repeating admissible splittings as possible, E_1 denotes the set of split edges, and $G_1 = (V, E \cup E_1)$; the p_1-extension H_1 of G_1 has no admissible pair at s, where $p_1(X) = \max\{0, R(X) - d_{G_1}(X)\}$ for every $\emptyset \neq X \subset V$ and $p_1(\emptyset) = p_1(V) = 0$.

Theorem 16 implies that $d_{H_1}(s) \in \{0, 3, 4\}$. If $d_{H_1}(s) = 3$, then we can add one edge between s and V so that the resulting graph has a complete admissible splitting at s, by Theorem 16(i). If $d_{H_1}(s) = 4$, then after adding one edge connecting two components C_1 and C_2 satisfying (a) and (b) in Theorem 16(ii), we can obtain a complete admissible splitting at s (note that in the graph H' resulting from adding the edge, all neighbors of s is contained in one component in $H' - s$, and hence Theorem 16 ensures the existence of a complete admissible splitting in H'). Thus, in any case, after adding at most one edge in G or making the odd degree of s even, there is a complete admissible splitting at s.

(ii) Assume that $d_H(s)$ is even, because the case of odd $d_H(s)$ has been already seen in the above case of $d_{H_1}(s) = 3$. Since at least one of (P2*)–(P4*) does not hold, there are the following four possible cases:

(I) Every component C of G satisfies $d_H(s, C) \neq 1$.
(II) There are at least two components C of G with $d_H(s, C) = 1$.
(III) There is exactly one component C of G with $d_H(s, C) = 1$ where $\{(s, u)\} = E_H(s, C)$ holds. In H, $\{(s, u), (s, v)\}$ is admissible for some $(s, v) \in E_H(s, V - V_1) - \{(s, u)\}$.
(IV) There is exactly one component C of G with $d_H(s, C) = 1$ where $\{(s, u)\} = E_H(s, C)$ holds. There is no set $X \subseteq C$ with $u \in X$ and $d_H(s, X) = p(X)$.

Claim 18 *In the case (IV), there is a p-extension $H' = (V \cup \{s\}, E \cup (F - \{(s, u)\}) \cup \{(s, x)\})$ of G such that x is a vertex in some component $C' \neq C$ of G with $d_H(s, C') > 0$; H' belongs to the case (I).*

PROOF. Let $X_u \subset V$ be a tight set containing u such that no set $X' \subset X_u$ with $u \in X'$ is tight (such X_u exists since H is a critical p-extension). From $X_u - C \neq \emptyset$ and Lemma 14, we have $X_u \in \mathcal{B}^*$.

Then $X_u \cap (V_1 \cup V_2 - C) \neq \emptyset$ holds since otherwise $(V - V_1 - V_2) \cup C$ belongs to $\mathcal{B}$ by Lemma 3(ii) and hence $1 = d_H(s, (V - V_1 - V_2) \cup C) \geq p((V - V_1 - V_2) \cup C) = R((V - V_1 - V_2) \cup C) \geq 2$ holds by $d_G((V - V_1 - V_2) \cup C) = 0$ and (4.2), a contradiction. Let H_1 be the graph obtained from H by replacing the edge (s, u) with (s, x) with some $x \in X_u \cap (V_1 \cup V_2 - C)$.

We claim that H_1 is also a p-extension of G. Assume by contradiction that this does not hold. Then H has a tight set $X' \subset V$ with $u \in X' \cap X_u$ and

$x \in X_u - X'$. Note that $X' \in \mathcal{B}^*$ holds since $X' - C \neq \emptyset$ also holds from the assumption. We have $X' - X_u \neq \emptyset$ from the minimality of X_u and hence X_u and X' cross each other in H. Lemma 15 implies that $d_H(s, X_u) = p(X_u) + 1$, contradicting that X_u is tight.

Let $C' \subseteq V_1 \cup V_2 - C$ be the component of G with $x \in C'$. By the assumption, $d_H(s, C') \geq 2$ holds and hence $d_{H_1}(s, C') \geq 3$ holds. $\square$

In the case (IV), according to this claim, replace H with H' which belongs to the case (I), and redenote H' by H. Assume by contradiction that H has no complete splitting at s. Repeat admissible splittings as possible in H, and again consider H_1 defined as the above (i). Note that since $d_H(s)$ is even, $d_{H_1}(s) = 4$.

Then we have only to consider the cases where

$$G_1[V - C_1] \text{ contains no split edge in } E_1. \tag{5.1}$$

Consider the cases where $G_1[V - C_1]$ has a split edge $e \in E_1$. The graph H_2 obtained from H_1 by hooking up e has an admissible pair $\{e_1, e_2\}$ with $e_1 \in E_{H_2}(s, C_1)$ and $e_2 \in E_{H_2}(s, V - C_1)$ by Lemma 17. From the assumption, the graph H_3 obtained from H_2 by splitting e_1 and e_2 has no complete splitting, and has two components C_1' and C_2' satisfying (a) and (b) in Theorem 16. By $C_1 \subset C_1'$, we can see that the number of split edges in $H_3[V - C_1']$ is less than that in $H_1[V - C_1]$. By repeating this observation, we can assume that $G_1[V - C_1]$ contains no split edge in E_1.

In the case (I), $d_H(s, C_2) = 1$ implies that $G[C_2]$ contains a split edge in E_1 and hence such H_1 satisfying (5.1) does not exist; in this case, H has a complete admissible splitting.

Consider the case (II). Let C', C'' denote components of G with $d_H(s, C') = d_H(s, C'') = 1$. By (5.1), $C' = C_2$ and $C'' \subseteq C_1$ without loss of generality. Then $d_H(C'') = 1 < 2$ contradicts Theorem 16(ii)(b). Hence also in the case (II), such H_1 does not exist.

Consider the case (III). Let H' denote the graph obtained from splitting (s, u) and (s, v) in H, and C' denote the component containing v in H. If $d_{H'}(s, C \cup C') \neq 1$ in the graph H' obtained from H by splitting (s, u) and (s, v), then H' has no component C'' of $H' - s$ with $d_{H'}(s, C'') = 1$ and belongs to the case (I), which indicates that H' has a complete admissible splitting at s. Consider the case of $d_{H'}(s, C \cup C') = 1$; $d_H(s, C') = 2$. From the choice of (s, v), C' is not proper, since if C' is proper, then $C' \in \mathcal{C}_1$ would hold. By (5.1), in H_1, we have $C' \subseteq C_1$ and $d_{H_1}(C') = 2$, contradicting Theorem 16(ii)(c). Hence also in this case, such H_1 does not exist.

Consequently, in any case of (I)–(IV) such H_1 does not exist; H has a complete admissible splitting. $\square$

In the rest of this subsection, we give proofs of Theorem 16 and Lemma 17. In [12, Proposition 5.3], it was shown that a critical extension of G which has no admissible pair has the following property if p is a symmetric skew-supermodular.

Theorem 19 [12] *Let $p : 2^V \to Z^+$ be a symmetric skew-supermodular set-function on V, and H be a critical p-extension. If there is no admissible pair in H, then p is $\{0,1\}$-valued.* $\square$

For a graph $G = (V, E)$, every three sets X, Y, and Z satisfy the following inequality.

$$\begin{aligned} d_G(X) + d_G(Y) + d_G(Z) \geq\ & d_G(X - Y - Z) + d_G(Y - X - Z) \\ & + d_G(Z - X - Y) + d_G(X \cap Y \cap Z) \\ & + 2d_G(X \cap Y \cap Z, V - (X \cup Y \cup Z)). \end{aligned} \tag{5.2}$$

PROOF of Theorem 16: Lemma 2 and Theorem 19 imply that p is $\{0,1\}$-valued, and hence the following claim holds (note that H is critical).

Claim 20 *(i) Every set $X \subseteq V$ satisfies $d_G(X) \geq R(X) - 1$. In particular, if X is dangerous, then $d_G(X) = R(X) - 1$ and $d_H(s, X) = 2$.*
(ii) $d_H(s, u) \leq 1$ holds for every $u \in V$. $\square$

Observe that $d_H(s) \geq 3$ since $d_H(s) = 1$ would contradict the criticality of H and $d_H(s) = 2$ would contradict that no pair is admissible. There are the following two possible cases: (Case-1) $d_H(s) = 3$ and (Case-2) $d_H(s) \geq 4$.

(Case-1) Let u_0, u_1, u_2 be three distinct neighbours of s in H (these vertices exist by Claim 20 (ii)). Let H_1 be the graph obtained from H by adding one edge connecting s and u_0; $d_{H_1}(s, u_0) = 2$. Then we claim that $\{(s, u_0), (s, u_1)\}$ is admissible in H_1. Indeed, for any set Y containing u_0 and u_1 which is dangerous in H, we have $d_{H_1}(s, Y) = d_H(s, Y)+1 = p(Y)+2$, since Claim 20(i) implies that $d_G(Y) = R(Y) - 1$ and $d_H(s, Y) = p(Y) + 1$. Therefore, H_1 has a complete admissible splitting at s; the statement (i) is proved.

(Case-2) Let $u_0, u_1, u_2, u_3 \in V$ be four distinct neighbours of s in H. Let Y_i denote a dangerous set with $\{u_0, u_i\} \subseteq Y_i$, $i = 1, 2, 3$. Note that $E_H(s, Y_i) = \{(s, u_0), (s, u_i)\}$ by Claim 20, and hence we have $u_1 \in Y_1 - Y_2 - Y_3$, $u_2 \in$

$Y_2 - Y_3 - Y_1$, and $u_3 \in Y_3 - Y_1 - Y_2$.

Claim 21 *(i) Each $Y_i \in \mathcal{B}^*$ holds and we have $d_G(Y_1 \cap Y_2 \cap Y_3) = 0$ and $d_H(s, Y_1 \cap Y_2 \cap Y_3) = 1$, or (ii) $\{Y_1, Y_2\} \subseteq \mathcal{A}^*$ and $Y_3 \in \mathcal{B}^*$ without loss of generality, $d_G(Y_3 - Y_1 - Y_2) = 0$, and $R(Y_1) = R(Y_2) = R(Y_3)$.*

PROOF. Without loss of generality, there are the following four possible cases:
(I) $Y_1, Y_2, Y_3 \in \mathcal{A}^*$.
(II) $Y_1, Y_2, Y_3 \in \mathcal{B}^*$.
(III) $Y_1 \in \mathcal{A}^*$, $Y_2, Y_3 \in \mathcal{B}^*$.
(IV) $Y_1, Y_2 \in \mathcal{A}^*$, $Y_3 \in \mathcal{B}^*$.

(I) Lemma 3(i) and Claim 20(i) imply that $d_G(Y_1 - Y_2 - Y_3) \geq R(Y_1 - Y_2 - Y_3) - 1 \geq R(Y_1) - 1$, $d_G(Y_2 - Y_3 - Y_1) \geq R(Y_2 - Y_3 - Y_1) - 1 \geq R(Y_2) - 1$, $d_G(Y_3 - Y_1 - Y_2) \geq R(Y_3 - Y_1 - Y_2) - 1 \geq R(Y_3) - 1$, and $d_G(Y_1 \cap Y_2 \cap Y_3) \geq R(Y_1 \cap Y_2 \cap Y_3) - 1 \geq R(Y_1) - 1$. By (5.2) and Claim 20(i), it follows that $R(Y_1) - 1 + R(Y_2) - 1 + R(Y_3) - 1 = d_G(Y_1) + d_G(Y_2) + d_G(Y_3) \geq d_G(Y_1 - Y_2 - Y_3) + d_G(Y_2 - Y_3 - Y_1) + d_G(Y_3 - Y_1 - Y_2) + d_G(Y_1 \cap Y_2 \cap Y_3) \geq 2R(Y_1) + R(Y_2) + R(Y_3) - 4$. Hence $R(Y_1) \leq 1$, contradicting (4.2). The case (I) does not occur.

(II) By $Y_1 \in \mathcal{B}^*$, $V - Y_1 \in \mathcal{A}^*$ holds and Lemma 3(i) implies that $Y_2 - Y_3 - Y_1 \in \mathcal{A}$ and $d_G(Y_2 - Y_3 - Y_1) \geq R(Y_2 - Y_3 - Y_1) - 1 \geq R(V - Y_1) - 1 = R(Y_1) - 1$. Similarly, $d_G(Y_3 - Y_1 - Y_2) \geq R(Y_2) - 1$ and $d_G(Y_1 - Y_2 - Y_3) \geq R(Y_3) - 1$. Again by (5.2), it follows that $\sum_{i=1}^{3}(R(Y_i) - 1) = \sum_{i=1}^{3} d_G(Y_i) \geq d_G(Y_1 - Y_2 - Y_3) + d_G(Y_2 - Y_3 - Y_1) + d_G(Y_3 - Y_1 - Y_2) + d_G(Y_1 \cap Y_2 \cap Y_3) \geq R(Y_1) + R(Y_2) + R(Y_3) - 3$. Thus, every inequality turns out to be an equality, and hence $d_G(Y_1 \cap Y_2 \cap Y_3) = 0$. By $d_H(s, Y_1) = 2$ and $u_0 \in Y_1 \cap Y_2 \cap Y_3$, $d_H(s, Y_1 \cap Y_2 \cap Y_3) = 1$.

(III) Similarly to the above case, we have $d_G(Y_1 - Y_2 - Y_3) \geq R(Y_1) - 1$ and $d_G(Y_1 \cap Y_2 \cap Y_3) \geq R(Y_1) - 1$ by $Y_1 \in \mathcal{A}^*$ and $d_G(Y_3 - Y_1 - Y_2) \geq R(Y_2) - 1$ and $d_G(Y_2 - Y_3 - Y_1) \geq R(Y_3) - 1$ by $Y_2, Y_3 \in \mathcal{B}^*$. Again by (5.2), it follows that $\sum_{i=1}^{3}(R(Y_i) - 1) = \sum_{i=1}^{3} d_G(Y_i) \geq d_G(Y_1 - Y_2 - Y_3) + d_G(Y_2 - Y_3 - Y_1) + d_G(Y_3 - Y_1 - Y_2) + d_G(Y_1 \cap Y_2 \cap Y_3) \geq 2R(Y_1) + R(Y_2) + R(Y_3) - 4$. Hence $R(Y_1) \leq 1$, contradicting (4.2). Thus, the case (III) does not occur.

(IV) Similarly to the above cases, we can observe that $d_G(Y_1 - Y_2 - Y_3) \geq \max\{R(Y_1), R(Y_3)\} - 1$, $d_G(Y_2 - Y_3 - Y_1) \geq \max\{R(Y_2), R(Y_3)\} - 1$, and $d_G(Y_1 \cap Y_2 \cap Y_3) \geq \max\{R(Y_1), R(Y_2)\} - 1$. Again by (5.2), it follows that $\sum_{i=1}^{3}(R(Y_i) - 1) = \sum_{i=1}^{3} d_G(Y_i) \geq d_G(Y_1 - Y_2 - Y_3) + d_G(Y_2 - Y_3 - Y_1) + d_G(Y_3 - Y_1 - Y_2) + d_G(Y_1 \cap Y_2 \cap Y_3) \geq \max\{R(Y_1), R(Y_2)\} + \max\{R(Y_2), R(Y_3)\} + \max\{R(Y_3), R(Y_1)\} - 3$. Thus, every inequality turns out to be an equality, and hence $d_G(Y_3 - Y_1 - Y_2) = 0$ and $R(Y_1) = R(Y_2) = R(Y_3)$. $\square$

Claim 22 *There is at least one dangerous set of $\mathcal{A}^*$ in H.*

PROOF. Assume by contradiction that every dangerous set in H belongs to $\mathcal{B}^*$. By Claim 21, for every $(s,u) \in E_H(s,V)$, there is a component C_u of G with $E_H(s,C_u) = \{(s,u)\}$. Let Y be a dangerous set and u,v be two neighbors of s with $u,v \in V-Y$ (such u,v exist because $d_H(s) \geq 4$ and $d_H(s,Y) = 2$ by Claim 20(i)). Then $Y \cap C_u \neq \emptyset$ holds, since otherwise $Y \in \mathcal{B}^*$ implies that $C_u \in \mathcal{A}$ and $1 = d_H(s,C_u) \geq p(C_u) = R(C_u)$, contradicting (4.2). Hence $d_G(Y \cup C_u) = d_G(Y) - d_G(Y, C_u - Y) \leq d_G(Y) - 1 = R(Y) - 2$ by $d_G(C_u) = 0$ and Claim 20(i). On the other hand, by $v \notin Y \cup C_u$ and $Y \in \mathcal{B}^*$, Lemma 3 implies that $R(Y \cup C_u) \geq R(Y)$. It follows that $d_G(Y \cup C_u) \leq R(Y \cup C_u) - 2$, contradicting Claim 20(i). $\square$

Rechoose u_i and Y_i so that $Y_1 \in \mathcal{A}^*$. Then $d_H(s) = 4$ holds. Indeed, if $d_H(s) \geq 5$, then some three dangerous sets containing u_0 satisfy the cases (I) or (III) in the proof of Claim 21, in both cases of $Y_4 \in \mathcal{A}^*$ and $Y_4 \in \mathcal{B}^*$, where Y_4 denotes a dangerous set containing u_0 and u_4 with some neighbor $u_4 \notin \{u_0,u_1,u_2,u_3\}$ of s. According to Claim 21, let $Y_2 \in \mathcal{A}^*$ and $Y_3 \in \mathcal{B}^*$ without loss of generality. Let $Y_{ij} = V - Y_k$ with $\{i,j,k\} = \{1,2,3\}$ and $i<j$. Then Y_{ij} is also dangerous because Y_{ij} is clearly proper and satisfies $d_H(s,Y_{ij}) = 4 - d_H(s,Y_k) = 2$ and $d_G(Y_{ij}) = d_G(Y_k) = R(Y_k) - 1 = R(Y_{ij}) - 1$. Hence, Y_{12} is a dangerous set of $\mathcal{A}^*$ and Y_{23} and Y_{13} are dangerous sets of $\mathcal{B}^*$.

Lemma 14 implies that $G[Y_i]$ connects u_0 and u_i for $i = 1,2$ and $G[Y_{12}]$ connects u_1 and u_2. Hence, $Y_1 \cup Y_2 \cup Y_{12} = V - (Y_3 - Y_1 - Y_2)$ induces a connected graph. Claim 21 implies that $d_G(Y_3 - Y_1 - Y_2) = 0$. It follows that $Y_1 \cup Y_2 \cup Y_{12}$ is a component of G containing $\{u_0,u_1,u_2\}$, and that $Y_1 \cup Y_2 \cup Y_{12}$ and the component of G containing u_3 correspond to C_1 and C_2 of the statement of this theorem, respectively.

We next show the statement (ii)(b); every set $X \subseteq C_1$ satisfies $d_H(X) \geq 2$. Let $C_1 = Y_1 \cup Y_2 \cup Y_{12}$. We first claim that $C_1 = Y_1 \cup Y_2$.

Claim 23 $C_1 = Y_1 \cup Y_2$.

PROOF. Assume by contradiction that $Z = Y_{12} - Y_1 - Y_2 \neq \emptyset$. Now $0 = d_H(s,Z) \geq p(Z) \geq R(Z) - d_G(Z)$. Thus, Lemma 3(i) and $Y_{12} \in \mathcal{A}^*$ imply that $d_G(Z) \geq R(Z) \geq R(Y_{12}) = R(Y_3)$. Since $Y_1 \cup Y_2 \cup Y_{12}$ is a component of G, $d_G(Z) = d_G(Y_1 \cup Y_2)$. By (2.1), it follows that $R(Y_1) - 1 + R(Y_2) - 1 = d_G(Y_1) + d_G(Y_2) \geq d_G(Y_1 \cap Y_2) + d_G(Y_1 \cup Y_2) \geq R(Y_1 \cap Y_2) - 1 + R(Y_3)$. Now Lemma 3(i) indicates that $R(Y_1 \cap Y_2) \geq R(Y_1)$. Hence, $R(Y_2) - 1 \geq R(Y_3)$ holds, contradicting that $R(Y_1) = R(Y_2) = R(Y_3)$ (by Claim 21(ii)). $\square$

For proving (ii)(b), assume by contradiction that there is a set $X \subseteq C_1$ with $d_H(X) = 1$. Clearly, $d_H(s,X) = 0$ and $d_G(X) = 1$ since C_1 induces a connected graph. Moreover, X is not proper since otherwise $0 = d_H(s, X) \geq p(X) \geq R(X) - d_G(X) = R(X) - 1$, contradicting (4.2). Hence, X is not contained in any of Y_1 and Y_2; $X \cap (Y_1 - Y_2) \neq \emptyset \neq X \cap (Y_2 - Y_1)$ by Claim 23. Now by applying Lemma 15 to Y_1 and Y_2, both of $Y_1 - Y_2$ and $Y_2 - Y_1$ are tight sets of $\mathcal{A}^*$ (note that $d_H(s, Y_1 - Y_2) = p(Y_1 - Y_2) > 0, d_H(s, Y_2 - Y_1) = p(Y_2 - Y_1) > 0$). Lemma 14 implies that $G[Y_1 - Y_2]$ and $G[Y_2 - Y_1]$ are both connected. Then it is not difficult to see that $d_G(X) = 1$ would contradict the connectedness of $G[Y_1 - Y_2]$ or $G[Y_2 - Y_1]$.

We finally show (ii)(c); every set $\emptyset \neq X \subseteq C_1$ with $d_H(X) = 2$ belongs to $\mathcal{A}$. Assume by contradiction that $X \subseteq C_1$ does not belong to $\mathcal{A}$. By Lemma 3(i), X cannot be included in any of Y_1 and Y_2. Hence, we can assume that $X \cap (Y_1 - Y_2) \neq \emptyset \neq X \cap (Y_2 - Y_1)$. By $d_H(X) = 2$ and $d_H(C_1) \geq 3$, we have $C_1 - X \neq \emptyset$. Since $G[C_1]$ is connected, it follows that $d_G(X) \geq 1$, from which $d_H(s, X) \leq 1$. This implies that X and Y_1 cross each other in H. From (2.2), $X - Y_1 \subseteq Y_2$, and Lemma 3, we have $(R(Y_1) - 1 + 2) + 2 = d_H(Y_1) + d_H(X) = d_H(Y_1 - X) + d_H(X - Y_1) + 2d_H(X \cap Y_1, (V \cup s) - X - Y_1) \geq R(Y_1 - X) + R(X - Y_1) + 2d_H(X \cap Y_1, (V \cup s) - X - Y_1) \geq R(Y_1) + R(Y_2) + 2d_H(X \cap Y_1, (V \cup s) - X - Y_1)$ (note that $d_H(X') = d_H(s, X') + d_G(X') \geq R(X')$ holds for every $X' \subseteq V$ by $d_H(s, X') \geq p(X')$). Now observe that $R(Y_2) \geq 2$ by (4.2) and that $d_H(X - Y_1) \geq R(Y_2) = R(Y_1)$ by Claim 21. It follows that $d_H(X \cap Y_1, V \cup \{s\} - X - Y_1) = 0$ and $d_H(Y_1 - X) \leq 3$. Hence we have $Y_1 - Y_2 - X \neq \emptyset \neq (Y_1 \cap Y_2) - X$ by $d_H(s, Y_1 - Y_2) > 0$ and $d_H(s, Y_1 \cap Y_2) > 0$. By these and $X \cap (Y_1 - Y_2) \neq \emptyset$, $Y_1 - X$ and $Y_1 - Y_2$ cross each other in H. From (2.2) and $d_H(Y_1 - X) \leq 3$, it follows that $d_H(Y_1 - Y_2) + 3 \geq d_H(Y_1 - Y_2) + d_H(Y_1 - X) \geq d_H((Y_1 - Y_2) \cap X) + d_H(Y_1 \cap Y_2 - X) + 2d_H(s, Y_1 - Y_2 - X) \geq R((Y_1 - Y_2) \cap X) + R(Y_1 \cap Y_2 - X) + 2 \geq R(Y_1 - Y_2) + R(Y_2) + 2$ (note that $d_H(s, Y_1 - Y_2 - X) > 0$ by $d_H(s, Y_1 - Y_2) > 0$ and $d_H(s, X \cap Y_1) = 0$ and that $R((Y_1 - Y_2) \cap X) \geq R(Y_1 - Y_2)$ and $R(Y_1 \cap Y_2 - X) \geq R(Y_2)$ by $Y_1 - Y_2, Y_2 \in \mathcal{A}^*$). It follows from $R(Y_2) \geq 2$ that $d_H(Y_1 - Y_2) \geq R(Y_1 - Y_2) + 1$. Now as observed in the above, $d_H(s, Y_1 - Y_2) = p(Y_1 - Y_2) > 0$ and hence $d_H(Y_1 - Y_2) = R(Y_1 - Y_2)$, a contradiction. $\Box$

PROOF of Lemma 17: Let $E_H(s, C_1) = \{(s, u_0), (s, u_1), (s, u_2)\}$ and $E_H(s, C_2) = \{(s, u_3)\}$. From the above proof of Theorem 16, observe that there is a dangerous set $Y_i \subseteq C_1$ with $\{u_0, u_i\} \subseteq Y_i$ for $i = 1, 2$. Hence, also in the graph H_1 obtained from H by hooking up the edge e, Y_1 and Y_2 remain dangerous. Assume by contradiction that $\{(s, x), (s, u_0)\}$ is not admissible for any $x \in \{u, v, u_3\}$ in H_1; denote by Y_x a dangerous set containing x and u_0. Lemma 14 implies that each $Y_x \in \mathcal{B}^*$ holds.

Claim 24 *In H, (a) $v \notin Y_3$ and $u_3 \notin Y_v$ or (b) $u \notin Y_3$ and $u_3 \notin Y_u$.*

PROOF. Note that H_1 has no dangerous set containing both of u and v, since H is a p-extension of G. Hence $\{u,v\} - Y_3 \neq \emptyset$, $v \notin Y_u$, and $u \notin Y_v$. Without loss of generality, assume that $v \notin Y_3$. If $u_3 \notin Y_v$, then we are done.

Assume that $u_3 \in Y_v$. If Y_3 and Y_v cross each other in H_1, then Lemma 15 implies that $d_{H_1}(s, Y_3 \cap Y_v) \leq 1$, contradicting that $E_{H_1}(s, Y_3 \cap Y_v) = \{(s,u_0),(s,u_3)\}$. Hence, $Y_3 \subseteq Y_v$. Now, since Y_u and Y_v cross each other in H_1, again by Lemma 15, we can observe that $d_{H_1}(s, Y_u \cap Y_v) = 1$ and hence $u_3 \notin Y_u \cap Y_v$. Hence, $u_3 \notin Y_u$. Moreover, $u \notin Y_3$ holds by $Y_3 \subseteq Y_v$. $\Box$

Without loss of generality, assume that $v \notin Y_3$ and $u_3 \notin Y_v$. Now note that Y_3 is dangerous also in H, since even if $u \in Y_3$, then $d_H(s, Y_3) = d_{H_1}(s, Y_3) - 1 \leq (\max\{0, R(Y_3) - d_{G_1}(Y_3)\} + 1) - 1 \leq \max\{0, R(Y_3) - d_G(Y_3)\} + 1 = p(Y_3) + 1$, where $G_1 = G - (u,v)$. Hence, Claim 20(i) implies that $E_H(s, Y_3) = \{(s,u_0),(s,u_3)\}$.

Claim 25 *We have* $Y_v \cap \{u_1, u_2\} = \emptyset$, $d_G(Y_v) \leq R(Y_v)$, *and* $d_G(Y_v - Y_1 - Y_3) \geq R(Y_v - Y_1 - Y_3)$.

PROOF. Assume by contradiction that Y_v contains u_1. Then Y_v is dangerous also in H, since $d_H(s, Y_v) = d_{H_1}(s, Y_v) - 1 \leq (\max\{0, R(Y_v) - d_{G_1}(Y_v)\} + 1) - 1 \leq \max\{0, R(Y_v) - d_G(Y_v)\} + 1$. Lemma 13 implies that $\{u_2, u_3\} \cap Y_v = \emptyset$. Then three dangerous sets Y_2, Y_3, and Y_v satisfy the case (III) in the proof of Claim 21, a contradiction. Similarly, $u_2 \notin Y_v$ can be seen. It follows that $2 = d_{H_1}(s, Y_v) \leq \max\{0, R(Y_v) - d_{G_1}(Y_v)\} + 1 = R(Y_v) - (d_G(Y_v) - 1) + 1$; $d_G(Y_v) \leq R(Y_v)$. Moreover, $\{u_0, u_3\} \subseteq Y_3$ indicates that $0 = d_H(s, Y_v - Y_1 - Y_3) \geq p(Y_v - Y_1 - Y_3) \geq R(Y_v - Y_1 - Y_3) - d_G(Y_v - Y_1 - Y_3)$. $\Box$

Note that $u_1 \in Y_1 - Y_3 - Y_v$, $u_3 \in Y_3 - Y_v - Y_1$, $v \in Y_v - Y_1 - Y_3$, and $u_0 \in Y_1 \cap Y_3 \cap Y_v$. By Claim 20(i), $Y_1 \in \mathcal{A}^*$, and $Y_v \in \mathcal{B}^*$, we have $d_G(Y_1 - Y_3 - Y_v) \geq R(Y_1 - Y_3 - Y_v) \geq R(Y_1) - 1$, $d_G(Y_1 \cap Y_3 \cap Y_v) \geq R(Y_1 \cap Y_3 \cap Y_v) \geq R(Y_1) - 1$, and $d_G(Y_3 - Y_v - Y_1) \geq R(Y_3 - Y_v - Y_1) - 1 \geq R(V - Y_v) - 1 = R(Y_v) - 1$. Claim 25 and $Y_3 \in \mathcal{B}^*$ imply that $d_G(Y_v - Y_1 - Y_3) \geq R(Y_v - Y_1 - Y_3) \geq R(V - Y_3) = R(Y_3)$ and $d_G(Y_v) \leq R(Y_v)$. From (5.2), it follows that $R(Y_1) - 1 + R(Y_3) - 1 + R(Y_v) \geq d_G(Y_1) + d_G(Y_3) + d_G(Y_v) \geq d_G(Y_1 - Y_3 - Y_v) + d_G(Y_3 - Y_v - Y_1) + d_G(Y_v - Y_1 - Y_3) + d_G(Y_1 \cap Y_3 \cap Y_v) \geq 2R(Y_1) + R(Y_3) + R(Y_v) - 3$. Hence $R(Y_1) \leq 1$ holds, contradicting (4.2). $\Box$

Let $r : 2^V \to Z^+$ be a monotone function on V and H be a p-extension of $G = (V, E)$ with property (P*). In this subsection, we prove that G has property (P). By (P4*), for each $(s, v) \in E_H(s, V_2 - C^*)$ there is a dangerous set Y with $\{u^*, v\} \subseteq Y$, which will play a role as a cut Y_X in Definition 5 in the subsequent arguments. Note that any proper set X with $X \cap C^* = \emptyset$ belongs to $\mathcal{A}^*$, since if $X \in \mathcal{B}^*$, then $C^* \in \mathcal{A}$ and $1 = d_H(s, C^*) \geq p(C^*) = R(C^*) \geq 2$ by (4.2) and Lemma 3, a contradiction. Hence, each $C \in \mathcal{C}_1$ satisfies $C \in \mathcal{A}^*$. We first show properties of such dangerous sets in Lemma 26, and show by Lemma 27 that G has property (P).

Lemma 26 *Let H be a p-extension of $G = (V, E)$ with property (P^*), and $(s, v) \in E_H(s, V_2 - C^*)$ and Y_v be a dangerous set with $\{u^*, v\} \subseteq Y_v$ (such Y_v exists by the property ($P4^*$)). Then*
(i) $d_H(s, V_2 - Y_v) \geq 1$ holds.
(ii) For some $(s, w) \in E_H(s, V_2 - C^*) - \{(s, v)\}$, Y_v and Y_w cross each other in H, where Y_w denotes a dangerous cut with $\{u^*, w\} \subseteq Y_w$ in H. Moreover, $v \in Y_v - Y_w$ and $Y_v \subset V - V_1$ hold and $Y_v - Y_w$ is a tight set of $\mathcal{A}^*$ with $Y_v - Y_w \subseteq V_2$.
(iii) $Y_v \cup C^*$ is a dangerous set of $\mathcal{B}^*$.

PROOF. Note that $Y_v \in \mathcal{B}^*$ holds by Lemma 14 since Y_v does not induce a connected graph. Also note that $d_H(s, V_2) \geq 4$ holds since $d_H(s, V_2)$ is even by the property (P1*) and the property that $d_H(s, V_1)$ is even, and $d_H(s, V_2 - C^*) \neq 1$ holds by the property (P2*).

(i) Assume by contradiction that $d_H(s, V_2 - Y_v) = 0$ holds. Let Y'_v be a dangerous set with $Y_v \subseteq Y'_v$ such that no $Y'' \supset Y'_v$ is dangerous. Note that $Y'_v \in \mathcal{B}^*$ and $d_H(s, V_2 - Y'_v) = 0$ also hold. We have $p(Y'_v) \geq d_H(s, Y'_v) - 1 \geq d_H(s, V_2) - 1 \geq 3$ holds, from which $R(Y'_v) \geq 3$. Lemma 13 and $d_H(s, Y'_v) \geq 4$ imply that $d_H(s, V - Y'_v) \geq 3$. It follows that there exist at least two sets $C_1, C_2 \in \mathcal{C}_1$ with $d_H(s, C_i - Y'_v) > 0$ for $i = 1, 2$. We have $C_1 \cap Y'_v \neq \emptyset$, since otherwise $C_1 \subseteq V - Y'_v \in \mathcal{A}^*$ and Lemma 3(i) imply that $2 = R(C_1) \geq R(V - Y'_v) = R(Y'_v) \geq 3$, a contradiction. Now by $C_1 \in \mathcal{A}^*$, every $\emptyset \neq X \subseteq C_1$ satisfies $d_H(X) = d_H(s, X) + d_G(X) \geq R(X) \geq R(C_1) \geq 2$. This indicates that $d_H(Y'_v) = d_H(Y'_v \cap C_1) + d_H(Y'_v - C_1) \geq 2 + d_H(Y'_v - C_1) = d_H(Y'_v \cup C_1)$. It follows from Lemma 3(ii) and $d_H(s, C_2 - Y'_v) > 0$ that $Y'_v \cup C_1 \in \mathcal{B}$ and $R(Y'_v) \leq R(Y'_v \cup C_1)$. This indicates that $Y'_v \cup C_1$ is also dangerous by $d_H(Y'_v \cup C_1) \leq d_H(Y'_v) \leq R(Y'_v) + 1 \leq R(Y'_v \cup C_1) + 1$. This contradicts the maximality of Y'_v.

(ii) Let Y'_v be a dangerous set with $\{u^*, v\} \subseteq Y'_v$ and $Y_v \subseteq Y'_v$ such that no $Y'' \supset Y'_v$ is dangerous in H. By (i), $d_H(s, V_2 - Y'_v) > 0$ holds. Let $w \in V_2 - Y'_v$ be a vertex with $d_H(s, w) > 0$ and Y_w be a dangerous set with $\{u^*, w\} \subseteq Y_w$. Then Y'_v and Y_w cross each other in H since we have $u^* \in Y'_v \cap Y_w$, $w \in Y_w - Y'_v$, and $Y'_v - Y_w \neq \emptyset$ by the maximality of Y'_v. Note that $Y_w \in \mathcal{B}^*$. Lemma 15 implies that $d_H(s, Y'_v \cap Y_w) = 1$, and it follows from $u^* \in Y'_v \cap Y_w$ that $v \in Y_v - Y_w$. Hence, Y_v and Y_w also cross each other in H.

Again by Lemma 15, we have $p(Y_v - Y_w) = d_H(s, Y_v - Y_w) > 0$, and hence $Y_v - Y_w$ is a tight set of $\mathcal{A}^*$ and Lemma 14 implies that $G[Y_v - Y_w]$ is connected; $Y_v - Y_w \subseteq V_2$. Similarly, $G[Y_w - Y_v]$ is connected. Finally, we prove that $Y_v \cap Y_w \cap V_1 = \emptyset$ in order to show that $Y_v \subset V - V_1$ (note that $V - V_1 - Y_v \neq \emptyset$ holds by $d_H(s, V_2 - Y_v) > 0$). Assume by contradiction that $Y_v \cap Y_w \cap C \neq \emptyset$ holds for some $C \in \mathcal{C}_1$. From $d_H(s, V_2 - Y_v) > 0$, $d_H(s, V_2 - Y_w) > 0$, and the similar arguments in the above (i), it is not difficult to see that $Y_v \cup C$ and $Y_w \cup C$ are both dangerous sets of $\mathcal{B}^*$ and cross each other in H. Then $d_H(s, (Y_v \cap Y_w) \cup C) \geq 3$ would contradict Lemma 15.

(iii) Let $Y''_v = Y_v \cup C^*$. By (i) and $u^* \in Y_v$, we have $d_H(s, V - Y''_v) \geq 1$. Hence $V - Y''_v \neq \emptyset$ and Lemma 3(ii) imply that Y''_v belongs to $\mathcal{B}$ and $R(Y''_v) \geq R(Y_v)$. By $E_H(s, C^*) \subseteq E_H(s, Y_v)$ and $d_H(s, C^*) = d_H(C^*)$, we have $d_H(Y''_v) \leq d_H(Y_v)$. Hence $d_H(Y''_v) \leq d_H(Y_v) \leq R(Y_v) + 1 \leq R(Y''_v) + 1$. Moreover, $Y''_v \notin \mathcal{A}^*$ by Lemma 14 and it follows that $Y''_v \in \mathcal{B}^*$, which proves the lemma. $\square$

Lemma 27 *If $H = (V \cup \{s\}, E \cup F)$ is a p-extension of $G = (V, E)$ with property (P^*), then G has property (P).*

PROOF. Lemma 26 implies that for each $v \in V[F] - V_1 - \{s, u^*\}$, there are two proper sets $X_v \subset V - V_1$ and $Y_v \subset V - V_1$ with $v \in X_v \subseteq Y_v$ satisfying the following (a) and (b).

(a) X_v is a tight set of $\mathcal{A}^*$, and no set $\emptyset \neq X' \subset X_v$ with $v \in X'$ satisfies this property.

(b) Y_v satisfies $u^* \in Y_v$ and $C^* \subseteq Y_v \subset V - V_1$ (by (ii)(iii) in Lemma 26) and is a dangerous set of $\mathcal{B}^*$.

Let X_{u^*} be a tight set with $u^* \in X_{u^*} \subseteq C^*$ such that no set $X' \subset X_{u^*}$ satisfies this property (such X_{u^*} exists from the property (P3*)). Let $\mathcal{X}$ be the family of all sets X_v, $v \in V[F] - \{s\} - V_1$ such that $\cup_{X \in \mathcal{X}} X \supseteq V[F] - \{s\} - V_1$ and $X_v \in \mathcal{X}$ does not satisfy $X_v \subset X$ for any $X \in \mathcal{X}$, and $\mathcal{Y}$ be the family of the corresponding Y_v. We will show that $\alpha(G, r)$ is even, implying (P1), and the family $\mathcal{X} \cup \mathcal{C}_1$ is a subpartition of V satisfying $\sum_{X \in \mathcal{X} \cup \mathcal{C}_1} p(X) = \alpha(G, r)$ and (P2) and (P3), which proves the lemma.

We claim that

$$\mathcal{X} \text{ is a subpartition of } V - V_1. \tag{5.3}$$

Assume by contradiction that there are two sets $X_u, X_v \in \mathcal{X}$ which cross each other in H. By $X_u, X_v \in \mathcal{A}^*$ and Corollary 12, we have $0 \geq d_H(s, X_u) - p(X_u) + d_H(s, X_v) - p(X_v) \geq d_H(s, X_u - X_v) - p(X_u - X_v) + d_H(s, X_v - X_u) - p(X_v - X_u) + 2d_G(X_u \cap X_v, V - X_u - X_v) + 2d_H(s, X_u \cap X_v) \geq 0$. It follows that $d_H(s, X_u - X_v) = p(X_u - X_v)$, $d_H(s, X_v - X_u) = p(X_v - X_u)$, and $d_H(X_u \cap X_v, (V \cup \{s\}) - X_u - X_v) = 0$. Hence $u \in X_u - X_v$ and $p(X_u - X_v) = d_H(s, X_u - X_v) > 0$. Now $X_u - X_v \in \mathcal{A}$ holds by Lemma 3(i). As mentioned in the first paragraph of this subsection, every proper set disjoint with C^* belongs to $\mathcal{A}^*$, and hence it follows that $X_u - X_v \in \mathcal{A}^*$. Thus, $X_u - X_v$ is also tight of $\mathcal{A}^*$, contradicting the minimality of X_u.

Now each $C \in \mathcal{C}_1$ is tight since $2 = d_H(s, C) \geq p(C) = R(C) \geq 2$ holds by (4.2). Hence, by (5.3), $\mathcal{X} \cup \mathcal{C}_1$ is a subpartition of V and a family of tight sets such that $V[F] - \{s\} \subseteq \cup_{X \in \mathcal{X} \cup \mathcal{C}_1} X$. Since $|F| = \alpha(G, r)$ holds by Theorem 8, $\sum_{X \in \mathcal{X} \cup \mathcal{C}_1} p(X) = d_H(s, V) = |F| = \alpha(G, r)$. Since $|F|$ is even, $\alpha(G, r)$ is even. Moreover, $\mathcal{X} \cup \mathcal{C}_1$ is a subpartition of V satisfying (P2) by taking $X^* = X_{u^*}$. Now for every dangerous set $Y \in \mathcal{Y}$ which does not cross with any $X \in \mathcal{X}$ in H, we have $\sum_{X' \in \mathcal{X}, X' \subseteq Y} p(X') = \sum_{X' \in \mathcal{X}, X' \subseteq Y} d_H(s, X') = d_H(s, Y) \leq p(Y) + 1$. Moreover, note that each $Y \in \mathcal{Y}$ satisfies $V - V_1 - Y \neq \emptyset$ by $V_2 - Y \neq \emptyset$. Therefore, by regarding $\mathcal{C}_1$ as $\mathcal{X}_1$ in Definition 5, in order to show that $\mathcal{X} \cup \mathcal{C}_1$ satisfies (P3), it suffices to prove that for any $X_u \in \mathcal{X}$ with $u \neq u^*$, there is a set $Y_w \in \mathcal{Y}$ with $X_u \subseteq Y_w$ such that for any set $X \in \mathcal{X}$, Y_w and X do not cross each other in H (note that each $Y \in \mathcal{Y}$ satisfies $C \cap Y = \emptyset$ for any $C \in \mathcal{C}_1$ by $Y \subset V - V_1$). For this, we show that

$$\begin{aligned}&\text{if there is a set } Y_u \in \mathcal{Y} \text{ which crosses with}\\ &\text{some } X_v \in \mathcal{X} \text{ in } H,\ v \neq u^* \text{ and } Y_u \subseteq Y_v.\end{aligned} \tag{5.4}$$

Since each $Y \in \mathcal{Y}$ satisfies $X_{u^*} \subseteq C^* \subseteq Y$, $v \neq u^*$ holds. Assume by contradiction that $Y_u - Y_v \neq \emptyset$. By $X_v - Y_u \neq \emptyset \neq X_v \cap Y_u$, Y_u and Y_v cross each other in H. From Lemma 15, it follows that $Y_v - Y_u \in \mathcal{A}^*$, $d_H(s, Y_v - Y_u) = p(Y_v - Y_u)$, and $d_H(s, u^*) = d_H(Y_u \cap Y_v, V \cup \{s\} - Y_u - Y_v) = 1$. Hence we have $v \in X_v - Y_u$, from which $X_v \cap (Y_v - Y_u) \neq \emptyset$ holds and $Y_v - Y_u$ is tight. Note that $X_v - (Y_v - Y_u) \neq \emptyset$ holds since X_v and Y_u cross each other in H. Moreover, $(Y_v - Y_u) - X_v \neq \emptyset$ holds since if $Y_v - Y_u \subseteq X_v$ holds, then the tight set $Y_v - Y_u$ contradicts the minimality of X_v. This means that X_v and $Y_v - Y_u$ cross each other in H. Now $d_H(X_v \cap (Y_v - Y_u), V \cup \{s\} - X_v - (Y_v - Y_u)) > 0$ holds by $v \in X_v - Y_u$. By applying Lemma 15 to X_v and $Y_v - Y_u$, we have $d_H(s, X_v) = p(X_v) + 1$, contradicting that X_v is tight (note that X_v and $Y_v - Y_u$

are both tight sets of $\mathcal{A}^*$). Hence (5.4) holds. $\Box$

5.3 Step 2

According to the proof of Theorems 11, Step 2 of algorithm M-AUG is described as follows.

Step 2: (1) Check whether H has property (P*).

(2) The case where H has property (P*): Repeat admissible splittings as possible. In the resulting graph, after adding one edge between C_1 and C_2 according to the case of $d_{H_1}(s) = 4$ in the proof of Theorem 11(i), find a complete admissible splitting (note that $d_H(s)$ is even from the property (P1*)). Halt after outputting the set E^* of all edges added to G as an optimal solution, where $|E^*| = \lceil \alpha(G,r)/2 \rceil + 1$.

(3) The case where H does not have property (P*):

(3-1) If $d_H(s)$ is odd, then according to the proof of Theorem 11(i), find a complete admissible splitting by adding one edge incident to s and halt after outputting the set E^* of all edges added to G as an optimal solution, where $|E^*| = \lceil \alpha(G,r)/2 \rceil$.

(3-2) Otherwise one of the cases (I)–(IV) in the proof of Theorem 11(ii) hold. In the case of (IV), we replace one edge incident to s so that the resulting graph belongs to the case (I), according to Claim 18. In the case of (III), first split the edges (s,u) and (s,v) in H.

After that, in all cases repeat admissible splittings as possible. If the resulting graph H_1 still has an edge incident to s, then according to the statements immediately after (5.1), find a complete admissible splitting while hooking up some edges in $H_1[V - C_1]$ and resplitting (note that the proof of Theorem 11(ii) implies that $H_1[V - C_1]$ has a split edge and that hooking up and resplitting operations can find a complete admissible splitting). Halt after outputting the set E^* of all edges added to G, where $|E^*| = \lceil \alpha(G,r)/2 \rceil$. $\Box$

Finally we show that algorithm M-AUG can be implemented to run in $O(n^4(m + n \log n + q))$ time. In the following arguments about the time complexity of the algorithm, we regard k multiple edges in a graph as a single edge with capacity k; an addition/deletion of ℓ multiple edges means the increase/decrease of the capacity on the corresponding single edge by ℓ.

Note that H satisfies (4.1) if and only if $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{ d_H(X) - R(X) \} \geq 0$. We can prove the following lemma by using the family of all *extreme sets*

[10,11], where in G, a set $\emptyset \neq X \subset V$ is called *extreme* if any $\emptyset \neq X' \subset X$ satisfies $d_G(X') > d_G(X)$.

Lemma 28 *It can be checked in $O(n^2(m + n \log n + q))$ time whether a given H satisfies (4.1) or not. Moreover, if H violates (4.1), then $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}}\{ d_H(X) - R(X)\}$ can be obtained in the same time.*

PROOF. Let $\mathcal{Z}(H)$ denote the family of all extreme sets in H. It is known that $\mathcal{Z}(H)$ is laminar and hence $|\mathcal{Z}(H)| = O(n(H))$ holds. It was shown in [10,11] that $\mathcal{Z}(H)$ can be found in $O(m(H)n(H) + n(H)^2 \log n(H))$ time. Note that $m(H) \leq m(G) + n(G)$ and $n(H) = n(G) + 1$.

Let $H(v)$ denote the graph obtained from H by adding $\max\{r(u) \mid u \in V\}$ multiple edges to $E_H(s, v)$ for a vertex $v \in V$, and $\mathcal{Z}^s(H(v))$ denote the family of extreme sets $X \in \mathcal{Z}(H(v))$ in $H(v)$ with $s \in X$. For a given H, let $g(H) = \min\{0, \min\{d_H(X) - R(X) \mid X \in \mathcal{Z}(H), s \notin X\}, \min\{d_H(X) - R(X - s) \mid X \in \mathcal{Z}^s(H(v)), v \in V\}\}$. Note that given $\mathcal{Z}(H)$ and $\mathcal{Z}^s(H(v))$, $v \in V$, we obtain $g(H)$ by computing $d_H(X) - R(X)$ or $d_H(X) - R(X - s)$ $O(n^2)$ times; $O(n^2)$ times computation of r suffices. For proving this lemma, we will show that H satisfies (4.1) if and only if $g(H) = 0$, and that if H violates (4.1), then $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}}\{d_H(X) - R(X)\} = g(H) < 0$ (note that $g(H) \leq 0$ holds from the definition).

For this, we first show by Claims 29 and 30 that $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}}\{d_H(X) - R(X)\} \geq g(H)$.

Claim 29 *Every proper set $X \subset V$ of $\mathcal{A}$ satisfies $d_H(X) - r(X) \geq d_H(X') - r(X')$ for some $X' \in \mathcal{Z}(H)$ with $X' \subseteq X$.*

PROOF. From the definition of extreme sets, there is an extreme set $Y \in \mathcal{Z}(H)$ with $Y \subseteq X$ and $d_H(Y) \leq d_H(X)$. By the monotonicity of r, $r(Y) \geq r(X)$. Hence, $d_H(X) - r(X) \geq d_H(Y) - r(Y)$. $\square$

Claim 30 *Assume that $\min\{d_H(X) - r(V - X) \mid X \in \mathcal{B}\} < 0$. Then, every proper set $X \subset V$ of $\mathcal{B}$ satisfies (a) $d_H(X) - r(V - X) \geq d_H(X') - r(X')$ for some $X' \in \mathcal{Z}(H)$ with $X' \subseteq V - X$ or (b) $d_H(X) - r(V - X) \geq d_H(X') - r(X' - s)$ for some $X' \in \mathcal{Z}^s(H(v))$ and $v \in V$.*

PROOF. Let $X \subset V$ be a proper set of $\mathcal{B}$ such that $d_H(X) - r(V - X) = \min\{d_H(X') - r(V - X') \mid X' \in \mathcal{B}\}$ and any set $V \neq X'' \supset X$ satisfies $d_H(X'') - r(V - X'') > d_H(X) - r(V - X)$ (note that each $V \neq X'' \supset X$ belongs to $\mathcal{B}$). Let $\overline{X} = V - X$. Note that $\overline{X} \in \mathcal{A}$. By $d_H(X) = d_H(\overline{X} \cup \{s\})$, we have

$d_H(X)-r(V-X) = d_H(\overline{X}\cup\{s\})-r(\overline{X}) = \min\{d_H(X'\cup\{s\})-r(X') \mid X' \in \mathcal{A}\}$, and any set $\emptyset \neq X'' \subset \overline{X}$ satisfies $d_H(X''\cup\{s\})-r(X'') > d_H(\overline{X}\cup\{s\})-r(\overline{X})$.

First we consider the case where some $\emptyset \neq X' \subset \overline{X}$ satisfies $d_H(X') \leq d_H(\overline{X} \cup \{s\})$. Since $\overline{X} \in \mathcal{A}$, we have $X' \in \mathcal{A}$ and hence $r(X') \geq r(\overline{X})$ by the monotonicity of r. It follows that $d_H(\overline{X} \cup \{s\}) - r(\overline{X}) \geq d_H(X') - r(X')$. Claim 29 implies that $d_H(X') - r(X') \geq d_H(Y) - r(Y)$ for some $Y \in \mathcal{Z}(H)$ with $Y \subseteq X'$.

Next consider the case where some $\emptyset \neq X' \subset \overline{X}$ satisfies $d_H(X' \cup \{s\}) \leq d_H(\overline{X}\cup\{s\})$. Similarly to the above, $r(X') \geq r(\overline{X})$. Hence, $d_H(\overline{X}\cup\{s\})-r(\overline{X}) \geq d_H(X' \cup \{s\}) - r(X')$, contradicting the minimality of $\overline{X}$.

Finally, we consider the case where every $\emptyset \neq X' \subset \overline{X}$ satisfies $d_H(X') > d_H(\overline{X} \cup \{s\})$ and $d_H(X' \cup \{s\}) > d_H(\overline{X} \cup \{s\})$. Let $u \in \overline{X}$ (note that $\overline{X} \neq \emptyset$). We can observe that $\overline{X}\cup\{s\} \in \mathcal{Z}^s(H(u))$ or $d_H(X) = d_H(\overline{X}\cup\{s\}) \geq r(V-X)$. Indeed, if $d_{H(u)}(s) > d_H(\overline{X} \cup \{s\}) = d_{H(u)}(\overline{X} \cup \{s\})$, then every set $X' \subset \overline{X}\cup\{s\}$ satisfies $d_{H(u)}(X') > d_{H(u)}(\overline{X}\cup\{s\})$, and otherwise then $d_H(\overline{X}\cup\{s\}) \geq \max\{r(w) \mid w \in V\} \geq r(V - X)$ (note that from the monotonicity of r, $\max\{r(X) \mid X \subseteq V\} = \max\{r(v) \mid v \in V\}$). $\square$

Clearly, if $g(H) = 0$, H satisfies (4.1). Consider the case of $g(H) < 0$. Here, we claim that $\min_{\emptyset\neq X\subset V, X\in\mathcal{A}\cup\mathcal{B}}\{d_H(X)-R(X)\} = g(H)$. As observed in Claims 29 and 30, we have $\min_{\emptyset\neq X\subset V, X\in\mathcal{A}\cup\mathcal{B}}\{d_H(X) - R(X)\} \geq g(H)$. Notice that for each set $X \subseteq V$ with $X \notin \mathcal{A}\cup\mathcal{B}$, we have $R(X) = R(V-X) = 0$ and $d_H(X) - R(X) = d_H(V\cup\{s\}-X)-R(V-X) \geq 0$. Hence, if $d_H(X)-R(X) = g(H) < 0$ holds for some $X \in \mathcal{Z}(H)$ with $s \notin X$, then $X \in \mathcal{A}\cup\mathcal{B}$. If $d_H(X)-R(X-s) = g(H) < 0$ for some $X \in \mathcal{Z}^s(H)(v)$ with $v \in V$, then $d_H(X) - R(X - s) = d_H(V - X) - R(V - X) < 0$ holds by $s \in X$ and hence $V - X$ belongs to $\mathcal{A} \cup \mathcal{B}$. Thus, we have $\min_{\emptyset\neq X\subset V, X\in\mathcal{A}\cup\mathcal{B}}\{d_H(X) - R(X)\} \leq g(H)$. Therefore, we have $\min_{\emptyset\neq X\subset V, X\in\mathcal{A}\cup\mathcal{B}}\{d_H(X) - R(X)\} = g(H)$. Thus, we can observe that if $g(H) < 0$, then H violates (4.1) and $\min_{\emptyset\neq X\subset V, X\in\mathcal{A}\cup\mathcal{B}}\{d_H(X) - R(X)\}$ is also obtained. $\square$

It suffices to show that the following (A) (resp. (B)) can be done by computing $\min_{\emptyset\neq X\subset V, X\in\mathcal{A}\cup\mathcal{B}}\{d_H(X) - R(X)\}$ at most n times (resp. once):

(A) The computation of a critical p-extension of a given G.

(B) The computation of how many pairs of $\{(s, u), (s, v)\}$ are admissible for a given pair $\{u, v\} \subseteq V$ of two vertices in a p-extension H of G.

Indeed, Step 2(1) can be done by the computation (B) for $O(n)$ pairs, a sequence of greedy admissible splittings in Step 2(2)(3) can be done by the computation (B) for $O(n^2)$ pairs, and the hooking up operations in Step 2(3-2) are executed at most n times (since the statements immediately after (5.1) indicates that one hooking up decreases $|V - C_1|$ at least by one).

(A) A critical p-extension of G can be obtained as follows. First we add $\max\{r(v) \mid v \in V\}$ edges between s and each $v \in V$. From the monotonicity of r, $\max\{r(X) \mid X \subseteq V\} = \max\{r(v) \mid v \in V\}$, and hence the resulting graph H' is a p-extension of G. After that, for each $v \in V$, after deleting all edges between s and v, we check whether the resulting graph H'' satisfies (4.1) or not. If not, we add $-\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}}\{d_{H''}(X) - R(X)\}$ edges between s and v in H''. Thus, a critical p-extension of G can be found by computing $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}}\{d_H(X) - R(X)\}$ for some H at most n times.

(B) Given a p-extension H of G, we can check how many pairs of $\{(s,u),(s,v)\}$ can be split as follows. This can be done by checking whether the resulting graph H' satisfies (4.1) or not after splitting $\min\{d_H(s,u), d_H(s,v)\}$ pairs $\{(s,u),(s,v)\}$. If (4.1) is violated, we have only to hook up $\lceil -\frac{1}{2}\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}}$ $\{d_{H'}(X) - R(X)\}\rceil$ pairs in H'.

6 Concluding Remarks

In this paper, given a graph $G = (V, E)$ and a monotone function $r : 2^V \to Z^+$, we considered the problem of asking to augment G by adding a smallest number of new edges F such that the resulting graph $G + F$ satisfies $d_{G+F}(X) \geq r(X)$ for every $\emptyset \neq X \subset V$. We have shown that the problem can be solved in $O(n^4(m + n\log n + q))$ time under the assumption that $r(X) \geq 2$ holds for every $X \subseteq V$ whenever $r(X) > 0$. It is a future work to consider RECAP with a more general R, such as one including both of LECAP and NAECAP.

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References

[1] A. A. Benczúr, A. Frank, Covering symmetric supermodular functions by graphs, Mathematical Programming 84 (1999) 483–503.

[2] A. Frank, Augmenting graphs to meet edge-connectivity requirements, SIAM Journal on Discrete Mathematics 5 (1) (1992) 25–53.

[3] A. Frank, Connectivity augmentation problems in network design, in: J. Birge, K. Murty (eds.), Mathematical Programming: State of the Art 1994, Ann Arbor, MI, The University of Michigan, 1994, pp. 34–63.

[4] H. N. Gabow, Efficient splitting off algorithms for graphs, in: Proceedings of the 26th ACM Symposium on the Theory of Computing, 1994, pp. 696–705.

[5] M. Grötschel, C. L. Monma, M. Stoer, Design of survivable networks, in: Network Models, vol. 7 of Handbook in Operations Research and Management Science, North-Holland, Amsterdam, 1995, pp. 617–672.

[6] T. Ishii, M. Hagiwara, Minimum augmentation of local edge-connectivity between vertices and vertex subsets in undirected graphs, Discrete Applied Mathematics 154 (2006) 2307–2329.

[7] Z. Király, B. Cosh, B. Jackson, Local connectivity augmentation in hypergraphs is NP-complete, manuscript (1999).

[8] H. Miwa, H. Ito, NA-edge-connectivity augmentation problem by adding edges, Journal of Operations Research Society of Japan 47 (4) (2004) 224–243.

[9] H. Nagamochi, T. Ibaraki, Graph connectivity and its augmentation: applications of MA orderings, Discrete Applied Mathematics 123 (2002) 447–472.

[10] H. Nagamochi, Graph algorithms for network connectivity problems, Journal of Operations Research Society of Japan 47 (4) (2004) 199–223.

[11] H. Nagamochi, Minimum degree orderings, in: Proceedings of the 18th International Symposium on Algorithms and Computation, 2007, pp. 17–28.

[12] Z. Nutov, Approximating connectivity augmentation problems, in: Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms, 2005, pp. 176–185.

[13] T. Watanabe, A. Nakamura, Edge-connectivity augmentation problems, Journal of Computer System Sciences 35 (1987) 96–144.